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Abstract

In this work, we consider a random access Internet of Things IoT wireless network assisted by two aggregators collecting information from two disjoint groups of sensors. The nodes and the aggregators are transmitting in a random access manner under slotted time, the aggregators perform network-level cooperation for the data collection. The aggregators are equipped with queues to store data packets that are transmitted by the network nodes and relaying them to the destination node. We characterize the throughput performance of the IoT network and we obtain the stability conditions for the queues at the aggregators. We apply the theory of boundary value problems to analyze the delay performance. Our results show that the presence of the aggregators provides significant gains in the IoT network performance, in addition, we provide useful insights regarding the scalability of the IoT network.

Index Terms

Boundary Value Problems, Delay, IoT, Queueing, Random Access, Stability, Throughput.

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I. INTRODUCTION

The Internet of Things (IoT) is one of the most attractive concepts in the area of information and communication technology. IoT is expected to play an important role in our daily life by supporting massive connectivity with seamless service. It involves the interconnection of many, and possibly heterogeneous objects through the Internet using different communication technologies. The objects are equipped with communications capabilities and can vary from sensors, smart objects, etc. [2]–[4].

The total number of IoT connections is expected to grow tremendously the next years. The application of random access protocols in the IoT networks can potentially mitigate the co-channel interference caused by massive amount of IoT devices with low signaling overhead [5]. To support the massive connectivity in future IoT networks, practical techniques are required to collect data from a large set of devices and the traditional orthogonal multi-access schemes are not efficient. The works in [6]–[13] have considered data aggregation under different scenarios and have evaluated the benefits of such technique. The work in [14] proposes a cloud-assisted priority-based channel access and data aggregation scheme for irregularly deployed sensor nodes to minimize the network latency and to enhance the system reliability of IoT networks. The work in [15] considers an uplink IoT network where a cellular user operating as aggregator can assist IoT devices. IoT can leverage the unlicensed spectrum for machine-to-machine communications. The problem of spectrum allocation and device association in uplink two-hop IoT networks is studied in [16].

Network-level cooperation introduced in [17] and [18] can be an effective alternative method for data aggregation. It is plain relaying without any physical layer considerations and it has been shown to provide large gain in terms of throughput and delay performance. Recently, several works have investigated relaying at the network level [19]–[29]. The deployment of aggregators under network-level cooperation can improve the throughput and delay in IoT networks [30].
However, due to the queueing delay, the stability conditions at the aggregator queues need to be considered.

In this work, we consider a random access IoT network with two aggregators which can receive and forward data in form of packets from the network nodes. The nodes transmit in a random access manner, which is a common assumption in IoT technologies such as LoRa [31]. The aggregators use network-level cooperation and their receiving and transmitting modules are operating in different frequency bands, thus, they have out-of-band full duplex capabilities. Furthermore, we assume multi-packet reception (MPR) capabilities at the receivers. MPR is suitable to capture the SINR (Signal-to-Interference-plus-Noise-Ratio) based model and is more realistic to model the wireless transmission. Similar assumptions to our work can be found in [6], [10], [30].

A. Contributions

The contributions of our work can be summarized as follows. Our primary goal is to provide support for data collection in IoT networks by applying network-level cooperation. We provide the throughput analysis of an IoT network consisting of sensors that are assisted by two aggregators, from which we can gain insights on the scalability of the considered network. In addition, we study the stability conditions for the queues at the aggregators, which guarantee finite queueing delay. The main difficulty for characterizing the stability conditions lies on the interaction of the queues, which arises when the service rate of a queue depends on the state of the other. The technique of stochastic dominance is utilized to bypass this difficulty and characterize necessary and sufficient stability conditions for the queues at the aggregators.

Furthermore, we study the average delay of the packets possibly received and forwarded by the aggregators. Our system is modeled as a two-dimensional discrete time Markov chain, and we show that the generating function of the stationary joint queue length distribution can be obtained by solving a fundamental functional equation with the theory of boundary value
problems [32], [33]. The theory of boundary value problems was initially applied in queueing applications in [34]. There, the authors were strongly motivated by the general results for two-dimensional random walks that were obtained in [35]. A complete methodological approach to cope with queueing applications is established in [32], [33]. For the symmetric general MPR case, we obtain a lower and an upper bound for the average delay. In addition, we characterize in a closed form expression the gap between the lower and the upper bound. These bounds as it is also seen in the numerical results appear to be tight. For the model with capture effect, i.e., a subclass of MPR model, we provide the explicit expression for the average delay. The analysis in this work can act as a framework for other research directions that involve multiple aggregators with interacting queues.

The remainder of the paper is organized as follows: Section II describes the system model and in Section III we present the analysis related to the network-wide throughput. In Sections IV and V, we provide the analysis for the average delay per packet. The numerical evaluation of the theoretical results is presented in Section VI, and Section VII concludes the paper.

II. SYSTEM MODEL

A. Network Layer Model

We consider a wireless network consisting of IoT nodes/sensors/objects, which intend to communicate to a common destination/sink $D$, and two aggregators, denoted by $R_1$ and $R_2$, which can help aggregating and relaying messages from the IoT nodes to $D$. The network model is depicted in Fig. 1. The nodes are located in two non-overlapping regions. There are in total $M_1$ and $M_2$ nodes within the service range of the aggregators $R_1$ and $R_2$, respectively. Note that the transmissions in the first region cannot be overheard by the nodes in the second region and vice versa. This can be done by carefully planning for the placement of the aggregators in order to increase the coverage area without interfering with each other. *In the following, we will use the terms nodes, sensors, and objects interchangeably.*
The sensors intend to transmit packets to the destination node $D$ and they are assumed to be saturated, i.e., they always have packets to transmit. In case the transmission from a sensor to $D$ fails, the aggregator can help relaying the message to $D$ and the aggregators do not generate their own traffic. We consider using network-level cooperation at the aggregators [17], [18], which means that the aggregators are cooperating as relays in a decode-and-forward manner. The packets are assumed to have equal length and the time is divided into slots, which corresponds to the transmission time of a packet. We assume that the sensors access the wireless channel randomly without any coordination among them. We consider a full multi-packet reception (MPR) channel model, which allows the receivers to successfully receive more than one packet when there are multiple transmissions in the same slot [36]. As the result, the sink node $D$ can receive information simultaneously from the sensors and the aggregators. Note that when all nodes are transmitting, we can have in total up to $M_1 + M_2$ interfering devices at the sink. This assumption is common in the literature [6], [11], [30].

We assume that different frequency bands are allocated for the transmissions from the sensors and the aggregators, thus, there is no interference between them. On the other hand, the transmission of a node creates interference to the other nodes of the same kind, i.e., there is interference between the sensors, and between the aggregators. The transmitting and receiving units of the aggregators are operating in different channels/frequency bands to avoid self-interference, which can be considered as out-of-band full duplex mode [37]. The aggregators are equipped with queues that store possible packets from the sensors that failed to reach the destination. The queues are assumed to have infinite capacity, thus there is no packet dropping.\(^1\) This is a common assumption in the literature, in the IoT context [30].

\(^1\)In practice, the buffers have limited size, which is usually quite large. Our analysis based on the infinite buffer size assumption can capture this scenario with minor modifications and it is more general. Furthermore, our analysis can provide design guidelines for selecting the appropriate queue size in order to achieve specific performance requirements regarding throughput and delay. The arrival and service rates for the queues are defined in Section III.
Fig. 1. The IoT network considered in this work. The nodes are assisted by two aggregators that are equipped with queues, and they are operating in an out-of-band full duplex mode.

B. Physical Layer Model

At the beginning of a timeslot, sensor nodes that belong to the coverage area of $R_i$, attempt to transmit with probability $t_i$, $i = 1, 2$. The aggregator $R_i$ will attempt to transmit a packet with probability $\alpha_i$, if it has a non-empty queue. Note that we assume that all the sensors in the same area transmit with the same probability. Our analysis can be easily extended to handle the general case where each node has different transmit probabilities however, the expressions will be cumbersome, and it will be difficult to extract meaningful insights.

We assume that a packet transmitted by sensor $s$ in the first coverage area is successfully received by its aggregator $R_1$ if and only if $\text{SINR}(s, R_1) \geq \gamma$, where $\gamma$ is the SINR threshold. The wireless channel is subject to fading; let $P_{tx}(s)$ be the transmit power at sensor $s$ and $r(s, R_1)$ be the distance between the sensor and the aggregator. The received power at $R_1$, when $s$ transmits is $P_{rx}(s, R_1) = A(s, R_1)h(s, R_1)$ where $A(s, R_1)$ is a random variable representing small-scale fading. Under Rayleigh fading assumption, $A(s, R_1)$ is exponentially distributed. The received power factor $h(s, R_1)$ is given by $h(s, R_1) = P_{tx}(s)(r(s, R_1))^{-\theta}$ where $\theta$ is the path loss exponent. Then, the success probability between a sensor in the first coverage area and its
aggregator is denoted by $P_{i}^{R_1}$, when there are $i$ sensors from the same area transmitting in a
timeslot, and the expression is given by [38], [39]

$$P_{i}^{R_1} = \exp \left( -\frac{\gamma \eta_{R_1}}{v(s, R_1) h(s, R_1)} \right) \prod_{k \in \mathcal{T} \setminus \{s\}} \left( 1 + \frac{\gamma v(k, R_1) h(k, R_1)}{v(s, R_1) h(s, R_1)} \right)^{-1},$$  \hspace{1cm} (1)

where $\mathcal{T}$ is the set of transmitting sensors in the first area and $|\mathcal{T}| = i$; $v(s, R_1)$ is the parameter of the Rayleigh fading random variable, $\eta_{R_1}$ is the receiver noise power at $R_1$.

Note that transmitting sensors from the other coverage area do not create interference at the aggregator. However, the concurrently transmitting sensors from both areas interfere with each other at the destination $D$. Denote by $P_{i,j}^{1D}$ the success probability to the sink from a sensor in coverage area 1 when there are $i$ active transmitters from area 1 and $j$ active transmitters from area 2. Similarly we can define $P_{i,j}^{2D}$.

A packet transmission from a sensor in the first area fails to reach the destination with probability $P_{i,j}^{1D} = 1 - P_{i,j}^{1D}$, when there are $i$ active sensors in the first area and $j$ active sensors in the second area. In this case, that packet will be stored in the queue of aggregator $R_1$ with probability $P_{i}^{R_1}$. Otherwise, with probability $\bar{P}_{i}^{R_1}$, the aggregator fails to decode that packet and it has to be re-transmitted by the sensor in a future time slot.$^2$

Recall that if there are stored packets in the queues of the aggregator $R_i$, $i = 1, 2$, then $R_i$ transmits a packet with probability $\alpha_i$. If only one aggregator $R_i$ is active, then the packet will be successfully transmitted to $D$ with probability $p_{R_{i},(R_1)}^{D}$. If both aggregators transmit simultaneously, then with probability $p_{R_{i}/R_1,R_2}^{D}$ the packet from $R_i$ is successfully received by node $D$. If a transmitted packet from an aggregator fails to reach the $D$, that packet remains in the queue and will be retransmitted in a later time slot.

$^2$In this work we assume that after a transmission the intended receiver sends an instantaneous and error free ACK/NACK.
III. Throughput and Stability Analysis

In this section, we characterize the network throughput performance and provide the stability conditions for the queues at the aggregators.

A. Throughput Analysis

The throughput per node consists of the direct throughput from each sensor to the destination and the throughput contributed by the aggregator. Recall that the devices that are in coverage from the first aggregator cannot cause interference at the receiver of the second aggregator. Moreover, the devices that are covered by aggregator $R_i$, $i = 1, 2$, are transmitting with probability $t_i$. The direct throughput from a sensor in the first coverage area to the sink is given by

$$T_{1,D} = \sum_{i=0}^{M_1-1} \sum_{j=0}^{M_2} \binom{M_1 - 1}{i} \binom{M_2}{j} t_1^{i+1} t_1^{M_1-i-1} t_2^{j+1} t_2^{M_2-j} P_{i+1,j}^D. \quad (2)$$

Note that the direct throughput in this setup is equivalent to the throughput in the network without aggregators. The contributed throughput from a sensor to the aggregator $R_1$ is given by

$$T_{1,R_1} = \sum_{i=0}^{M_1-1} \sum_{j=0}^{M_2} \binom{M_1 - 1}{i} \binom{M_2}{j} t_1^{i+1} t_1^{M_1-i-1} t_2^{j+1} t_2^{M_2-j} T_{i+1,j}^D P_{i+1}. \quad (3)$$

In order to write the previous expressions, we need to consider all the possible combinations for the number of active nodes in both coverage areas. The number of active nodes affects the success probabilities, since the nodes interfere at the destination. The total throughput seen by a sensor in the first coverage area is $T_1 = T_{1,D} + T_{1,R_1}$. Similarly, we obtain the throughput seen by a sensor located in the second coverage area which is assisted by the second aggregator $R_2$.

Remark 1. The term $T_{1,D}$ can also be interpreted as the probability that a transmitted packet from a sensor in the first coverage area reaches the destination directly. Furthermore, $T_{1,R_1}$ is the probability of unsuccessful transmission from a sensor to the destination while the packet is received by the aggregator. The percentage of a sensor’s traffic that is being relayed is $\frac{T_{1,R_1}}{T_1}$. 

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In addition, we need to characterize the average arrival rates at the aggregators, denoted by \( \lambda_1 \) and \( \lambda_2 \). Since we assume full MPR capability at the receivers, the \( i \)-th aggregator can receive up to \( M_i \) packets in a timeslot. We define \( l_{k,i} \) as the probability that \( k \) packets will arrive in a timeslot at the \( i \)-th aggregator. The average arrival rate at the \( i \)-th aggregator is given by

\[
\lambda_i = \sum_{k=1}^{M_i} kl_{k,i}, \quad i = 1, 2.
\]  

The probability \( l_{k,1} \) where \( 1 \leq k \leq N_1 \) is given by

\[
l_{k,1} = \sum_{s=k}^{M_1} \sum_{m=0}^{M_2} \left( \binom{M_1}{s} \binom{M_2}{m} \right) t_1^{s} t_2^{M_1-s} t_2^{M_2-m} \left( P_{s,m}^{1D} P_{s}^{R_1} \right)^{k} \left( P_{s,m}^{1D} + P_{s,m}^{1D} P_{s}^{R_1} \right)^{s-k}.
\]  

Similarly we can obtain \( l_{k,2} \). The network-wide throughput is \( T = M_1 T_1 + M_2 T_2 \). The previous expressions for the throughput are valid when the queues at the aggregators are stable.

**Remark 2.** Previously we considered the case where the queues are stable, but in order to obtain the throughput when the queues are not stable we need to replace the sum of expressions \( T_{i,R_i} \) with the service rate of the aggregator, \( \mu_i \). In this case, the network-wide throughput will be given by \( T = M_1 T_{1,D} + M_2 T_{2,D} + \mu_1 + \mu_2 \), \( \mu_i \), \( i = 1, 2 \), is given by (6) in the next subsection.

**B. Stability Analysis at the Aggregators**

The average service rate for the aggregator \( i \) is given by

\[
\mu_i = Pr(N_j \neq 0) \left[ \alpha_i \bar{\alpha}_j p^{D}_{R_i \rightarrow (R_i)} + \alpha_i \alpha_j p^{D}_{R_i / R_i, R_j} \right] + Pr(N_j = 0) \alpha_i p^{D}_{R_i \rightarrow (R_i)}, \quad j = i \mod{2} + 1,
\]  

where \( N_j \) is the queue size at queue \( j \). The notation for the success probabilities used in (6) is the one introduced in Section II. The theorem below, provides the stability conditions for the considered setup.
TABLE I
BASIC NOTATION

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_i$</td>
<td>The queue size at aggregator $i$</td>
</tr>
<tr>
<td>$\lambda_i$</td>
<td>The average arrival rate at aggregator $i$</td>
</tr>
<tr>
<td>$t_i$</td>
<td>Transmission probability of a device in $i$-th coverage area</td>
</tr>
<tr>
<td>$\alpha_i$</td>
<td>Transmission probability of aggregator $i$</td>
</tr>
<tr>
<td>$P_{kD}^{i,j}$</td>
<td>Success probability from a node in coverage area $k$ to $D$, when $i$ nodes are active from area 1 and $j$ from area 2</td>
</tr>
<tr>
<td>$P_{R_k}^i$</td>
<td>Success probability from a node in coverage area $k$ to $R_k$, when $i$ nodes are active</td>
</tr>
<tr>
<td>$P_{R_i,{R}}^{D}$</td>
<td>Success probability of aggregator $i$, when only $R_i$ is active</td>
</tr>
<tr>
<td>$P_{R_i/R_i,R_j}^{D}$</td>
<td>Success probability of aggregator $i$, when both aggregators are active</td>
</tr>
</tbody>
</table>

**Theorem III.1.** The stability conditions for the queues at the aggregators for fixed transmission probabilities $\alpha_i, i = 1, 2$, are described by the region $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where $\mathcal{R}_i$ is given by

$$\mathcal{R}_i = \left\{ (\lambda_1, \lambda_2) : \lambda_i \leq \alpha_i p_{R_i,\{R\}}^D - \frac{\alpha_i \alpha_j (p_{R_i,\{R\}}^D - p_{R_i/R_i,R_j}^D)}{\alpha_j \alpha_i p_{R_i,\{R\}}^D + \alpha_i \alpha_j p_{R_j/R_i,R_j}^D} \lambda_j, \lambda_j \leq \alpha_j \alpha_i p_{R_j,\{R\}}^D + \alpha_i \alpha_j p_{R_j/R_i,R_j}^D \right\}, j = i \mod 2 + 1. \tag{7}$$

**Proof:** The proof is given in Appendix A.

A detailed overview of the system parameters can be found in Table I.

IV. DELAY ANALYSIS: THE GENERAL CASE

In this section, we focus on the investigation of the delay performance. Due to the high level of interaction among aggregators and among sensors-aggregators, and in order to enhance the readability of our paper we assume in the following that within the service region of each aggregator there is only one sensor. The following analysis can be extended to the case of an arbitrary number of sensors at the service region of each aggregator. However, in this case, since we may have more than one packet arrivals per slot at each aggregator, the resulting two-dimensional random walk is no more of nearest neighbour. Thus, the delay analysis must be
adapted following the lines in [33].

A. The fundamental functional equation and preliminary results

Let $N_{k,n}$ be the number of packets in the buffer of aggregator $R_k$, $k = 1, 2$, at the beginning of the $n$th slot. Then, $Y_n = (N_{1,n}, N_{2,n})$ is a discrete time Markov chain with state space $E = \{(i, j) : i, j = 0, 1, 2, \ldots\}$. The queues of both aggregators evolve as follows:

$$N_{k,n+1} = [N_{k,n} + F_{k,n}]^+, \ k = 1, 2,$$

(8)

where $F_{k,n}$ is the difference of the number of departed from the packets that enter the buffer of the $k$th aggregator at the beginning of slot $n$ ($F_{k,n}$ equals $0$ or $\pm 1$), and $[x]^+ := \max(0, x)$.

Before proceeding with the analysis, we will slightly modify the notation for the success probabilities presented in Section II in order to be more convenient for the delay analysis. If two sensors transmit a packet simultaneously, $p_{D, \{1,2\}}^i$ represents the probability that the packet from sensor $i$ is successfully received by node $D$ and the packet from sensor $j = i \mod 2 + 1$ failed to be received by $D$. $p_{1,2,\{1,2\}}^D$ represents the probability that the packets from both nodes are successfully received by $D$. Then we have $p_{\{1,2\}}^D = 1 - p_{1,2,\{1,2\}}^D - p_{1,\{1,2\}}^D - p_{2,\{1,2\}}^D$, which denotes the probability that both packets fail to be received by the node $D$. If both aggregators transmit simultaneously, then with probability $p_{R_k,\{R_1,R_2\}}^D$ the packet from $R_k$ is successfully received by node $D$, with probability $p_{R_1,R_2,\{R_1,R_2\}}^D$ the packets from both aggregators are successfully received by $D$, while with probability $\bar{p}_{\{R_1,R_2\}}^D = 1 - p_{R_1,R_2,\{R_1,R_2\}}^D - p_{R_1,\{R_1,R_2\}}^D - p_{R_2,\{R_1,R_2\}}^D$, both of them failed to be received by $D$.

Let $H(x, y)$ be the generating function of the joint stationary queue process, $H(x, y) = \lim_{n \to \infty} E(x^{N_{1,n}}y^{N_{2,n}})$, $|x| \leq 1, |y| \leq 1$. Note that the derivation of $H(x, y)$ is of paramount importance to investigate the queueing delay at aggregators. Using (8) we can write down the equilibrium equations, and by employing the generating function approach we come up with the
following functional equation,

\[ R(x, y)H(x, y) = A(x, y)H(x, 0) + B(x, y)H(0, y) + C(x, y)H(0, 0), \]  
(9)

where, \( R(x, y) := 1 - L(x, y)[1 - \alpha \widehat{\alpha}_2(1 - \frac{1}{x}) - \widehat{\alpha}_1 \alpha_2(1 - \frac{1}{y}) - \alpha_1 \alpha_2 p_{R_1, R_2, \{R_1, R_2\}}(1 - \frac{1}{xy})] \), is the kernel of functional equation (9) and

\[ A(x, y) = L(x, y)[d_1(1 - \frac{1}{x}) + \alpha_2 \widehat{\alpha}_1(1 - \frac{1}{y}) + \alpha_1 \alpha_2 p_{R_1, R_2, \{R_1, R_2\}}(1 - \frac{1}{xy})], \]
\[ B(x, y) = L(x, y)[d_2(1 - \frac{1}{y}) + \alpha_1 \alpha_2(1 - \frac{1}{x}) + a_1 a_2 p_{R_1, R_2, \{R_1, R_2\}}(1 - \frac{1}{xy})], \]
\[ C(x, y) = L(x, y)[d_1(\frac{1}{x} - 1) + d_2(\frac{1}{y} - 1) + \alpha_1 \alpha_2 p_{R_1, R_2, \{R_1, R_2\}}(\frac{1}{xy} - 1)], \]
\[ L(x, y) = 1 - (1 - x)t_1[\widehat{t}_2 p_{1,\{1\}}^D p_{R_1,\{1\}}^R] + t_2[\widehat{p}_{1,\{1\}}^D + p_{2,\{1,2\}}^D p_{1,\{1,2\}}^R] + (1 - y)t_2[\widehat{t}_1 p_{2,\{2\}}^D p_{2,\{2\}}^R]
+ t_1[p_{1,\{1\}}^D + p_{1,\{1\}}^2] + (1 - x)(1 - y)t_1 t_2 p_{1,\{1\}}^D p_{1,\{1\}}^R p_{2,\{1,2\}}^R, \]

\[ \widehat{\alpha}_k = \widehat{\alpha}_k p_{R_k,\{R_k\}} + \alpha_k p_{R_k,\{R_k\}}, k = 1, 2, \]
\[ d_1 = \alpha_1(\widehat{\alpha}_2 - p_{R_1,\{R_1\}}; d_2 = \alpha_2(\widehat{\alpha}_1 - p_{R_2,\{R_2\}}). \]

Our main focus in the rest of this section is to solve (9) and obtain \( H(x, y) \). Clearly, for every fixed \( y \) with \( |y| \leq 1 \), \( H(x, y) \) it is regular in \( x \) for \( |x| < 1 \), and continuous in \( x \) for \( |x| \leq 1 \); similar observation hold for the variable \( y \).

Some interesting relations are directly obtained using (9). In particular, by setting in (9) \( y = 1 \), dividing with \( x - 1 \), and taking the limit \( x \to 1 \), by using the L'Hospital rule, and vice-versa we obtain the following conservation of flow relations:

\[ \lambda_1 = \alpha_1 \widehat{\alpha}_2[1 - H(1, 0) - H(0, 1) + H(0, 0)] + \alpha_1 p_{R_1,\{R_1\}}^D[H(1, 0) - H(0, 0)], \]
\[ \lambda_2 = \alpha_2 \widehat{\alpha}_1[1 - H(1, 0) - H(0, 1) + H(0, 0)] + \alpha_2 p_{R_2,\{R_2\}}^D[H(0, 1) - H(0, 0)], \]

(10)

where, \( \widehat{\alpha}_k := \widehat{\alpha}_k + \alpha_k p_{R_k,\{R_k\}}, k = 1, 2, \)

\[ \lambda_1 := t_1 \widehat{t}_2 p_{1,\{1\}}^D p_{R_1,\{1\}}^R + t_1 t_2(p_{1,\{1\}}^D + p_{2,\{1,2\}}^D)p_{1,\{1,2\}}^R, \]
\[ \lambda_2 := t_2 \widehat{t}_1 p_{2,\{2\}}^D p_{2,\{2\}}^R + t_1 t_2(p_{1,\{1\}}^D + p_{2,\{1,2\}}^D)p_{2,\{1,2\}}^R. \]

Note that the previous equations are the same with the ones obtained in the previous section, but here we use the more convenient notation for the delay analysis regarding the success
probabilities. In order to facilitate presentation we present the case of two nodes, but clearly the analysis holds for the general case of $N$ nodes, just by replacing the right parts of $\lambda_1$ and $\lambda_2$.

The expressions in (10), equate the flow of jobs into an aggregator, with the flow of jobs out of the aggregator. Looking at (10) it is seen that the analysis is distinguished in two cases:

1) \[ \frac{\alpha_1 \tilde{\alpha}_2}{\alpha_1 P_{R_1}(R_1)} + \frac{\alpha_2 \tilde{\alpha}_1}{\alpha_2 P_{R_2}(R_2)} = 1. \]
Then, \[ H(0, 0) = 1 - \frac{\lambda_1}{\alpha_1 P_{R_1}(R_1)} - \frac{\lambda_2}{\alpha_2 P_{R_2}(R_2)} = 1 - \rho. \]

2) \[ \frac{\alpha_1 \tilde{\alpha}_2}{\alpha_1 P_{R_1}(R_1)} + \frac{\alpha_2 \tilde{\alpha}_1}{\alpha_2 P_{R_2}(R_2)} \neq 1. \]
Then, (10) yields \[ H(1, 0) = \frac{\alpha_2 \tilde{\alpha}_1 (\lambda_1 - \alpha_1 P_{R_1}(R_1)) - \tilde{d}_1 (\lambda_2 + H(0, 0) \alpha_2 P_{R_2}(R_2))}{d_1 d_2 - \alpha_1 \alpha_2 \tilde{\alpha}_1 \tilde{\alpha}_2}, \]
and \[ H(0, 1) = \frac{\alpha_2 \tilde{\alpha}_1 (\lambda_2 - \alpha_2 P_{R_2}(R_2)) - \tilde{d}_2 (\lambda_1 + H(0, 0) \alpha_1 P_{R_1}(R_1))}{d_1 d_2 - \alpha_1 \alpha_2 \tilde{\alpha}_1 \tilde{\alpha}_2}, \]
where \[ \tilde{d}_1 = \alpha_1 (\tilde{\alpha}_2 - \rho_{R_1}(R_1)), \]
and \[ \tilde{d}_2 = \alpha_2 (\tilde{\alpha}_1 - \rho_{R_2}(R_2)). \]

The solution of the fundamental functional equation (9), which provides $H(x, y)$, is the key element to perform the delay analysis. However, (9) contains additional unknown elements: the boundary functions $H(x, 0)$, $H(0, y)$ and the term $H(0, 0)$. Our aim in the following, is to obtain these unknowns, and substitute back in (9) to obtain $H(x, y)$. Our methodological tool to accomplish this goal is the theory of boundary value problems [32], [33]. Such an approach is quite technical and briefly summarized below:

Step 1 We provide a thorough investigation of the kernel equation $R(x, y) = 0$, by identifying its roots as well as their properties, which are crucial for the next steps (see IV-B).

Step 2 The functional equation (9), is then used to show that $H(x, 0)$, $H(0, y)$ satisfy certain boundary value problems of Riemann-Hilbert type [32], with boundary conditions on closed curves. Information on these curves are obtained from Step 1 (see subsection IV-B, and Lemmas IV.1, IV.2, IV.3; for the proofs of Lemmas IV.1, IV.3 see Appendices B, C respectively). Clearly, the functions $H(x, 0)$, $H(0, y)$ are analytic (by definition) inside the unit disc, but they might have poles in the region bounded by the unit disc and these closed curves. We then analytically continue $H(x, 0)$, $H(0, y)$ in the whole interiors of the curves; see [32, Chapter 3]. Then, the boundary conditions on these curves are given in (22) (see subsection IV-C1), (24) (see subsection IV-C2), respectively.
Step 3 Then, we transform (through conformal mappings [40]) these problems from the closed curved defined in Step 1, into boundary value problems of Riemann-Hilbert type on the unit disc; see (17). This conversion is motivated by the fact that the latter problems are more usual and by far more treated in the literature [33].

Step 4 Finally, we solve these new problems by providing an integral expression of the unknown boundary functions; see (11) and (12) in subsections IV-C1, and IV-C2, respectively. Having obtained the unknown boundary functions, we substitute in (9) to obtain $H(x, y)$.

B. Kernel analysis

The kernel $R(x, y)$ plays a crucial role in the following analysis and hereon we provide some important properties. For convenience, assume in the following that $p_{R_1, R_2, \{R_1, R_2\}} = 0$. It is readily seen that $R(x, y) = a(x)y^2 + b(x)y + c(x) = \tilde{a}(y)x^2 + \tilde{b}(y)x + \tilde{c}(y)$, where, for $L = t_1t_2p_{1,2}^{R_1}p_{1,2}^{R_2}$,

$$
\tilde{a}(y) = L(1 - \alpha_1\hat{\alpha}_2 - \alpha_2\hat{\alpha}_1)y^2 + y[\lambda_1(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 - 1) + L(\alpha_1\hat{\alpha}_2 + 2\alpha_2\hat{\alpha}_1 - 1)] - \alpha_2\hat{\alpha}_1(\lambda_1 + L),
$$

$$
\tilde{b}(y) = y^2[\lambda_2(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 - 1) + L(2\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 - 1)] + y[\lambda_1(1 - 2\alpha_1\hat{\alpha}_2 - \alpha_2\hat{\alpha}_1) + \lambda_2(1 - 2\alpha_1\hat{\alpha}_2 - \alpha_1\hat{\alpha}_2)] + L(1 - 2(\alpha_1\hat{\alpha}_2 + \alpha_1\hat{\alpha}_2)) + \alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 + \hat{\alpha}_1\alpha_2(\lambda_1 + \lambda_2 + L - 1),
$$

$$
\tilde{c}(y) = \alpha_1\hat{\alpha}_2y[\lambda_1 - 1 + (\lambda_2 + L)(1 - y)],
$$

$$
a(x) = L(1 - \alpha_1\hat{\alpha}_2 - \alpha_2\hat{\alpha}_1)x^2 + x[\lambda_2(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 - 1) + L(2\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 - 1)] - \alpha_1\hat{\alpha}_2(\lambda_2 + L),
$$

$$
b(x) = x^2[\lambda_1(\alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 - 1) + L(2\alpha_1\hat{\alpha}_2 + \alpha_1\hat{\alpha}_2 - 1)] + x[\lambda_1(1 - 2\alpha_1\hat{\alpha}_2 - \alpha_2\hat{\alpha}_1) + \lambda_2(1 - 2\alpha_1\hat{\alpha}_2 - \alpha_1\hat{\alpha}_2)] + L(1 - 2(\alpha_1\hat{\alpha}_2 + \alpha_1\hat{\alpha}_2)) + \alpha_1\hat{\alpha}_2 + \alpha_2\hat{\alpha}_1 + \hat{\alpha}_2\alpha_1(\lambda_1 + \lambda_2 + L - 1),
$$

$$
c(x) = \alpha_2\hat{\alpha}_1x[\lambda_2 - 1 + (\lambda_1 + L)(1 - x)].
$$
The roots of $R(x,y) = 0$ are $X_\pm(y) = \frac{-b(y)\pm\sqrt{D_y(y)}}{2\alpha(y)}$, $Y_\pm(x) = \frac{-b(x)\pm\sqrt{D_x(x)}}{2\alpha(x)}$, where $D_y(y) = \hat{b}(y)^2 - 4\alpha(y)\bar{c}(y)$, $D_x(x) = b(x)^2 - 4\alpha(x)c(x)$.

**Lemma IV.1.** For $|y| = 1$, $y \neq 1$, $R(x,y) = 0$ has exactly one root $x = X_0(y)$ such that $|X_0(y)| < 1$. When $\lambda_1 < \alpha_1\omega_2$, $X_0(1) = 1$. Similarly, $R(x,y) = 0$ has exactly one root $y = Y_0(x)$, such that $|Y_0(x)| \leq 1$, for $|x| = 1$.

*Proof:* See Appendix B. □

The lemma below provides information about the location of the branch points of the two-valued functions $Y(x)$, $X(y)$, its proof is based on algebraic arguments, thus it is omitted.

**Lemma IV.2.** The algebraic function $Y(x)$, defined by $R(x,Y(x)) = 0$, has four real branch points $0 < x_1 < x_2 < x_3 < x_4$. Moreover, $D_x(x) < 0$, $x \in (x_1, x_2) \cup (x_3, x_4)$ and $D_x(x) > 0$, $x \in (-\infty, x_1) \cup (x_2, x_3) \cup (x_4, \infty)$. Similarly, $X(y)$, is defined by $R(X(y), y) = 0$, it has four real branch points $0 < y_1 < y_2 < y_3 < y_4$, and $D_x(y) < 0$, $y \in (y_1, y_2) \cup (y_3, y_4)$ and $D_x(y) > 0$, $y \in (-\infty, y_1) \cup (y_2, y_3) \cup (y_4, \infty)$.

Let $\tilde{C}_x = \mathbb{C}_x - ([x_1, x_2] \cup [x_3, x_4])$, $\tilde{C}_y = \mathbb{C}_y - ([y_1, y_2] \cup [y_3, y_4])$, where $\mathbb{C}_x$, $\mathbb{C}_y$ the complex planes of $x$, $y$, respectively. In $\tilde{C}_x$ (resp. $\tilde{C}_y$), denote by $Y_0(x)$ (resp. $X_0(y)$) the root of $R(x,Y(x)) = 0$ (resp. $R(X(y), y) = 0$) with the smallest modulus, and $Y_1(x)$ (resp. $X_1(y)$) the other one. Define the image contours, $L = Y_0[x_1, x_2]$, $M = X_0[y_1, y_2]$, where $[u, v]$ stands for the contour traversed from $u$ to $v$ along the upper edge of the slit $[u, v]$ and then back to $u$ along the lower edge of the slit. The following lemma provides exact characterization for the contours $L$, $M$ respectively.

**Lemma IV.3.** The algebraic function $X(y)$, $y \in [y_1, y_2]$ lies on a closed contour $M$, which is symmetric with respect to the real line and defined by $|x|^2 = m(Re(x))$, $m(\delta) = \frac{\tilde{c}(\zeta(\delta))}{a(\zeta(\delta))}$, $|x|^2 \leq $
\[
\frac{\tilde{a}(y_2)}{a(y_2)^2}, \text{ where, } \zeta(\delta) = \frac{k_2(\delta) - \sqrt{k_2^2(\delta) - 4k_3(\delta)k_1(\delta)}}{2k_1(\delta)}, \text{ and }
\]

\[
k_1(\delta) := a_1\tilde{a}_2(\lambda_2 + 2L(1 - \delta)) - (\lambda_2 + L(1 - 2\delta))(1 - \tilde{a}_1a_2),
\]

\[
k_2(\delta) := 2\delta[a_1\tilde{a}_2(\lambda_1 + L) + \tilde{a}_1a_2(\lambda_1 + 2L) - \lambda_1 - L] + \lambda_1 + \lambda_2 + L + a_1\tilde{a}_2(1 - 2\lambda_1) + \tilde{a}_1a_2(1 - \lambda_1 - 2(\lambda_2 + L)),
\]

\[
k_3(\delta) := -[\lambda_2(\tilde{a}_1a_2 + a_1\tilde{a}_2) + \tilde{a}_1a_2(1 + (\lambda_1 + L)(1 + 2\delta))].
\]

Moreover, \(\beta_0 := \sqrt{\frac{\tilde{a}(y_2)}{a(y_2)}}\), \(\beta_1 := -\sqrt{\frac{\tilde{a}(y_1)}{a(y_1)}}\) are the extreme right and left points of \(M\), respectively.

Similarly, \(Y(x)\), \(x \in [x_1, x_2]\) lies on a closed contour \(L\). Its exact representation is derived as for \(M\), and further details are omitted.

Proof: See Appendix C.

C. The boundary value problems

Here, we proceed with the derivation of \(H(x, 0), H(0, y)\), which are crucial for the derivation of \(H(x, y)\). The analysis is distinguished in two cases according to the values of the parameters.

1) A Dirichlet boundary value problem: Let \(\frac{\alpha_1\tilde{a}_2}{a_1p_{R_1}(R_1)} + \frac{\alpha_2\tilde{a}_1}{a_2p_{R_2}(R_2)} = 1\). Then, \(A(x, y) = \frac{d_1}{\alpha_1\alpha_2}B(x, y) \Leftrightarrow A(x, y) = \frac{\alpha_1\tilde{a}_2}{a_1p_{R_1}(R_1)}B(x, y)\). Theorem IV.4 states the main result of this subsection.

**Theorem IV.4.** For \(\rho := \frac{\lambda_1}{\alpha_1p_{R_1}(R_1)} + \frac{\lambda_2}{\alpha_2p_{R_2}(R_2)} < 1\), \(H(x, 0)\) is obtained as the solution of a Dirichlet boundary value problem on \(M\). In particular, for \(x \in G_M\),

\[
H(x, 0) = (1 - \rho) \left\{ 1 + \frac{2\gamma(x)i}{\pi} \int_0^\pi \frac{f(e^{i\phi}) \sin(\phi) d\phi}{1 - 2\gamma(x) \cos(\phi) - \gamma(x)^2} \right\},
\]

where \(G_M\) is the interior domain bounded by the closed contour \(M\), and \(\gamma(.)\) is a conformal mapping, see Appendix D. An integral expression for \(H(0, y)\) is obtained similarly by solving another Dirichlet problem on \(L\). Substituting in (9) we uniquely obtain \(H(x, y)\).

Proof: See Appendix E.
2) A homogeneous Riemann-Hilbert boundary value problem: If \( \frac{a_1 \tilde{a}_2}{a_1^* p_{R_1}^{D} (R_1)} + \frac{a_2 \tilde{a}_1}{a_2^* p_{R_2}^{D} (R_2)} \neq 1 \), consider the following transformation:

\[
G(x) := H(x, 0) + \frac{\alpha_1 p_{R_1}^{D} (R_1) d_2 H(0, 0)}{d_1 d_2 - \alpha_1 \tilde{a}_2 \alpha_2 \tilde{a}_1}, \quad L(y) := H(0, y) + \frac{\alpha_2 p_{R_2}^{D} (R_2) d_1 H(0, 0)}{d_1 d_2 - \alpha_1 \tilde{a}_2 \alpha_2 \tilde{a}_1}.
\]

Theorem IV.5 states the main result of this subsection.

**Theorem IV.5.** Under stability conditions stated in Theorem III.1, for \( x \in G_M \),

\[
H(x, 0) = \frac{\lambda_1 d_2 + \alpha_1 \tilde{a}_2 (t_1 t_2 \alpha_2 \alpha_1 - d_1 d_2)}{(x - \bar{x})^r} \exp \left[ \frac{\gamma(x) - \gamma(1)}{2 \pi} \int_{|t| = 1} \log J(t) \frac{dt}{(t - \gamma(x))(t - \gamma(1))} \right] + \frac{\alpha_1 p_{R_1}^{D} (R_1) d_2 \bar{x}^r}{(x - \bar{x})^r} \exp \left[ \frac{-\gamma(1)}{2 \pi} \int_{|t| = 1} \log J(t) \frac{dt}{t(t - \gamma(1))} \right],
\]

where \( \bar{x} \) is the only zero (if exists) of \( S_x := G_M \cap \{x : |x| > 1\} \), and \( \gamma(.) \) a conformal mapping, see Appendix D. Similarly, we obtain \( H(0, y) \) by solving another Riemann-Hilbert boundary value problem on the closed contour \( \mathcal{L} \). Then, using (9) we uniquely obtain \( H(x, y) \).

**Proof:** See Appendix F. \( \blacksquare \)

By solving another Riemann-Hilbert boundary value problem on the closed contour \( \mathcal{L} \), we can similarly obtain the other unknown boundary function \( H(0, y) \). Then, substituting back in (9), we uniquely obtain \( H(x, y) \).

**D. Performance metrics & some comments on the numerical evaluation**

Having obtained the boundary functions \( H(x, 0) \), \( H(0, y) \) in terms of the solution of the boundary value problems (see Theorems IV.4, IV.5), we can use (9) to derive the unknown \( H(x, y) \). This is the key ingredient to derive useful performance metrics such as the expected number of packets, and the expected delay experienced in aggregator \( i \), say \( E_i \), \( D_i \), respectively, where \( i = 1, 2 \).

Let \( H_i(x, y) \) the derivatives of \( H(x, y) \) with respect to \( x \) and \( y \), respectively. Then, \( E_i = H_i(1, 1) \) and using Little’s law, \( D_i = \frac{E_i}{\lambda_i} \), \( i = 1, 2 \). After, tedious but standard calculations, the
expected delay at each aggregator is

\[ D_1 = \frac{\lambda_1 + d_1 H_1(1,0)}{\lambda_1 \hat{a}_1 \hat{a}_2}, \quad D_2 = \frac{\lambda_2 + d_2 H_2(0,1)}{\lambda_2 \hat{a}_2 \hat{a}_1}. \]

(13)

Thus, the computation of the basic performance metrics, require the computation of \( H_1(1,0) \), \( H_2(0,1) \). For the case where \( \frac{a_1 \hat{a}_2}{a_1^2 P_{R_1}^1} + \frac{a_2 \hat{a}_1}{a_2^2 P_{R_2}^2} \neq 1 \), tedious calculations yield

\[ H_1(1,0) = \frac{\lambda_1 d_2 + a_1 \hat{a}_2 (t_1 t_2 a_2^2 \gamma_2^2 - \lambda_2)}{t_1 t_2 (a_1 a_2 \hat{a}_1 \hat{a}_2 - d_1 d_2)} \left[ \frac{\gamma'(1)}{2\pi i} \log \left( \frac{t}{t - \gamma(1)} \right) \right]. \]

(14)

A similar expression can be derived for \( H_2(0,1) \).

Note that the computation of the performance metrics requires the computation of the integral expression (14), which is done numerically, using the trapezoidal rule. On top of that, we further require the computation of the conformal mapping, i.e., to obtain \( \gamma(1) \), \( \gamma'(1) \). An explicit expression of the conformal mapping will substantially improve the computational complexity of the integral expressions. The task of deriving explicit expressions for the conformal mappings is quite challenging in such problems, and most of the times, we are unable to explicitly obtain them. Therefore, to proceed, we further need to numerically obtain the conformal mapping using the Newton-Raphson method and the trapezoidal rule; see Appendix D for more details. Therefore, the numerical computation of the exact conformal mappings is generally time consuming. Alternatively, by exploiting the fact that \( M, L \) are nearly circular, we can approximate them with conformal mappings that map the interior of ellipses to \( G_C \); for further details see Appendix D, and [41].

Another point of concern, refers to the position of \( \beta_0 \) given in Lemma IV.3. In particular, based on values of the system parameters, \( \beta_0 = 1 \) (resp. \( \beta_0 < 1 \), \( \beta_0 > 1 \)) implies \( 1 \in M \) (resp. \( 1 \in M \cup G_M, 1 \in G_M \)). Thus, in order to evaluate the integrals, \( \gamma(1) \) should lie on or within the unit circle, which in turn means that we have to assure that \( 1 \) lie on or within the contour \( M \). This is a common problem in queueing applications where the boundary value approach is applied. In our case, \( \beta_0 > 1 \), i.e., \( 1 \in M \cup G_M \) when \( \frac{\gamma(y_2)}{y_2} > 1 \). For such system parameters, the
integrals can be evaluated. To obtain results for system parameters, for which \( \frac{\tilde{c}(y_2)}{\tilde{a}(y_2)} < 1 \), we might consider analytic continuations for the functions (11), (12). In such a case, it is most probable to arise additional numerical difficulties. Similarly, the computation of the integral expression \( H(0, y) \) is allowed when \( 1 \in L \cup G_L \), whereas for system parameters under which \( 1 \notin L \cup G_L \), the analytic continuation of \( H(0, y) \), \( H_2(0, y) \) may result in additional numerical difficulties.

V. EXPLICIT BOUNDS FOR THE MEAN DELAY IN THE SYMMETRICAL SYSTEM

As stated above, although we are able to determine the expected delay for the general model through (13), these expressions are given in terms of contour integrals, whereas we also need to numerically obtain the conformal mappings, which is time consuming; see Appendix D. In the following, we consider the symmetrical model, and obtain explicit bounds for the average delay.

In particular, symmetry implies for \( k = 1, 2 \) let \( \alpha_k = \alpha \), \( t_k = t \), \( p_{R_k,\{k\}} = s_1^R \), \( p_{L,R_{k,\{k\}}} = s_1^D \), \( p_{R_k,\{1,2\}} = s_2^R \), \( p_{L,R_{k,\{1,2\}}} = s_2^D \), \( p_{R_k,\{1,2\}} = s_0^D \), \( p_{R_k,\{R_k\}} = r_1^D \), \( p_{R_{k,\{R_k\}}(R_k)} = r_1^D \), \( p_{R_{k,\{R_k\}}(R_k)} = r_2^D \), \( p_{R_{k,\{R_k\}}(R_k)} = r_2^D \), \( p_{R_{k,\{R_k\}}(R_k)} = r_0^D \), and as a result \( d_k = d := \alpha^2(r_2^D - r_1^D) \), \( \lambda_k = \lambda \), \( k = 1, 2 \), \( \tilde{\alpha} = \alpha r_1^D + \alpha r_2^D \), and \( \lambda = t s_2^R (s_0^D + s_2^D) + t \bar{s}_1^D s_1^R \). The following theorem summarizes the basic result in this section.

**Theorem V.1.** Under the stability condition \( \alpha [\tilde{\alpha} + \alpha r_0^D] > \lambda \),

1) If \( d + \alpha^2 r_0^D < 0 \), the upper and lower delay bound, say \( D_1^{\text{low}} \), \( D_1^{\text{up}} \) respectively are,

\[
D_1^{\text{low}} = S, \quad D_1^{\text{up}} = D_1^{\text{low}} - \frac{\alpha^2 r_0^D (d + \alpha^2 r_0^D)}{2 \lambda \alpha (\alpha [\tilde{\alpha} + \alpha r_0^D] - \lambda)}.
\]

2) If \( d + \alpha^2 r_0^D > 0 \), then, \( D_1^{\text{up}} = S \), \( D_1^{\text{low}} = D_1^{\text{up}} - \frac{\alpha^2 r_0^D (d + \alpha^2 r_0^D)}{2 \lambda \alpha (\alpha [\tilde{\alpha} + \alpha r_0^D] - \lambda)}.
\]

**Proof:** See Appendix G

The following remark shows that the gap between the upper and the lower bound of the average delay has a closed form expression.

**Remark 3.** We observe that the gap between the lower and the upper average delay is given by \( D_1^{\text{up}} - D_1^{\text{low}} = \left| \frac{\alpha^2 r_0^D (d + \alpha^2 r_0^D)}{2 \lambda \alpha (\alpha [\tilde{\alpha} + \alpha r_0^D] - \lambda)} \right| \). Note that when the arrival rate \( \lambda \) increases, then the gap
tends to zero. Furthermore, when $\lambda \to 0$, then the average delay is very close to zero thus, the lower bound on the delay is tight.

**Remark 4.** In the case of a collision channel model assumption, which is equivalent to $r^D_0 = r^D_2 = 0$, we have that $D_1^{up} = D_1^{low}$. This is also true for the case where $r^D_0 = 0$, which means that the destination can successfully receive at most one packet from the aggregators when both of them transmit. This is known as the capture effect which is common in LoRaWAN [42]. For the case of the general MPR model where $r^D_0 \neq 0$, we can easily obtain bounds for the average delay at aggregators based on the sign of $\phi$. Since $Pr(N_1 > 0, N_2 > 0) > 0$, the sign of $\phi$ coincides with the sign of $d + \alpha^2 r^D_0$.

**VI. Numerical Results**

In this section, we provide numerical results to evaluate the presented theoretical performance analysis. For exposition convenience, we consider the case where all sensors have the same link characteristics and transmission probabilities.

We consider the following network setup, the distance between the sensors and the sink is $130m$, the distance between the sensors and the aggregator is $60m$, and the distance between the aggregators and the destination is $80m$. The path loss exponent is assumed 4, and the transmission power for each sensor is $1mW$ and for each aggregator is $10mW$. We assume Rayleigh block fading channel model. The values for the SINR threshold are considered in this section are 0.2, 0.5, 1.2, and 2. In addition, regarding the transmission probabilities, for the sensors we consider the values $t = 0.1$, $t = 0.2$, and for the aggregators $\alpha = 0.8$.

**A. Stability Region**

Here we present the closure of the stability region for the queues at the aggregators for all the possible random access probabilities $\alpha$. We consider two cases for the SINR threshold, the case where $\text{SINR} = 0.5$, which is depicted in Fig. 2(a). We observe that this region is a convex set,
thus, the system performs better than time sharing schemes and this is an indication of strong MPR capabilities at the destination. The stability region for SINR = 1.2 is depicted in Fig. 2(b). In this case, a time sharing scheme has higher performance since the destination has weak MPR capabilities.

![Stability region for SINR = 0.5 and SINR = 1.2](image)

(a) SINR = 0.5  
(b) SINR = 1.2

Fig. 2. The stability region for the queues at the aggregators.

B. Network Throughput

In this subsection, we present the performance in terms of throughput per sensor and the aggregate throughput of the considered system. We consider four cases for the SNR/SINR threshold, two cases for the transmission probabilities of the sensors, \( t = 0.1 \) and \( t = 0.2 \), and for the aggregators \( \alpha = 0.8 \). We also plot the throughput of the IoT network without the presence of the aggregators. The case where \( \text{SINR} < 1 \), is depicted in Fig. 3, in this case, we have strong MPR capabilities at the receivers, thus it is more likely to have more concurrent successful transmissions. Furthermore, we observe that the performance of the IoT network without the assistance of the aggregators can be sufficient for the case with low number of sensors.
The case where SINR $> 1$, is depicted in Fig. 4. In this case, the benefits of the aggregators deployment are prominent. The throughput of the IoT network without the aggregators is almost zero. The presence of the aggregators that deploy network-level cooperation provides significant gains in the throughput performance of the network.
In Fig. 5, we present the percentage of the traffic that is being relayed. We observe that in almost all cases the majority of the traffic is relayed. Furthermore, in Fig. 5(b) it becomes apparent that almost all the traffic that ends to the common destination is relayed traffic.

![Graph](image.png)

(a) SINR < 1  
(b) SINR > 1

Fig. 5. Percentage of relayed traffic per sensor.

Table II provides what is the number of sensors per aggregator that can cause its queue stable or unstable. As expected, when SINR < 1, we have a clear transition from a stable queue to unstable queue as the number of sensors increase. However, the more interesting case is when SINR > 1. We observe that for low number of sensors, we start with a stable queue, then the queue becomes unstable, and finally becomes stable again. The reason behind this is that initially, the incoming traffic increases up to a point the queue becomes unstable. Then due to the increased interference, the incoming traffic is reduced significantly thus, the queues are stable. The shape of the curves presented in Fig. 5 can be explained by Table II. These results provide useful guidelines on the number of sensors that can be supported by an aggregator in order to achieve a required network performance regarding throughput.
TABLE II
STATE OF THE QUEUE PER AGGREGATOR AND THE NUMBER OF SENSORS.

<table>
<thead>
<tr>
<th>SINR = 0.2, t = 0.2</th>
<th>Stable</th>
<th>Unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>{1, ..., 6}</td>
<td>{7, ..., 30}</td>
<td></td>
</tr>
<tr>
<td>SINR = 0.5, t = 0.2</td>
<td>{1, ..., 4}</td>
<td>{5, ..., 30}</td>
</tr>
<tr>
<td>SINR = 0.2, t = 0.1</td>
<td>{1, ..., 13}</td>
<td>{14, ..., 30}</td>
</tr>
<tr>
<td>SINR = 0.5, t = 0.1</td>
<td>{1, ..., 9}</td>
<td>{10, ..., 30}</td>
</tr>
<tr>
<td>SINR = 1.2, t = 0.2</td>
<td>{1, ..., 3}, {19, ..., 30}</td>
<td>{4, ..., 18}</td>
</tr>
<tr>
<td>SINR = 2, t = 0.2</td>
<td>{1, 2}, {14, ..., 30}</td>
<td>{3, ..., 13}</td>
</tr>
<tr>
<td>SINR = 1.2, t = 0.1</td>
<td>{1, ..., 7}</td>
<td>{8, ..., 30}</td>
</tr>
<tr>
<td>SINR = 2, t = 0.1</td>
<td>{1, ..., 6}, {28, ..., 30}</td>
<td>{7, ..., 27}</td>
</tr>
</tbody>
</table>

C. Delay Performance

In Fig. 6, the lower and the upper bounds for the average delay per packet versus the arrival rate at the aggregator are depicted for the four cases of SINR threshold, 0.2, 0.5, 1.2, and 2 respectively. The values of arrival rates are obtained by the stability conditions and we can connect the value of arrival rate with the number of users and their transmission probability through expression (4). Furthermore, we observe that the average delay increases with the increase of the SINR threshold. It is also important to notice that the upper and the lower bounds are very close, as an indication that the obtained bounds are tight.

VII. Conclusions and Future Directions

In this work, we considered a random access IoT wireless network assisted by two aggregators providing support for data collection by applying network level cooperation. We characterized the throughput performance of the IoT network and obtained the stability conditions for the queues at the aggregators, which guarantee finite queueing delay. Furthermore, by applying the theory of boundary value problems we provided a detailed analysis for the delay. We showed that the benefits for the deployment of aggregators are more profound for higher SINR threshold and when the MPR capability of the network is weak. Our analytical results provide useful design
guidelines for deploying aggregators in random access wireless IoT networks.

The suggested framework in this work can be extended to capture the case of cognitive aggregators that are adapting according to the incoming traffic from the sensors and also the channel conditions [43], [44]. Considering adaptation of the operation of the aggregator according to the incoming traffic, can provide also performance improvements related to power consumption and energy efficiency, which is also another important factor for the design of IoT networks. In addition, it will be of further interest to consider delay-aware operation protocols for the IoT network and using successive interference cancellation to mitigate interference among the aggregators at the destination. The case of multiple aggregators in the same coverage area is an interesting future direction, our approach can also be applied but with some modifications.

APPENDIX A

PROOF OF THEOREM III.1

To determine the stability region of the aggregators in our network, we apply the stochastic dominance technique [45]. More specifically, we construct hypothetical dominant systems, in
which an aggregator transmits dummy packets for the packet queue that is empty, while for the non-empty queue it transmits according to its traffic. Under this approach, we consider the $D_1$, and $D_2$-dominant systems. In the $D_k$ dominant system, whenever the queue of aggregator $k$, $k = 1, 2$ empties, it transmits a dummy packet instead.

Thus, in $D_1$, the first aggregator never empties, and hence, the second aggregator sees a constant service rate, while the service rate of aggregator 1 depends on the state of aggregator 2, i.e., empty or not. We proceed with the queue at aggregator 1, its service rate is given by

$$
\mu_1 = Pr(N_2 \neq 0) \left[ \alpha_1 \alpha_2 p_{R_1, \{R_i\}}^D + \alpha_1 \alpha_2 p_{R_1/R_1, R_2}^D \right] + Pr(N_2 = 0) \alpha_1 p_{R_1, \{R_i\}}^D. \quad (15)
$$

The service rate of the second aggregator is given by

$$
\mu_2 = \alpha_2 \alpha_1 p_{R_2, \{R_2\}}^D + \alpha_1 \alpha_2 p_{R_2/R_1, R_2}^D. \quad (16)
$$

By applying Loyne’s criterion, the second node is stable if and only if the average arrival rate is less than the average service rate, $\lambda_2 < \alpha_2 \alpha_1 p_{R_2, \{R_2\}}^D + \alpha_1 \alpha_2 p_{R_2/R_1, R_2}^D$. We can obtain the probability that the second node is empty and is given by $Pr(N_2 = 0) = 1 - \frac{\lambda_2}{\mu_2}$. After replacing $Pr(N_2 = 0)$ into (15), and applying Loynes criterion we can obtain the stability condition for the first node. Then, we have the stability region $\mathcal{R}_1$ given by (7) for $i = 1$. Similarly, we can obtain the stability region for the second dominant system $\mathcal{R}_2$. For a detailed treatment of dominant systems please refer to [45].

An important observation made in [45] is that the stability conditions obtained by the stochastic dominance technique are not only sufficient but also necessary for the stability of the original system. The indistinguishability argument [45] applies here as well. Based on the construction of the dominant system, we can see that the queue sizes in the dominant system are always greater than those in the original system, provided they are both initialized to the same value and the arrivals are identical in both systems. Therefore, given $\lambda_2 < \mu_2$, if for some $\lambda_1$, the queue at the first user is stable in the dominant system, then the corresponding queue in the original system must be stable. Conversely, if for some $\lambda_1$ in the dominant system, the queue at the first node
saturates, then it will not transmit dummy packets, and as long as the first user has a packet to transmit, the behavior of the dominant system is identical to that of the original system since dummy packet transmissions are eliminated as we approach the stability boundary. Therefore, the original and the dominant system are indistinguishable at the boundary points.

**APPENDIX B**

**Proof of Lemma IV.1**

It is readily seen that \( R(x, y) = \frac{xy - \Psi(x, y)}{xy} \), where \( \Psi(x, y) = L(x, y)[xy + y(1-x)\alpha_1\hat{\alpha}_2 + x(1-y)\alpha_2\hat{\alpha}_1] \), where for \(|x| \leq 1, |y| \leq 1\), \( \Psi(x, y) \) is a generating function of a proper probability distribution. Now, for \(|y| = 1\), \( y \neq 1 \) and \(|x| = 1\) it is clear that \(|\Psi(x, y)| < 1 = |xy|\). Thus, a direct application of Rouché’s theorem states that, \( xy - \Psi(x, y) \) has exactly one zero inside the unit circle. Therefore, \( R(x, y) = 0 \) has exactly one root \( x = X_0(y) \), such that \( |x| < 1 \). For \( y = 1 \), \( R(x, 1) = 0 \) implies \((1-x)\left(\lambda_1 + \lambda_1 \frac{\alpha_1\hat{\alpha}_2(1-x)}{x} - \frac{\alpha_1\hat{\alpha}_2}{x}\right) = 0\). Therefore, for \( y = 1 \), and since \( \lambda_1 < \alpha_1\hat{\alpha}_2 \), the only root of \( R(x, 1) = 0 \) for \(|x| \leq 1\), is \( x = 1 \).

**APPENDIX C**

**Proof of Lemma IV.3**

For \( y \in [y_1, y_2] \), \( D_y(y) \) is negative, so \( X_0(y), X_1(y) \) are complex conjugates. Therefore, \( |X(y)|^2 = \frac{\hat{c}(y)}{\hat{a}(y)} = g(y) \). Clearly, \( g(y) \) is an increasing function for \( y \in [0, 1] \) and thus, \( |X(y)|^2 \leq g(y_2) = \beta_0 \). Using simple algebraic considerations we can prove that, \( X_0(y_1) := \beta_1 = -g(y_1) \) is the extreme left point of \( \mathcal{M} \). Finally, \( \zeta(\delta) \) is derived by solving \( \text{Re}(X(y)) = -\hat{b}(y)/2\hat{a}(y) \) for \( y \) with \( \delta = \text{Re}(X(y)) \), and taking the solution such that \( y \in [0, 1] \).

**APPENDIX D**

**Construction of the Conformal Mappings**

To proceed, with the construction of \( \gamma_0(z) \) we represent \( \mathcal{M} \) in polar coordinates, i.e., \( \mathcal{M} = \{x : x = \rho(\phi)\exp(i\phi), \phi \in [0, 2\pi]\} \). In the following, we summarize the basic steps: Since \( 0 \in G_{\mathcal{M}} \),
for each $x \in \mathcal{M}$ we have $|x|^2 = m(\text{Re}(x))$ (see Lemma IV.3). Given the angle $\phi$ of some point on $\mathcal{M}$, its real part, say $\delta(\phi)$, is the zero of $\delta - \cos(\phi)\sqrt{m(\delta)}$, $\phi \in [0, 2\pi]$. Since $\mathcal{M}$ is a smooth, egg-shaped contour, the solution is unique. Clearly, $\rho(\phi) = \frac{\delta(\phi)}{\cos(\phi)}$, and the parametrization of $\mathcal{M}$ in polar coordinates is fully specified.

Then, the mapping from $z \in G_C$ to $x \in G_M$, where $z = e^{i\phi}$ and $x = \rho(\psi(\phi))e^{i\psi(\phi)}$, satisfying $\gamma_0(0) = 0$, $\gamma_0(z) = \overline{\gamma_0(z)}$ is uniquely determined by (see [33], Section I.4.4),

$$\begin{align*}
\gamma_0(z) &= z \exp\left[\frac{1}{2\pi} \int_0^{2\pi} \log\{\rho(\psi(\omega))\} \frac{e^{i\omega + \gamma_0(\omega)}}{e^{i\omega} - z} d\omega\right], |z| < 1, \\
\psi(\phi) &= \phi - \int_0^{2\pi} \log\{\rho(\psi(\omega))\} \cot\left(\frac{\omega - \phi}{2}\right) d\omega, \ 0 \leq \phi \leq 2\pi,
\end{align*}$$

(17)
i.e., $\psi(.)$ is uniquely determined as the solution of a Theodorsen integral equation with $\psi(\phi) = 2\pi - \psi(2\pi - \phi)$. This integral equation has to be solved numerically by an iterative procedure. For the numerical evaluation of the integrals we split the interval $[0, 2\pi]$ into $M$ parts of length $2\pi/M$, by taking $M$ points $\phi_k = \frac{2k\pi}{M}$, $k = 0, 1, \ldots, M - 1$. For the $M$ points given by their angles $\{\phi_0, \ldots, \phi_M-1\}$ we should solve the second in (17) to obtain the corresponding points $\{\psi(\phi_0), \ldots, \psi(\phi_{M-1})\}$, iteratively from,

$$\begin{align*}
\psi_0(\phi_k) &= \phi_k, \psi_{n+1}(\phi_k) = \phi_k - \frac{1}{2\pi} \int_0^{2\pi} \log\left\{\frac{\delta(\psi_n(\omega))}{\cos(\psi_n(\omega))}\right\} \cot\left(\frac{\omega - \phi_k}{2}\right) d\omega, \\
\end{align*}$$

(18)

where $\lim_{n \to \infty} \psi_{n+1}(\phi) = \psi(\phi)$, and $\delta(\psi_n(\omega))$ is determined by, $\delta(\psi_n(\omega)) = \cos(\psi_n(\omega))\sqrt{m(\delta(\psi_n(\omega)))}$, using the Newton-Raphson root finding method. For each step, the integral in (18) is numerically determined by again using the trapezium rule with $M$ parts of equal length $2\pi/M$. For the iteration, we may use the stopping criterion $\max_{k \in \{0, 1, \ldots, M-1\}} |\psi_{n+1}(\phi_k) - \psi_n(\phi_k)| < 10^{-6}$. 

After obtaining $\psi(\phi)$ numerically, the values of the conformal mapping $\gamma_0(z)$, $|z| \leq 1$, can be calculated by applying the Plemelj-Sokhotski formula to the first in (17) for $0 \leq \phi \leq 2\pi$,

$$\begin{align*}
\gamma_0(e^{i\phi}) = \frac{e^{i\psi(\phi)}\delta(\psi(\phi))}{\cos(\psi(\phi))} = \delta(\psi(\phi)) [1 + i \tan(\psi(\phi))].
\end{align*}$$

We can find $\gamma(1)$, $\gamma'(1)$ by applying the Newton’s method and solving $\gamma_0(z_0) = 1$, in $[0, 1]$, i.e., $z_0$ is the zero in $[0, 1]$ of $\gamma_0(z) = 1$. Then, $\gamma(1) = z_0$. Moreover, using the first in (17)

$$\begin{align*}
\gamma'(1) = (\gamma_0(z_0))^{-1} = \left\{\frac{1}{\gamma(1)} + \frac{1}{2\pi i} \int_0^{2\pi} \log\{\rho(\psi(\omega))\} \frac{2e^{i\omega}}{(e^{i\omega} - \gamma(1))^2} d\omega\right\}^{-1},
\end{align*}$$

(19)
which is obtained numerically by using the trapezoidal rule.

Alternatively, one can use the nearly circular approximation. Clearly, the numerical computation of the exact conformal mappings is generally time consuming. Since $\mathcal{M}$, $\mathcal{L}$ are close to ellipses, alternatively, we can approximate them by conformal mappings that map the interior of ellipses to $G_C$ [41]. Therefore, we approximate $\mathcal{M}$ by the ellipse $\mathcal{E}$ with semi-axes $\rho(0)$, $\rho(\pi/2)$.

Then, $\epsilon(x)$ maps $G_\mathcal{E}$ to $G_C$ [41], where
\[
\epsilon(x) = \sqrt{k} sn \left( \frac{2\Omega}{\pi} \sin^{-1} \left( \frac{x}{\sqrt{\rho^2(0) - \rho^2(\pi/2)}} \right); k^2 \right), \quad k = 16q \prod_{n=1}^{\infty} \left( \frac{1+q^{2n}}{1+q^{2n-1}} \right)^{8},
\]
\[
q = \left( \frac{\rho(0) - \rho(\pi/2)}{\rho(0) + \rho(\pi/2)} \right)^2, \quad Q = \int_0^1 \frac{dt}{\sqrt{(1+t^4)(1-k^2t^2)}},
\]
where $sn(w;l)$ is the Jacobian elliptic function. Our approximation for $\gamma(x)$ is $\epsilon(x)$, $x \in \mathcal{M} \cup G_\mathcal{M}$.

**APPENDIX E**

**PROOF OF THEOREM IV.4**

For $y \in D_y = \{ y \in \mathcal{C} : |y| \leq 1, |X_0(y)| \leq 1 \}$,
\[
\alpha_2 \hat{\alpha}_1 H(X_0(y),0) + d_2 H(0,y) + \frac{\alpha_2 \hat{\alpha}_1 (1-\rho) C(X_0(y),y)}{A(X_0(y),y)} = 0. \tag{20}
\]
Both $H(X_0(y),0)$, and $H(0,y)$, where $y \in D_y - [y_1,y_2]$, are analytic functions. Using analytic continuation arguments we consider (20) for $x \in \mathcal{M}$
\[
\alpha_2 \hat{\alpha}_1 H(x,0) + d_2 H(0,Y_0(x)) + \frac{\alpha_2 \hat{\alpha}_1 (1-\rho) C(x,Y_0(x))}{A(x,Y_0(x))} = 0. \tag{21}
\]
By noticing that $H(0,Y_0(x))$ is real for $x \in \mathcal{M}$, i.e., $Y_0(x) \in [y_1, y_2]$, we have
\[
Re(iH(x,0)) = w(x), \quad x \in \mathcal{M}, \tag{22}
\]
where $w(x) := Re \left( -i \frac{C(x,Y_0(x))}{A(x,Y_0(x))} \right) (1-\rho)$. To proceed we have to investigate the possible poles of $H(x,0)$ (i.e., the zeros of $A(x,Y_0(x))$), $x \in S_x := G_\mathcal{M} \cap \bar{D}_x^c$, where $G_\mathcal{U}$ be the interior domain bounded by $\mathcal{U}$, and $D_x = \{ x : |x| < 1 \}$, $\bar{D}_x = \{ x : |x| \leq 1 \}$, $\bar{D}_x^c = \{ x : |x| > 1 \}$. Moreover, to solve (22) we first transform the problem from $\mathcal{M}$ to the unit circle; see Appendix D for details.

Let the conformal mapping, $z = \gamma(x) : G_\mathcal{M} \to G_C$, and its inverse $x = \gamma_0(z) : G_C \to G_\mathcal{M}$.
By applying the transformation, the problem is reduced to the determination of function $\tilde{T}(z) = H(\gamma_0(z), 0)$ regular for $z \in \mathbb{C}$, continuous for $z \in \mathbb{C} \cup \mathbb{C}$ such that, $\text{Re}(i\tilde{T}(z)) = w(\gamma_0(z))$, $z \in \mathbb{C}$. The solution of the Dirichlet problem with boundary condition (22) is:

$$H(x, 0) = -\frac{1 - \rho}{2\pi} \int_{|t| = 1} f(t) \frac{t^{1 + \gamma(x)} dt}{t^{-\gamma(x)} I} + C, \quad x \in \mathcal{M},$$  \hspace{1cm} (23)

where $f(t) = \text{Re} \left(-\frac{iC(\gamma_0(t), Y_0(\gamma_0(t)))}{A(\gamma_0(t), Y_0(\gamma_0(t)))}\right)$, $C$ a constant to be defined by setting $x = 0 \in \mathcal{M}$ in (23) and using the fact that $H(0, 0) = 1 - \rho, \gamma(0) = 0$. In case $H(x, 0)$ has a pole, say $\bar{x}$, we still have a Dirichlet problem for the function $(x - \bar{x})H(x, 0)$. Setting $t = e^{i\phi}, \gamma_0(e^{i\phi}) = \rho(\psi(\phi))e^{i\psi(\phi)}$, we have $f(e^{i\phi}) = d_1^2 \alpha_2 \sin(\psi(\phi))(1 - Y_0(\rho(\psi(\phi)))^{-1})/\rho(\psi(\phi))\psi(\phi)$, which is an odd function of $\phi$, and

$$k(\phi) = \left[\alpha_2 \bar{\alpha}_1 (1 - Y_0^{-1}(\rho(\psi(\phi)))) + d_1 (1 - \frac{\cos(\psi(\phi))}{\rho(\psi(\phi))})\right]^2 + \left(d_1 \frac{\sin(\psi(\phi))}{\rho(\psi(\phi))}\right)^2.$$

Thus, $C = 1 - \rho$. Substituting in (23) we obtain after simple calculations an integral representation of $H(x, 0)$ on a real interval for $x \in \mathcal{M}$, i.e.,

$$H(x, 0) = (1 - \rho) \left\{1 + \frac{2\gamma(x)i}{\pi} \int_0^\pi \frac{f(e^{i\phi}) \sin(\phi) d\phi}{1 - 2\gamma(x)\cos(\phi) - \gamma(x)^2}\right\}.$$

Similarly, we can determine $H(0, y)$ by solving another Dirichlet boundary value problem on the closed contour $\mathcal{L}$. Then, substituting back in (9), we uniquely obtain $H(x, y)$.

**APPENDIX F**

**PROOF OF THEOREM IV.5**

For $y \in \mathcal{D}_y$, $A(X_0(y), y)G(X_0(y)) = -B(X_0(y), y)L(y)$. Using similar arguments as in previous subsection, we have for $x \in \mathcal{M}$, $A(x, Y_0(x))G(x) = -B(x, Y_0(x))L(Y_0(x))$. Clearly, $G(x)$ is holomorphic in $D_x$, continuous in $\bar{D}_x$, but it might have poles in $S_x$, based on the values of the system parameters, i.e., the zeros of $A(x, Y_0(x))$ in $S_x$. It can be shown that there might be at most one such a zero, say $\bar{x}$. For $y \in [y_1, y_2]$, let $X_0(y) = x \in \mathcal{M}$ and realize that $Y_0(X_0(y)) = y$ so that $y = Y_0(x)$. Taking into account the possible poles of $G(x)$, and noticing that $L(Y_0(x))$ is real for $x \in \mathcal{M}$, since $Y_0(x) \in [y_1, y_2]$, we have

$$\text{Re}[iU(x)\tilde{G}(x)] = 0, \quad x \in \mathcal{M}, U(x) = \frac{A(x, Y_0(x))}{(x - \bar{x})^rB(x, Y_0(x))}, \quad \tilde{G}(x) = (x - \bar{x})^rG(x),$$  \hspace{1cm} (24)
where $r = 0, 1$. Thus, $\tilde{G}(x)$ is regular for $x \in G_M$, continuous for $x \in \mathcal{M} \cup G_M$, and $U(x)$ is a non-vanishing function on $\mathcal{M}$. By conformally transform the problem (24) from $\mathcal{M}$ to the unit circle, using $z = \gamma(x) : G_M \to G_C$, and its inverse given by $x = \gamma_0(z) : G_C \to G_M$, the problem in (24) is reduced to the following: find a function $F(z) := \tilde{G}(\gamma_0(z))$, regular in $G_C$, continuous in $G_C \cup \mathcal{C}$ such that, $Re[iU(\gamma_0(z))F(z)] = 0$, $z \in \mathcal{C}$.

To solve this problem, we must determine its index $\chi = \frac{-1}{\pi}[arg\{U(x)\}]_{x \in \mathcal{M}}$, where $[arg\{U(x)\}]_{x \in \mathcal{M}}$ denotes the variation of the argument of the function $U(x)$ as $x$ moves along the closed contour $\mathcal{M}$ in the positive direction, provided that $U(x) \neq 0$, $x \in \mathcal{M}$. Following [32] we have:

**Lemma F.1.** 1) If $\lambda_2 < \alpha_2 \hat{\alpha}_1$, then $\chi = 0$ is equivalent to

$$\left.\frac{dA(x,Y_0(x))}{dx}\right|_{x=1} < 0 \Leftrightarrow \lambda_1 < \frac{\alpha_1 p^D_{R_1}(R_1)}{\alpha_2 \hat{\alpha}_1} + \frac{d_1 \gamma_2}{\alpha_2 \hat{\alpha}_1},$$

$$\left.\frac{dB(x,Y_0(x))}{dy}\right|_{y=1} < 0 \Leftrightarrow \lambda_2 < \frac{\alpha_2 p^D_{R_2}(R_2)}{\alpha_1 \hat{\alpha}_2} + \frac{d_2 \gamma_1}{\alpha_1 \hat{\alpha}_2}.$$ 

2) If $\lambda_2 \geq \alpha_2 \hat{\alpha}_1$, $\chi = 0$ is equivalent to

$$\left.\frac{dB(x,Y_0(x),y)}{dy}\right|_{y=1} < 0 \Leftrightarrow \lambda_2 < \frac{\alpha_2 p^D_{R_2}(R_2)}{\alpha_1 \hat{\alpha}_2} + \frac{d_2 \gamma_1}{\alpha_1 \hat{\alpha}_2}.$$

Thus, under stability conditions the problem defined in (24) has a unique solution given by,

$$H(x,0) = K(x - \bar{x})^{-r} \exp \left\{ \frac{1}{2i\pi} \int_{|t|=1} \frac{\log J(t)}{t - \gamma(x)} dt \right\} - \frac{\alpha_1 p^D_{R_1}(R_1)}{d_1 d_2 - \alpha_1 \hat{\alpha}_2 \alpha_2 \hat{\alpha}_1} H(0,0), \ x \in G_M \tag{25}$$

where $K$ is a constant and $J(t) = \frac{U_1(t)}{U_1(1)}$, $U_1(t) = U(\gamma_0(t))$, $|t| = 1$. Setting $x = 0$ in (25) we derive a relation between $K$ and $H(0,0)$. Then, for $x = 1 \in G_M$, and using the first in (10) we can obtain $K$ and $H(0,0)$. Substituting back in (25) we obtain for $x \in G_M$,

$$H(x,0) = \frac{\lambda_1 d_2 + \alpha_1 \hat{\alpha}_2 (l_1 l_2 \alpha_2^2 p^D_{R_2}(R_2) - \lambda_2)}{(x - \bar{x})^r \gamma_1 t_1 t_2 (\alpha_1 \hat{\alpha}_2 \alpha_2 \hat{\alpha}_1 - d_1 d_2)} \times$$

$$\left( (x - \bar{x})^r \exp \left[ \frac{\gamma(x) - \gamma(1)}{2i\pi} \int_{|t|=1} \frac{\log J(t)}{t - \gamma(x)(t - \gamma(1))} dt \right] + \frac{\alpha_1 p^D_{R_2}(R_2)}{\alpha_1 \hat{\alpha}_2 \alpha_2 \hat{\alpha}_1} d_2 x^r \exp \left[ \frac{-\gamma(1)}{2i\pi} \int_{|t|=1} \frac{\log J(t)}{t - \gamma(1)} dt \right] \right).$$

The other unknown function $H(0, y)$ is determined similarly, by solving another Riemann-Hilbert boundary value problem on the closed contour $\mathcal{C}$. Then, Then, substituting back in (9), we uniquely obtain $H(x, y)$.
APPENDIX G
PROOF OF THEOREM V.1

Due to the symmetry, \( H(0, 1) = H(1, 0) \) and \( H_1(1, 1) = H_2(1, 1), H_1(1, 0) = H_2(0, 1) \), where \( H_1(x, y) \) (resp. \( H_2(x, y) \)) denotes the first-order derivative of \( H(x, y) \) with respect to \( x \) (resp. \( y \)). The proof is outlined in the following steps:

Step 1 Since \( H(1, 1) = 1 \), set \( y = 1 \) in (9), divide with \( x - 1 \), and take the limit \( x \to 1 \), by using the L'Hospital rule, to obtain

\[
\alpha[\hat{\alpha} + \alpha r_0^D] - \lambda = (d + \alpha \hat{\alpha} + 2\alpha^2 r_0^D)H(1, 0) - (d + \alpha^2 r_0^D)H(0, 0),
\]

Denote by \( M_k = H_1(1, 1), k = 1, 2 \). Differentiating (9) with respect to \( x \), setting \((x, y) = (1, 1)\), and using (26) we obtain for \( \bar{\lambda} = 1 - \lambda \),

\[
M_1 = \frac{\bar{\lambda}(d + \alpha \hat{\alpha} + 2\alpha^2 r_0^D)H(1, 0) - (d + \alpha^2 r_0^D)P(N_1 > 0, N_2 > 0)}{\alpha[\hat{\alpha} + \alpha r_0^D] - \lambda}.
\]  

(27)

Step 2 Now set \( x = y \) in (9) to obtain

\[
2(\alpha[\hat{\alpha} + \alpha r_0^D] - \lambda) \frac{d}{dx} H(x, x)|_{x=1} = 2(1 + 2\lambda)(\alpha[\hat{\alpha} + \alpha r_0^D] - \lambda) + \alpha^2 r_0^D P(N_1 > 0, N_2 > 0)
\]

\[+ 2H_1(1, 0)(d + \alpha \hat{\alpha} + 2\alpha^2 r_0^D) - 2\alpha(\hat{\alpha} + \alpha r_0^D) + t^2 \bar{s}_0^D (s_0^D)^2 + 4\lambda(1 - \alpha \hat{\alpha} - \alpha^2 r_0^D).
\]

Using (27), (28), and realizing that due to the symmetry \( \frac{d}{dx} H(x, x)|_{x=1} = 2M_1 \), we obtain

\[
M_1 = M_2 = \frac{\bar{\lambda}(d + \alpha \hat{\alpha} + 2\alpha^2 r_0^D) - (d + \alpha^2 r_0^D)[2\lambda(1 + 2\lambda)] + t^2 \bar{s}_0^D (s_0^D)^2 + \alpha^2 r_0^D P(N_1 > 0, N_2 > 0)}{\alpha[\hat{\alpha} + \alpha r_0^D] - \lambda}.
\]

(28)

Step 3 Using Little’s law, the average delay at each aggregator equals \( D_1 = D_2 := S - \phi \), where \( S = \frac{\lambda \bar{\lambda}(d + \alpha \hat{\alpha} + 2\alpha^2 r_0^D) - (d + \alpha^2 r_0^D)[2\lambda(1 + 2\lambda) + t^2 \bar{s}_0^D (s_0^D)^2]}{2\alpha[\hat{\alpha} + \alpha r_0^D] - \lambda} \)

and \( \phi = \frac{(d + \alpha^2 r_0^D) \alpha^2 r_0^D P(N_1 > 0, N_2 > 0)}{2\lambda \alpha[\hat{\alpha} + \alpha r_0^D] - \lambda} \).

To conclude, we distinguish the analysis in the following two cases:

a) If \( d + \alpha^2 r_0^D < 0 \), then \( 0 \leq \phi \leq \frac{\alpha^2 r_0^D (d + \alpha^2 r_0^D)}{2\lambda \alpha[\hat{\alpha} + \alpha r_0^D] - \lambda} \). Thus,

\[
D_1^{\text{low}} = S, \quad D_1^{\text{up}} = D_1^{\text{low}} + \frac{\alpha^2 r_0^D (d + \alpha^2 r_0^D)}{2\lambda \alpha[\hat{\alpha} + \alpha r_0^D] - \lambda}.
\]

b) If \( d + \alpha^2 r_0^D > 0 \), then \( \frac{\alpha^2 r_0^D (d + \alpha^2 r_0^D)}{2\lambda \alpha[\hat{\alpha} + \alpha r_0^D] - \lambda} \leq \phi \leq 0 \). In such a case,

\[
D_1^{\text{up}} = S, \quad D_1^{\text{low}} = D_1^{\text{up}} - \frac{\alpha^2 r_0^D (d + \alpha^2 r_0^D)}{2\lambda \alpha[\hat{\alpha} + \alpha r_0^D] - \lambda}.
\]
REFERENCES


