Decomposition Methods for Combinatorial Optimization

Uledi Ngulo

Linköping Studies in Science and Technology.
Licentiate Thesis No. 1910

Decomposition Methods for Combinatorial Optimization

Uledi Ngulo

Department of Mathematics
Linköping, 2021

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.

<https://creativecommons.org/licenses/by-nc/4.0/>

Linköping Studies in Science and Technology. Licentiate Thesis No. 1910

Decomposition Methods for Combinatorial Optimization

Copyright © Uledi Ngulo, 2021

Department of Mathematics

Linköping University

SE-581 83 Linköping, Sweden

Email: `uledi.ngulo@liu.se`

ISSN 0280-7971

ISBN 978-91-7929-624-7

Printed by LiU-Tryck, Linköping, Sweden, 2021

Abstract

This thesis aims at research in the field of combinatorial optimization. Problems within this field often possess special structures allowing them to be decomposed into more easily solved subproblems, which can be exploited in solution methods. These structures appear frequently in applications. We contribute with both research on the development of decomposition principles and on applications. The thesis consists of an introduction and three papers.

In Paper I, we develop a Lagrangian meta-heuristic principle, which is founded on a primal-dual global optimality condition for discrete and non-convex optimization problems. This condition characterizes (near-)optimal solutions in terms of near-optimality and near-complementarity measures for Lagrangian relaxed solutions. The meta-heuristic principle amounts to constructing a weighted combination of these measures, thus creating a parametric auxiliary objective function (which is a close relative to a Lagrangian function), and embedding a Lagrangian heuristic in a search procedure in the space of the weight parameters. We illustrate and assess the Lagrangian meta-heuristic principle by applying it to the generalized assignment problem and to the set covering problem. Our computational experience shows that the meta-heuristic extension of a standard Lagrangian heuristic principle can significantly improve upon the solution quality.

In Paper II, we study the duality gap for set covering problems. Such problems sometimes have large duality gaps, which make them computationally challenging. The duality gap is dissected with the purpose of understanding its relationship to problem characteristics, such as problem shape and density. The means for doing this is the above-mentioned optimality condition, which is used to decompose the duality gap into terms describing near-optimality in a Lagrangian relaxation and near-complementarity in the relaxed constraints. We analyse these terms for numerous problem instances, including some large real-life instances, and conclude that when the duality gap is large, the near-complementarity term is typically large and the near-optimality term small. The large violation of complementarity is due to extensive over-coverage. Our observations have implications for the design of solution methods, especially for the design of core problems.

In Paper III, we study a bi-objective covering problem stemming from a real-world application concerning the design of camera surveillance systems for large-scale outdoor areas. It is prohibitively costly to surveil the entire area, and therefore relevant to be able to present a decision-maker with trade-offs between total cost and the portion of the area that is surveilled. The problem is stated as a set covering problem with two objectives, describing cost and portion of covering constraints that are fulfilled, respectively. Finding the Pareto frontier for these objectives is very computationally demanding and we therefore develop a method for finding a good approximate frontier in a reasonable computing time. The method is based on the ϵ -constraint reformulation, an established heuristic for set covering problems, and subgradient optimization.

Sammanfattning

Denna avhandling behandlar lösningsmetoder för stora och komplexa kombinatoriska optimeringsproblem. Sådana problem har ofta speciella strukturer som gör att de kan dekomponeras i en uppsättning mindre delproblem, vilket kan utnyttjas för konstruktion av effektiva lösningsmetoder. Avhandlingen omfattar både grundforskning inom utvecklingen av dekompositionsprinciper för kombinatorisk optimering och forskning på tillämpningar inom detta område. Avhandlingen består av en introduktion och tre artiklar.

I den första artikeln utvecklar vi en "Lagrange-meta-heuristik-princip". Principen bygger på primal-duala globala optimalitetsvillkor för diskreta och icke-konvexa optimeringsproblem. Dessa optimalitetsvillkor beskriver (när)optimala lösningar i termer av när-optimalitet och när-komplementaritet för Lagrange-relaxerade lösningar. Den meta-heuristiska principen bygger på en ihopviktnig av dessa storheter vilket skapar en parametrisk hjälpmålfunktion, som har stora likheter med en Lagrange-funktion, varefter en traditionell Lagrange-heuristik används för olika värden på viktparametrarna, vilka avsöks med en meta-heuristik. Vi illustrerar och utvärderar denna meta-heuristiska princip genom att tillämpa den på det generaliserade tillordningsproblemet och övertäckningsproblemet, vilka båda är välkända och svårlösta kombinatoriska optimeringsproblem. Våra beräkningsresultat visar att denna meta-heuristiska utvidgning av en vanlig Lagrange-heuristik kan förbättra lösningskvaliteten avsevärt.

I den andra artikeln studerar vi egenskaper hos övertäckningsproblem. Denna typ av optimeringsproblem har ibland stora dual-gap, vilket gör dem beräkningskrävande. Dual-gapet analyseras därför med syfte att förstå dess relation till problemegenskaper, såsom problemstorlek och täthet. Medlet för att göra detta är de ovan nämnda primal-duala globala optimalitetsvillkoren för diskreta och icke-konvexa optimeringsproblem. Dessa delar upp dual-gapet i två termer, som är när-optimalitet i en Lagrange-relaxation och när-komplementaritet i de relaxerade bivillkoren, och vi analyserar dessa termer för ett stort antal probleminstanser, däribland några storskaliga praktiska problem. Vi drar slutsatsen att när dual-gapet är stort är vanligen den när-komplementära termen stor och den när-optimala termen liten. Vidare observeras att när den när-komplementära termen är stor så beror det på en stor överflödig övertäckning. Denna förståelse för problemets inreboende egenskaper går att använda vid utformningen av lösningsmetoder för övertäckningsproblem, och speciellt för konstruktion av så kallade kärnproblem.

I den tredje artikeln studeras tvåmålsproblem som uppstår vid utformningen av ett kameraövervakningssystem för stora områden utomhus. Det är i denna tillämpning alltför kostsamt att övervaka hela området och problemet modelleras därför som ett övertäckningsproblem med två mål, där ett mål beskriver totalkostnaden och ett mål beskriver hur stor del av området som övervakas. Man önskar därefter kunna skapa flera lösningar som har olika avvägningar mellan total kostnad och hur stor del av området som övervakas. Detta är dock mycket beräkningskrävande och vi utvecklar därför en metod för att hitta bra approximationer av sådana lösningar inom rimlig beräkningstid.

Acknowledgments

I would like to express my sincere gratitude to my supervisors Prof. Torbjörn Larsson, Ass. Prof. Nils-Hassan Quttineh and Dr. Egbert Mujuni for their constant support, guidance, continuous encouragements, and constructive ideas throughout my research work. They have helped me to understand various concepts in optimization. They have been both, an example and a sense of inspiration.

I am grateful to Sida bilateral programme, in collaboration with University of Dar es Salaam, for the funding I have received that enabled me do my studies. I am particularly thankful to Dr. Bengt-Ove Turesson, Theresa Lagali, Meaza Abebe, and Dr. Sylvester Rugeihyamu for making my visits in Sweden smooth and conducive.

I am thankful to my fellow students and other colleagues, present and past, at the Department of mathematics, Linköping University for the time we spent together during the whole period of my studies.

I am thankful to my wife Grace and our children Salome and Esther for their patience, prayers, and love. I am most thankful to my parents for the encouragements, moral and emotional support. I am forever thankful to the true God, JEHOVAH (Psalm 83:18), for his abundant love and support.

Contents

Abstract	i
Sammanfattning	iii
Acknowledgments	v
Contents	vii
1 Introduction	1
1.1 General introduction	1
1.2 Thesis structure	1
1.3 Summary of papers	2
1.4 Publication status	2
1.5 Presentations	3
2 Theoretical background	5
2.1 Lagrangian relaxation	5
2.2 Application of Lagrangian relaxation	5
2.3 Properties of the Lagrangian dual problem	6
2.4 Subgradients and subdifferential	6
2.5 Optimality condition	7
2.6 Subgradient optimization	9
2.7 Types of subgradient optimization	10
2.8 Choice of step length	11
2.9 Lagrangian heuristics	11
3 Pareto frontier	13
3.1 Introduction	13
3.2 Formulation of multi-objective problem	13
3.3 Solution methods	13
4 Applications	17
4.1 Assignment problem	17
4.2 Lagrangian heuristic for GAP	18
4.3 Set covering problem	19
4.4 Lagrangian heuristic for SCP	20
4.5 Bi-objective set covering problem	20
5 Bibliography	23
Paper I	29
Paper II	69
Paper III	77

1 – Introduction

1.1 General introduction

Many large complex combinatorial optimization problems (such as allocations, multiple knapsacks and, traveling salesman problems) are computationally demanding optimization problems (e.g., NP-hard problems). Thus, there is no hope to find polynomial time algorithms (i.e., no exact methods) for these problems [12]. All known algorithms for these problems require exponential time in a worst-case [19]. A large number of combinatorial problems consist of a set of constraints with a special structure, which can be exploited by decomposition approaches, and a set of difficult constraints. These problems become easy to solve if the difficult constraints are removed, see for example [3]. One of the techniques that make use of this special structure of constraints is Lagrangian relaxation principle. Lagrangian relaxation approximates a difficult problem of constrained optimization by a simpler problem called Lagrangian relaxed problem (Lagrangian subproblem). The Lagrangian subproblem is created as follows: A set of complicating constraints, attached with Lagrangian dual variables, are penalized into an objective function of the original problem, see, e.g., [28, 27, 45, 1, 60, 6]. This principle provides an optimistic bound on an optimal objective value of the original problem. The tightest optimistic bound on the objective value of the original problem with respect to the choice of Lagrangian dual variables is obtained by solving a Lagrangian dual problem, see for example [38].

In this thesis, we investigate decomposition methods for solving combinatorial optimization problems. We develop a Lagrangian meta-heuristic principle which is founded on a primal-dual optimality condition for discrete optimization problems. The meta-heuristic principle is embedding a Lagrangian heuristic in a meta-heuristic search with respect to weight parameters on the near-optimality and near-complementarity in the relaxed constraints. The meta-heuristic principle provides an optimal or near-optimal solution of the original problem.

1.2 Thesis structure

The remaining part of the thesis is mainly organized into two parts. The first part provides a theoretical background, Pareto frontier, and applications. In the theoretical background, we give a detailed discussion of Lagrangian relaxation and its applications on a general combinatorial optimization problem. We study the properties of the Lagrangian dual problem and give a primal-dual global optimality condition. We discuss the solution methods, i.e., a subgradient algorithm for the dual problem and a Lagrangian heuristic for finding a feasible solution to the original problem. In the Pareto frontier, we give a formulation of the multiobjective

problem and its solution methods. Lastly, we discuss the main problems, i.e., generalized assignment, set covering, and bi-objective set covering, which we use as a test example in this thesis. The second part of this thesis consists of three papers as discussed in the summary of papers.

1.3 Summary of papers

- **Paper I: A Lagrangian Meta-Heuristic Principle—Derivation and Application.**

This paper presents a development of a decomposition principle for combinatorial optimization. A Lagrangian Meta-Heuristic (LMH) principle is developed by embedding a Lagrangian heuristic in a meta-heuristic search with respect to certain weight parameters in the objective function of the Lagrangian relaxation. The development is founded on the generalized global optimality condition derived by Larsson and Patriksson [40]. The LMH principle is implemented on two test problems, a generalized assignment problem and a set covering problem.

- **Paper II: A Dissection of the Duality Gap of Set Covering Problems**

In this paper, we dissect the duality gap for a non-convex problem in two terms: degree of near-optimality in a Lagrangian relaxation and degree of near-complementarity in the relaxed constraints. The reason for doing this is to understand its relationship to problem characteristics, such as problem shape and density. We analyse these terms for numerous problem instances, including some large real-life instances.

- **Paper II: Approximating the Pareto Frontier for a Challenging Real-world Bi-Objective Covering Problem**

In this paper, we study a bi-objective covering problem stemming from a real-world application concerning the design of a camera surveillance systems for large-scale outdoor areas. Since it is prohibitively costly to surveil the entire area, we strive to find a Pareto front of solutions for the trade-off between the total cost and the portion of the area that is surveilled. Therefore, we develop a method for finding a good approximate frontier in a reasonable computing time. The method is based on the ϵ -constraint reformulation, an established heuristic for set covering problems, and subgradient optimization.

1.4 Publication status

The status of the appended papers are as follows:

- **Paper I** with the title “A Lagrangian Meta-Heuristic Principle—Derivation and Application” to be submitted.

- **Paper II** with the title “A Dissection of the Duality Gap of Set Covering Problems” is a slightly extended version of the article that is published in the Operations Research Proceedings 2019, pp. 175-181. Springer (2020).
- **Paper III** with the title “Approximating the Pareto Frontier for a Challenging Real-world Bi-Objective Covering Problem ” to be submitted.

1.5 Presentations

During my doctoral studies I have attended and made presentations at the following conferences.

- **Meta-Heuristic Summer School (MESS2018)**. It took place in Taormina-Sicily, Italy, July 21–25, 2018. I presented an early version of paper I.
- **Third Sida-ISP networking meeting**. It took place in August 2018, Kampala, Uganda. I presented an early version of paper I.
- **OR2019, The annual international conference of the German Operations Research Society (GOR)**. It took place in Dresden, Germany, September 4–6, 2019. I presented paper II.
- **SOAK2019, The bi-annual conference of the Swedish Operations Research Association (SOAF)**. It took place in Nyköping, Sweden, October 23–24, 2019. I presented paper II.

2 – Theoretical background

2.1 Lagrangian relaxation

Lagrangian relaxation principle is a universal technique in combinatorial optimization for finding optimistic bounds to an optimal objective value of the original problem. The principle is suitable if some constraints in the original problem make it difficult to solve. The complicating constraints, multiplied with Lagrangian dual variables, are penalized to the objective function, giving a Lagrangian relaxed problem which is easy to solve with respect to the non-relaxed constraints. The Lagrangian objective function value obtained is an optimistic bound of the optimal objective value to the original problem [20]. A solution to the Lagrangian relaxed problem is generally not feasible with respect to the relaxed constraints.

The literature on Lagrangian relaxation goes back to early 1970s when Held and Karp [30, 31] used a Lagrangian relaxation principle on their work of a traveling salesman problem and minimum spanning trees. Many other researchers have been motivated to apply the principle in solving mathematical programming models, see for example [26, 15, 52, 19, 36, 29, 5, 57, 42, 35, 46].

2.2 Application of Lagrangian relaxation

Consider a general primal problem of finding

$$\begin{aligned} f^* &= \min f(x) \\ \text{s.t. } g(x) &\leq 0, \end{aligned} \tag{2.1a}$$

$$x \in X, \tag{2.1b}$$

where the set $X \subset \mathbb{R}^n$ is finite and the functions $f : X \rightarrow \mathbb{R}$ and $g : X \rightarrow \mathbb{R}^m$, are continuous.

Letting $u \in \mathbb{R}^m$ be a vector of Lagrangian dual variables associated with constraints (2.1a), the Lagrangian dual function $h : \mathbb{R}^m \rightarrow \mathbb{R}$ is given as

$$h(u) = \min_{x \in X} f(x) + u^T g(x). \tag{2.2}$$

The objective value $h(u)$ is a lower bound on the optimal objective value of problem (2.1), and $X(u) = \arg \min_{x \in X} f(x) + u^T g(x)$ is the set of solutions of problem (2.2).

The Lagrangian dual problem is defined by

$$h^* = \max_{u \in \mathbb{R}_+^m} h(u). \tag{2.3}$$

It provides a tightest lower bound to f^* . The duality gap of the primal-dual pair is defined as $\Gamma = f^* - h^*$. Typically, $\Gamma > 0$ for problem (2.1) and its relaxation

(2.3). Denoting the set of points in X by x^p , $p \in P_X$, where P_X is a set of indices, the Lagrangian dual objective function h given by $h(u) = \min_{p \in P_X} f(x^p) + u^T g(x^p)$, $u \in \mathbb{R}^m$ is piecewise linear.

2.3 Properties of the Lagrangian dual problem

Theorem 1 gives some of properties of Lagrangian dual problems.

Theorem 1 *For the primal problem (2.1) and its Lagrangian dual problem (2.3), we have the following properties*

- i. *The Lagrangian dual function h is finite, piecewise linear and concave on $\mathbb{R}^m$ [51, Lemma 5.1].*
- ii. *(Weak duality) For any $u \in \mathbb{R}_+^m$, $h(u) \leq f^*$, [51, Theorem 5.2].*
- iii. *(Strong duality) If $x^* \in X$ is a primal feasible solution and $u^* \in \mathbb{R}_+^m$ is a dual solution satisfying $h(u^*) = f(x^*)$, then x^* is optimal for primal problem (2.1) and u^* is optimal for Lagrangian dual problem (2.3), [58, Proposition 2.6].*

2.4 Subgradients and subdifferential

Subgradients are a generalization of gradients. Formally, subgradients can be defined as in Definition 1.

Definition 1 *A $\bar{\xi} \in \mathbb{R}^m$ is a subgradient of h if $h(u) \leq h(\bar{u}) + \bar{\xi}^T(u - \bar{u})$ holds for all $u \in \mathbb{R}^m$.*

If $x(\bar{u})$ solves the Lagrangian relaxed problem (2.2), then $g(x(\bar{u}))$ is a subgradient of h at $\bar{u}$, denoted by $\bar{\xi}$. The subgradients exist in all Lagrangian dual solutions, since the function h is concave. If there is a unique optimal solution to problem (2.3) (i.e., there is only one subgradient), then a gradient exist.

Definition 2 *The set of all subgradients of h at $\bar{u}$ is called subdifferential, i.e., $\xi \in \partial h(\bar{u})$ where $\partial h(\bar{u})$ is the subdifferential.*

Theorem 2 [38] *Let $X(\bar{u}) \subseteq X$ be a set of feasible solutions to the problem (2.2) and the vector $\bar{\xi} = g(x(\bar{u}))$ be the subgradient of h at $\bar{u}$, then a subdifferential is equal to the convex hull of the set of all subgradients, that is, $\partial h(\bar{u}) = \text{Conv}\{\bar{\xi} \mid x(\bar{u}) \in X(\bar{u})\}$.*

It means that for every $\bar{u} \in \mathbb{R}_+^m$, the set of all subgradients of the Lagrangian dual function h at $\bar{u}$, $\partial h(\bar{u})$, is non-empty, convex and compact.

Theorem 3 [51] *Let u^* solve problem (2.3), let $\bar{u} \in \mathbb{R}_+^m$ and $\xi \in \partial h(\bar{u})$. Then $u^* \in \{u : \xi^T(u - \bar{u}) \geq 0\}$.*

Interpretation: Every subgradient of the concave function h at $\bar{u}$ points into the half-space containing all optimal Lagrangian dual solutions of the Lagrangian dual problem (2.3), that is, $\bar{\xi}$ and $(u^* - \bar{u})$ for $\bar{u} \neq u^*$ form an acute angle. If a sufficiently small step size is chosen, in Euclidean norm sense, a step direction will allow a search method to move towards an optimal Lagrangian dual solution.

In section 2.5 we present an optimality condition which characterizes a primal-dual solutions by means of primal and dual feasibility, complementarity and primal Lagrangian optimality.

2.5 Optimality condition

The optimality condition for optimal or near-optimal primal solutions is known, see for example [40]. If X is convex, f and g are continuous on X , then there exist a primal-dual pair $(x, u) \in X \times \mathbb{R}_+^m$ such that $g(x) \leq 0$ implies that $f(x) - h(u) = 0$ holds. The classic optimality condition is stated as follows.

Theorem 4 [51, Theorem 5.1] *If $u^* \in \mathbb{R}_+^m$ is an optimal solution of problem (2.3) and $\Gamma = 0$, then $x^* \in X$ is an optimal solution of problem (2.1) if and only if*

$$f(x^*) + u^{*\text{T}}g(x^*) \leq h(u^*), \quad (2.4a)$$

$$g(x^*) \leq 0, \quad (2.4b)$$

$$u^{*\text{T}}g(x^*) = 0. \quad (2.4c)$$

Here, (2.4a) define optimality in a Lagrangian relaxation, $x^* \in X$ such that x^* solves $\min_{x \in X} f(x) + u^{*\text{T}}g(x)$, (2.4b) define primal feasibility in the relaxed constraints, and (2.4c) define complementarity in the relaxed constraints. A generalized global optimality condition for optimal primal-dual solutions is stated as in Theorem 5.

Theorem 5 [40, Theorem 3] *If $u^* \in \mathbb{R}_+^m$ is an optimal solution of problem (2.3), then $x^* \in X$ is an optimal solution of the problem (2.1) if and only if there are ε and δ such that*

$$f(x^*) + u^{*\text{T}}g(x^*) \leq h(u^*) + \varepsilon, \quad (2.5a)$$

$$g(x^*) \leq 0, \quad (2.5b)$$

$$u^{*\text{T}}g(x^*) \geq -\delta, \quad (2.5c)$$

$$\varepsilon + \delta \leq f^* - h^*, \quad (2.5d)$$

$$\varepsilon, \delta \geq 0. \quad (2.5e)$$

Here, (2.5a) is ε -optimality in a Lagrangian relaxation, (2.5b) is primal feasibility in relaxed constraints, (2.5c) is δ -complementarity in the relaxed constraints. It reduces to (2.4) for a case where the duality gap is zero. Theorem 6 gives a generalized global optimality condition for any possibly non-optimal solutions x and u .

Theorem 6 [40, Proposition 5] *For any given $u \in \mathbb{R}_+^m$, $x \in X$ is β -optimal in problem (2.1) if and only if there are ε and δ such that*

$$f(x) + u^T g(x) \leq h(u) + \varepsilon, \quad (2.6a)$$

$$g(x) \leq 0, \quad (2.6b)$$

$$u^T g(x) \geq -\delta, \quad (2.6c)$$

$$\varepsilon + \delta \leq f^* + \beta - h(u), \quad (2.6d)$$

$$\varepsilon, \delta \geq 0. \quad (2.6e)$$

Here, (2.6a) is near-optimality in a Lagrangian relaxation, (2.6b) is primal feasibility in the relaxed constraints and (2.6c) is near-complementarity in the relaxed constraints. System (2.6) reduces to (2.5) if x and u are primal and dual optimal solutions, respectively, and $\beta = 0$.

A main interest in Theorems 5 and 6 are the quantities ε and δ in relation to Γ , the duality gap. The optimality condition in the above mentioned theorems does not provide a required information about the values of these quantities. We need an optimality condition which gives enough information about the values of $\varepsilon(x, u)$ and $\delta(x, u)$ for derivation of methods that finds optimal or near-optimal solution to the primal problem (2.1). This is achieved by introducing the functions $\varepsilon : X \times \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ and $\delta : X \times \mathbb{R}_+^m \rightarrow \mathbb{R}_+$ with

$$\varepsilon(x, u) := f(x) + u^T g(x) - h(u), \quad (2.7)$$

and

$$\delta(x, u) := \sum_{i=1}^m \max\{0, -u_i g_i(x)\}, \quad (2.8)$$

where $\varepsilon(x, u)$ is the degree of near-optimality of an $x \in X$ in a Lagrangian relaxation, and $\delta(x, u)$ is the degree of near-complementarity of $x \in X$ with respect to the Lagrangian dual solution u . Using the functions $\varepsilon(x, u)$ and $\delta(x, u)$, Theorem 6 can be restated as

Proposition 1 *Given $u \in \mathbb{R}_+^m$, then $x \in X$ with $g(x) \leq 0$ is β -optimal in problem (2.1) if and only if*

$$\varepsilon(x, u) + \delta(x, u) \leq f^* + \beta - h(u). \quad (2.9)$$

Figure 2.1 illustrates the results of Proposition 1. If $x^* \in X$ and $u^* \in \mathbb{R}_+^m$ are optimal solutions in problems (2.1) and (2.3), respectively, and if $\beta = 0$, then $\Gamma = f^* - h^* = \varepsilon(x^*, u^*) + \delta(x^*, u^*)$.

Figure 2.1: A plot of a Lagrangian dual function with an illustration of $\varepsilon(x, u)$ and $\delta(x, u)$ in relation to the duality gap Γ for β -optimal solutions.

An application of a Lagrangian relaxation has to a great extent encouraged a discipline of nondifferentiable problems to using subgradients. There are various techniques that can be employed for solving the Lagrangian dual problems, such as subgradient optimization, multiplier adjustment, bundle methods, and cutting planes [19]. In this thesis, we use subgradient optimization to find the dual solutions of the Lagrangian dual problem.

2.6 Subgradient optimization

Subgradient optimization is an iterative approach that can be used, starting with an initial dual solution, to generate a sequence of Lagrangian dual solutions in a systematic way. It is a generalization of the gradient search methods for differentiable functions in which the gradients are replaced by the subgradients. Subgradient optimization has become the most applied technique [19] in the mathematical programming field, especially in a context of Lagrangian relaxation. It is a simplest technique of all dual search methods.

Subgradient optimization, together with Lagrangian relaxation, is often used to find a solution to a Lagrangian dual problem. This algorithm has been, to a large extent, studied for nondifferentiable optimization problems. It has been, practically, useful technique for solving various difficult optimization problems, such as integer programs, see e.g. [32, 37, 24, 42]. A general subgradient optimization is given as follows:

Algorithm 1: Subgradient Optimization

input : Initialize all dual variables $u^0 \in \mathbb{R}_+^m$.

Construct a sequence of dual variables $\{u^k\} \subseteq \mathbb{R}_+^m$ using the rule

$$u^{k+1} = P_{\mathbb{R}_+^m}(u^k + \lambda_k v^k), \quad (2.10)$$

where $P_{\mathbb{R}_+^m}(\cdot)$ is the projection operator on set $\mathbb{R}_+^m$, $\lambda_k > 0$ is a step length and a vector v^k is a step direction obtained at each iteration.

Until stopping criterion.

output: Best Lagrangian dual variables, $\bar{u}$, and the best Lagrangian dual objective value $h(\bar{u})$.

2.7 Types of subgradient optimization

Subgradient optimization algorithms are categorized depending on step direction techniques. Choosing a step direction that affects a computational performances of the algorithm is a challenging task. Here, we present three types of the subgradient optimization with respect to the step direction are presented as traditional subgradient optimization, deflected subgradient optimization, and conditional subgradient optimization [41].

Traditional subgradient optimization is an iterative method used for finding a solution to a Lagrangian dual problem (2.3) in which a subgradient, $\xi^k = g(u^k)$, $u^k \in \mathbb{R}_+^m$, is used as step direction. In (2.10), the step direction v^k is replaced by ξ^k . The Lagrangian dual variables for next iteration is computed according to (2.11).

$$u^{k+1} = P_{\mathbb{R}_+^m}(u^k + \lambda_k \xi^k). \quad (2.11)$$

The use of ξ^k as a step direction leads to zigzagging phenomenon that slows down the convergence of the algorithm. This phenomenon is due to a formation of an obtuse angle between current step direction and the previous step direction. To overcome this situation, another version, called deflected subgradient optimization, is employed.

Deflected subgradient optimization adopts a step direction strategy from the conjugate gradient method, see for example [59], in order to deflect a traditional subgradient direction. This eliminates the zigzagging phenomenon by taking a suitable compromise of a previous step direction and the current step direction. The deflected step direction d^k , at u^k , is calculated as

$$d^k = \xi^k + \vartheta_k d^{k-1}, \quad k \geq 1, \quad (2.12)$$

where $\vartheta_k \geq 0$ is a deflection parameter. The step direction v^k in (2.10) is replaced by d^k in (2.12). Then, Lagrangian dual variables are updated as in equation (2.13)

$$u^{k+1} = P_{\mathbb{R}_+^m}(u^k + \lambda_k d^k). \quad (2.13)$$

Conditional subgradient optimization is used when the convergence of a traditional subgradient optimization becomes slow, especially when an angle formed between a previous and the current step directions is acute. It generalizes the traditional subgradient optimization by taking a set of feasible solutions into account [41].

2.8 Choice of step length

Step length selection depends on some rules which guarantee the convergence of the algorithms [4, 20, 41]. There are different types of step length rules, see for example [48, 7, 2]. Some of these rules are

- i. A constant step size, $\lambda_k = \varphi$ where φ is constant for all k .
- ii. A constant step length, $\lambda_k = \varphi / \|\xi^k\|_2$ where $\varphi = \|u^{k+1} - u^k\|_2$.
- iii. A step length satisfying

$$\lambda_k \rightarrow 0, \quad \text{as } k \rightarrow \infty, \quad \sum_{k=1}^{\infty} \lambda_k = \infty. \quad (2.14)$$

A typical example is $\lambda_k = a/\sqrt{k}$, where $a > 0$ and $k > 0$.

- iv. A step length satisfying

$$\lambda_k \rightarrow 0, \quad \text{as } k \rightarrow \infty, \quad \sum_{k=1}^{\infty} \lambda_k^2 < \infty, \quad \sum_{k=1}^{\infty} \lambda_k = \infty. \quad (2.15)$$

For example $\lambda_k = a/(b+k)$, where $a > 0$ and $b \geq 0$.

- v. A Polyak step length

$$\lambda_k = \theta_k \frac{(\text{UBD} - h(u))}{\xi^k}, \quad 0 < \epsilon_1 \leq \theta_k \leq 2 - \epsilon_2 < 2, \quad \forall k, \quad (2.16)$$

where θ_k is the step length parameter, and UBD is an upper bound to the Lagrangian dual objective value.

2.9 Lagrangian heuristics

In Lagrangian heuristics, a Lagrangian relaxation, together with subgradient optimization method, finds a near-optimal Lagrangian dual solution and an optimistic bound to the optimal objective value of the problem (2.1). The solution found is used in the construction of an optimal or near-optimal solution to problem (2.1). In Lagrangian relaxation, some constraints are relaxed and, the solution generated in each iteration may not be feasible to the primal problem (2.1). Lagrangian

heuristics are designed, using information from the Lagrangian relaxation, to identify a feasible solution for the primal problem (2.1). Note that, the heuristic may not succeed in finding feasible solution.

In [40], Larsson and Patriksson defined a Lagrangian heuristic as follows: “Initiated at a vector in the set defined by the non-relaxed constraints, it adjusts this vector by executing a finite number of steps that have the properties that (a) they utilize information from the Lagrangian dual problem, (b) the sequence of primal vectors generated remains within the set of non-relaxed constraints, and (c) the terminal vector is, if possible, primal feasible and hopefully also near-optimal in the primal problem”.

A Lagrangian heuristics can be implemented within a subgradient optimization framework. Taking the feasible solution with respect to non-relaxed constraint but infeasible to the relaxed constraints, the heuristic tries to convert into feasible solution in the relaxed constraints while maintaining the non-relaxed constraints fulfilled. The heuristic can be applied at every subgradient iteration or for some selected iterations, see Figure 2.2 for an illustration. Every time the Lagrangian heuristics succeeds, the upper bound is recorded. The best out of the recorded upper bounds is a pessimistic bound. The design of a Lagrangian heuristic depends much on applications. Many applications of Lagrangian heuristics can be found in the literature, for example [24, 5, 39, 10, 34, 22].

Figure 2.2: Lagrangian-based heuristic in which the Lagrangian heuristic is implemented within a subgradient optimization framework.

3 – Pareto frontier

3.1 Introduction

Multi-Objective Optimization (MOO) problem deal with solving problems having multiple, often conflicting, objective functions. Usually, it is not within reach to find the optimal solution of the conflicting objective functions simultaneously. The only hope is to find the trade-off solution of these multiple competing objective functions. Optimizing the trade-off of the conflicting objective functions produces the set of solutions called Pareto optimal solution set or efficient solution set. The projection of the Pareto optimal solution set in an objective space is called Pareto front or nondominated set. The Pareto front is bounded, see for example [44, 9].

The MOO problem is crucial as it appears in many real-life problems. They have been applied in many fields of science such as in engineering, optimal design, process optimization, finance and logistics, see e.g., [43, 25, 23, 53, 54, 47].

3.2 Formulation of multi-objective problem

Consider a general MOO problem

$$\begin{aligned} (\text{P}) \quad & \min f(x) = \{f_1(x), \dots, f_k(x)\} \\ & \text{s.t. } x \in \Omega := \{x \in X \mid g_i(x) \leq 0, i = 1, \dots, m\}, \end{aligned} \quad (3.1)$$

where $k \geq 2$. Assume that the functions $f_1(x), \dots, f_k(x)$ are bounded below on constraint set Ω . Let objective space, F , be defined as $F := \{f(x) \mid x \in \Omega\}$.

Definition 3 [8] *A feasible solution $\bar{x} \in \Omega$ is said to be Pareto optimal for problem (3.1) if and only if there is no $x \in \Omega$ such that $f_j(x) \leq f_j(\bar{x})$, for all $j \in \{1, \dots, k\}$ and with at least one strictly inequality.*

A set of Pareto solutions of problem (3.1) in Ω is denoted by $E(\text{P})$. A projection of $E(\text{P})$ in F is called nondominated solution set [49].

3.3 Solution methods

A purpose of solving multiobjective problems is to provide a decision maker with a set of Pareto optimal solutions to choose from according to his/her personal preferences. The MOO methods for selecting best trade-off solutions from all nondominated alternatives can be categorized into three classes, namely a priori methods (such as weighted-sum, ϵ -constraint and utility function), see e.g., [50], a posteriori methods (such as evolutionary algorithm and mathematical programming-based)

[21, 13] and interactive methods [55]. More information of these solution methods can be found in, for example [11, 17, 16, 14, 44].

Considering a biobjective problem with the objective space given by $F := \{(f_1(x), f_2(x)) \mid x \in \Omega\}$, where X is finite, the graphical representation of the Pareto front on F is

Figure 3.1: A star represents dominated points, and the black dot represents Pareto front.

In this thesis, we focus on two priori methods, which are weighted sum method, and ϵ -constraint method.

3.3.1 Weighted-sum method

This method scalarizes problem (3.1) by attaching a nonnegative weights to each objective function and then minimizes the weighted-sum of the objective functions. For any given weights $w \in W$ where

$$W := \left\{ w \in \mathbb{R}_+^k \mid w_j \geq 0, \sum_{j=1}^k w_j = 1 \right\},$$

the weighted-sum problem is given by

$$\begin{aligned} \min \quad & \sum_{j=1}^k w_j f_j(x) \\ \text{s.t.} \quad & x \in \Omega. \end{aligned} \tag{3.2}$$

The solution to problem (3.2) is on the Pareto front for any choice of $w \in W$, see for example [18]. However, the weighted-sum method cannot find certain Pareto optimal solutions in case of a nonconvex objective space. The weights are chosen in proportion to the relative importance of the objective functions. Consider the objective space F as in Figure 3.1. Figure 3.2 shows that the weighted-sum method provides the Pareto optimal front of the weighted-sum objective functions, that is, $w_1 f_1(x) + w_2 f_2(x)$.

Figure 3.2: The Pareto optimal front obtained by the weighted-sum method

3.3.2 ϵ -constraint method

This method keep one objective function to be optimized and convert the rest of the objective functions into additional constraints, see e.g. [18]. Each of the additional constraint is restricted within user-specified value. Given a constants $\epsilon_j \in \mathbb{R}$, $j = 1, \dots, k$, ϵ -constraint problem is given by

$$\begin{aligned} \min \quad & f_1(x) \\ \text{s.t.} \quad & f_j(x) \leq \epsilon_j, \quad j = 1, \dots, k-1, \\ & x \in \Omega. \end{aligned} \tag{3.3}$$

Different points can be obtained on the Pareto front by varying $\epsilon_j \in \mathbb{R}$, $j = 1, \dots, k-1$. Considering again a biobjective problem, the method is illustrated in Figure 3.3. It shows that the method provides all points on the Pareto front.

Figure 3.3: The points on the Pareto optimal front are obtained by varying ϵ_2 .

4 – Applications

The main problems that we use as a test examples in this thesis are assignment problem, set covering problem and biobjective set covering problem.

4.1 Assignment problem

Consider a Generalized Assignment Problem (GAP) with m machines and n jobs,

$$z^* = \min z(x) = \sum_{i=1}^m \sum_{j=1}^n c_{ij} x_{ij}$$

$$\text{s.t. } \sum_{j=1}^n a_{ij} x_{ij} \leq b_i, \quad i = 1, \dots, m, \quad (4.1a)$$

$$\sum_{i=1}^m x_{ij} = 1, \quad j = 1, \dots, n, \quad (4.1b)$$

$$x_{ij} = 0/1, \quad i = 1, \dots, m, \quad j = 1, \dots, n. \quad (4.1c)$$

The constraints (4.1b) are relaxed with Lagrangian dual variables $u \in \mathbb{R}^n$. The Lagrangian dual function is $h : \mathbb{R}^n \rightarrow \mathbb{R}$ with

$$h(u) = \sum_{j=1}^n u_j + \sum_{i=1}^m h_i(u), \quad (4.2)$$

where $h_i(u) = \min_{x_{ij} \in \{0,1\}^n} \sum_{j=1}^n (c_{ij} - u_j) x_{ij}$, $i = 1, \dots, m$. The Lagrangian dual problem becomes

$$h^* = \max_{u \in \mathbb{R}_+^n} h(u). \quad (4.3)$$

A generalized global optimality condition where the duality gap $\Gamma : z^* - h^* > 0$ for problem (4.1) is given in Proposition 2.

Proposition 2 *Suppose that x and u are non-optimal solutions in problems (4.1) and (4.3) respectively. For any given $u \in \mathbb{R}^n$, $x \in \Sigma$ is β -optimal in problem*

(4.1) if and only if there are ε such that

$$\sum_{j=1}^n u_j + \sum_{j=1}^n (c_{ij} - u_j) x_{ij} \leq h_i(u) + \varepsilon_i, \quad i = 1, \dots, m, \quad (4.4a)$$

$$1 - \sum_{i=1}^m x_{ij} = 0, \quad j = 1, \dots, n, \quad (4.4b)$$

$$\sum_{i=1}^m \varepsilon_i \leq z^* + \beta - h(u), \quad (4.4c)$$

$$\varepsilon_i \geq 0 \quad i = 1, \dots, m. \quad (4.4d)$$

4.2 Lagrangian heuristic for GAP

The Lagrangian Heuristic (LH) using solutions of the Lagrangian relaxation problem strive, if possible, to generate feasible solutions for the original GAP. Given a vector $\bar{u} \in \mathbb{R}_+^m$, found by subgradient optimization method on the Lagrangian dual problem (4.3), the LH is presented in Algorithm 2.

Algorithm 2: LH for GAP

```

input :  $\bar{u}, x(\bar{u})$ 
let  $\bar{x} = x(\bar{u})$ 
for each  $k$  such that  $\sum_{i=1}^m \bar{x}_{ik} > 1$  do
    find  $i_k \in \arg \min_{i: \bar{x}_{ik}=1} \frac{(c_{ik} - \bar{u}_k)}{a_{ik}}$ 
    set  $\bar{x}_{ik} = 0, i \neq i_k$ 
end
for each  $k$  such that  $\sum_{i=1}^m \bar{x}_{ik} = 0$  do
    let  $I_k = \{i = 1, \dots, m \mid \sum_{j=1}^n a_{ij} \bar{x}_{ij} - a_{ik} \leq b_i\}$ 
    if  $I_k \neq \emptyset$  then
        let  $i_k \in \arg \min_{i \in I_k} \frac{(c_{ik} - \bar{u}_k)}{a_{ik}}$ 
        set  $\bar{x}_{i_k, k} = 1$ 
    end
    if  $I_k = \emptyset$  then
        heuristic fails
    end
end
 $x_h(\bar{u}) = \bar{x}$ 
if heuristic succeeds then
     $z_h(\bar{u}) = \sum_{i=1}^m \sum_{j=1}^n c_{ij} \bar{x}_{ij}$ 
else
     $z_h(\bar{u}) = +\infty$ 
end
output:  $z_h(\bar{u})$ .

```

4.3 Set covering problem

The Set Covering Problem (SCP) is one of the most well-known combinatorial optimization problems, with many applications, within a field of operations research, see for example [33, 56]. Consider a general SCP.

$$\begin{aligned} z^* = \min z(x) &= \sum_{j=1}^n c_j x_j \\ \text{s.t. } \sum_{j=1}^n a_{ij} x_j &\geq 1, \quad i = 1, \dots, m, \end{aligned} \quad (4.5a)$$

$$x_j = \{0, 1\}, \quad j = 1, \dots, n. \quad (4.5b)$$

Let $u \in \mathbb{R}_+^m$ be a vector of Lagrangian dual variables associated with constraints (4.5b), $M = \{1, \dots, m\}$, $N = \{1, \dots, n\}$, $S_j = \{i \in M \mid a_{ij} = 1\} \subseteq M$. For notational convenience, for each row $i \in M$, let $N_i = \{j \in N \mid a_{ij} = 1\}$. The Lagrangian dual function is $h : \mathbb{R}_+^m \rightarrow \mathbb{R}$ with

$$h(u) = \sum_{i=1}^m u_i + \sum_{j=1}^n h_j(u), \quad (4.6)$$

where $h_j(u) = \min_{x \in \{0,1\}^n} (c_j - \sum_{i=1}^m u_i a_{ij}) x_j$. The Lagrangian dual problem for (4.5) is defined as

$$h^* = \max_{u \in \mathbb{R}_+^m} h(u). \quad (4.7)$$

A generalized global optimality condition in which the duality gap $\Gamma : z^* - h^* > 0$ for problem (4.5) is stated in Proposition 3.

Proposition 3 *Suppose that x and u are non-optimal solutions in problems (4.5) and (4.7) respectively. For any given $u \geq 0$, $x \in \{0, 1\}^n$ is β -optimal in (4.5) if and only if there are ε and δ such that*

$$\left(c_j - \sum_{i=1}^m u_i a_{ij} \right) x_j + \sum_{i=1}^m u_i \leq h_j(u) + \varepsilon_j, \quad j = 1, \dots, n, \quad (4.8a)$$

$$1 - \sum_{j=1}^n a_{ij} x_j \leq 0, \quad i = 1, \dots, m, \quad (4.8b)$$

$$u_i \left(1 - \sum_{j=1}^n a_{ij} x_j \right) \geq -\delta_i, \quad i = 1, \dots, m, \quad (4.8c)$$

$$\sum_{j=1}^n \varepsilon_j + \sum_{i=1}^m \delta_i \leq z^* + \beta - h(u), \quad (4.8d)$$

$$\varepsilon, \delta \geq 0. \quad (4.8e)$$

4.4 Lagrangian heuristic for SCP

The LH used to construct a feasible solution for the original SCP (4.5) from the solution of the Lagrangian relaxation problem (4.6) is given in Algorithm 3. This heuristic uses standard reduced cost to set a variable to one. Let a vector $\bar{u} \in \mathbb{R}_+^m$, $\bar{c}_j = c_j - \sum_{i=1}^m u_i a_{ij}$, for all $j \in N$, $x(\bar{u})$, $N' = \{j \in N \mid x_j(\bar{u}) = 0\}$, and $M' = M - \{S_j\}$, for all $j \in N'$. The LH that uses partial reduced cost to set a variable to one is given in Algorithm 4.

Algorithm 3: LH for SCP with standard reduced cost

input : $\bar{u}$, $\bar{x} = x(\bar{u})$, $\bar{c}$, M' , N' .

while $M' \neq 0$ **do**

compute $j^* := \arg \min_{j \in N'} (\bar{c}_j / |S_j \cap M'|)$

Set $\bar{x}_{j^*} = 1$, $N' = N' - \{j^*\}$ and $M' = M' - \{S_{j^*}\}$

end

if *Redundant column j exists* **then**

set $\bar{x}_j = 0$

end

result: $\bar{x}$

calculate $z(\bar{x}) = \sum_{j=1}^n c_j \bar{x}_j$.

output: $\bar{x}$, and $z(\bar{x})$.

Algorithm 4: LH for SCP with partial reduced cost

input : $\bar{u}$, $\bar{x} = x(\bar{u})$, $\bar{c}$, M' , N' .

while $M' \neq 0$ **do**

compute $j^* := \arg \min_{j \in N'} \left(\left(c_j - \sum_{i \in S_j \cap M'} u_i \right) / |S_j \cap M'| \right)$

Set $\bar{x}_{j^*} = 1$, $N' = N' - \{j^*\}$ and $M' = M' - \{S_{j^*}\}$

end

if *Redundant column j exists* **then**

set $\bar{x}_j = 0$

end

result: $\bar{x}$

calculate $z(\bar{x}) = \sum_{j=1}^n c_j \bar{x}_j$.

output: $\bar{x}$, and $z(\bar{x})$.

4.5 Bi-objective set covering problem

The Bi-Objective Set Covering Problem (BiSCP) has the same structure as the ordinary set covering problem, except that BiSCP has two objective functions,

that is, each column is associated with two costs, see for example [49]. A BiSCP is given by

$$\begin{aligned} \min z_1(x) &= \sum_{j=1}^n c_j^1 x_j \\ \min z_2(x) &= \sum_{j=1}^n c_j^2 x_j \\ \text{s.t. } \sum_{j=1}^n a_{ij} x_j &\geq 1, \quad i = 1, \dots, m, \end{aligned} \tag{4.9a}$$

$$x_j = \{0, 1\}, \quad j = 1, \dots, n, \tag{4.9b}$$

where n is the number of decision variables, and m is the number of constraints.

The weighted-sum method: This method converts the objective functions in problem (4.9) into single-objective function by associating the weights w_1 and w_2 to $z_1(x)$ and $z_2(x)$, respectively, and then minimizes the weighted-sum of the objective functions. The weighted-sum problem is defined by

$$\begin{aligned} \min w_1 z_1(x) + w_2 z_2(x) &= w_1 \sum_{j=1}^n c_j^1 x_j + w_2 \sum_{j=1}^n c_j^2 x_j \\ \text{s.t. } \sum_{j=1}^n a_{ij} x_j &\geq 1, \quad i = 1, \dots, m, \end{aligned} \tag{4.10a}$$

$$x_j = \{0, 1\}, \quad j = 1, \dots, n, \tag{4.10b}$$

where $w_1, w_2 \in W = \{w \in \mathbb{R}_+^2 \mid w_1, w_2 \geq 0, w_1 + w_2 = 1\}$. Problem (4.10) can be optimized using the optimization methods for the single-objective function.

ϵ -constraint method: The method converts a BiSCP into a constraint single-objective optimization problem. Given an $\epsilon_2 \in \mathbb{R}$, the ϵ -constraint problem becomes

$$\begin{aligned} \min z_1(x) &= \sum_{j=1}^n c_j^1 x_j \\ \text{s.t. } \sum_{j=1}^n c_j^2 x_j &\leq \epsilon_2 \end{aligned} \tag{4.11a}$$

$$\sum_{j=1}^n a_{ij} x_j \geq 1, \quad i = 1, \dots, m, \tag{4.11b}$$

$$x_j = \{0, 1\}, \quad j = 1, \dots, n. \tag{4.11c}$$

The parameter $\epsilon_2 \in \mathbb{R}$ is specified by the decision maker according to his/her preferences.

5 – Bibliography

Bibliography

- [1] Altay, N., Robinson Jr, P.E., Bretthauer, K.M.: Exact and heuristic solution approaches for the mixed integer setup knapsack problem. *European Journal of Operational Research* **190**(3), 598–609 (2008)
- [2] Anstreicher, K.M., Wolsey, L.A.: Two “well-known” properties of subgradient optimization. *Mathematical Programming* **120**(1), 213–220 (2009)
- [3] Bazaraa, M.S., Goode, J.J.: A survey of various tactics for generating Lagrangian multipliers in the context of lagrangian duality. *European Journal of Operational Research* **3**(4), 322–338 (1979)
- [4] Bazaraa, M.S., Sherali, H.D.: On the choice of step size in subgradient optimization. *European Journal of Operational Research* **7**(4), 380–388 (1981)
- [5] Beasley, J.E.: Lagrangean heuristics for location problems. *European Journal of Operational Research* **65**(3), 383–399 (1993)
- [6] Bertsekas, D.P.: *Constrained optimization and Lagrange multiplier methods*. Academic press (2014)
- [7] Boyd, S., Xiao, L., Mutapcic, A.: Subgradient methods. *Lecture Notes of EE392o, Stanford University, Autumn Quarter* **2004**, 2004–2005 (2003)
- [8] Burachik, R.S., Kaya, C.Y., Rizvi, M.: A new scalarization technique to approximate pareto fronts of problems with disconnected feasible sets. *Journal of Optimization Theory and Applications* **162**(2), 428–446 (2014)
- [9] Burachik, R.S., Kaya, C.Y., Rizvi, M.M.: Algorithms for generating pareto fronts of multi-objective integer and mixed-integer programming problems. *arXiv preprint arXiv:1903.07041* (2019)
- [10] Caprara, A., Fischetti, M., Toth, P.: A heuristic method for the set covering problem. *Operations Research* **47**(5), 730–743 (1999)
- [11] Ceria, S., Nobili, P., Sassano, A.: A Lagrangian-based heuristic for large-scale set covering problems. *Mathematical Programming* **81**(2), 215–228 (1998)
- [12] Chhajer, D., Lowe, T.J.: Solving a selected class of location problems by exploiting problem structure: A decomposition approach. *Naval Research Logistics (NRL)* **45**(8), 791–815 (1998)
- [13] Coello, C.A.C., Lamont, G.B., Van Veldhuizen, D.A.: *Evolutionary algorithms for solving multi-objective problems*, vol. 5. Springer (2007)

- [14] Cohon, J.L.: Multiobjective programming and planning, vol. 140. Courier Corporation (2004)
- [15] Cornuejols, G., Fisher, M.L., Nemhauser, G.L.: Exceptional paper-location of bank accounts to optimize float: An analytic study of exact and approximate algorithms. *Management Science* **23**(8), 789–810 (1977)
- [16] Das, I.: On characterizing the “knee” of the pareto curve based on normal-boundary intersection. *Structural Optimization* **18**(2-3), 107–115 (1999)
- [17] Das, I., Dennis, J.E.: Normal-boundary intersection: A new method for generating the pareto surface in nonlinear multicriteria optimization problems. *SIAM Journal on Optimization* **8**(3), 631–657 (1998)
- [18] Emmerich, M.T., Deutz, A.H.: A tutorial on multiobjective optimization: fundamentals and evolutionary methods. *Natural Computing* **17**(3), 585–609 (2018)
- [19] Fisher, M.L.: The Lagrangian relaxation method for solving integer programming problems. *Management Science* **27**(1), 1–18 (1981)
- [20] Fisher, M.L.: An applications oriented guide to Lagrangian relaxation. *Interfaces* **15**(2), 10–21 (1985)
- [21] Fonseca, C.M., Fleming, P.J.: Multiobjective optimization and multiple constraint handling with evolutionary algorithms. i. a unified formulation. *IEEE Transactions on Systems, Man, and Cybernetics-Part A: Systems and Humans* **28**(1), 26–37 (1998)
- [22] Fonseca, G.B., Nogueira, T.H., Ravetti, M.G.: A hybrid Lagrangian meta-heuristic for the cross-docking flow shop scheduling problem. *European Journal of Operational Research* **275**(1), 139–154 (2019)
- [23] Galceran, E., Carreras, M.: A survey on coverage path planning for robotics. *Robotics and Autonomous Systems* **61**(12), 1258–1276 (2013)
- [24] Galvão, R.D., Santibanez-Gonzalez, E.d.R.: A Lagrangean heuristic for the pk-median dynamic location problem. *European Journal of Operational Research* **58**(2), 250–262 (1992)
- [25] Ganesan, T., Elamvazuthi, I., Shaari, K.Z.K., Vasant, P.: Swarm intelligence and gravitational search algorithm for multi-objective optimization of synthesis gas production. *Applied Energy* **103**, 368–374 (2013)
- [26] Geoffrion, A.M.: Lagrangean relaxation for integer programming. In: *Approaches to Integer Programming*, pp. 82–114. Springer (1974)
- [27] Grigoriadis, M.D., Khachiyan, L.G.: Coordination complexity of parallel price-directive decomposition. *Mathematics of Operations Research* **21**(2), 321–340 (1996)

- [28] Guignard, M., Kim, S.: Lagrangean decomposition: A model yielding stronger Lagrangean bounds. *Mathematical Programming* **39**(2), 215–228 (1987)
- [29] Guignard, M., Rosenwein, M.B.: An improved dual based algorithm for the generalized assignment problem. *Operations Research* **37**(4), 658–663 (1989)
- [30] Held, M., Karp, R.M.: The traveling-salesman problem and minimum spanning trees. *Operations Research* **18**(6), 1138–1162 (1970)
- [31] Held, M., Karp, R.M.: The traveling-salesman problem and minimum spanning trees: Part ii. *Mathematical Programming* **1**(1), 6–25 (1971)
- [32] Held, M., Wolfe, P., Crowder, H.P.: Validation of subgradient optimization. *Mathematical Programming* **6**(1), 62–88 (1974)
- [33] Hoffman, K.L., Padberg, M.: Set covering, packing and partitioning problems. *Encyclopedia of Optimization* **5** (2009)
- [34] Holmberg, K., Yuan, D.: A Lagrangian heuristic based branch-and-bound approach for the capacitated network design problem. *Operations Research* **48**(3), 461–481 (2000)
- [35] Imai, A., Nishimura, E., Current, J.: A Lagrangian relaxation-based heuristic for the vehicle routing with full container load. *European Journal of Operational Research* **176**(1), 87–105 (2007)
- [36] Jörnsten, K., Näsberg, M.: A new Lagrangian relaxation approach to the generalized assignment problem. *European Journal of Operational Research* **27**(3), 313–323 (1986)
- [37] Klastorin, T.D.: An effective subgradient algorithm for the generalized assignment problem. *Computers & Operations Research* **6**(3), 155–164 (1979)
- [38] Larsson, T., Liu, Z.: A Lagrangean relaxation scheme for structured linear programs with application to multicommodity network flows. *Optimization* **40**(3), 247–284 (1997)
- [39] Larsson, T., Migdalas, A., Rönnqvist, M.: A Lagrangean heuristic for the capacitated concave minimum cost network flow problem. *European Journal of Operational Research* **78**(1), 116–129 (1994)
- [40] Larsson, T., Patriksson, M.: Global optimality conditions for discrete and nonconvex optimization—with applications to Lagrangian heuristics and column generation. *Operations Research* **54**(3), 436–453 (2006)
- [41] Larsson, T., Patriksson, M., Strömberg, A.B.: Conditional subgradient optimization—theory and applications. *European Journal of Operational Research* **88**(2), 382–403 (1996)

- [42] Larsson, T., Patriksson, M., Strömberg, A.B.: Ergodic, primal convergence in dual subgradient schemes for convex programming. *Mathematical Programming* **86**(2), 283–312 (1999)
- [43] Mendoza Baeza, J., Lopez, M.E., Coello Coello, C.A., Lopez, E.A.: Microgenetic multiobjective reconfiguration algorithm considering power losses and reliability indices for medium voltage distribution network. *IET Generation, Transmission & Distribution* **3**(9), 825–840 (2009)
- [44] Mueller-Gritschneider, D., Graeb, H., Schlichtmann, U.: A successive approach to compute the bounded pareto front of practical multiobjective optimization problems. *SIAM Journal on Optimization* **20**(2), 915–934 (2009)
- [45] Nishi, T., Konishi, M.: An augmented lagrangian approach for scheduling problems. *JSME International Journal Series C Mechanical Systems, Machine Elements and Manufacturing* **48**(2), 299–304 (2005)
- [46] Önnheim, M., Gustavsson, E., Strömberg, A.B., Patriksson, M., Larsson, T.: Ergodic, primal convergence in dual subgradient schemes for convex programming, ii: the case of inconsistent primal problems. *Mathematical Programming* **163**(1-2), 57–84 (2017)
- [47] Pllana, S., Memeti, S., Kolodziej, J.: Customizing pareto simulated annealing for multi-objective optimization of control cabinet layout. In: 2019 22nd International Conference on Control Systems and Computer Science (CSCS), pp. 78–85. IEEE (2019)
- [48] Polyak, B.T.: Minimization of unsmooth functionals. *USSR Computational Mathematics and Mathematical Physics* **9**(3), 14–29 (1969)
- [49] Prins, C., Prodhon, C., Calvo, R.W.: Two-phase method and lagrangian relaxation to solve the bi-objective set covering problem. *Annals of Operations Research* **147**(1), 23–41 (2006)
- [50] Sanchis, J., Martínez, M.A., Blasco, X.: Integrated multiobjective optimization and a priori preferences using genetic algorithms. *Information Sciences* **178**(4), 931–951 (2008)
- [51] Shapiro, J.F.: *Mathematical Programming: Structures and Algorithms*. Wiley (1979)
- [52] Shepardson, F., Marsten, R.E.: A Lagrangean relaxation algorithm for the two duty period scheduling problem. *Management Science* **26**(3), 274–281 (1980)
- [53] Shirazi, A., Najafi, B., Aminyavari, M., Rinaldi, F., Taylor, R.A.: Thermal-economic-environmental analysis and multi-objective optimization of an ice thermal energy storage system for gas turbine cycle inlet air cooling. *Energy* **69**, 212–226 (2014)

- [54] Smimou, K.: International portfolio choice and political instability risk: A multi-objective approach. *European Journal of Operational Research* **234**(2), 546–560 (2014)
- [55] Thiele, L., Miettinen, K., Korhonen, P.J., Molina, J.: A preference-based evolutionary algorithm for multi-objective optimization. *Evolutionary Computation* **17**(3), 411–436 (2009)
- [56] Umetani, S., Yagiura, M.: Relaxation heuristics for the set covering problem. *Journal of the Operations Research Society of Japan* **50**(4), 350–375 (2007)
- [57] Wentges, P.: Weighted Dantzig-Wolfe decomposition for linear mixed-integer programming. *International Transactions in Operational Research* **4**(2), 151–162 (1997)
- [58] Wolsey, L.A.: *Integer programming*. Wiley (1998)
- [59] Wright, S., Nocedal, J.: *Numerical optimization*. Springer Science **35**(67-68), 7 (1999)
- [60] Xie, C., Turnquist, M.A., Waller, S.T.: A hybrid Lagrangian relaxation and tabu search method for interdependent-choice network design problems. In: *Hybrid Algorithms for Service, Computing and Manufacturing Systems: Routing and Scheduling Solutions*, pp. 294–324. IGI Global (2012)

Papers

The papers associated with this thesis have been removed for copyright reasons. For more details about these see:

<http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-175896>

FACULTY OF SCIENCE AND ENGINEERING

Linköping Studies in Science and Technology, Licentiate
Thesis No. 1910, 2021
Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden

www.liu.se