Smooth Schubert varieties and boolean complexes of involutions

Vincent Umutabazi

Linköping Studies in Science and Technology.
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Department of Mathematics
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Abstract

This thesis is composed of two papers both in algebraic combinatorics and Coxeter groups.

In Paper I, we concentrate on smoothness of Schubert varieties indexed by involutions from finite simply laced types. We show that if a Schubert variety indexed by an involution of a finite and simply laced Coxeter group is smooth, then that involution must be the longest element of a parabolic subgroup.

Given a Coxeter system (W, S) , we introduce in Paper II the boolean complex of involutions of W as an analogue of the boolean complex of W studied by Ragnarsson and Tenner. By using discrete Morse Theory, we compute the homotopy type for a large class of W , including all finite Coxeter groups. In all cases, the homotopy type is that of a wedge of spheres of dimension $|S| - 1$. In addition, we provide a recurrence formula for the number of spheres in the wedge.

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1 – Introduction

This chapter introduces basic facts and definitions on Coxeter groups, Schubert varieties, and boolean complexes of a Coxeter system.

1.1 Coxeter groups and Bruhat graphs

Let W be a group with generating set S , where the relations in W are given by $(ss')^{m(s,s')} = e$, where e is the identity element, $m(s, s) = 1$, $m(s, s') = m(s', s) \in \{2, 3, \dots\} \cup \{\infty\}$ for all $s \neq s' \in S$. Then W is called a *Coxeter group* and the pair (W, S) is called a *Coxeter system*. The number of elements of S (i.e., $|S|$) is called the *rank* of (W, S) . Every element $w \in W$ can be written as $w = s_1 s_2 \cdots s_k$ for some $s_j \in S$. If k is minimal among all such expressions for w then it is called the *length* of w , and is denoted $\ell(w)$. Any expression $s_1 s_2 \cdots s_j$ for w such that $j = \ell(w)$ is called a *reduced expression*. If W is finite, it has a unique element w_0 called *longest element* for which $\ell(w_0) \geq \ell(w)$ for every $w \in W$, and $\ell(sw_0) < \ell(w_0)$ for every $s \in S$.

Definition 1.1.1 Let $J \subseteq S$. The subgroup $W_J \subseteq W$ generated by J is called a parabolic subgroup, and the pair (W_J, J) is called a Coxeter subsystem.

Denote the longest element of a finite parabolic subgroup W_J by $w_0(J)$.

Definition 1.1.2 The Coxeter graph of (W, S) , denoted G_W , is the graph whose set of vertices is S , and whose edges are pairs $\{s, s'\}$ such that $m(s, s') \geq 3$. If $m(s, s') = 2$, i.e., s and s' commute then there is no edge between s and s' in G_W . An edge $\{s, s'\}$ with $m(s, s') = 3$ is unlabelled, but an edge with $m(s, s') \geq 4$ is labelled by that number.

A Coxeter system (W, S) is *irreducible* if its Coxeter graph is connected. Irreducible finite Coxeter groups have been classified, see [2] and [10]. In Figure 1.1, we present these groups by their associated Coxeter graphs. There are three classical families of groups of types A_n ($n \geq 1$), B_n ($n \geq 2$), and D_n ($n \geq 4$), six exceptional groups of types E_6 , E_7 , E_8 , F_4 , H_3 , H_4 and the dihedral group of type $I_2(m)$ ($m \geq 3$).

Figure 1.1: Finite irreducible Coxeter groups.

The set $T := \{wsw^{-1} | w \in W, s \in S\}$ is called the set of *reflections* in W .

Definition 1.1.3 For $u, v \in W$, let us write:

1. $u \rightarrow w$ if $ut = w$ where $t \in T$ and $\ell(u) < \ell(w)$,
2. $u \leq w$ if $u = u_0 \rightarrow u_1 \cdots \rightarrow u_k = w$ for some $u_i \in W$.

The Bruhat graph of (W, S) denoted $Bg_S(W)$ is the directed graph whose set of vertices is W and whose set of edges is $\{(u, w) | u \rightarrow w\}$. The Bruhat order is the partial order on W given by (2).

Example 1.1.4 Consider the symmetric group S_4 whose generating set S is the set of adjacent transpositions $s_i = (i, i+1)$, $1 \leq i \leq 3$. Every permutation $w \in S_4$ is a composition of adjacent transpositions $s_i \in S$. The set T of reflections of S_4 is the set of all transpositions $T = \{(i, j) \in S_4 : 1 \leq i < j \leq 4\}$. In fact, since we have relations of the form $s_j s_k = s_k s_j$ if $|j - k| \geq 2$, and $s_j s_k s_j = s_k s_j s_k$ if

$|j - k| = 1$, then (S_4, S) is a Coxeter system of type A_3 . On Figure 1.2, T is marked by permutations of yellow color, the Bruhat order is indicated by the poset with black edges only while the Bruhat graph $Bg_S(S_4)$ is indicated by the whole poset with both black and red edges.

Figure 1.2: Bruhat graph and Bruhat order of S_4 .

Theorems 1.1.5 and 1.1.6, which can be found in [2], are fundamental facts of the Bruhat order.

Theorem 1.1.5 (Subword Property) *Let $s_1s_2 \cdots s_m$ be a reduced expression for $w \in W$. Then, $v \leq w$ if and only if there exists a reduced expression $s_{i_1}s_{i_2} \cdots s_{i_k}$ for v such that $1 \leq i_1 \leq \cdots i_k \leq m$.*

Theorem 1.1.6 (Chain Property) *Let $v, w \in W$ and $v < w$. Then there exists a chain $v = v_0 < v_1 < \cdots < v_k = w$ such that $\ell(v_j) = \ell(v) + j$, for $1 \leq j \leq k$.*

Define the *left descent set* of w by $D_L(w) = \{s \in S : \ell(sw) < \ell(w)\}$ and the *right descent set* of w by $D_R(w) = \{s \in S : \ell(ws) < \ell(w)\}$, respectively. Lemma 1.1.7, which can be found in [2], is used to characterise the Bruhat order.

Lemma 1.1.7 (Lifting property) *Let $u < w$ and $s \in D_L(w) \setminus D_L(u)$. Then $u \leq sw$ and $su \leq w$.*

1.1.1 Reflection subgroups of Coxeter groups

Let W' be a subgroup of W . Then W' is called a *reflection subgroup* of W if $W' = \langle W' \cap T \rangle$. The reflection subgroup W' is called *dihedral* if $W' = \langle t, t' \rangle$ for distinct reflections t, t' in T . For more on reflection subgroups, see [6].

Let (W, S) be a Coxeter system with $m(s, s') \leq 3$ for all $s, s' \in S$. We call such a system *simply laced*.

Lemma 1.1.8 *Let W be finite and simply laced. Then every reflection subgroup of W is simply laced.*

Define the set of *inversions* of $w \in W$ by $N(w) := \{t \in T : \ell(tw) < \ell(w)\}$, and for a reflection subgroup W' of W define also $X := \{t \in T : N(t) \cap W' = \{t\}\}$.

Dyer in [7] proved that (W', X) is a Coxeter system. For any $V \subseteq W$, let $\text{Bg}_S(V)$ be the Bruhat graph of V (i.e., the directed subgraph of the Bruhat graph of W induced by V).

Theorem 1.1.9 [7] *Let W' be a reflection subgroup of W . Then $\text{Bg}_S(W') = \text{Bg}_X(W')$.*

1.1.2 Schubert varieties

Let G be an algebraic group over $\mathbb{C}$. Fix $\mathcal{T} \subset B \subset G$ where $\mathcal{T}$ is a maximal torus, and B a Borel subgroup of G . Then, G/B is called the *flag variety*. Let $\mathcal{N}(\mathcal{T})$ be the normalizer of $\mathcal{T}$ in G . We have $W = \mathcal{N}(\mathcal{T})/\mathcal{T}$ where W is a Coxeter group. For every $w \in W$, the set BwB/B is called a *Schubert cell* and its closure $X_w := \overline{BwB}/B$ is called a *Schubert variety*. The flag variety G/B is the disjoint union of Schubert cells. In fact, for w_0 the longest element of W , $X_{w_0} = G/B$ is the full flag variety, see [1].

Let $[e, w] := \{x \in W : e \leq x \leq w\}$, $\text{Bg}_S(w) := \text{Bg}_S([e, w])$, and z be a vertex in $\text{Bg}_S(w)$.

Definition 1.1.10 *The degree of z in $\text{Bg}_S(w)$ is the number of edges that are incident to z . Here the direction of edges is not considered.*

Theorem 1.1.11 is due to Dyer in [8]. When W is a (finite) Weyl group, this theorem is also proved in [3] and [12].

Theorem 1.1.11 [8] *Let $w \in W$. Then the degree of any vertex in $\text{Bg}_S(w)$ is at least $\ell(w)$.*

Bruhat graphs are used to detect the singularities of Schubert varieties.

Theorem 1.1.12 [3] *The Schubert variety X_w is rationally smooth if and only if $\text{Bg}_S(w)$ is regular, i.e., every vertex has the same number of edges.*

Let $\deg_w(v)$ denote the degree of v in $\text{Bg}_S(w)$. We have that $\deg_w(w) = \ell(w)$. Hence $\text{Bg}_S(w)$ is regular if and only if $\deg_w(v) = \ell(w)$ for every vertex $v \in \text{Bg}_S(w)$.

By Carrell and Kuttler [4], over simply laced Coxeter groups every rationally smooth Schubert variety is smooth.

Corollary 1.1.13 *Let W be a finite Coxeter group of simply laced type and $w \in W$. Then, the Schubert variety X_w is smooth if and only if $\text{Bg}_S(w)$ is regular.*

Example 1.1.14 *Suppose W is of type A_2 . Its Bruhat graph is depicted in Figure 1.3. Using Corollary 1.1.13, we see that X_w is smooth for every $w \in W$ in this case.*

Figure 1.3: $\text{Bg}_S(W(A_2))$

1.2 Matchings and cell complexes

Let P be a poset and $a, b \in P$ and $a < b$. We say that the element b *covers* a , denoted $a < b$, if there is no $t \in P$ such that $a < t < b$.

Definition 1.2.1 *Let P be a poset with cover relation $<$. A matching M on P is an involution $M : P \rightarrow P$ such that for every $a \in P$, either $a < M(a)$, or $M(a) < a$ or $a = M(a)$. An element $a \in P$ for which $M(a) = a$ is called critical.*

Define the *Hasse diagram* of a poset P to be the directed graph whose vertices are the poset elements and whose edges are $b \rightarrow a$ if $a < b$ for $a, b \in P$. A matching M on P is called *acyclic* if the directed graph $G_M(P)$ constructed from the Hasse diagram of P by reversing each arrow $b \rightarrow a$ to $a \rightarrow b$ if $b = M(a)$ has no directed cycles. In $G_M(P)$, the reversed arrows from the Hasse diagram of P are said to point *up*, the non reversed arrows are said to point *down*.

We now present some preliminaries on CW complexes and regular cell complexes that are needed in this thesis. For more details on CW complexes and regular cell complexes, see [11] and [14]. The set $B^n := \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is called the *unit n -ball*, and the *interior* of B^n is the subset of B^n given by $\text{Int}B^n := \{x \in \mathbb{R}^n : \|x\| < 1\}$. The *unit $(n-1)$ -sphere* denoted by S^{n-1} is $S^{n-1} := \{x \in \mathbb{R}^n : \|x\| = 1\}$ for all $n \geq 0$. In fact $S^{-1} = \emptyset$, $B^0 = \{\text{a point}\}$, and $S^0 = \{\text{two points}\}$.

Definition 1.2.2 *A space σ which is homeomorphic to $\text{Int}B^n$ is called an n -cell (or a cell of dimension n).*

Let $\bar{\sigma}$ denote the closure of a cell σ , and $\dot{\sigma} = \bar{\sigma} \setminus \sigma$. If the dimension of σ is 0 we set $\sigma = \bar{\sigma} = \{\text{a point}\}$.

Definition 1.2.3 A finite CW complex (or a finite cell complex) is a Hausdorff space Δ together with a finite collection of disjoint cells σ whose union is Δ such that for every n -cell σ_k in the collection, there is a continuous map $g_k : B^n \rightarrow \Delta$ which maps $\text{Int} B^n$ homeomorphically onto σ_k , and carries ∂B^n into a union of cells of dimension less than n .

The map g_k from Definition 1.2.3 is called the *characteristic map* for the open cell σ_k .

Definition 1.2.4 Let Δ be a finite CW complex. We call Δ a *finite regular cell complex* if each characteristic map g_k is a homeomorphism, and each σ is a union of cells of Δ .

Let Δ be a finite regular cell complex.

Definition 1.2.5 Let $P(\Delta)$ be the poset of all cells of Δ ordered by set inclusion of their closures together with an artificial minimal element. We call $P(\Delta)$ the *face poset* of Δ .

A poset having a unique minimum element in which every principal order ideal is isomorphic to a boolean algebra is called a *simplicial poset*.

Definition 1.2.6 A finite regular cell complex Δ is called a *boolean cell complex* if its face poset $P(\Delta)$ is a simplicial poset.

In fact, $P(\Delta)$ determines Δ up to isomorphism. That is, given a finite simplicial poset P , there is a boolean cell complex Δ (unique up to isomorphism) such that $P = P(\Delta)$.

In this thesis we sometimes consider the bottom element of a simplicial poset $P(\Delta)$ to be a cell in the finite regular cell complex Δ and refer to it as the “empty cell”.

Theorem 1.2.7 [9, Theorem 6.3] Let Δ be a boolean cell complex and let M be an acyclic matching on the face poset $P(\Delta)$. If there are c critical cells, all of the same dimension m , then Δ is homotopy equivalent to a wedge of c spheres of dimension m .

In fact, Theorem 1.2.7 also holds for arbitrary regular cell complexes. For further reading on boolean cell complexes see [5].

Boolean complexes of Coxeter systems were first introduced in [13]. We now recall some definitions and known results related to the boolean complex of a Coxeter system (W, S) .

Definition 1.2.8 Let (W, S) be a Coxeter system and W be considered as a poset under Bruhat order. An element $w \in W$ is called *boolean* if its lower principal order ideal is isomorphic to a boolean algebra.

Let $\mathbb{B}(W)$ denote the subposet of W induced by the boolean elements.

Definition 1.2.9 [13] The *boolean cell complex* $\Delta(W)$ whose face poset is $\mathbb{B}(W)$ is called the *boolean complex* of (W, S) .

For a Coxeter graph G_W of (W, S) and an edge e in G_W , let $G_W - e$ be the subgraph obtained after deleting e , G_W/e the subgraph obtained after contracting e , and $G_W - [e]$ the subgraph obtained after removing e and its incident edges from G_W , respectively.

Theorem 1.2.10 [13, Theorem 3.4] *Let (W, S) be a Coxeter system of rank n . There exists a nonnegative integer $\beta(G_W)$ such that $\Delta(W)$ is homotopy equivalent to the wedge of $\beta(G_W)$ spheres, all of dimension $n - 1$. The values of $\beta(G_W)$ are computed recursively by using the following equations:*

- (i) $\beta(G_W) = \beta(G_W - e) + \beta(G_W/e) + \beta(G_W - [e])$ if e is an edge in $\beta(G_W)$,
- (ii) $\beta(G_W) = 0$ if G_W is a nonempty graph without edges,
- (iii) $\beta(G_W) = 1$ if G_W is the empty graph.

The integer $\beta(G_W)$ from Theorem 1.2.10 is called the *boolean number*.

Let I be the set of involutions in W , $\text{Br}(I)$ the subposet of the Bruhat order on W induced by I , and $B(w)$ the principal order ideal generated by $w \in \text{Br}(I)$.

Definition 1.2.11 *An element $w \in I$ is a boolean involution if $B(w)$ is isomorphic to a boolean algebra.*

Let $P(\Delta_{\text{inv}}(W))$ denote the subposet of $\text{Br}(I)$ induced by all boolean involutions.

Definition 1.2.12 *We call $P(\Delta_{\text{inv}}(W))$ a boolean involution ideal.*

Note that $P(\Delta_{\text{inv}}(W))$ is a simplicial poset which determines a boolean cell complex $\Delta_{\text{inv}}(W)$ whose face poset is $P(\Delta_{\text{inv}}(W))$.

Example 1.2.13 From Example 1.1.4:

1. The subposet $\mathbb{B}(S_4)$ of the symmetric group S_4 induced by the boolean elements is illustrated in Figure 1.4. So by Theorem 1.2.10, $\beta(G_{S_4}) = 1$ and $\Delta(S_4)$ is homotopy equivalent to the sphere S^2 .
2. The subposet $\text{Br}(I)$ of the Bruhat order on S_4 induced by the set of involutions I is illustrated in Figure 1.5 and the boolean involution ideal $P(\Delta_{\text{inv}}(S_4))$ induced by boolean involutions is the subposet indicated by permutations of yellow color together with all blue edges.

Figure 1.4: The subset of S_4 induced by the boolean elements.

Figure 1.5: The subset $\text{Br}(I)$ induced by I in S_4 .

2 – Summary of papers

2.1 Paper I: Smoothness of Schubert varieties indexed by involutions in finite simply laced types

Let (W, S) be finite and simply laced. The main result of this paper is about smoothness of Schubert varieties X_w , when w is an involution in W . Namely, if w is an involution which is not a longest element in some parabolic subgroup of W , then X_w is singular. This means that X_w is smooth if and only if $w = w_0(J)$ for some $J \subseteq S$. By way of contrast, there are counterexamples whenever W is not simply laced. The proof is based on Corollary 1.1.13.

2.2 Paper II: Boolean complexes of involutions

Let (W, S) be a Coxeter system of rank n . In [13], Ragnarsson and Tenner constructed the boolean complex $\Delta(W)$ of W and proved that $\Delta(W)$ is homotopy equivalent to a wedge of spheres of dimension $n - 1$. In our paper, we construct an analogue of Ragnarsson and Tenner's boolean complexes for involutions in W . Let $\Delta_{\text{inv}}(W)$ be the boolean complex of involutions in W . We compute the homotopy type of $\Delta_{\text{inv}}(W)$ for many Coxeter groups including all finite Coxeter groups. The main tool is Theorem 1.2.7.

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Papers

The papers associated with this thesis have been removed for copyright reasons. For more details about these see:

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