Edge Precoloring Extension of Trees

Fikre Bogale Petros

Linköping Studies in Science and Technology.
Licenciate Thesis No. 1921

Edge Precoloring Extension of Trees

Fikre Bogale Petros

Department of Mathematics
Linköping, 2022

This work is licensed under a Creative Commons Attribution-NonCommercial 4.0 International License.

<https://creativecommons.org/licenses/by-nc/4.0/>

Linköping Studies in Science and Technology. Licentiate Thesis No. 1921

Edge Precoloring Extension of Trees

Copyright © Fikre Bogale Petros, 2022

*Department of Mathematics
Linköping University
SE-581 83 Linköping, Sweden*

Email: fikre.bogale.petros@liu.se

ISSN 0280-7971

ISBN 978-91-7929-210-2 (print)

ISBN 978-91-7929-211-9 (PDF)

URL <https://doi.org/10.3384/9789179292119>

Printed by LiU-Tryck, Linköping, Sweden, 2022

This is a Swedish Licentiate Thesis.

The Licentiate degree comprises 120 ECTS credits of postgraduate studies.

Abstract

Given a set of k colors and a graph G with a subset S of precolored edges (a partial k -edge coloring of G), we consider the problem of determining whether G has a proper edge coloring of G with the same k colors (an extension of the partial coloring) where the colors of edges in S are not changed. If such a coloring exists, then the partial k -coloring is called extendable.

Some scheduling problems as well as some combinatorial problems can be reformulated as partial edge coloring extension problems for corresponding graphs. Partial edge coloring extension problems seem to have been first considered in connection with the problem of completing partial Latin squares. In 1960 Evans stated his conjecture that any partial Latin square of size n with at most $n - 1$ non-empty cells can be completed to a Latin square of size n . In terms of edge colorings this is equivalent to the statement that any proper partial n -edge coloring of the balanced complete bipartite graph $K_{n,n}$ with at most $n - 1$ precolored edges is extendable. This classical conjecture was proved by Smetaniuk (1981), and also independently by Andersen and Hilton (1983). Moreover, Andersen and Hilton completely characterized which partial Latin squares of size n with n non-empty cells that cannot be completed to a Latin square of size n . In addition, Andersen (1985) characterized partial Latin squares of size n with $n + 1$ non-empty cells that are completable to Latin squares of size n .

More recently, the problem of extending a partial edge coloring where the precolored edges form a matching has been considered by Edwards et al. (2018). Casselgren, Markstrom and Pham (2020) studied questions on extending partial edge colorings of the n -dimensional hypercubes Q_n . In particular, they obtained an analogue of the positive solution to Evans' conjecture on completing partial Latin squares by proving that every proper partial edge coloring of at most $n - 1$ edges of Q_n can be extended to a proper n -edge coloring of Q_n . They also characterized which partial edge colorings of Q_n with precisely n precolored edges are extendable to proper n -edge colorings of Q_n .

In this thesis we study similar partial edge coloring extension problems for trees. Let T be a tree with maximum degree $\Delta(T)$. First, we characterize which partial edge colorings with at most $\Delta(T)$ precolored edges in T that are extendable to proper $\Delta(T)$ -edge colorings, thereby proving an analogue of the aforementioned result by Andersen and Hilton for Latin squares. Then, we prove an analogue for trees of the result of Andersen by characterizing exactly which precolorings of at most $\Delta(T) + 1$ precolored edges in a tree T that are extendable to $\Delta(T)$ -edge colorings of T . We also prove sharp conditions on when it is possible to extend a precolored matching or a collection of connected precolored subgraphs of a tree T to a $\Delta(T)$ -edge coloring of T . Finally, we consider the problem of avoiding a given (not necessarily proper) partial edge coloring.

*To my wife Tizita, and children Azarias and Ananiya,
for their encouragement and love!*

Acknowledgments

I would like to express great gratitude to my main supervisor Carl Johan Casselgren for his continuous support, great mentor, expert advice, constructive feedbacks, hard work, and providing me all the necessary reading materials during my stuiies at Linöping University. I am very grateful to my co-supervisor Armen Asratian for useful comments and technical advice whenever needed.

I would also like to thank Department of Mathematics, Addis Ababa University, Ethiopia, and Linköping University, Sweden, for being accepted as PhD student between these universities. A profound appreciation go to my supervisor Samuel Asefa for his support and encouragement during my studies.

I would like to express my sincere gratitude to the International Science Program(ISP) for their financial and material support. I offer my special thanks to Leif Abrahamson, Chris Fabian Bengtsson, Theresa Lagali, and Tilahun Abebaw for making my visists in Sweden smooth and conducive.

Next I would like to thank my family for their encouragement and support, especially, my beloved wife and children for their patience and love.

Finally, I would like to use this opportunity to show respect and thank all staff members, students, and lecturers with whom I enjoyed during my entire study.

Contents

Abstract	i
Acknowledgments	iii
Contents	v
1 Introduction	1
2 Fundamentals	3
2.1 Common Families of Graphs	4
2.2 Matchings	5
2.3 Forests and Trees	7
3 Graph Coloring	9
3.1 Vertex-Coloring	9
3.2 Edge-coloring	9
3.3 Graph Modeling Applications	11
4 Extending Partial Edge Colorings	13
5 Summary of Papers	17
References	19
PAPER I	21
PAPER II	35

1 – Introduction

A proper k -edge coloring of a graph G is an assignment of k colors to the edges of G , one to each edge, so that any two edges with a common endpoint receive different colors. Given a graph G , the classical edge coloring problem is to minimize the number of colors required to properly color the edges of G ; this number is denoted by $\chi'(G)$ and is known as the *chromatic index* of G . The two classical theorems on edge coloring are Vizing's theorem, which states that the chromatic index of a simple graph G is either $\Delta(G)$ or $\Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G , and König's theorem which asserts that $\Delta(G)$ colors suffices for a proper edge coloring if G is a bipartite graph.

The edge coloring dates back to Peter Tait's attempts around 1880 to prove the Four-Color Conjecture. This conjecture posed by F. Guthrie in 1852 states that any map drawn in the plane can be colored using four colors in such a way that regions sharing a common boundary (other than a single point) do not share the same color. P. Tait showed that the Four-Color Conjecture is equivalent to the statement that $\chi'(G) = 3$ for every planar, bridgeless, 3-regular, simple graph G . The statement became known as Tait's Theorem, but remained unproved until the Four-Color Conjecture was finally proved in 1976 by Appel and Haken.

It is known that many scheduling problems as well as combinatorial problems can be formulated in terms of proper edge colorings with some additional constraints for a corresponding graph.

In this thesis, our main interest is partial edge coloring extension problems. A *partial edge coloring* (or *precoloring*) of a graph G is a coloring of some subset $S \subseteq E(G)$. A partial k -edge coloring φ of G is called *extendable* if there is a proper k -edge coloring f of G such that every edge $e \in S$ receives under f the same color that it receives under φ ; f is called an extension of φ . Questions on extending partial edge colorings have immediate applications in scheduling problems where some activities have been prescheduled and we ask for a schedule of minimum length which respects the prescheduled activities.

Partial edge coloring extension problems seem to have been first considered in connection with the problem of completing partial Latin squares and the well-known Evans' conjecture posed in 1960 (see e.g. [25, 14, 4]). This conjecture states that any partial Latin square of size n with at most $n - 1$ non-empty cells can be completed to a Latin square of size n . In terms of edge colorings this is equivalent to the statement that any proper partial n -edge coloring of the balanced complete bipartite graph $K_{n,n}$ with at most $n - 1$ precolored edges is extendable. The conjecture was proved for large n by Häggkvist [18] and later for all n by Smetaniuk [25]. This was also proved independently by Andersen and Hilton [4]. Moreover, Andersen and Hilton [4] completely characterized which partial Latin squares of size n with n non-empty cells that cannot be completed to a Latin square of size n .

Later some results on edge precoloring extensions were obtained by Marcotte and Seymour [22]. More recently, the problem of extending a precoloring of a matching has been considered in [12]. In particular it is conjectured that for every graph G , if φ is an edge precoloring of a matching M in G using $\Delta(G) + 1$ colors, and any two edges in M are at distance at least 2 from each other, then φ can be extended to a proper $(\Delta(G) + 1)$ -edge coloring of G . In addition, with motivation from results on completing partial Latin squares, questions on extending partial edge colorings of d -dimensional hypercubes Q_d was studied in [9]. In particular, a characterization of which partial edge colorings with at most d precolored edges are extendable to d -edge colorings of Q_d is obtained, thereby establishing an analogue for hypercubes of the characterization by Andersen and Hilton [4] of which $n \times n$ partial Latin squares with at most n non-empty cells are completable to Latin squares.

In this thesis we study partial edge coloring extension problems for trees. First, we characterize which partial edge colorings with at most $\Delta(T)$ precolored edges in a tree T that are extendable, thereby proving an analogue of the aforementioned result by Andersen and Hilton [4] for Latin squares. Then, we prove an analogue for trees of the result of Andersen [3] by characterizing exactly which precolorings of at most $\Delta(T) + 1$ precolored edges in a tree T that are extendable to $\Delta(T)$ -edge colorings of T . We also prove sharp conditions on when it is possible to extend a precolored matching or a collection of connected precolored subgraphs of a tree T to a $\Delta(T)$ -edge coloring of T . Finally, we consider the problem of *avoiding* a (not necessarily proper) partial edge coloring.

2 – Fundamentals

This chapter will describe some of the fundamental concepts and theorems we will encounter in the thesis as well as some common terminology and notation in graph theory. Note that the terminology of graph theory can differ significantly between authors. For this work we use the book, Introduction to Graph Theory by Douglas West [27].

Definition 2.0.1. A *graph* G is a triple consisting of a vertex set $V(G)$, an edge set $E(G)$, and a relation ϕ that associates with each edge two vertices (not necessarily distinct) called its endpoints.

Vertices are sometimes called points or nodes, while edges are sometimes referred to as lines or links.

Definition 2.0.2. A *loop* is an edge whose endpoints are equal. *Multiple edges* are edges having the same pair of endpoints. A *simple* graph is a graph having no loops or multiple edges. We specify a simple graph by its vertex set and edge set, treating the edge set as a set unordered pairs of vertices and writing $e = uv$ (or $e = vu$) for an edge e with endpoints u and v .

A graph G is *finite* if its vertex set and edge set are finite, otherwise it is *infinite*. However, we consider only finite graphs.

Definition 2.0.3. The *order* of a graph G , written $n(G)$, is the number of vertices in G . An n -vertex graph is a graph of order n . The *size* of a graph G , written $e(G)$, is the number of edges in G .

If uv is an edge of G , then u and v are *adjacent* vertices. Two adjacent vertices are referred to as *neighbors* of each other. The set of neighbors of a vertex v is called the *neighborhood* of v and is denoted by $N(v)$. If uv and vw are distinct edges in G with a common endpoint, then uv and vw are adjacent edges. The vertex u and the edge uv are said to be incident with each other.

Definition 2.0.4. The *degree* of a vertex v in a graph G , written $d_G(v)$ or $d(v)$, is the number of edges incident to v , except that each loop at v counts twice. The *maximum* degree is denoted by $\Delta(G)$, the *minimum* degree is denoted by $\delta(G)$, and G is *regular* if $\Delta(G) = \delta(G)$. It is *k-regular* if the common degree is k .

If G is a simple graph, then the degree of a vertex v is the number of vertices in its neighborhood $N(v)$. Equivalently, the degree of v is the number of vertices in G that are adjacent to v . A vertex of degree 0 is referred to as an *isolated vertex* and a vertex of degree 1 is an *end-vertex* or a *leaf*. An edge incident with an end-vertex is called a *pendant* edge. Thus if v is a vertex of a graph G of order n , then

$$0 \leq \delta(G) \leq d(v) \leq \Delta(G) \leq n - 1.$$

A graph is *trivial* if it has no edges; otherwise it is *nontrivial*.

Definition 2.0.5. The *complement* $\overline{G}$ of a simple graph G is the simple graph with vertex set $V(G)$ and edge set defined by $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$.

Definition 2.0.6. A *clique* in a graph is a set of pairwise adjacent vertices. An *independent set* (or *stable set*) in a graph is a set of pairwise nonadjacent vertices.

Definition 2.0.7. A *subgraph* of a graph G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We then write $H \subseteq G$ and say that “ G contains H ”. The subgraph of G induced by a subset S of vertices of G denoted by $G[S]$, is the subgraph whose vertex set is S and whose edges are all the edges of G with both ends in S .

Note that any graph is both a subgraph and an induced subgraph of itself.

2.1 Common Families of Graphs

Certain types of graphs play prominent roles in graph theory.

Definition 2.1.1. A *complete* graph is a simple graph, all vertices of which are pairwise adjacent; the complete graph with n vertices is denoted by K_n .

Since the degree of each node in a complete graph with n vertices is $n - 1$, it is $(n - 1)$ -regular.

Definition 2.1.2. A graph is *bipartite* if $V(G)$ is the union of two disjoint (possibly empty) independent sets called *partite sets* of G . An X, Y -*bigraph* is a bipartite graph with partite sets X, Y .

Definition 2.1.3. A *complete bipartite* graph or *biclique* is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets. When the sets have size r and s , respectively, it is denoted by $K_{r,s}$.

Definition 2.1.4. The k -*dimensional cube* or *hypercube* Q_k is the simple graph whose vertices are the k -tuples $t_1 t_2 \dots t_k$ with entries $\{0, 1\}$ and whose edges are the pairs of k -tuples that differ in exactly one position. A j -dimensional subcube of Q_k is a subgraph of Q_k isomorphic to Q_j .

The parity of a vertex (k -tuple) $t_1 t_2 \dots t_k$ in Q_k is the parity of the number of $t_1 + t_2 + \dots + t_k$.

Each edge of Q_k has an even vertex and an odd vertex as endpoints. Hence the even vertices form an independent set, as do the odd vertices, so Q_k is bipartite.

The following theorem characterizes bipartite graphs.

Theorem 2.1.5. (König[1936]) *A graph is bipartite if and only if it has no odd cycle.*

Definition 2.1.6. A *path* is a simple graph whose vertices can be ordered in a list so that two vertices are adjacent if and only if they are consecutive in the list. A *cycle* is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle.

The *length* of a path or a cycle is the number of its edges. A path or cycle of length k is called a k -*path* or k -*cycle*, respectively; the path or cycle is *odd* or *even* according to the parity of k . A *maximal path* in a graph G is a path P in G that is not contained in a longer path.

Definition 2.1.7. A graph G is *connected* if it has a u, v -path whenever $u, v \in V(G)$ (otherwise, G is *disconnected*). If G has a u, v -path, then u is *connected* to v in G .

Definition 2.1.8. The *components* of a graph G are its maximal connected subgraphs of G , that is, connected subgraphs which are not contained in a larger connected subgraph of G .

2.2 Matchings

There are numerous problems concerning graphs that include sets of edges or sets of vertices possessing a property of particular interest. We are often interested in partitioning the edge set or vertex set of a graph into subsets so that each subset possesses this property. In this section, we describe some of the known examples of such sets and the subgraphs they induce [10].

Suppose that A and B are finite nonempty sets with $|A| = s$ and $|B| = t$, say

$$A = \{a_1, a_2, \dots, a_s\} \text{ and } B = \{b_1, b_2, \dots, b_t\}.$$

Clearly, a one-to-one function from A to B exists if and only if $s \leq t$. But what if the image of each element a_i ($1 \leq i \leq s$) must be selected from some nonempty list $L(a_i)$ of elements of B ? Then this question cannot be answered without knowing more about the lists $L(a_i)$.

Example 2.2.1. Let's consider an example of this situation. If $A = \{a_1, a_2, a_3, a_4\}$ and $B = \{b_1, b_2, b_3, b_4\}$, where

$$L(a_1) = \{b_1, b_2\}, L(a_2) = \{b_1, b_4\}, L(a_3) = \{b_2, b_3\}, L(a_4) = \{b_3, b_4\},$$

then the function $f : A \rightarrow B$ defined by

$$f(a_1) = b_1, f(a_2) = b_4, f(a_3) = b_2, f(a_4) = b_3 \quad (1)$$

is one-to-one. This situation can be modeled by a bipartite graph G with partite sets A and B , where $a_i b_j \in E(G)$ if $b_j \in L(a_i)$ as shown in Figure 2.1. To show that there is a one-to-one function f from A to B , it is sufficient to show that G contains a set of pairwise nonadjacent edges such that each vertex of A is incident with one of these edges. The edges corresponding to the function f defined in (1) are shown in bold in Figure 2.1.

Figure 2.1: A bipartite graph modeling a function problem

Example 2.2.2. Consider an additional example. Suppose that $A = \{a_1, \dots, a_5\}$ and $B = \{b_1, \dots, b_6\}$, where

$$L(a_1) = \{b_3, b_5\}, L(a_2) = \{b_1, b_2, b_3, b_4, b_6\},$$

$$L(a_3) = \{b_1, b_2, b_4, b_5, b_6\}, L(a_4) = \{b_3, b_5\}, L(a_5) = \{b_3, b_5\}$$

Again we ask whether there is a one-to-one function from A to B . This situation can also be modeled by a bipartite graph. In this case, however, notice that the image of each of the elements a_1, a_4 and a_5 of A must be chosen from the same two elements of B (namely b_3 and b_5). Consequently, there is no way to construct a function from A to B such that every two elements of $\{a_1, a_4, a_5\}$ have distinct images and so no one-to-one function from A to B exists.

The problem mentioned above pertains to an important concept in graph theory that is encountered in a variety of circumstances.

Definition 2.2.3. A *matching* in a graph G is a set of non-loop edges with no share endpoints. The vertices incident to the edges of a matching M are *saturated* (or covered) by M ; the others are *unsaturated* (we say *M -saturated* and *M -unsaturated*). A *perfect matching* in a graph is a matching that saturates every vertex.

We saw that the bipartite graph of Figure 2.1 contains a matching of size 4 (and so there is a one-to-one function from A to B under the given restrictions) but that there is no matching of size 5 in a bipartite graph modeling of Example 2.2.2 (and so there is no one-to-one function from A to B under the given restrictions).

If a matching M saturates X , then for every $S \subseteq X$ there must be at least $|S|$ vertices that have neighbors in S , because the vertices matched to S must be chosen from that set. Thus $|N(S)| \geq |S|$ is a necessary condition. P. Hall proved that this necessary condition is also sufficient.

Theorem 2.2.4. (*Hall's Theorem-P.Hall [1935]*) An X, Y -bigraph G has a matching that saturates X if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.

As in Example 2.2.2, if an X, Y -bigraph has no matching saturating X , then Theorem 2.2.4 implies that we can exhibit a subset $S \subseteq X$ such that $|N(S)| < |S|$.

Definition 2.2.5. A *maximal matching* in a graph is a matching that cannot be enlarged by adding an edge. A *maximum matching* is a matching of maximum size among all matchings in the graph.

A matching M is maximal if every edge not in M is incident to an edge already in M . Every maximum matching is a maximal matching, but the converse need not hold.

Example 2.2.6. *Maximal $\neq$ maximum.* The smallest graph having a maximal matching that is not a maximum matching is P_4 . If we take the middle edge, then we can add no other, but the two end edges form a larger matching. In Figure 2.2 we show this phenomenon in P_4 and P_6 . In Figure 2.2, replacing the bold edges by the solid edges yields a larger matching. This gives us a way to look for larger matchings.

Figure 2.2: Graphs containing maximal matchings that are not maximum

Definition 2.2.7. Given a matching M , an M -*alternating path* is a path that alternates between edges in M and edges not in M . An M -alternating path whose endpoints are unsaturated by M is an M -*augmenting path*.

Given an M -augmenting path P , we can replace the edges of M in P with the other edges of P to obtain a new matching M' with one more edge. Thus when M is a maximum matching, there is no M -augmenting path.

Theorem 2.2.8. (Berge [1957]) A matching M in a graph G is a maximum matching in G if and only if G has no M -augmenting path.

2.3 Forests and Trees

Definition 2.3.1. A graph with no cycle is *acyclic*. A *forest* is an acyclic graph. A *tree* is a connected acyclic graph. A *spanning tree* of a graph G is a spanning subgraph of G that is a tree.

A tree is a connected forest, and every component of a forest is a tree. A graph with no cycles has no odd cycles; hence trees and forests are bipartite. Paths are trees. A tree is a path if and only if its maximum degree is 2. A *star* is a tree consisting of one vertex adjacent to all the others. The n -vertex star is the biclique $K_{1,n-1}$.

Theorem 2.3.2. [27] For an n -vertex graph G (with $n \geq 1$), the following are equivalent (and characterize the trees with n vertices).

- A) G is connected and has no cycles
- B) G is connected and has $n - 1$ edges.
- C) G has $n - 1$ edges and no cycles.
- D) For $u, v \in V(G)$, G has exactly one u, v -path.

3 – Graph Coloring

In this chapter, we present some classical results concerning vertex and edge colorings of graphs. A vertex coloring (edge coloring) of a graph G is an assignment of colors to the vertices (edges) of G , one to each vertex (edge). The particular variant which we are interested here is that adjacent vertices (edges) receive different colors. Such colorings have many applications in scheduling problems.

3.1 Vertex-Coloring

Vertex coloring can also be defined using functions:

Definition 3.1.1. A k -coloring of a graph G is a map $f : V(G) \rightarrow S$, where $|S| = k$ (often we use $S = \{1, 2, \dots, k\}$). The set S consists of colors; the vertices of one color form a color class. A k -coloring is *proper* if adjacent vertices have different colors. A graph is k -colorable if it has a proper k -coloring. The *chromatic number* $\chi(G)$ is the least k such that G is k -colorable.

There is no general formula for the chromatic number of a graph. Certainly, for every graph G of order n ,

$$1 \leq \chi(G) \leq n.$$

Algorithm 3.1.2. (Greedy coloring) *The greedy coloring relative to a vertex ordering $v_1, \dots, v_n$, assigns to v_i the smallest positive integer which is not already used on its neighbors.*

Proposition 3.1.3. $\chi(G) \leq \Delta(G) + 1$.

Proof. In a vertex ordering, each vertex has at most $\Delta(G)$ earlier neighbors, so the greedy coloring cannot be forced to use more than $\Delta(G) + 1$ colors. This proves constructively that $\chi(G) \leq \Delta(G) + 1$. $\square$

This upper bound has been improved as follows.

Theorem 3.1.4. (Brooks [1941]) [27] *If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.*

3.2 Edge-coloring

Edge coloring can also be defined using functions:

Definition 3.2.1. A k -edge-coloring of G is a map $f : E(G) \rightarrow S$, where $|S| = k$ (often we use $S = \{1, 2, \dots, k\}$). The set S consists of colors; the edges of one color form a color class. A k -edge-coloring is *proper* if adjacent edges have different colors; that is, if each color class is a matching. A graph is k -edge-colorable if it has a proper k -edge-coloring. The *edge-chromatic number* $\chi'(G)$ of a loopless graph G is the least k such that G is k -edge-colorable.

Chromatic index is another name for $\chi'(G)$. Since edges sharing a vertex need different colors, $\chi'(G) \geq \Delta(G)$. Vizing [1964] and Gupta [1966] independently proved that $\Delta(G) + 1$ colors suffice when G is simple.

Theorem 3.2.2. (Vizing [1964, 1965], Gupta [1966]) [27] If G is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$.

For bipartite graphs, König proved the following theorem.

Theorem 3.2.3. (König [1916]) If a graph G is bipartite, then $\chi'(G) = \Delta(G)$.

Definition 3.2.4. A simple graph G is class 1 if $\chi'(G) = \Delta(G)$. It is class 2 if $\chi'(G) = \Delta(G) + 1$.

Definition 3.2.5. In a graph G with multiple edges, we say that a vertex pair x, y is an edge of *multiplicity* m if there are m edges with endpoints x, y . We write $\mu(xy)$ for the multiplicity of the pair, and we write $\mu(G)$ for the maximum of the edge multiplicities in G .

For loopless graphs with multiple edges, $\chi'(G)$ may exceed $\Delta(G) + 1$. Shannon [1949] proved the following result:

Theorem 3.2.6. (Shannon [1949]) If G is a loopless graph, then $\chi'(G) \leq \frac{3}{2}\Delta(G)$.

Vizing and Gupta independently proved that $\chi'(G) \leq \Delta(G) + \mu(G)$, where $\mu(G)$ is the maximum edge multiplicity. The graph in Figure 3.1 achieves both bounds. All edges in this graph are pairwise intersecting and hence require distinct colors. Thus $\chi'(G) = \frac{3\Delta(G)}{2} = \Delta(G) + \mu(G)$.

Figure 3.1: Fat Triangle

3.3 Graph Modeling Applications

Next, we give some explicit applications of graph colorings. Different kinds of graphs arise in modeling of real-world problems [5].

Assigning Broadcasting Frequencies: When the radio transmission in a geographical region are assigned broadcasting frequencies, some pairs of transmitters require different frequencies to avoid interference. A graph model can be used for the problem of minimizing the number of different frequencies assigned.

Suppose that the seven radio transmitters $A, B, \dots, G$ should be assigned frequencies. For simplicity, assume that if two transmitters are less than 100 miles apart, they must broadcast at different frequencies. Consider a graph whose vertices represent the transmitters, and whose edges indicate those pairs that are less than 100 miles apart. Figure 3.2 shows that a table of distances for the seven transmitters and the corresponding graph on seven vertices. The problem of assigning

Figure 3.2: A simple graph for a radio-frequency-assignment problem.

radio frequencies to avoid interference is equivalent to the problem of coloring the vertices of the graph so that adjacent vertices get different colors. The minimum number of frequencies is equal to the minimum number of different colors required for such a coloring.

The Timetable Problem. Suppose in a school there are r teachers, $T_1, T_2, \dots, T_r$, and s classes, $C_1, C_2, \dots, C_s$. Each teacher T_i is expected to teach the class C_j for p_{ij} periods. It is presumed that during any particular period, no more than one teacher can handle a particular class and no more than one class can be engaged by any teacher. Our aim is to draw up a timetable for the day that requires only the minimum number of periods. This problem is known as the "timetable problem." We form a bipartite graph $G = G(T, C)$ with bipartition (T, C) , where T represents the set of teachers T_i and C represents the set of classes C_j . Further, T_i is adjacent to C_j in G with p_{ij} parallel edges if and only if teacher T_i is to teach class C_j for p_{ij} periods. Now color the edges of G so that no two adjacent edges receive the same color. Then the edges in a particular color class form a matching in G and correspond to a schedule of work for a particular time period. Hence, the minimum number of time periods required is the minimum number of colors in an

edge coloring of G in which adjacent edges receive distinct colors; in other words, it is the edge-chromatic number of G .

4 – Extending Partial Edge Colorings

Let G be a graph and $\mathcal{K} = [k] = \{1, 2, 3, \dots, k\}$ be a palette of available colors. We are looking for a proper k -edge coloring of G , but with the constraint that some edges have already been colored and cannot be changed. We will formulate the problem more formally below.

A *partial edge coloring* (or *precoloring*) of a graph G is a coloring of some subset $S \subseteq E(G)$. A partial k -edge coloring φ of G is called *extendable* if there is a proper k -edge coloring f of G such that $f(e) = \varphi(e)$ for every edge e that is colored under φ ; f is called an *extension* of φ .

Now our problem can be formulated as follows: Is a given partial k -edge coloring of a graph G extendable?

Questions on extending partial edge colorings have immediate applications in scheduling problems where we do not have full control over the schedule; rather the assignments of certain activities have been prescheduled, that is, some of the edges of the model graph have a preassigned color, and we have to solve the pre-coloring extension problem; i.e., extend the coloring of these edges to the whole graph, preferably using a minimum number of colors (corresponding to a schedule of minimum length).

A *list-edge-assignment* of G is the assignment of “list” (sets) $L(e)$ of colors to every edge e of G ; a *k -list-assignment* is a list-assignment such that $|L(e)| \geq k$ for every edge e . If L is a list-assignment for G , then an *L -coloring* of G is a coloring (not necessarily proper) in which each edge e receives a color from its own list $L(e)$; we loosely talk of coloring G from its lists. The graph G is *k -edge-choosable* or *k -list-edge-colorable*, if it is properly L -colorable for every k -list-assignment L to G . The smallest k for which G is k -edge-choosable is called the *list chromatic index* of G and is denoted by $\chi'_L(G)$.

Note that edge precoloring extension is a special case of list edge coloring, where a single color is allowed for every precolored edge e . The following conjecture is the most famous open problem in list edge coloring theory.

Conjecture 4.0.1. (*List Coloring Conjecture (LCC)*) For any multigraph G , $\chi'_L(G) = \chi'(G)$.

The conjecture was proved by Galvin [16] for bipartite multigraphs, and remains very much open, even for complete and cubic graphs.

Edge precoloring extension problems seem to have been first considered in connection with the problem of completing partial Latin squares and the well-known Evans’ conjecture (see e.g. [4, 14, 25]).

A *partial Latin square* of size n is an $n \times n$ array filled with n different symbols,

each occurring at most once in each row and each column. A partial Latin square of size n is called a *Latin square* if each of its cells is non-empty.

Evans [14] stated a conjecture that any partial Latin square of size n with at most $n - 1$ non-empty cells can be completed to a Latin square of size n . In terms of edge colorings this is equivalent to the statement that any proper partial n -edge coloring of the balanced complete bipartite graph $K_{n,n}$ with at most $n - 1$ precolored edges is extendable. This classical conjecture was proved by Smetaniuk [15], and also independently by Andersen and Hilton [4]. Moreover, Andersen and Hilton [4] completely characterized which partial Latin squares of size n with n nonempty cells that cannot be completed to a Latin square of size n .

More recently, the problem of extending a precoloring of a matching have been considered in [12].

Question 4.0.2. *Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$, let $\mathcal{K} = [\Delta(G) + \mu(G)]$, and let M be a matching of G precolored from the palette $\mathcal{K}$. What conditions on G and M ensure that the precoloring of M extends to a proper $(\Delta(G) + \mu(G))$ -edge coloring of all of G ?*

Here the *distance* between two edges e and e' we mean the number of edges in a shortest path between an endpoint of e and an endpoint of e' ; a *distance- t matching* is a matching where any two edges are at distance at least t from each other. A distance-2 matching is also called an *induced matching*. The following proposition follows from a result of Albertson [1] and Vizing's theorem.

Proposition 4.0.3. *Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Using the palette $\mathcal{K} = [\Delta(G) + \mu(G) + 1]$, any precolored distance-3 matching can be extended to a proper edge coloring of all of G .*

Albertson and Moore [2] conjectured that when G is a simple graph, any precolored distance-3 matching can be extended to a proper edge-coloring of G using the palette $\mathcal{K} = [\Delta(G) + 1]$. Edwards et al. [12] propose a stronger conjecture as follows.

Conjecture 4.0.4. *Let G be a multigraph with maximum degree $\Delta(G)$ and maximum multiplicity $\mu(G)$. Using the palette $\mathcal{K} = [\Delta(G) + \mu(G)]$, any precolored distance-2 matching can be extended to a proper edge coloring of all of G .*

Conjecture 4.0.4 strengthens Proposition 4.0.3 in two ways:

- imposes a weaker constraint on the distance between precolored edges, and
- uses a smaller palette size.

Conjecture 4.0.4 becomes false if we are allowed to precolor a distance-1 rather than a distance-2 matching; cf. [12]. Note that it becomes false, even for trees, if we replace the palette $\mathcal{K}$ by $[\Delta(G)] = [\chi'(G)]$. For instance, consider stars with each edge subdivided exactly once; see Figure 4.1.

Note that a strong enough distance on the precolored matching does not permit us to take a smaller palette, like $[\Delta(G)]$ or $[\chi'(G)]$ as described in [12].

The first result of Edwards et al [12] is an edge-precoloring extension of König's theorem that any bipartite multigraph G is $\Delta(G)$ -edge colorable.

Figure 4.1: A representative G of a class of trees, with a non-extendable precolored (distance-2) matching.

Theorem 4.0.5. *Let G be a bipartite multigraph with maximum degree $\Delta(G)$. With the palette $\mathcal{K} = [\Delta(G) + 1]$, any precolored matching can be extended to a proper edge-coloring of all of G .*

Note that as indicated in Figure 4.1, the palette $\mathcal{K}$ in Theorem 4.0.5 is sharp.

In [12], they also obtained a precoloring extension variant of Shannon's theorem by showing that for any multigraph G , any precolored matching can be extended to a proper $(\lfloor \frac{3}{2}\Delta(G) + \frac{1}{2} \rfloor)$ -edge colorings of all of G . Due to the Shannon multigraphs, the last statement is sharp if $\Delta(G)$ is even, and within 1 of being sharp if $\Delta(G)$ is odd. Moreover, Edwards, et al [12] obtained results that confirm Conjecture 4.0.4 for subcubic multigraphs (i.e., of maximum degree at most 3), and planar graphs of large enough maximum degree.

The problem of extending partial edge colorings of hypercubes was recently studied by Casselgren, Markstrom and Pham [9]. They obtained an analogue of the positive solution to Evans' conjecture on completing partial Latin squares by proving that every proper partial edge coloring of at most $n - 1$ edges of the n -dimensional hypercube Q_n can be extended to a proper n -edge coloring of Q_n .

Theorem 4.0.6. [9] *Let $n \geq 2$ be a positive integer. If φ is an edge precoloring of at most $n - 1$ edges of the hypercube Q_n , then φ can be extended to a proper n -edge coloring of Q_n .*

The proof of this theorem proceeds by induction on n and relies heavily on the fact that Q_n decomposes into n so-called dimensional matchings, that is, perfect matchings M such that $Q_n - M$ is isomorphic to two copies of Q_{n-1} .

Next, Casselgren et al [9] established an analogue for hypercubes of the characterization by Andersen and Hilton [4] of which $n \times n$ partial Latin squares with exactly n non empty cells are completable. They proved that a proper precoloring of at most n edges in Q_n is always extendable unless the precoloring ϕ satisfies any of the following conditions.

(C1) there is an uncolored edge uv in Q_n such that u is incident with edges of $k \leq n$ distinct colors and v is incident to $n - k$ edges colored with $n - k$ other

distinct colors (so uv is adjacent to edges of n distinct colors);

(C2) there is a vertex u that is incident with edges of $n - k$ distinct colors $c_1, \dots, c_{n-k}$, and k vertices $v_1, \dots, v_k$ such that for $i = 1, \dots, k$, uv_i is uncolored but v_i is incident with an edge colored $c \notin \{c_1, \dots, c_{n-k}\}$;

(C3) there is a vertex u of degree $\Delta(T)$ such that every edge incident with u is uncolored but there is a fixed color c satisfying that every edge incident with u is adjacent to another edge colored c ,

(C4) $n = 3$ and the three precolored edges use different colors and form a subset of a dimensional matching.

For $i = 1, 2, 3, 4$, we denote by C_i the set of all colorings of T , $\Delta(T) \geq 2$, satisfying the corresponding condition above, and we set $C = \cup C_i$.

Theorem 4.0.7. [9] *If φ is a proper n -edge precoloring of Q_n with exactly n precolored edges and $\varphi \notin C$, then φ is extendable to a proper n -edge coloring of Q_n .*

Finally, Casselgren et al [9] also suggested that the following conjecture where G^n denotes the n th power of Cartesian product of G with itself.

Conjecture 4.0.8. *If m and n are positive integers, and φ is a proper edge precoloring of $(K_{m,m})^n$ with at most $mn - 1$ precolored edges, then φ extends to a proper mn -edge coloring of $(K_{m,m})^n$.*

Note that this is a generalization of both Evans' conjuncture and the results obtained for hypercubes; Evans' conjecture is the case $n = 1$, and the results obtained for hypercubes resolve the cases when $m = 1$ and $m = 2$; thus this conjecture is open whenever $n \geq 2$ and $m \geq 3$.

Given that a precoloring of at most $n - 1$ precolored edges of Q_n or $K_{n,n}$ is always extendable, one might ask how many precolored edges of a general n -regular bipartite graph allow for an extension. Note that any precoloring of at most one edge of a graph G can be extended to $\chi'(G)$ -edge coloring of G . For larger sets of precolored edges, we have the following [9]:

Proposition 4.0.9. *For any $n \geq 2$, there is an n -regular bipartite graph with a precoloring f of only 2 edges, such that f cannot be extended to a proper n -edge coloring.*

Proof. Let $r > 1$ be a positive integer, and let $G_1, \dots, G_r$ be r copies of $K_{n,n} - e$, that is the complete bipartite graph with $n + n$ vertices with exactly one edge removed. From $G_1, \dots, G_r$ we form n -regular graph H for $i = 1, \dots, r$ by joining a vertex in G_i of degree $n - 1$ with a vertex in G_{i+1} of degree $n - 1$ by an edge so that all added edges have distinct endpoints (indices taken modulo r). Let e_1 and e_2 be two edges in H joining vertices in distinct copies of $K_{n,n} - e$. We color e_1 with color 1, and e_2 with color 2. Since any perfect matching in H that contains e_1 also contains e_2 , this precoloring cannot be extended to a proper n -edge coloring of H . $\square$

5 – Summary of Papers

This section contains a summary of the work done and the results obtained in the papers that are included in the thesis.

Paper I: Edge precoloring extension of trees

In 1960 Evans [14] stated a classical conjecture that any partial Latin square of size $n \times n$ with at most $n - 1$ non-empty cells can be completed to a Latin square of size $n \times n$. The conjecture was proved by Smetaniuk [15], and also independently by Andersen and Hilton [4]. Moreover, Andersen and Hilton [4] completely characterized which partial Latin squares of size $n \times n$ with n non-empty cells that cannot be completed to a Latin square of size $n \times n$. More recently, the problem of extending a precoloring of a matching have been considered in [12]. In addition, questions on extending partial edge colorings of hypercubes was studied in [9].

In this paper, we consider similar problem of extending partial edge colorings of trees. First, we characterize which partial edge colorings with at most $\Delta(T)$ precolored edges in a tree T that are extendable, thereby proving an analogue of the result by Andersen and Hilton [4] for Latin squares. Then we prove sharp conditions on when it is possible to extend a precolored matching or a collection of connected precolored subgraphs of a tree T to a $\Delta(T)$ -edge coloring of T . Finally, we consider the problem of avoiding a given (not necessarily proper) partial edge coloring.

Paper II: Edge precoloring extension of trees II

Andersen [3] characterized which $n \times n$ partial Latin squares with $n + 1$ non-empty cells are completable to Latin squares.

Building on results for hypercubes [9], Casselgren et al. [7] proved a “mixed” result by characterizing, for all $1 \leq k \leq d$, for which configurations consisting of a partial coloring φ of $d - k$ edges and a partial coloring ψ of k edges, there is an extension of φ that avoids ψ .

In this paper, we consider the problem of avoiding and extending partial edge colorings of trees. First, we prove an analogue for trees of the result of Andersen [3] by characterizing exactly which precolorings of at most $\Delta(T) + 1$ precolored edges in a tree T that are extendable to $\Delta(T)$ -edge colorings of T . Next, we use this result for proving a characterization, for all $1 \leq r \leq \Delta(T) + 1$, of which configurations consisting of a precoloring φ of a tree T with $\Delta(T) + 1 - r$ colored edges, and a precoloring ψ of T with r arbitrary (i.e., not necessarily proper) colored edges, there is an extension of φ that avoid ψ . Finally, we consider the problem of extending a precolored matching subject to the condition that the

extension should avoid a given partial coloring.

Bibliography

- [1] M. O. Albertson, *You can't paint yourself into a corner*. J. Combin. Theory Ser. B, 73(2): 189-194, 1998.
- [2] M. O. Albertson, E. H. Moore, *Extending graph colorings using no extra colors*. Discrete Math., 234(1-3): 125-132, 2001.
- [3] L. D. Andersen, *Completing partial Latin squares*, The Royal Danish Academy of Sciences and Letters, Matematisk-fysiske Meddelelser 41(1-10): 23-69, Copenhagen 7.X.1985.
- [4] L. D. Andersen, A. J. W. Hilton, *Thanks Evans!*, Proc. London Math. Soc. 47: 507-522, 1983.
- [5] R. Balakrishnan, K. Ranganathan, *A Textbook of Graph Theory*, Springer, Second Edition, 2012.
- [6] O. V. Borodin, A. V. Kostochka, D. R. Woodall, *List edge and list total colorings of multigraphs*. J. Combin. Theory Ser. B, 71(2): 184-204, 1997.
- [7] C. J. Casselgren, P. Johansson, K. Markström, *Avoiding and extending partial edge colorings of hypercubes*, submitted.
- [8] C. J. Casselgren, *On avoiding some families of arrays*, Discrete Math. 312 (2012) 963-972.
- [9] C. J. Casselgren, K. Markström, L.A. Pham, *Edge Precoloring extension of hypercubes*, Journal of Graph Theory, Vol. 95, no. 3, 410-444, 2020.
- [10] G. Chartrand, P. Zhang, *Chromatic Graph Theory*, CRC Press, Taylor & Francis Group, 2009.
- [11] J. Denes, A. D. Keedwell, *Latin Squares and their Applications*, Bull. Amer. Math. Soc. 82(3): 468-471, 1976.
- [12] K. Edwards, A. Girao, J. van den Heuvel, R. J. Puleo, J. -S. Serini, *Extension from Precoloured Sets of Edges*, Electron. J. Combin, 25(3): Paper 3.1, 28, 2018.
- [13] P. Erdős, A. Rubin, H. Taylor, *Choosability in graphs*, Congr. Numer. 26: 125-157, 1980.
- [14] T. Evans, *Embedding incomplete Latin squares*, Amer. Math. Monthly 67: 958-96, 1960.
- [15] J. Fiala, *NP-completeness of the edge precoloring extension problem on bipartite graphs*, Journal of Graph Theory 43: 156-160, 2003.
- [16] F. Galvin, *The list chromatic index of a bipartite multigraph*, J. Combin. Theory Ser. B 63: 153-158, 1995.

- [17] A. Girao, R. J. Kang, *A precoloring extension of Vizing's theorem*, Journal of Graph Theory, 92: 255-260, 2019.
- [18] R. Häggkvist, *A solution to the Evans conjecture for Latin squares of large size*, Colloqu. Math. Soc. Janos Bolyai 18: 405–513, 1978.
- [19] A. V. Kostochka, *The total coloring of a multigraph with maximal degree 4*, Discrete Math. 17: 161-163, 1977.
- [20] A. V. Kostochka, *The total coloring of a multigraph with maximal degree 5 is at most seven*, manuscript, 1994.
- [21] A. Kumar, *A study on Euler Graphs and it's Applications*, International Journal of Mathematics Trends and Technology (IJMTT), Vol. 43, Num. I, 2017.
- [22] O. Marcotte, P. Seymour, *Extending an edge-coloring*. Journal of Graph Theory, 14(5): 565-573, 1990.
- [23] M. Rosenfeld, *On the total coloring of certain graphs*, Israel J. Math. 9: 396-402, 1971.
- [24] D. P. Sanders, Y. Zhao, *On total 9–coloring planar graphs of maximum degree seven*. Journal of Graph Theory, 31(1): 67-73, 1999.
- [25] B. Smetaniuk, *A new construction on Latin squares. I. A proof of the Evans conjecture*. Ars Combinatoria, 11: 155-172, 1981.
- [26] V. G. Vizing, *Critical graphs with given chromatic class (in Russian)*, Diskret. Analiz, 5: 9-17, 1965.
- [27] D. B. West, *Introduction to Graph Theory*, Prentice Hall, 2nd edition, 2000.

Papers

The papers associated with this thesis have been removed for copyright reasons. For more details about these see:

<https://doi.org/10.3384/9789179292119>

FACULTY OF SCIENCE AND ENGINEERING

Linköping Studies in Science and Technology. Licenciate Thesis No. 1921, 2022
Department of Mathematics

Linköping University
SE-581 83 Linköping, Sweden

www.liu.se