# Regularizing An Ill-Posed Problem with Tikhonov's Regularization 

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LiTH-MAT-EX-2021/09-SE

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Level: G2
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Linköping: April 2021

## Abstract

This thesis presents how Tikhonov's regularization can be used to solve an inverse problem of Helmholtz equation inside of a rectangle. The rectangle will be met with both Neumann and Dirichlet boundary conditions. A linear operator containing a Fourier series will be derived from the Helmholtz equation. Using this linear operator, an expression for the inverse operator can be formulated to solve the inverse problem. However, the inverse problem will be found to be ill-posed according to Hadamard's definition. The regularization used to overcome this ill-posedness (in this thesis) is Tikhonov's regularization. To compare the efficiency of this inverse operator with Tikhonov's regularization, another inverse operator will be derived from Helmholtz equation in the partial frequency domain. The inverse operator from the frequency domain will also be regularized with Tikhonov's regularization. Plots and error measurements will be given to understand how accurate the Tikhonov's regularization is for both inverse operators. The main focus in this thesis is the inverse operator containing the Fourier series.

A series of examples will also be given to strengthen the definitions, theorems and proofs that are made in this work.

## Keywords:

Inverse problem, Ill-posed problems, Tikhonov's regularization, Fourier series, Helmholtz equation

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## Sammanfattning

Denna uppsats presenterar hur Tikhonovs regulariseringsmetod kan användas för att lösa ett inverst problem från Helmholtz ekvation i en rektangel. Rektangeln besitter randvillkor av både Neumann och Dirichlet. En linjär operator som innehåller en Fourier Serie kommer erhållas från Helmholtz ekvation. Genom denna linjära operator kan ett uttryck formas för att lösa det inversa problemet. Dock, är det inversa problemet illa-ställt enligt Hadamards definition. För att överkomma illa-ställdheten (i denna uppsats) kommer Tikhonovs regulariseringsmetod att användas. För att jämföra effektiviteten för den regulariserade inversoperatorn, kommer även en annan inversoperator härledas från Helmholtz ekvation ifrån den partiella frekvensdomänen. Även den inversoperatorn kommer att regulariseras med Tikhonovs regularisering. Grafer och feluppskattningar kommer studeras för att få en förståelse om hur ackurat Tikhonovs regularisering är för de både. Huvudfokuset i denna uppsats är för den regulariserade inversoperatorn som innehåller en Fourier Serie.

Ett antal exempel kommer ges för att förstärka alla definitioner, satser och bevis som görs i denna uppsatts.

## Nyckelord:

Inversa Problem, Illa-ställda problem, Tikhonov's regularisering, Fourier serier, Helmholtz ekvation

## URL för elektronisk version:

Theurltothethesis

Singh, 2021.

## Acknowledgements

This bachelor's work in mathematical analysis would be impossible if it wasn't for my supervisor, Johan Thim, working during the summer of 2021. His instructive answers to my many questions made the flow of this work smooth and sound. Another person worth of mention is my examiner, Fredrik Berntsson, for suggesting me to do a bachelor's work in mathematics. So I would like to send my sincerest gratitude and appreciation to my supervisor and my examiner.

I'd also like to thank my mother, Parmjeet Kaur, for her continuous support in all my endeavours throughout my life.

## Contents

1 Introduction and Concepts ..... 1
1.1 Introduction ..... 1
1.2 Normed Vector Spaces ..... 2
1.3 Inverse Problems and Ill-posed Problems ..... 4
1.4 Regularization ..... 12
1.5 Helmholtz Equation ..... 14
1.5.1 Maxwell's equations ..... 14
2 Solving an Inverse Problem With Regularization ..... 17
2.1 An Example Involving Fourier Series ..... 17
3 Solving Helmholtz Equation in a Rectangle ..... 29
3.1 Boundary Conditions ..... 30
3.2 Solution to Helmholtz Equation ..... 31
3.3 Expansion of a Solution Using Fourier Series ..... 32
3.4 The Linear Operator ..... 35
3.5 The Inverse Operator ..... 38
3.6 Tikhonov's Regularization ..... 39
3.7 Operator From Partial Fourier Transform ..... 40
3.8 Error Estimation ..... 43
4 Conclusion ..... 49
A Proving Divergence With Maclaurin Expansion ..... 53
B Applying Boundary Conditions ..... 55

## Chapter 1

## Introduction and Concepts

In this chapter we will develop the mathematical theory needed for solving the inverse problem of Helmholtz equation in a rectangle, in which a linear operator will be extracted, so relevant definitions and theorems of linear operators will also be presented. The last section of this Chapter provides a brief explanation of regularization. The complete definition of Tikhonov's regularization will be presented in Chapter 3.

### 1.1 Introduction

As the title of this thesis reveals, this work is about using Tikhonov's regularization to solve ill-posed problems. More specifically, Helmholtz equation will be solved in a rectangle with both Neumann and Dirichlet boundary conditions. The direct problem in this thesis is to determine the Dirichlet data of the bottomside of the rectangle caused by the Dirichlet data inserted in the topside of the rectangle. Solving this problem yields a linear operator in the form of a Fourier series. Then, by using this linear operator, we can get an expression for the inverse operator and try to solve the inverse problem. It turns out that this inverse problem is ill-posed and thereby justifying the use of Tikhonov's regularization to solve the problem.

The work of this thesis is similiar to the work of [6], where an inverse problem for an elliptic equation is solved using a Fourier-sine series, and of [7], where an ill-posed problem is solved with mollification.

### 1.2 Normed Vector Spaces

When someone hears the word "space," they might think of the space between two objects, the space-bar on your keyboard, some might even think of outer space. In mathematics, a space represents a set of elements with some added structures. Some spaces are function spaces where each element is a function, sequential spaces where each element is a sequence and vector spaces where each element is a vector. Since the solution of Helmholtz equation will be in a vector space it seems fit to define what a vector space truely is.

Definition 1.2.1. A vector space or (linear space) over a field $K$ is a nonempty set $X$ of elements $x, y, \ldots$ (called vectors) together with two algebraic operations ${ }^{1}$ These operations are called vector addition and multiplication of vectors by scalars, that is, by elements of $K$.

The following examples defines the operations of vector addition and scalar multiplication:

- Vector addition has the following properties for summation of vectors $x, y$ and $z$ :

$$
\begin{equation*}
x+y=y+x \tag{1.1}
\end{equation*}
$$

and

$$
\begin{equation*}
(x+y)+z=x+(y+z) \tag{1.2}
\end{equation*}
$$

In mathematics, these properties are formally known as commutive and associative laws.

- Multiplication by scalars is defined such that for every vector $x, y$ and scalar $\alpha, \beta$, the following properties must be valid:

$$
\begin{equation*}
\alpha(x+y)=\alpha x+\alpha y \tag{1.3}
\end{equation*}
$$

and

$$
\begin{equation*}
(\alpha+\beta) x=\alpha x+\beta x \tag{1.4}
\end{equation*}
$$

In mathematics, these properties are formally known as distributive laws.
Furthermore, there must exist a zero vector such that when $x$ and $-x$ are added together, the zero vector emerges:

$$
\begin{equation*}
x+(-x)=0 \tag{1.5}
\end{equation*}
$$

[^0]In order to interpret the length of elements in a vector space a norm must be defined on it. A norm essentially transforms a vector from the space $X$ to a scalar ${ }^{2}$ The norm has the following definition (according to [5]).

Definition 1.2.2. A normed vector space $X$ is a vector space with a norm defined on it. A Banach space is a complete normed space (complete in the metric defined by the norm). Here a norm on a (real or complex) vector space $X$ is a real-valued function on $X$ whose value at $x \in X$ is denoted by

$$
\begin{equation*}
\|x\| \tag{1.6}
\end{equation*}
$$

and which has the following properties:
1.

$$
\begin{equation*}
\|x\| \geq 0 \tag{1.7}
\end{equation*}
$$

2. 

$$
\begin{equation*}
\|x\|=0 \Longleftrightarrow x=0 \tag{1.8}
\end{equation*}
$$

3. 

$$
\begin{equation*}
\|\alpha x\|=|\alpha|\|x\| \tag{1.9}
\end{equation*}
$$

4. 

$$
\begin{equation*}
\|x+y\| \leq\|x\|+\|y\| \tag{1.10}
\end{equation*}
$$

Here $x$ and $y$ are arbitrary vectors in $X$ and $\alpha$ is any scalar.
Remark 1.2.1. A normed vector space is also referred to as a normed space or normed linear space.

Remark 1.2.2. Not all vector spaces allow a norm, when a topology has been choosen.

For more information regarding the subjects in this chapter, see [5].

[^1]
### 1.3 Inverse Problems and Ill-posed Problems

Imagine you have a bow and arrow, and you are given instructions to shoot the arrow with the intial velocity, $v_{0}$, and with the intial angle, $\alpha$, at position, $\left(x_{0}, y_{0}\right)$. There is no air resistance present.


Figure 1.1: A representation of the bow and arrow problem where the arrow is shot from the intial position $\left(x_{0}, y_{0}\right)$ with velocity $v_{0}$ and with the angle $\alpha_{0}$.

You are asked to solve the problem of finding out where the arrow would land before verifying it by shooting the arrow. Since there only exists one unique solution to this problem we can solve it by laws of mechanics. A small change in the angle or velocity yields a small change to the arrows landing. These types of problems are referred to as direct problems and are commonly known to be well-posed.

Imagine now that you are in world which is not properly idealized and you're shooting an arrow over a surface which is both impenetrable and rough. Since the arrow won't be able to penetrate the surface, the arrow would hit the surface and glide away from the initial hit, see Figure 1.2 . In other words we can't guarantee that a small change in intial angle, $\alpha_{0}$, yields a small change in the arrows placement on the field. This breaks the uniqueness and continuity from the direct problem.


Figure 1.2: A visualization of the multiple ways an arrow could theortically land, making it impossible to know from which position it was shot from.

Now, consider the following. The only information you know is the initial velocity $v_{0}$ and the angle $\alpha$ the arrow had. You are now asked to find out at what point, $\left(x_{0}, y_{0}\right)$, the arrow was shot from by just studying the arrow's orientation and placement on the field. Since there exists multiple ways this arrow could have gotten to this position, (see Figure 1.2), the initial point is nearly impossible to find out. This problem is called an inverse problem and happens to be ill-posed because no unique solution exists to this problem.

Now we shall mathematically define every important term that was used. According to [2], the mathmatical definition of a well-posed problem is given by the following.

Definition 1.3.1. Hadamard's definition of well-posedness occurs if and only if a problem has the following characteristics:

1. For all admissible data, a solution exists.
2. For all admissible data, the solution is unique.
3. The solution depend continuously on the data.

To understand this definition, we must further examine the well-posedness characteristics. Suppose a linear operator, $T$, is given in the form

$$
\begin{equation*}
T x=y \tag{1.11}
\end{equation*}
$$

where $T: X \rightarrow Y$ is a bounded linear operator between the normed vector spaces $X$ and $Y$. The first condition is met if all $y \in Y$ are also in $\mathcal{R}(T)$. The second condition is met if and only if $\mathcal{N}(T)=\{0\}$. In other words we need to provide sufficient evidence that the solution is unique. Suppose that $y_{1}-y_{2}=0$. By equation 1.11) we get $T x_{1}-T x_{2}=0$ and through linearity we obtain,

$$
\begin{equation*}
T\left(x_{1}-x_{2}\right)=0 \Longleftrightarrow x_{1}=x_{2} \tag{1.12}
\end{equation*}
$$

Thus the solution is unique if $\mathcal{N}(T)=\{0\}$.

Remark 1.3.1. If the linear operator, $T$, satisfies both condition one and condition two of Hadamard's condition, then $T^{-1}$ exists.

In order to comprenhend condition three we need to define continuity. A definition and theorem from [5] states the following.

Definition 1.3.2. Let $X=(X,\|\cdot\|)$ and $Y=(Y,|\cdot|)$ be normed vector spaces. A mapping $T: X \rightarrow Y$ is said to be continuous at a point $x_{0} \in X$ if for every $\epsilon>0$ there is a $\delta>0$ such that

$$
\begin{equation*}
\left|\left(T x-T x_{0}\right)\right|<\epsilon, \quad \forall x \in\left\|x-x_{0}\right\|<\delta \tag{1.13}
\end{equation*}
$$

$T$ is said to be continuous if it is continuous at every point of $X$.
Definition 1.3.3. The Euclidean norm of a vector $u$, where $u \in \mathbb{C}^{n}$, and of an $L^{2}$-function, $f$, are given by:

$$
\|u\|_{2}=\sqrt{u_{1}^{2}+u_{2}^{2}+\ldots+u_{n}^{2}}
$$

and

$$
\|f\|_{2}=\left(\int_{-\infty}^{\infty}|f(x)|^{2} d x\right)^{\frac{1}{2}}
$$

Definition 1.3.4. The Uniform norm of a vector, $u \in \mathbb{C}^{n}$, and of an $L^{\infty}$ function, $f$, are given by:

$$
\|u\|_{\infty}=\max _{1 \leq k \leq n}\left|u_{k}\right|
$$

and

$$
\begin{equation*}
\|f\|_{\infty}=\operatorname{ess} \sup _{x \in D}|f(x)| \tag{1.14}
\end{equation*}
$$

respectively, where $D$ is the domain of $f$.
Theorem 1.3.1. (Continuity and boundedness) Let $T: \mathcal{D}(T) \longrightarrow Y$ be a linear operator, where $\mathcal{D}(T) \subset X$ and $X, Y$ are normed vector spaces. Then it follows that:

1. $T$ is continuous if and only if $T$ is bounded.
2. If $T$ is continuous at a single point, it is continuous.

From (1) in Theorem 1.3.1 we understand that proving a linear operators continuity is the same as proving it's boundedness. Hence we need to show that the linear operator $T$ is bounded. The definition of a bounded linear operator is given by the following,

Definition 1.3.5. Let $X$ and $Y$ be normed vector spaces and $T: \mathcal{D}(T) \longrightarrow Y a$ linear operator, where $\mathcal{D}(T) \subset X$. The operator $T$ is said to be bounded if there is a real number, $c$, such that for all $x \in \mathcal{D}(T)$,

$$
\begin{equation*}
\|T x\| \leq c\|x\| \tag{1.15}
\end{equation*}
$$

To deepen our understanding of this concept and well-posedness in general, a demonstration through an example seems appropriate.

Example 1.3.1. Suppose a linear operator is given on $\mathbb{R}^{n}$ as a matrix transformation, $T: n \times n$ where $n \in \mathbb{N}^{+}$, and satisfies the following relations:

$$
\begin{equation*}
T x=y \tag{1.16}
\end{equation*}
$$

where $x, y \in \mathbb{R}^{n}$. Then,

$$
\begin{equation*}
\|T x\| \leq c\|x\| \tag{1.17}
\end{equation*}
$$

where $c \in \mathbb{R}$. The problem is to find out if the inverse problem,

$$
\begin{equation*}
x=T^{-1} y \tag{1.18}
\end{equation*}
$$

is well-posed.

The first and second condition is met if the solution is unique. In order to have uniqueness, the condition $\mathcal{N}(T)=\{0\}$ must be valid. If $\mathcal{N}(T)=\{0\}$ is true then it follows that $T^{-1}$ exists $(T x=0 \Longleftrightarrow x=0)$.

If the solution is unique this would satisfy Hadamard's first and second condition. Because $T$ consists of $n \times n$ elements and $n<\infty$, this means that $T$ is bounded in every norm. If $T$ is bounded and $T x=0 \Longleftrightarrow x=0$ is true then it follows that $T^{-1}$ exists and is bounded as well. (Since also $T^{-1}$ is given by an $n \times n$ matrix.) This notion implies that $T^{-1}$ is continuous and thereby validating Hadamard's third condition for the inverse problem.

In conclusion, if $T^{-1}$ exists the problem is well-posed and if $T^{-1}$ does not exist the problem becomes ill-posed.

Example 1.3 .1 introduces the concept that a problem can be found to be illposed. To completely understand what ill-posedness means, a definition will be given to explain it, followed by examples to demonstrate it.

Definition 1.3.6. A problem is ill-posed if one, or more, of the conditions for well-posedness are not satisfied, that is, a problem is ill-posed if it is not well-posed.

Example 1.3.2. The dot product with one factor kept fixed defines a functional on $\mathbb{R}^{3}$. Let $\odot$ be this dot products functional, $\odot: \mathbb{R}^{3} \rightarrow \mathbb{R}$,

$$
\begin{equation*}
\odot(x)=\langle\bar{x}, \bar{\lambda}\rangle=3 x_{1}+4 x_{2}+5 x_{3} \tag{1.19}
\end{equation*}
$$

where $\bar{\lambda}=\left[\begin{array}{l}3 \\ 4 \\ 5\end{array}\right]$.
The problem is to explain why the equation, $\odot(x)=S: S \in \mathbb{R}$, is found as ill-posed rather than well-posed.

Firstly, we shall investigate whether this functional is bounded or not. By Cauchy's inequality,

$$
\begin{equation*}
|\odot(\bar{x})|=|\langle\bar{x}, \bar{\lambda}\rangle| \leq\|\bar{x}\|\|\bar{\lambda}\| \tag{1.20}
\end{equation*}
$$

Let $c=\|\bar{\lambda}\|$ and use the following definition for the norm:

$$
\begin{equation*}
\|\odot\|=\sup _{\substack{x \in \mathcal{D}(\odot) \\\|x\|=1}} \frac{|\odot(\bar{x})|}{\|\bar{x}\|} \tag{1.21}
\end{equation*}
$$

By applying this definition of the norm we get the following inequality:

$$
\begin{equation*}
\|\odot\| \leq c \tag{1.22}
\end{equation*}
$$

Thus the operator can be concluded to be bounded, yielding that the third condition of Definition 1.3.1 is met. For the first- and the second conditions to be met, a unique solution to equation (1.23) must exist. Let the scalar product be denoted as the real constant, $S$. In order to have uniquesness the functional equation must have one unique solution:

$$
\begin{equation*}
3 x_{1}+4 x_{2}+5 x_{3}=S \tag{1.23}
\end{equation*}
$$

In linear algebra this equation represents a plane in $\mathbb{R}^{3}$. It is well known that a plane has infinitely many points, which means there are an infinite amount of solutions. This satisfies Hadamard's first condition but not Hadamard's second condition, thus this problem can be concluded to be ill-posed.

To further strengthen our understanding of well-posedness two more examples are given.

Example 1.3.3. In this example we will define an operator, $T$, as a convolution integral $T: C^{0}[0,1] \rightarrow G$ by

$$
\begin{equation*}
T f(t)=\int_{0}^{t} f(\tau)(t-\tau) d \tau \tag{1.24}
\end{equation*}
$$

where

$$
\begin{equation*}
g=T f \tag{1.25}
\end{equation*}
$$

and $G$ is defined as

$$
\begin{equation*}
G \in\left\{g \in C^{2}[0,1]: \quad g(0)=g^{\prime}(0)=0\right\} . \tag{1.26}
\end{equation*}
$$

Proof. Let $F(t)$ be a primitive function of $f(t)$ and let $\mathbb{F}(t)$ be a primitive function of $F(t)$. Then

$$
\begin{align*}
g(t) & =\int_{0}^{t} f(\tau)(t-\tau) d \tau \\
& =[F(\tau)(t-\tau)]_{0}^{t}-\int_{0}^{t} F(\tau) d \tau  \tag{1.27}\\
& =-F[0] t-[\mathbb{F}(\tau)]_{0}^{t} \\
& =-F[0] t-\mathbb{F}(t)+\mathbb{F}(0) .
\end{align*}
$$

Since the double derivative of $\mathbb{F}(t)$ yields the $C^{0}[0,1]$-function $f(t)$, it's clear that $g \in C^{2}[0,1]$. Through equation 1.27 we also see that $g(0)=g^{\prime}(0)=0$.

The problem is now to show that this example, $g=T f$, is well-posed.
Remark 1.3.2. In this example, finding $\mathcal{R}(T)$ is easy. However, for many other operators' it can be incredibly difficult to find $\mathcal{R}(T)$.

According to Hadamard's first and second condition, we need to provide evidence that a unique solution exists. To show this, the Laplace transform will be used. We shall use the uniqueness for the unilateral Laplace transform, which states that for every continuous function in the $t$-domain there exists a transform of it in the $s$-domain if the integral in equation 1.28 is absolutely convergent. Furthermore, if $u$ and $v$ are continuous and absolutely integrable with $\mathscr{L}_{+} u=\mathscr{L}_{+} v$ for $\operatorname{Re}(s)>a>0$, then $u=v$ for $t \geq 0$.

From [1], the unilateral Laplace transform has the following definition:

$$
\begin{equation*}
\hat{f}(s)=\left(\mathscr{L}_{+} f\right)(s)=\int_{0}^{\infty} f(t) e^{-s t} d t \tag{1.28}
\end{equation*}
$$

where $f \in C(\mathbb{R})$ and whose domain consists of $s \in \mathbb{C}$ for which the integral is absolutely convergent. We shall now reformulate the problem with the unilateral Laplace transform. We write

$$
\begin{equation*}
g(t)=\int_{0}^{t} f(\tau)(t-\tau) d \tau=f(t) * t \tag{1.29}
\end{equation*}
$$

and the unilateral Laplace transform becomes:

$$
\begin{equation*}
\left(\mathscr{L}_{+} g\right)(s)=\mathscr{L}_{+}((f * t))(s) \Longleftrightarrow \hat{g}(s)=\hat{f}(s) \frac{1}{s^{2}} \tag{1.30}
\end{equation*}
$$

so

$$
\begin{equation*}
\hat{f}(s)=\hat{g}(s) s^{2} \tag{1.31}
\end{equation*}
$$

and since $g(0)=g^{\prime}(0)=0$ the inverse Laplace transform $f(t)$ is given by:

$$
\begin{equation*}
\mathscr{L}_{+}^{-1}(\hat{f}(s))=\mathscr{L}_{+}^{-1}\left(\hat{g}(s) s^{2}\right) \Longleftrightarrow f(t)=\frac{d^{2}}{d t^{2}} g(t) . \tag{1.32}
\end{equation*}
$$

From equation 1.32 we see that $f(t)$ is unique for every $g \in G$ for the unilateral Laplace operator. This ensures that the inverse of the unilateral Laplace operator exists for the space $\mathcal{R}\left(\mathscr{L}_{+}\right)$. Since the derivate of a function yields one
and only one answer the conclusion could be drawn that this operator equation has a unique solution, thereby satisfying Hadamard's first and second condition. Since we know that $t$ and $f(t)$ are continuous fucntions on $[0,1]$, the norms are given by:

$$
\begin{equation*}
\|t\|_{\infty}=\max _{0 \leq t \leq 1}|t|=1 \tag{1.33}
\end{equation*}
$$

and

$$
\begin{equation*}
\|f(t)\|=\max _{0 \leq t \leq 1}|f(t)|=\|f\|_{\infty} \tag{1.34}
\end{equation*}
$$

Hence

$$
\begin{align*}
\|g(t)\|=\|T f(t)\| & =\max _{0 \leq t \leq 1}\left|\int_{0}^{t} f(\tau)(t-\tau) d \tau\right| \\
& \leq \max _{0 \leq t \leq 1} \int_{0}^{t}|f(\tau) \|(t-\tau)| d \tau  \tag{1.35}\\
& \leq\|f\|_{\infty} \int_{0}^{1} d \tau \\
& \leq\|f\|_{\infty}
\end{align*}
$$

The result is that $\|T f\| \leq\|f\|_{\infty}$. Thus the linear operator is bounded. If it's bounded, it's continuous and thereby satisfying Hadamard's third condition. Thus this problem is well-posed.

For more information regarding the subject, see 4] and 5].

### 1.4 Regularization

Recall the "bow and arrow" problem in the previous section. Imagine now that we try to transform this ill-posed problem into a well-posed problem. To do this we must adjust the problem itself. Let's say we introduce an adjustable constant that makes the impenetrable surface more rough as the constant gets larger. It is still the same problem, only now when the arrow hits the surface it will reveal the direction from whence it came. By doing so the problem has potentially become well-posed.


Figure 1.3: This figure demonstrates how the problem becomes well-posed, when the surface friction changes. By calculating the arrrow's angle with the xy-plane, one can understand from which direction it was shot from.

This type of idea, where an adjustment can make ill-posed problems become well-posed is what we are trying to do in this thesis by applying Tikhonov's regularization to the inverse operator for the inverse problem of Helmholtz equation with certain boundary data.

Lets say we have the inverse problem, $T^{-1} f=g$, where the first- and second conditions of Hadamard's definition (Definition 1.3.1) are met but the third condition is not. In other words the inverse operator, $T^{-1}$, is unbounded, making the problem ill-posed. We shall now introduce the functional $J_{\alpha}$ as an approximation to $T^{-1}$. The $\alpha$ in $J_{\alpha}$ is a quantity measurement of how accurate the approximation is. According to [2], $J_{\alpha}$ has the following definition:

Definition 1.4.1. Let $T: F \rightarrow G$ be a bounded linear operator between the normed vector spaces $F$ and $G$, and let $\left.\alpha_{0} \in\right] 0, \infty[$. For every $\alpha \in] 0, \alpha_{0}[$, let

$$
\begin{equation*}
J_{\alpha}: G \rightarrow F \tag{1.36}
\end{equation*}
$$

be a continuous operator. The family $\left\{J_{\alpha}\right\}$ is called a regularization or a regularization operator if, for all $g \in D\left(T^{-1}\right)$, there exists a parameter choice rule $\alpha=\alpha\left(\delta, g^{\delta}\right)$ such that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \sup \left\{\left\|J_{\alpha\left(\delta, g^{\delta}\right)} g^{\delta}-T^{-1} g\right\|: g^{\delta} \in G,\left\|g^{\delta}-g\right\| \leq \delta\right\}=0 \tag{1.37}
\end{equation*}
$$

holds. Here,

$$
\begin{equation*}
\left.\alpha: \mathbb{R}^{+} \times G \rightarrow\right] 0, \alpha_{0}[ \tag{1.38}
\end{equation*}
$$

is such that

$$
\begin{equation*}
\lim _{\delta \rightarrow 0} \sup \left\{\alpha\left(\delta, g^{\delta}\right): g^{\delta} \in G,\left\|g^{\delta}-g\right\| \leq \delta\right\}=0 \tag{1.39}
\end{equation*}
$$

For a specific $g \in D\left(T^{-1}\right)$, a pair $\left(J_{\alpha}, \alpha\right)$ is called a (convergent) regularization method (for solving $T f=g$ ) if 1.37 ) and 1.39 hold.

Note that $\alpha$ is dependent on both $\delta$ and $g^{\delta}$. From [2], we can describe when $\alpha$ strictly depenedes only on $\delta$ :

Definition 1.4.2. Let $\alpha$ be a parameter choice rule according to Definition 1.4.1. If $\alpha$ does not depend on $g^{\delta}$, but only on $\delta$, then we call $\alpha$ an a-priori parameter choice rule and write $\alpha=\alpha(\delta)$. Otherwise, we call $\alpha$ an a-posteriori parameter choice rule.

To understand how we can use these definitions in practice, see Chapter 2 where an inverse problem is solved using a regularization method that satisfies Definition 1.4.1 and 1.4.2

### 1.5 Helmholtz Equation

Ever since Isaac Newton introduced the idea of creating a descriptive method to study the nature through mathematical models, e.g. differential equations, ideas have continued to evole and later even becoming fundemental laws in science. These mathematical models are the foundation of the future, they give an explaination to the previously thought as unexplainable. One of these mathematical models that changed the world was formed by Hermann von Helmholtz and was called the Helmholtz equation:

$$
\begin{equation*}
\nabla^{2} \psi(\bar{r})+k^{2} \psi(\bar{r})=0 \tag{1.40}
\end{equation*}
$$

where the Laplacian ${ }^{3}$ is defined as $\nabla^{2}=\frac{\partial^{2}}{\partial x^{2}}+\frac{\partial^{2}}{\partial y^{2}}+\frac{\partial^{2}}{\partial z^{2}}$ and $\bar{r}=\left[\begin{array}{l}x \\ y \\ z\end{array}\right]$.
The reason why this equation has been fascinating for many scientists is because this equation appears naturally from conservation laws in physics and can even be used to interpret the wave equation for monochromatic waves. When Schrödinger first heard of the concept of wave-particle dualism, that all quantum particles possess both wave and particle properties, he created his famous Schrödinger's equation:

$$
\begin{equation*}
i \hbar \frac{\partial}{\partial t} \Psi(\mathbf{r}, t)=\hat{H} \Psi(\mathbf{r}, t) \tag{1.41}
\end{equation*}
$$

It turns out that by even studying this equation, the Helmholtz equation could still be derived from it. However one of the most famous examples of the Helmholtz equation are Maxwell's equations.

### 1.5.1 Maxwell's equations

In this section we will understand how Maxwell's equations could be derived through Helmholtz equation 1.40 .

Suppose we have a medium where no free charge exists nor are any imposed currents present then we get the following Maxwell's equations:

$$
\begin{equation*}
\nabla \times E=-\mu \frac{\partial H}{\partial t} \tag{1.42}
\end{equation*}
$$

[^2]\[

$$
\begin{gather*}
\nabla \times H=\epsilon \frac{\partial E}{\partial t}  \tag{1.43}\\
\nabla \cdot E=0  \tag{1.44}\\
\nabla \cdot H=0 \tag{1.45}
\end{gather*}
$$
\]

Where H and E are magnetic- and electric field vectors and the electric permittivity and magnetic permeability are denoted as $\epsilon$ and $\mu$. We have the following relationships in vaccuum:

$$
\begin{equation*}
\mu=\mu_{0}, \quad \epsilon=\epsilon_{0}, \quad c=\left(\epsilon_{0} \mu_{0}\right)^{-1 / 2} \tag{1.46}
\end{equation*}
$$

where $c \simeq 3 \times 10^{8} \mathrm{~m} / \mathrm{s}$.
With equation $1.42,(1.43,1.44$ and 1.45 we obtain the identity:

$$
\begin{equation*}
\nabla^{2} E=\nabla(\nabla \cdot E)-\nabla \times \nabla \times E=-\nabla \times \nabla \times E \tag{1.47}
\end{equation*}
$$

By taking the curl of equation 1.42 we get the following equations:

$$
\begin{gather*}
\nabla \times \nabla \times E=-\mu \frac{\partial}{\partial t} \nabla \times H  \tag{1.48}\\
-\nabla^{2} E=-\epsilon \mu \frac{\partial^{2} E}{\partial t^{2}}  \tag{1.49}\\
\nabla^{2} E=\frac{1}{c^{2}} \frac{\partial^{2} E}{\partial t^{2}}  \tag{1.50}\\
\nabla^{2} E-\frac{1}{c^{2}} \frac{\partial^{2} E}{\partial t^{2}}=0 \tag{1.51}
\end{gather*}
$$

Similary we obtain $H$ as:

$$
\begin{equation*}
\nabla^{2} H-\frac{1}{c^{2}} \frac{\partial^{2} H}{\partial t^{2}}=0 \tag{1.52}
\end{equation*}
$$

The relevancy of the Helmholtz equation in many branches of physics is astonishing.

## Chapter 2

## Solving an Inverse Problem With Regularization

This section offers an instructive and comprehensible example where an inverse problem is first proved to be ill-posed and later becomes well-posed through regularization. It should be noted that the regularization in this example will not be of Tikhonov's kind.

### 2.1 An Example Involving Fourier Series

## Example 2.1.1.

Suppose a linear operator, $T$, transforms a $C^{1}[0, \pi]$-function into a sine series. If the given function is denoted as $f$ then $(T f)$ produces a sine series (whose index goes from 1 to $\infty$ ) multiplied with the Fourier coefficient of $f$ divided by the index. Since the Fourier coefficient will be divided by the index, the series won't reproduce $f$ instead it gives rise to a new $C^{2}[0, \pi]$-function. Let's call this function $g$.

The problem in this example is to figure out what $f$ is, if we can only measure $g$, and does $f$ depend continuously on $g$ ? Is this inverse problem well-posed? If the inverse problem is ill-posed then can we regularize the problem? How good is a truncated Fourier series as a regularization method?

Definition 2.1.1. Let the linear operator, $T: \mathbb{W} \rightarrow \mathbb{W}$, have the following definition:

$$
\begin{equation*}
(T f)(x)=\sum_{n=1}^{\infty} \frac{2 \sin (n x)}{n \pi} \int_{0}^{\pi} f(\tau) \sin (\tau n) d \tau \tag{2.1}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbb{W} \in\left\{w \in C^{1}[0, \pi]: \quad w(0)=w(\pi)=0\right\} . \tag{2.2}
\end{equation*}
$$

We will show that:

$$
\begin{equation*}
\mathcal{R}(T) \subset C^{2}[0, \pi], \tag{2.3}
\end{equation*}
$$

where $\mathcal{R}(T)$ is the range of the operator. Mathematically our problem is considered as,

$$
\begin{equation*}
(T f)(x)=g(x) \tag{2.4}
\end{equation*}
$$

where $f, g \in \mathbb{W}$.

Now we shall show that the linear operator, $T$, is bounded in $L^{2}$ and $L^{\infty}$. We can substitute the mean value integral with the inner product:

$$
\begin{equation*}
\frac{2}{\pi} \int_{0}^{\pi} f(\tau) \sin (\tau n) d \tau=\left\langle f, s_{n}\right\rangle ; \quad s_{n}=\{\sin (\tau n)\}_{n=1}^{\infty} \tag{2.5}
\end{equation*}
$$

Then we get:

$$
\begin{equation*}
T f(x)=\sum_{n=1}^{\infty}\left\langle f, s_{n}\right\rangle \cdot \frac{\sin (n x)}{n} \tag{2.6}
\end{equation*}
$$

so

$$
\begin{equation*}
|T f| \leq \sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right| \cdot \frac{1}{n} \tag{2.7}
\end{equation*}
$$

By the Cauchy-Schwarz inequality:

$$
\begin{equation*}
|T f| \leq \sqrt{\sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right|^{2}} \cdot \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{2}}} \tag{2.8}
\end{equation*}
$$

By applying Bessel's inequality for the left factor of the right hand side of equation (2.8) we get:

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right|^{2} \leq\|f\|_{2}^{2} \tag{2.9}
\end{equation*}
$$

Thus the sum can be concluded to be bounded. The right factor of equation (2.8) is known as the Basel problem in the math community, and

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{1}{n^{2}}=\frac{\pi^{2}}{6} \tag{2.10}
\end{equation*}
$$

A fact formulated in the 1644 by Pietro Mengoli and solved by Leonard Euler in the 1734. By applying these equalities and inequailties to equation 2.8, we get the following inequality:

$$
\begin{align*}
\|T f\|_{\infty} & \leq \sqrt{\|f\|_{2}^{2}} \cdot \sqrt{\frac{\pi^{2}}{6}}  \tag{2.11}\\
& =\|f\|_{2} \cdot \frac{\pi}{\sqrt{6}}
\end{align*}
$$

which gives the final expression,

$$
\begin{equation*}
\|T f\|_{\infty} \leq \frac{1}{\sqrt{6}} \int_{0}^{\pi}|f(x)|^{2} d x \tag{2.12}
\end{equation*}
$$

From Definition 1.3 .3 and Definition 1.3 .4 we see that $\|f\|_{2} \leq\|f\|_{\infty}$ so,

$$
\begin{equation*}
\|T f\|_{\infty} \leq \frac{\pi}{\sqrt{6}} \cdot\|f\|_{\infty} \tag{2.13}
\end{equation*}
$$

Let $\frac{c_{n}}{n}$ denote the Fourier coefficients of $T f$, where $c_{n}$ is the Fourier coefficients of $f$. Then, by Parseval's theorem, we get,

$$
\begin{equation*}
\frac{1}{\pi} \int_{0}^{\pi}|T f(x)|^{2} d x=\sum_{n=1}^{\infty}\left|\frac{c_{n}}{n}\right|^{2} \leq \sum_{n=1}^{\infty}\left|c_{n}\right|^{2}=\frac{1}{\pi} \int_{0}^{\pi}|f(x)|^{2} d x \tag{2.14}
\end{equation*}
$$

Thus,

$$
\begin{equation*}
\|T f\|_{2} \leq\|f\|_{2} \tag{2.15}
\end{equation*}
$$

Therefore the operator can be concluded as $L^{2}$ - and $L^{\infty}$-bounded. Thereby satisfying Hadamard's third condition in both spaces.
Now we need to show that the direct problem $T f=g$ has a unique solution. To demonstrate Hadamard's first and second condition we shall use Dirchlet convergence theorem. If Theorem 4.16 from [1] gets adjusted to our problem we get the following formula for Dirichlet's convergence theorem:

$$
\begin{equation*}
\sum_{n=1}^{\infty} \hat{u}(n) \sin (n \Omega a)=\frac{u\left(a^{+}\right)+u\left(a^{-}\right)}{2}=u(a) \tag{2.16}
\end{equation*}
$$

where $\hat{u}(n)$ represents the Fourier series coefficients. Here we see that for every $a$ the series will converge. Since the functions $f$ and $g$ are continuous, the values from $g\left(a^{+}\right)$and $g\left(a^{-}\right)$can easily be generalized. To prove its uniqueness more direct we need to show that,

$$
T f=0 \Longrightarrow f=0
$$

Lemma 2.1.1. The operator $T$ is injective.
Proof. Let's consider the complex form of,

$$
\text { (Real) } \quad T f(x)=u(x)=\sum_{n=1}^{\infty} \frac{\sin (n x)}{n \pi} \int_{0}^{\pi} f(\tau) \sin (n \tau) d \tau
$$

that is,
(Complex) $\quad T f(x)=u(x)=\sum_{n=-\infty}^{\infty} \frac{e^{i n x}}{2 n \pi} \int_{-\pi}^{\pi} f(\tau) e^{i n \tau} d \tau, \quad n \neq 0$.

Let $C_{n}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} f(\tau) e^{i n \tau} d \tau$. Then the complex form can be simplified to,

$$
\begin{equation*}
u(x)=\sum_{n=-\infty}^{\infty} C_{n} \frac{e^{i n x}}{n}, \quad n \neq 0 \tag{2.18}
\end{equation*}
$$

By differentiation we get,

$$
\begin{equation*}
\frac{d u(x)}{d x}=\sum_{n=-\infty}^{\infty} i \cdot C_{n} e^{i n x}, \quad n \neq 0 \tag{2.19}
\end{equation*}
$$

where the Fourier coefficient, $C_{n}$, can be expressed as,

$$
\begin{equation*}
C_{n}=\frac{1}{2 \pi} \int_{-\pi}^{\pi} \frac{d u(\tau)}{d \tau} e^{i n \tau} d \tau, \quad\left(C_{0}=0\right) \tag{2.20}
\end{equation*}
$$

By identification of 2.20 and $C_{n}$ in 2.18, we get the following equality,

$$
\begin{equation*}
\frac{d u(\tau)}{d \tau}=f(\tau) \Longrightarrow u(x)=\int_{0}^{x} f(\tau) d \tau \tag{2.21}
\end{equation*}
$$

since $u$ is smooth enough. Thus,

$$
\begin{equation*}
T f(x)=\int_{0}^{x} f(\tau) d \tau \tag{2.22}
\end{equation*}
$$

where its apparent that $T f(x)=0$ for all $x$ if and only if $f(x)=0$ for all $x$. Hence the operator is injective.

This essentially means that if two functions have the same Fourier coefficients, they are the same function. Thus the operator $T$ is unique and thereby satisfying all of Hadamard's conditions and making the direct problem wellposed.

In order to find out that there is a solution $f$ to $T f=g$, we need to define the inverse operator, $T^{-1}$. If

$$
\begin{equation*}
T f=g \tag{2.23}
\end{equation*}
$$

reasonably makes every Fourier coefficient of $f$ be divided by $n$,

$$
\begin{equation*}
T f(x)=\sum_{n=1}^{\infty} \frac{2 \sin (n x)}{n \pi} \int_{0}^{\pi} f(\tau) \sin (\tau n) d \tau=g(x) \tag{2.24}
\end{equation*}
$$

then that means that

$$
\begin{equation*}
T^{-1} g=f \tag{2.25}
\end{equation*}
$$

reasonably must mulitply every Fourier coefficent with $n$, where the odd function $g(x) \in C^{2}[0, \pi]$ satisfies

$$
\begin{gather*}
\text { (Real) } \quad T^{-1} g(x)=\sum_{n=1}^{\infty} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} g(\tau) \sin (\tau n) d \tau=f(x),  \tag{2.26}\\
\text { (Complex) } \\
T^{-1} g(x)=\sum_{n=-\infty}^{\infty} \frac{n e^{i n x}}{\pi} \int_{-\pi}^{\pi} g(\tau) e^{i \tau n} d \tau=f(x)
\end{gather*}
$$

Theorem 2.1.1. Let $T$ be as in Definition 2.1.1, then the inverse operator, $T^{-1}$, can be expressed as:

$$
\begin{equation*}
T^{-1} g=\sum_{n=1}^{\infty} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} g(\tau) \sin (\tau n) d \tau=f \tag{2.27}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathbb{W} \in\left\{w \in C^{2}[0, \pi]: \quad w(0)=w(\pi)=0\right\},  \tag{2.28}\\
\mathcal{R}(T) \subset \mathbb{W} \tag{2.29}
\end{gather*}
$$

and $g \in \mathcal{R}(T)$, where $\mathcal{R}(T)$ is the range of the operator.
By studying equation 2.26 and 2.24 , we see that the inverse, $T^{-1}$, and the linear operator, $T$, have many similarities between each other. The only difference being that one multiplies each Fourier coefficient with $n$ and the other divides them by $n$. So many properties of $T$ is shared by $T^{-1}$ as well. One of
these properties is the uniqueness demonstrated in equation 2.16). That means that $T^{-1}$ also satisfies Hadamard's first and second condition. Equation 2.26 gives a mathematical description of how the inverse linear operator, $T^{-1}$, acts when given a function $g$. We shall return to this expression later on, but for now consider we have $g$ with it's corresponding real and complex Fourier series:

$$
\text { (Real) } \begin{align*}
& g(x)
\end{align*}=\sum_{n=1}^{\infty} \frac{2}{\pi} \int_{0}^{\pi} g(\tau) \sin (n \tau) d \tau \cdot \sin (n x)
$$

We shall now take a closer look at the complex series' and show that the derivative of $g(x)$ basically yields our inverse operation $T^{-1} g$. Furthermore showing that our inverse operator is actually the differential operator $\frac{d}{d x}$.

Proof. Consider that $g \in C^{3}[0, \pi]$ is an odd function in $C^{3}[-\pi, \pi]$ and therefore

$$
\begin{equation*}
g(x)=\sum_{n=-\infty}^{\infty} \frac{1}{2 \pi} \int_{-\pi}^{\pi} g(\tau) e^{i n \tau} d \tau \cdot e^{i n x} \tag{2.31}
\end{equation*}
$$

so

$$
\begin{align*}
\frac{d g(x)}{d x} & =\sum_{n=-\infty}^{\infty} \frac{1}{2 \pi} \int_{-\pi}^{\pi} g(\tau) e^{i n \tau} d \tau \cdot i n e^{i n x} \\
& =i \sum_{n=-\infty}^{\infty} \frac{1}{2 \pi} \int_{-\pi}^{\pi} g(\tau) e^{i n \tau} d \tau \cdot n e^{i n x} \tag{2.32}
\end{align*}
$$

where we are allowed to differentiate term-wise since $g \in C^{3}$. Now inserting equation 2.30 we get:

$$
\begin{equation*}
T^{-1} g=\frac{1}{i} \frac{d g(x)}{d x} \tag{2.33}
\end{equation*}
$$

The differential operator, $D$, is unbounded. To support this claim we let $h_{k} \in C^{3}[0, \pi]$ be a succession of functions such that $h_{k}(0)=h_{k}(\pi)=0$ and $\frac{\left\|D\left(h_{k}\right)\right\|_{\infty}}{\left\|h_{k}\right\|_{\infty}} \rightarrow \infty$ when $k \rightarrow \infty$.

Proof. Suppose that

$$
\begin{equation*}
h_{k}(x)=\sin (k x) \tag{2.34}
\end{equation*}
$$

where the first- and second derivative of $h_{k}(x)$ becomes:

$$
\begin{align*}
D\left(h_{k}(x)\right) & =k \cos (k x) \\
D^{2}\left(h_{k}(x)\right) & =-k^{2} \sin (k x) . \tag{2.35}
\end{align*}
$$

The norm of $h_{k}$ is given by:

$$
\begin{equation*}
\left\|h_{k}\right\|_{\infty}=\|\sin (k x)\|_{\infty}=1 \tag{2.36}
\end{equation*}
$$

Now to calculate the norm for $D\left(h_{k}\right)$, we put the second derivative to zero:

$$
\begin{align*}
0 & =-k^{2} \sin (k x) \\
\Longrightarrow 0 & =\sin (k x)  \tag{2.37}\\
\Longrightarrow x & =\frac{\pi m}{k}, \quad m \in \mathbb{Z} .
\end{align*}
$$

Through equation 2.37 we get the norm of $D\left(h_{k}\right)$ as:

$$
\begin{equation*}
\left\|D\left(h_{k}\right)\right\|_{\infty}=\left|D\left(h_{k}\left(\frac{\pi m}{k}\right)\right)\right|=|k \cos (\pi m)|=k, \quad \forall m \in \mathbb{Z} \tag{2.38}
\end{equation*}
$$

Hence we get if $k>0$,

$$
\begin{equation*}
\frac{\left\|D\left(h_{k}\right)\right\|_{\infty}}{\left\|h_{k}\right\|_{\infty}}=k \tag{2.39}
\end{equation*}
$$

Thus the fraction in equation 2.39 diverges when $k \rightarrow \infty$, thus showing that the operator is unbounded.

Hence $T^{-1}$ is unbounded and therefore Hadamard's third condition does not hold. Which according to Definition 1.3 .6 makes this problem ill-posed. It turns out that this example qualifies for regularization according to Definition 1.4.1 (see the proof below). In other words, we can approximate $T^{-1}$ with $J_{\alpha}$ where $\alpha$ is an a-priori parameter; see Definition 1.4.2.

Lemma 2.1.2. The inverse operator, $T^{-1}$, can be regularized.
Proof. Suppose the noisy data, $g^{\delta}$, has the following form:

$$
\begin{equation*}
g^{\delta}(x)=g(x)+\delta(x), \quad x \in[0, \pi] \tag{2.40}
\end{equation*}
$$

where $g \in C^{2}[0, \pi]$, meaning that the approximation data, $f^{\delta}$, gets the following formal expression:

$$
\begin{align*}
f^{\delta}(x) & =T^{-1} g^{\delta}(x) \\
& =\sum_{n=1}^{\infty} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} g^{\delta}(\tau) \sin (\tau n) d \tau \\
& =\sum_{n=1}^{\infty} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} g(\tau) \sin (\tau n) d \tau+\sum_{n=1}^{\infty} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (\tau n) d \tau \\
& =\sum_{n=1}^{\infty} n \sin (n x) \cdot\left\langle g, s_{n}\right\rangle+\sum_{n=1}^{\infty} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (\tau n) d \tau \\
& =f(x)+\tilde{\delta}(x) . \tag{2.41}
\end{align*}
$$

The error in the approximation will lie in $\tilde{\delta}=\sum_{n=1}^{\infty} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (\tau n) d \tau$ so we need to prove that this tends to zero when $\delta \rightarrow 0$. First we shall simplify the inner product in equation 2.41 to get the following representation:

$$
\begin{align*}
\left\langle g, s_{n}\right\rangle & =\frac{2}{\pi} \int_{0}^{\pi} g(\tau) \sin (n \tau) d \tau \\
& =\frac{2}{\pi}\left[\frac{-g(\tau) \cos (\tau n)}{n}\right]_{0}^{\pi}+\frac{2}{\pi} \int_{0}^{\pi} \frac{g^{\prime}(\tau) \cos (\tau n)}{n} d \tau \tag{2.42}
\end{align*}
$$

From equation 3.37 we can see that $g(0)=g(\pi)=0$. Therefore,

$$
\begin{equation*}
\frac{2}{\pi} \int_{0}^{\pi} \frac{g^{\prime}(\tau) \cos (\tau n)}{n} d \tau=\frac{2}{\pi}\left[\frac{g^{\prime}(\tau) \sin (\tau n)}{n^{2}}\right]_{0}^{\pi}-\frac{2}{\pi} \int_{0}^{\pi} \frac{g^{\prime \prime}(\tau) \sin (\tau n)}{n^{2}} d \tau \tag{2.43}
\end{equation*}
$$

and since $\sin (n \pi)=\sin (0)=0$ for all $n \in \mathbb{N}^{+}$we get the following identity,

$$
\begin{equation*}
\left\langle g, s_{n}\right\rangle=-\frac{\left\langle g^{\prime \prime}, s_{n}\right\rangle}{n^{2}}, \quad n \in \mathbb{N}^{+} \tag{2.44}
\end{equation*}
$$

The regularization in this example is the truncation of the series:

$$
\begin{equation*}
J_{\alpha} g^{\delta}(x)=\sum_{n=1}^{N(\alpha)}\left\langle g^{\delta}, s_{n}\right\rangle \cdot n \sin (x n)=f_{\alpha}^{\delta}(x) \tag{2.45}
\end{equation*}
$$

Let $\alpha$ be a a-priori parameter choice rule according to Definition 1.4 .2 , where $N(\alpha)=\frac{1}{\alpha(\delta)}$. The parameter, $\alpha(\delta)$, shall be chosen as such that $\frac{\sqrt{\delta}}{\alpha(\delta)} \rightarrow 0$ when $\delta \rightarrow 0$.

Now to prove that $\left\|T^{-1} g-J_{\alpha} g^{\delta}\right\| \rightarrow 0, \delta \rightarrow 0$ in Definition 1.4.1 holds, we note that

$$
\begin{equation*}
\left\|T^{-1} g-J_{\alpha} g^{\delta}\right\|=\left\|f-f_{\alpha}^{\delta}\right\| \tag{2.46}
\end{equation*}
$$

where the $L_{2}$-norm will be used. So,

$$
\begin{align*}
\left\|f-f_{\alpha}^{\delta}\right\| & =\left\|\sum_{n=1}^{\infty}\left\langle g, s_{n}\right\rangle \cdot n \sin (x n)-\sum_{n=1}^{N(\alpha)}\left\langle g^{\delta}, s_{n}\right\rangle \cdot n \sin (x n)\right\| \\
& =\left\|\sum_{n=N(\alpha)+1}^{\infty}\left\langle g, s_{n}\right\rangle \cdot n \sin (x n)-\sum_{n=1}^{N(\alpha)} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (n \tau) d \tau\right\| \\
& =\left\|\sum_{n=N(\alpha)+1}^{\infty}-\frac{\left\langle g^{\prime \prime}, s_{n}\right\rangle}{n} \cdot \sin (x n)+\sum_{n=1}^{N(\alpha)} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (n \tau) d \tau\right\| . \tag{2.47}
\end{align*}
$$

By the triangle inequality we get:
$\left\|f-f_{\alpha}^{\delta}\right\| \leq\left\|\sum_{n=N(\alpha)+1}^{\infty}-\frac{\left\langle g^{\prime \prime}, s_{n}\right\rangle}{n} \cdot \sin (x n)\right\|+\left\|\sum_{n=1}^{N(\alpha)} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (n \tau) d \tau\right\|$
Remember that $\alpha$ is a a-priori. Let $\alpha$ be an arbitrary function of $\delta$ that satisfies the following conditions:

- $\alpha(\delta) \rightarrow 0$ when $\delta \rightarrow 0$.
- Let $\alpha$ be chosen as such that $\frac{\sqrt{\delta}}{\alpha(\delta)} \rightarrow 0$ when $\delta \rightarrow 0$.

This gives us the following expression for the right term in equation 2.48.

$$
\begin{align*}
\left\|\sum_{n=1}^{N(\alpha)} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (n \tau) d \tau\right\| & \leq \sum_{n=1}^{N(\alpha)}\left\|\frac{2 n \sin (n x)}{\pi}\right\| \cdot\left\|\int_{0}^{\pi} \delta(x) \sin (n \tau) d \tau\right\| \\
& \leq \sum_{n=1}^{N(\alpha)} \frac{2 n}{\pi} \delta \\
& \leq \frac{2}{\pi} \delta \frac{N(\alpha)(N(\alpha)+1)}{2} \\
& \leq \frac{\delta \cdot\left(N^{2}(\alpha)+N(\alpha)\right)}{\pi} \tag{2.49}
\end{align*}
$$

Since $N(\alpha)=\frac{1}{\alpha(\delta)}$ we get the final inequality as,

$$
\begin{equation*}
\left\|\sum_{n=1}^{N(\alpha)} \frac{2 n \sin (n x)}{\pi} \int_{0}^{\pi} \delta(x) \sin (n \tau) d \tau\right\| \leq \frac{1}{\pi}\left(\frac{\delta}{\alpha(\delta)}+\frac{\delta}{\alpha^{2}(\delta)}\right) \tag{2.50}
\end{equation*}
$$

This ensures that the condition in Definition 1.4.1 is met. Here we can see that $\frac{1}{\pi}\left(\frac{\delta}{\alpha(\delta)}+\frac{\delta}{\alpha^{2}(\delta)}\right)$ will converge to 0 when $\delta(x) \rightarrow 0$. Through the left term in equation $2.48,\left\|\sum_{n=N(\alpha)+1}^{\infty}-\frac{\left\langle g^{\prime \prime}, s_{n}\right\rangle}{n} \cdot \sin (x n)\right\|$ we get;

$$
\begin{equation*}
\left\|\sum_{n=N(\alpha)+1}^{\infty}-\left\langle g^{\prime \prime}, s_{n}\right\rangle \cdot \frac{\sin (x n)}{n}\right\|_{\infty} \leq \sum_{n=N(\alpha)+1}^{\infty}\left|\left\langle g^{\prime \prime}, s_{n}\right\rangle \cdot \frac{1}{n}\right| \tag{2.51}
\end{equation*}
$$

and applying Cauchy-Schwartz inequality,

$$
\begin{align*}
\left|\sum_{n=N(\alpha)+1}^{\infty}-\left\langle g^{\prime \prime}, s_{n}\right\rangle \cdot \frac{1}{n}\right| & \leq \sqrt{\sum_{n=N(\alpha)+1}^{\infty}\left|\left\langle g^{\prime \prime}, s_{n}\right\rangle\right|^{2}} \cdot \sqrt{\sum_{N(\alpha)+1}^{\infty} \frac{1}{n^{2}}} \\
& \leq \sqrt{\sum_{n=1}^{\infty}\left|\left\langle g^{\prime \prime}, s_{n}\right\rangle\right|^{2}} \cdot \sqrt{\sum_{n=1}^{\infty} \frac{1}{n^{2}}} \tag{2.52}
\end{align*}
$$

with Bessels inequality and the Basel problem we get,

$$
\begin{equation*}
\left|\sum_{n=N(\alpha)+1}^{\infty}-\left\langle g^{\prime \prime}, s_{n}\right\rangle \cdot \frac{1}{n}\right| \leq \frac{\left\|g^{\prime \prime}\right\|_{\infty} \pi}{\sqrt{6}} \tag{2.53}
\end{equation*}
$$

So the series is convergent and therefore the tail tends to zero. Thus this problem can be regularized.

As mentioned before, the regularization will truncate the infinite series into a finite sum. Thereby making the problem itself well-posed. By doing so we ensure a unique solution exists. Though it might not be exact it's still considered a solution. Suppose we have an odd function $f_{\text {odd }}$ and where $f \in \mathbb{W}$ such that,

$$
f_{o d d}=\left\{\begin{array}{l}
-f(x), \quad x \in[-\pi, 0]  \tag{2.54}\\
f(x), \quad x \in[0, \pi]
\end{array}\right.
$$

In order to find the optimal solution, the truncation number $N(\alpha)$ will be choosen through studying the error estimates below.

(a) This figure illustrates the function $f(x)$.

(b) This figure illustrates the function $f_{\text {odd }}(x)$.

Figure 2.1: This will be the value of $f_{\text {odd }}(x)$ thorughout this regularization.


Figure 2.2: Error estimation

From the error estimation, in Figure 2.2d it's understood that the error increases almost proportional to the noise level. However the error estimation, in Figure 2.2b, proves that $N(\alpha)=17$ is the minimum error the function can get. In other words the optimal solution becomes the expansion of Fourier series

[^3]to the 17 :th term.


Figure 2.3: A plot comparing the true function, $f(x)$, with $f_{40}^{\delta}$ and $f_{17}^{\delta}$.

Here we can visually see how $N(\alpha)=17$ is a far better approximation than $N(\alpha)=40$. In theory the more Fourier coeffiecients one uses the higher the accuracy of the approximation. The Fourier coeffiecients converges to zero fast making the last few terms in the sum unimportant. However in our case the more Fourier coefficients used the more noise is retrived. One need to find the perfect balence between noise and accuracy, in this example the balence was found at $N(\alpha)=17$. We have now demonstrated how an inverse problem which is ill-posed can be solved by transforming the problem into a well-posed one through regularization.

For more information regarding the subject see (4) and [5].

## Chapter 3

## Solving Helmholtz Equation in a Rectangle

This chapter will produce a solution for Helmholtz equation. The problem will be met with complete boundary conditions, this to ensure that a unique solution exists. The aim with this chapter is to demonstrate how ill-posed problems can turn into well-posed ones through Tikhonov's regularization. The main problem can be seen as two problems. The first, a direct problem, where given Dirichlet data on $\Lambda_{1}$ shall predict the Dirichlet data on $\Lambda_{2}$. The second, an inverse problem, that estimates the Dirichlet data on $\Lambda_{1}$ by only observing the Dirichlet data on $\Lambda_{2}$. In both cases, Neumann data will be given on $\Lambda_{2}$ and Dirichlet data on $\Lambda_{1}$.


Figure 3.1: This figure illustrates the boundary conditions of the problem.

### 3.1 Boundary Conditions

Before a complete and proper solution can be given for a problem, the conditions must be examined. Let $U \subset C^{1}([0,1] \times[0,1])$ be a normed vector space where $u \in U$ and where the following conditions for $u$ are met:

$$
\begin{align*}
u(x, b) & =f(x) & & \text { (Dirichlet Condition) }  \tag{3.1}\\
u(0, y) & =0 & & \text { (Dirichlet Condition) }  \tag{3.2}\\
u(1, y) & =0 & & \text { (Dirichlet Condition) }  \tag{3.3}\\
\nabla u(x, 0) \cdot\left[\begin{array}{c}
0 \\
-1
\end{array}\right] & =0 & & \text { (Neumann Condition) } \tag{3.4}
\end{align*}
$$

Remark 3.1.1. If an equation has both Neumann and Dirichlet boundary conditions, the conditions are denoted as Cauchy's boundary conditions.

The Helmholtz equation to solve is given as:

$$
\begin{equation*}
\nabla^{2} u(x, y)+k^{2} u(x, y)=0 \tag{3.5}
\end{equation*}
$$

where $k \in \mathbb{R}$.
An efficient way to find solutions to partial differential equations is to seperate the variables into two distinct functions,

$$
\begin{equation*}
u(x, y)=X(x) Y(y) \tag{3.6}
\end{equation*}
$$

when this is reasonable. By seperation of variables the boundary conditions becomes more precise:

## Condition 2.1

$$
\begin{equation*}
u(x, b)=X(x) Y(b)=f(x) \tag{3.7}
\end{equation*}
$$

Condition 2.2

$$
\begin{equation*}
u(0, y)=X(0) Y(y)=0 \Longrightarrow X(0)=0 \tag{3.8}
\end{equation*}
$$

## Condition 2.3

$$
\begin{equation*}
u(1, y)=X(1) Y(y)=0 \Longrightarrow X(1)=0 \tag{3.9}
\end{equation*}
$$

## Condition 2.4

$$
\nabla u(x, 0) \cdot\left[\begin{array}{c}
0  \tag{3.10}\\
-1
\end{array}\right]=\left[\begin{array}{l}
X^{\prime}(x) Y(0) \\
X(x) Y^{\prime}(0)
\end{array}\right] \cdot\left[\begin{array}{c}
0 \\
-1
\end{array}\right]=0 \Longrightarrow Y^{\prime}(0)=0
$$

Now we shall apply the conditions (3.31, (3.32, (3.33), (3.34) to $u(x, y)$ once a complete and proper expression is given to it.

### 3.2 Solution to Helmholtz Equation

By inserting the seperation functions, $X(x)$ and $Y(y)$, into Helmholtz equation we get:

$$
\begin{equation*}
X^{\prime \prime}(x) Y(y)+X(x) Y^{\prime \prime}(y)=-k^{2} X(x) Y(y) \tag{3.11}
\end{equation*}
$$

so

$$
\begin{equation*}
\frac{X^{\prime \prime}(x)}{X(x)}+\frac{Y^{\prime \prime}(y)}{Y(y)}=-k^{2} \Longleftrightarrow \frac{X^{\prime \prime}(x)}{X(x)}=-\frac{1}{Y(y)}\left(Y^{\prime \prime}(y)+k^{2} Y(y)\right) \tag{3.12}
\end{equation*}
$$

Both sides in equation (3.12 need to be equal to each other for a solution to exist and since the left hand side is a function of $x$ and the right hand side is a function of $y$, they must be equal to a constant. We shall now denote $-\lambda^{2}$ as our seperation constant, so

$$
\begin{equation*}
\frac{X^{\prime \prime}(x)}{X(x)}=-\frac{1}{Y(y)}\left(Y^{\prime \prime}(y)+k^{2} Y(y)\right)=-\lambda^{2} \tag{3.13}
\end{equation*}
$$

where $\lambda \in \mathbb{R}$. Therefore, we can write,

$$
\left\{\begin{array}{l}
X^{\prime \prime}(x)+\lambda^{2} X(x)=0  \tag{3.14}\\
Y^{\prime \prime}(y)-\left(\lambda^{2}-k^{2}\right) Y(y)=0
\end{array}\right.
$$

These equations are ordinary differential equations that have the following solutions:

$$
\left\{\begin{array}{l}
X(x)=S_{x} \sin (\lambda x)+C_{x} \cos (\lambda x)  \tag{3.15}\\
Y(y)=S_{y} \sinh (d y)+C_{y} \cosh (d y) \quad k \leq \lambda
\end{array}\right.
$$

where $S_{x}, S_{y}, C_{x}, C_{y} \in \mathbb{R}$ and $d^{2}=\lambda^{2}-k^{2}$, with $0 \leq d$.
Without any boundary conditions applied to equation (3.15), we get the solution:

$$
\begin{equation*}
u(x, y)=\left(S_{x} \sin (\lambda x)+C_{x} \cos (\lambda x)\right) \cdot\left(S_{y} \sinh (d y)+C_{y} \cosh (d y)\right) \tag{3.16}
\end{equation*}
$$

By applying the boundary conditions (3.31), (3.32, (3.33) given in section 3.1 , the expression for $u(x, y)$ can be reduced to:

$$
\begin{equation*}
u(x, y)=\Gamma \sin \left(\lambda_{n} x\right) \cosh \left(d_{n} y\right) \tag{3.17}
\end{equation*}
$$

where $\lambda_{n}=n \pi$ and $d_{n}=\sqrt{\lambda_{n}^{2}-k^{2}}$. Here we require that $k \leq \pi$ for all $n \in \mathbb{N}^{+}$, and $\Gamma \in \mathbb{R}$. The full derivation of this equation can be found in Appendix $B$.

Now we have found a concrete expression for $u(x, y)$. Observably there are an infinte amount of solutions to $u(x, y)$. Depending on the $n$ value, a different frequency with a different amplitude is given. Since they're all valid solutions we form a linear combination. In order to excute this we need to use Fourier series.

### 3.3 Expansion of a Solution Using Fourier Series

According to [1] the Fourier coefficients can be derived with this theorem.
Theorem 3.3.1. The Fourier series for $u \in L_{T}^{1}$ is the trignometric series,

$$
\begin{equation*}
u(t)=\sum_{n=-\infty}^{\infty} W_{n} e^{i n \Omega t}, \quad t \in \mathbb{R} \tag{3.18}
\end{equation*}
$$

where $W_{n} \in \mathbb{C}$ and the real part of $u$ is given by:

$$
\begin{equation*}
u(t)=\frac{C_{0}}{2}+\sum_{n=1}^{\infty} S_{n} \sin (n \Omega t)+C_{n} \cos (n \Omega t) \tag{3.19}
\end{equation*}
$$

where

$$
\begin{equation*}
S_{n}=2 f_{T} u(t) \sin (n \Omega t) d t \tag{3.20}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{n}=2 f_{T} u(t) \cos (n \Omega t) d t \tag{3.21}
\end{equation*}
$$

Equation (3.17) suggests that all the trigonometric functions satisfying the equation are needed to form $u(x, y)$. The series will contain all the possible frequencies with their corresponding amplitudes:

$$
\begin{equation*}
u(x, y)=\sum_{n=1}^{\infty} \Gamma_{n} \sin \left(\lambda_{n} x\right) \cosh \left(d_{n} y\right) \tag{3.22}
\end{equation*}
$$

Remark 3.3.1. If the period goes instead from $x: 0 \rightarrow 1$ to $x: 0 \rightarrow \pi$ the equations would become slightly more pleasent to work with,

$$
\begin{equation*}
\lambda_{n}=n, \tag{3.23}
\end{equation*}
$$

and

$$
\begin{equation*}
d_{n}=\sqrt{n^{2}-k^{2}}, \quad k \leq 1 \tag{3.24}
\end{equation*}
$$

where $n \in \mathbb{N}^{+}$.

By applying Condition 3.31 to the function we get

$$
\begin{align*}
u(x, b) & =f(x)=\sum_{n=1}^{\infty} \Gamma_{n} \sin \left(\lambda_{n} x\right) \cosh \left(d_{n} b\right)  \tag{3.25}\\
& \Longrightarrow f(x)=\sum_{n=1}^{\infty} \Gamma_{n}^{\prime} \sin \left(\lambda_{n} x\right) \tag{3.26}
\end{align*}
$$

where $\Gamma_{n}^{\prime}=\Gamma_{n} \cosh \left(d_{n} b\right)$.
From Theorem 3.3.1 it's understood that the Fourier coeffcients, $\Gamma_{n}^{\prime}$, can be expressed with equation (3.20), and $f(x)$ is chosen as an odd function, thereby yielding,

$$
\begin{equation*}
\Gamma_{n}^{\prime}=2 f_{T} f(x) \sin \left(\lambda_{n} x\right) d x=\int_{0}^{1} f(x) \sin \left(\lambda_{n} x\right) d x \tag{3.27}
\end{equation*}
$$

Thus we get $u(x, y)$ as:

$$
\begin{align*}
u(x, y) & =\sum_{n=1}^{\infty} \Gamma_{n} \sin \left(\lambda_{n} x\right) \cosh \left(d_{n} y\right) \\
& =\sum_{n=1}^{\infty} \Gamma_{n}^{\prime} \frac{\sin \left(\lambda_{n} x\right) \cosh \left(d_{n} y\right)}{\cosh \left(d_{n} b\right)}  \tag{3.28}\\
& =\sum_{n=1}^{\infty}\left[\frac{\sin \left(\lambda_{n} x\right) \cosh \left(d_{n} y\right)}{\cosh \left(d_{n} b\right)} \int_{0}^{1} f(x) \sin \left(\lambda_{n} x\right) d x\right]
\end{align*}
$$

An expression for $u(x, y)$ has been created for every point inside the rectangle, $[0,1] \times[0, b]$. The second problem is to decide what value is recieved at the opposite side of $f(x)$ in the rectangle. Since $f(x)=u(x, b)$, we get the other side as $g(x)=u(x, 0)$. The complete solution to the direct problem is:

$$
\begin{equation*}
u(x, 0)=\sum_{n=1}^{\infty}\left[\frac{\sin \left(\lambda_{n} x\right)}{\cosh \left(d_{n} b\right)} \int_{0}^{1} f(x) \sin \left(\lambda_{n} x\right) d x\right]=g(x) \tag{3.29}
\end{equation*}
$$

Now we have an expression for the linear operator, $T$ :

$$
\begin{align*}
(T f)(x) & =\sum_{n=1}^{\infty}\left[\frac{\sin \left(\lambda_{n} x\right)}{\cosh \left(d_{n} b\right)} \int_{0}^{1} f(x) \sin \left(\lambda_{n} x\right) d x\right]  \tag{3.30}\\
& =\sum_{n=1}^{\infty}\left\langle f, s_{n}\right\rangle \cdot \frac{\sin \left(\lambda_{n} x\right)}{\cosh \left(b d_{n}\right)}
\end{align*}
$$

Remark 3.3.2. If there exist another solution, unrelated to our solution, it indicates that the problem does not have a unique solution. To prove that Hadamard's first and second condition are also satisfied for some values of $k$, one need to study the homogeneous solutions for the Helmholtz equation.

A Theorem is given to strengthen the uniqueness of our solution.
Theorem 3.3.2. If $k=n \pi$, then equation (3.5) has infinitely many solutions with Dirichlet data equal to zero.

Proof. To find the homogeneous solutions to Helmholtz equation the boundary conditions are set to zero,


The boundary conditions therefore becomes:
Condition 3.4

$$
\begin{equation*}
u_{h}(x, b)=X(x) Y(b) \Longrightarrow Y(b)=0 \tag{3.31}
\end{equation*}
$$

## Condition 3.2

$$
\begin{equation*}
u_{h}(0, y)=X(0) Y(y)=0 \Longrightarrow X(0)=0 \tag{3.32}
\end{equation*}
$$

## Condition 3.3

$$
\begin{equation*}
u_{h}(1, y)=X(1) Y(y)=0 \Longrightarrow X(1)=0 \tag{3.33}
\end{equation*}
$$

## Condition 3.4

$$
\begin{equation*}
u_{h}(x, 0)=X(x) Y(0)=0 \Longrightarrow Y(0)=0 \tag{3.34}
\end{equation*}
$$

The seperations solutions to Helmholtz equation becomes:

$$
\Longrightarrow\left\{\begin{array}{l}
X(x)=S_{x} \sin (\lambda x)+C_{x} \cos (\lambda x)  \tag{3.35}\\
Y(y)=S_{y} \sinh (d y)+C_{y} \cosh (d y) \quad k \leq \lambda
\end{array}\right.
$$

Applying the boundary condition yields the complete solution:

$$
u(x, y)= \begin{cases}\Gamma \sin (\lambda x) \sinh (d y), & k=n \pi  \tag{3.36}\\ 0, & k \neq n \pi\end{cases}
$$

Which means if $k=n \pi$ then the solution cannot be unique and thereby do not satisfy Hadamard's second condition.

If $k \neq n \pi$ (say if $k$ is close to zero), the Helmholtz equation with our conditions is known to have a unique solution. This means in particular that our representation of $T$ is the unique operator mapping Dirichlet data from $\Lambda_{1}$ to $\Lambda_{2}$.

Now we shall show that this linear operator, 3.30, satisfies all of Hadamard's conditions.

### 3.4 The Linear Operator

Similarily with Chapter 2 we shall show that the linear operator, $T$, that was derived from Helmholtz equation satisfies all of Hadamard's conditions. To do this, we must first give a brief definition of our linear operator:

Definition 3.4.1. Let the linear operator, $T: \mathbb{P} \rightarrow \mathbb{P}$, have the following definition:

$$
\begin{equation*}
T f(x)=\sum_{n=1}^{\infty}\left\langle f, s_{n}\right\rangle \cdot \frac{\sin \left(\lambda_{n} x\right)}{\cosh \left(b d_{n}\right)} \tag{3.37}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbb{P} \in\left\{p \in C^{1}[0,1]: \quad p(0)=p(1)=0\right\} \tag{3.38}
\end{equation*}
$$

and $f \in \mathbb{P}$.
Considering that the linear operator is expressed as a Fourier series, we can use an analogous proof of uniqueness as the one in Chapter 2. In other words, to show that Hadamard's first and second condition of well-posedness are valid, we can use Dirchlet's convergence theorem, see equation (2.16). According to our linear operator definition, $f$ and $g$ ought to be continuous on $[0,1]$, thereby easily expressing the values from $g\left(a^{+}\right)$and $g\left(a^{-}\right)$as $g(a)$. Thus proving that the problem $T f=g$ has a unique solution.
Hadamard's third condition states that the operator needs to be continuous. Similarily as the proof showed in Chapter 2 we shall prove its continuity through its boundedness.

Lemma 3.4.1. The linear operator $T$ defined in 3.4.1 is $L^{\infty}$ - and $L^{2}$-bounded for all $p \in \mathbb{P}$.

There are many ways to prove this lemma. In this proof we shall use the Maclaurin Expansion of $\cosh \left(d_{n} b\right)$, since it is defined over the whole $\mathbb{R}$.

Proof.

$$
\begin{equation*}
|T f| \leq \sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right| \cdot \frac{|\sin (n x)|}{\cosh \left(d_{n} b\right)} \leq \sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right| \cdot \frac{1}{\cosh \left(d_{n} b\right)} \tag{3.39}
\end{equation*}
$$

the Maclaurin Expansion of $\cosh (x)$ :

$$
\begin{equation*}
\cosh (x)=\sum_{k=0}^{\infty} \frac{x^{2 k}}{(2 k)!}=1+\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\cdots \quad(x \in \mathbb{R}) \tag{3.40}
\end{equation*}
$$

so

$$
\begin{align*}
\frac{1}{\cosh (x)} & =\frac{1}{1+\frac{x^{2}}{2!}+\frac{x^{4}}{4!}+\cdots} \\
& \leq \frac{1}{1+\frac{x^{2}}{2!}}=\frac{2}{2+x^{2}}  \tag{3.41}\\
& \leq \frac{2}{x\left(\frac{2}{x}+x\right)} \leq \frac{2}{x}
\end{align*}
$$

where $x>0$. By letting $k \leq \sqrt{\pi^{2}-1}$, the inequality from equation 3.39 becomes:

$$
\begin{align*}
|T f| & \leq \sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right| \cdot \frac{2}{b d_{n}}=\sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right| \cdot \frac{2}{b \sqrt{\left(\pi^{2} n^{2}-k^{2}\right)}} \\
& \leq \sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right| \cdot \frac{2}{b n \sqrt{\left(\pi^{2}-\frac{k^{2}}{n^{2}}\right)}},  \tag{3.42}\\
& \leq \sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right| \cdot \frac{2}{b n} .
\end{align*}
$$

By the Cauchy-Schwartz inequality:

$$
\begin{equation*}
|T f| \leq \sqrt{\sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right|^{2}} \cdot \sqrt{\sum_{n=1}^{\infty} \frac{4}{b^{2} n^{2}}} \tag{3.43}
\end{equation*}
$$

and applying Bessel's inequality for the left factor in equation (3.43):

$$
\begin{equation*}
\sum_{n=1}^{\infty}\left|\left\langle f, s_{n}\right\rangle\right|^{2} \leq\|f\|_{2}^{2} \tag{3.44}
\end{equation*}
$$

Similarly as in equation 2.10, the right factor of equation (3.43) becomes:

$$
\begin{equation*}
\sum_{n=1}^{\infty} \frac{4}{b^{2} n^{2}}=\frac{4}{b^{2}} \sum_{n=1}^{\infty} \frac{1}{n^{2}}=\frac{2 \pi^{2}}{3 b^{2}} \tag{3.45}
\end{equation*}
$$

This gives us the following inequality,

$$
\begin{align*}
\|T f\|_{\infty} & \leq\|f\|_{2} \cdot \sqrt{\frac{2 \pi^{2}}{3 b^{2}}}  \tag{3.46}\\
& \leq\|f\|_{2} \cdot \sqrt{\frac{2}{3}} \frac{\pi}{b}
\end{align*}
$$

By Parseval's identity,

$$
\begin{equation*}
\|f\|_{2}=\sum_{n=1}^{\infty}\left|c_{n}\right|^{2}=\int_{0}^{1}|f(x)|^{2} d x \tag{3.47}
\end{equation*}
$$

which gives the final expression,

$$
\begin{equation*}
\|T f\|_{\infty} \leq \sqrt{\frac{2}{3}} \frac{\pi}{b} \int_{0}^{1}|f(x)|^{2} d x \tag{3.48}
\end{equation*}
$$

Since $\|f\|_{2} \leq\|f\|_{\infty}$, we get the final inequality,

$$
\begin{equation*}
\|T f\|_{\infty} \leq \sqrt{\frac{2}{3}} \frac{\pi}{b} \cdot\|f\|_{\infty} \tag{3.49}
\end{equation*}
$$

Thus proving that this operator is $L_{2}$-bounded and $L_{\infty}$-bounded, thereby $T$ is continuous.

We have proved that the linear operator, $T$, is continuous and that the problem $T f=g$ has unique solution. The operator satisfies all of Hadamard's conditions. However just because the linear operator satisfies them doesn't mean the inverse operator, $T^{-1}$, will.

### 3.5 The Inverse Operator

In this section we present the proof that the inverse operator is unbounded, making the whole problem ill-posed.

Recall that the linear operator divided each Fourier coefficient with $\cosh \left(b d_{n}\right)$, so by applying the same reasoning as in Chapter 2, the inverse operator is found when each Fourier coefficient is multiplied by $\cosh \left(b d_{n}\right)$. Due to how large the function space $\mathbb{P}$ is, it exceeds $\mathcal{R}(T)$. For more on this claim, see Appendix A. This give us the following Theorem:

Theorem 3.5.1. Let $T$ be as in Definition 3.4.1, then the inverse operator can be expressed as:

$$
\begin{equation*}
T^{-1} g(x)=\sum_{n=1}^{\infty}\left\langle g, s_{n}\right\rangle \cdot \sin \left(\lambda_{n} x\right) \cosh \left(b d_{n}\right) \tag{3.50}
\end{equation*}
$$

where

$$
\begin{gather*}
\mathbb{P} \in\left\{p \in C^{1}[0,1]: \quad p(0)=p(1)=0\right\}  \tag{3.51}\\
\mathcal{R}(T) \subset \mathbb{P} \tag{3.52}
\end{gather*}
$$

and $g \in \mathcal{R}(T)$, where $\mathcal{R}(T)$ is the range of the operator.
The inverse operator does not satisfy Hadamard's third condition and therefore the inverse problem is ill-posed.

Lemma 3.5.1. The inverse operator $T^{-1}$ defined in Theorem 3.5.1 is unbounded on $\mathcal{R}(T)$.

Proof. Firstly, let $z$ be a $C^{3}[0,1]$-function. Then the complex Fourier series is given as

$$
\begin{equation*}
z(x)=\sum_{n=-\infty}^{\infty}\left\langle z, e_{n}\right\rangle \cdot e^{i n \pi x} \tag{3.53}
\end{equation*}
$$

and let

$$
\begin{equation*}
\left\langle z, e_{n}\right\rangle=\frac{1}{2} \int_{-1}^{1} z(t) e^{i n \pi t} d t=\Phi_{n} \tag{3.54}
\end{equation*}
$$

Then the derivitive of $z$ becomes,

$$
\begin{equation*}
\frac{d z}{d x}=i \pi \sum_{n=-\infty}^{\infty} \Phi_{n} \cdot n e^{i n \pi x} \tag{3.55}
\end{equation*}
$$

From [8] we understand that if $z$ is of class $C^{3}$ then $\Phi_{n}$ tends to zero faster than $|n|^{-3}$ as $n \rightarrow \pm \infty$. We shall return to equation (3.55) later. For now, consider a function $g \in \mathcal{R}(T)$ such that,

$$
\begin{equation*}
\left\langle g, e_{n}\right\rangle=\frac{n \cdot \Phi_{n}}{\cosh \left(b d_{n}\right)} \tag{3.56}
\end{equation*}
$$

This is a reasonable sequence of Fourier coefficients, considering that both $\Phi_{n} \rightarrow$ 0 and $\frac{n}{\cosh \left(b d_{n}\right)} \rightarrow 0$ rapidly when $n \rightarrow \pm \infty$.
The inverse operation in complex form is given as,

$$
\begin{align*}
T^{-1} g & =\sum_{n=-\infty}^{\infty}\left\langle g, e_{n}\right\rangle \cdot e^{i n \pi x} \cosh \left(b d_{n}\right) \\
& =\sum_{n=-\infty}^{\infty} \frac{n \cdot \Phi_{n}}{\cosh \left(b d_{n}\right)} \cdot e^{i n \pi x}  \tag{3.57}\\
& =\sum_{n=-\infty}^{\infty} \Phi_{n} \cdot n e^{i n \pi x}
\end{align*}
$$

Inserting equation 3.55 in equation (3.57),

$$
\begin{equation*}
T^{-1} g=\frac{1}{i \pi} \frac{d z}{d x} \tag{3.58}
\end{equation*}
$$

This suggests that the inverse operator performs a differential operation on $z$. Similarly as Proof 2.1, one can show that the differential operator is unbounded on $C^{\infty}$. Therefore it can be concluded that this operator, $T^{-1}$, is unbounded yielding that the inverse problem is ill-posed.

Acknowledging that the inverse operator is unbounded point us to the direction of regularization, more specifically Tikhonov's regularization.

### 3.6 Tikhonov's Regularization

Inverse problems are generally unstable, which poses significant challenges to their accurate and stable numerical solution. That's why specialized techniques are required. One of the strongest regularization techniques was proposed by Andrey Tikhonov, hence the name Tikhonov's regularization. Since this is a well known regularization method, it should be understood that this regularization method yields methods satisfying Definition 1.4.1. In the search for sufficient evidence for this claim see [4].

Now, we shall introduce Tikhonov's functional, $J_{\alpha}$ (Note that this $J$ is different from the $J$ in Definition 1.4.1, and find the function that minimizes this functional:

$$
\begin{equation*}
\min _{f \in \mathbb{P}}\left\{J_{\alpha}(f)=\left\|T f-g^{\delta}\right\|_{2}^{2}+\alpha^{2}\|f\|_{2}^{2}\right\} \tag{3.59}
\end{equation*}
$$

where it's minimizer, denoted by $f_{\alpha}^{\delta}$, is the final solution. By equation 3.59 the following value for the minimizer, $f_{\alpha}^{\delta}$ could be obtained,

$$
\begin{equation*}
f_{\alpha}^{\delta}=\left(T^{*} T+\alpha^{2} I\right)^{-1} T^{*} g^{\delta} \tag{3.60}
\end{equation*}
$$

Through a matrix discretization perspective, we understand that the right side, $T^{T} g^{\delta}$ is bounded and the left side $T^{*} T+\alpha^{2} I$ could be written as $T^{T} T+\alpha^{2} I$ since the $T^{T} T$ yields a positive semidefinite matrix we get that the norm of the inverse is bounded by $\frac{1}{\alpha}$. Hence $g \rightarrow f$ is a bounded operation. Thereby satisfying Hadamard's third condition. If all the boundary condition of Helmholtz equation in a rectangle are set to zero (Dirichlet Condition), the Dirichlet data from $f(x)$ to $g(x)$ won't be altered. Before numerically testing this we will consider a different approach for comparison.

### 3.7 Operator From Partial Fourier Transform

In this section we shall present the partial Fourier transform operator which will be compared with our Fourier series operator.

To find this operator we shall first construct the Helmholtz equation in the frequency domain. The boundary conditions of the problem will also be transformed for the partial frequency domain. Rather than as a variable, $y$ will be considered as a constant thus making the equation into an ordinary differential equation. Let the partial Fourier transform of $u(x, y)$ be denoted as $U\left(\omega_{n}, y\right)=U$. Then the Helmholtz equation in the partial frequency domain becomes:

$$
\begin{equation*}
\nabla^{2} u(x, y)+k^{2} u(x, y)=0 \Longrightarrow-\omega_{n}^{2} U+\frac{d^{2} U}{d y^{2}}+k^{2} U=0 \tag{3.61}
\end{equation*}
$$

so

$$
\begin{equation*}
\frac{d^{2} U}{d y^{2}}=\left(\omega_{n}^{2}-k^{2}\right) U, \quad k<\omega_{n} \tag{3.62}
\end{equation*}
$$

This gives us the following solution,

$$
\begin{equation*}
U\left(\omega_{n}, y\right)=S\left(\omega_{n}\right) \sinh \left(\sqrt{\omega_{n}^{2}-k^{2}} y\right)+C\left(\omega_{n}\right) \cosh \left(\sqrt{\omega_{n}^{2}-k^{2}} y\right) \tag{3.63}
\end{equation*}
$$

where $C$ and $S$ are functions of $\omega$. The boundary conditions in the frequency domain becomes,

$$
\begin{align*}
u(x, b)=f(x) & \Longrightarrow \hat{u}\left(\omega_{n}, b\right)=F\left(\omega_{n}\right) \\
u(x, 0)=g(x) & \Longrightarrow \hat{u}\left(\omega_{n}, 0\right)=G\left(\omega_{n}\right)  \tag{3.64}\\
\nabla u(x, 0) \cdot\left[\begin{array}{c}
0 \\
-1
\end{array}\right]=0 & \Longrightarrow \hat{u}_{y}^{\prime}\left(\omega_{n}, 0\right)=0
\end{align*}
$$

and applying these conditions we get the following:

$$
\begin{equation*}
U_{y}^{\prime}\left(\omega_{n}, 0\right)=0 \Longrightarrow S\left(\omega_{n}\right)=0 \tag{3.65}
\end{equation*}
$$

Then it follows that

$$
\begin{align*}
& F\left(\omega_{n}\right)=\hat{u}\left(\omega_{n}, b\right)=C\left(\omega_{n}\right) \cosh \left(\sqrt{\omega_{n}^{2}-k^{2}} b\right),  \tag{3.66}\\
& G\left(\omega_{n}\right)=\hat{u}\left(\omega_{n}, 0\right)=C\left(\omega_{n}\right)
\end{align*}
$$

Recall that in the direct problem we wanted to determine $G\left(\omega_{n}\right)$ through only $F\left(\omega_{n}\right)$, so

$$
\begin{equation*}
G\left(\omega_{n}\right)=\frac{F\left(\omega_{n}\right)}{\cosh \left(\sqrt{\omega_{n}^{2}-k^{2}} b\right)} \tag{3.67}
\end{equation*}
$$

vice versa the inverse problem is given by the following.

$$
\begin{equation*}
F\left(\omega_{n}\right)=G\left(\omega_{n}\right) \cosh \left(\sqrt{\omega_{n}^{2}-k^{2}} b\right) \tag{3.68}
\end{equation*}
$$

Now that the expression for the linear operator and it's inverse has been expressed, we shall give them a proper definition. The linear operator is defined as follows.

Definition 3.7.1. The linear operator,

$$
T: \ell^{2}(\mathbb{N}) \rightarrow \ell^{2}(\mathbb{N})
$$

is defined by

$$
\begin{equation*}
(T F)\left(\omega_{n}\right)=\frac{F\left(\omega_{n}\right)}{\cosh \left(b \sqrt{\omega_{n}^{2}-k^{2}}\right)} \tag{3.69}
\end{equation*}
$$

where $F$ is a $\ell^{2}$-function and where $\omega_{n}=n \pi, n \in \mathbb{N}^{+}$.
The uniqueness in Fourier Transform still applies here, and satisfies Hadamard's first and second conditions.

Lemma 3.7.1. The operator $T$ defined in (3.7.1) is bounded, satisfying the Hadamard's third condition.

Proof.

$$
\begin{equation*}
\left\|T F\left(\omega_{n}\right)\right\|_{2}=\left\|\frac{F\left(\omega_{n}\right)}{\cosh \left(b \sqrt{\omega_{n}^{2}-k^{2}}\right)}\right\|_{2} \leq\|F\|_{2} \tag{3.70}
\end{equation*}
$$

Using the formulation made in equation (3.68), the inverse gets the following definition.

Definition 3.7.2. The inverse operator,

$$
T^{-1}: \ell^{2}(\mathbb{N}) \rightarrow \ell^{2}(\mathbb{N})
$$

this gives us,

$$
\begin{equation*}
\left(T^{-1} G\right)\left(\omega_{n}\right)=\cosh \left(b \sqrt{\omega_{n}^{2}-k^{2}}\right) \cdot G\left(\omega_{n}\right) \tag{3.71}
\end{equation*}
$$

where $G$ is an $\ell^{2}$-function.
In Chapter 2 of [6] it's proven that this inverse operator is unbounded. Acknowledging that the inverse operator is unbounded, yields that the inverse problem is ill-posed. Now, to solve this we shall implement Tikhonov's regularization to this inverse problem as well. However since the inverse operator in this case is just a multiplication of $\cosh \left(b \sqrt{\omega_{n}^{2}-k^{2}}\right)$, the equation 3.60 becomes

$$
\begin{equation*}
F_{\alpha}^{\delta}=\frac{\cosh \left(b \sqrt{\omega_{n}^{2}-k^{2}}\right)}{\cosh ^{2}\left(b \sqrt{\omega_{n}^{2}-k^{2}}\right)+\alpha^{2}} \cdot G^{\delta} \tag{3.72}
\end{equation*}
$$

Now all the necessary theory needed to regularize this inverse problem has been provided. A representation of the accuracy of each inverse operator will be presented in the next section.

### 3.8 Error Estimation

This section provides a series of figures demonstrating the stability of each operator. It should be noted that the operators of interest are as follows.

- The Partial Fourier Transform with Tikhonov's regularization, PFT-T, from section (3.7).
- The Complete Fourier Series with Tikhonov's regularization, CFS-T, from Section (3.5).
- The Truncated Fourier Series, TFS, from Section 2.1.1.

All of the operators stability will be demonstrated by plotting each operators predicted Dirichlet data on $\Lambda_{1}$ as the noise of the measured Dirichlet data on $\Lambda_{2}$ increases. Furthermore the real Dirichlet data on $\Lambda_{1}$ will be plotted before hand in order to know what the solution should look like and to decide the best operator, see Figure 3.2 .


Figure 3.2: This figure illustrates how the real Dirichlet data of $f(x)$.

Additional noise from a normally distributed source will be added to the Dirichlet data of $\Lambda_{2}$. The following plots contains the operators predicted Dirichlet data of $\Lambda_{1}$ from the measured Dirichlet data of $\Lambda_{2}$. The noise level will be presented in each plots.


Figure 3.3: Predicted $f(x)$ from all the operators. The red graph is CFS-T, blue dotted one is PFT-T and the black dotted one is TFS.

When the noise level is 0 the operator's predictions are pretty accurate. However, in the left corner of this plot and on the right top corner we find a difference.


Figure 3.4: Two zoomed in sections of interest, from Figure 3.3

Observably the TFS is far better than the other two. We can also conclude that the CFS-T operator is slightly better than PFT-T operator at the noise level 0 .


Figure 3.5: This figure illustrates how the three operators predicted the data of $f(x)$ with noise level $4 \cdot 10^{-6}$.


Figure 3.6: A zoomed in section of Figure 3.5. More specifically at $0.4<x<0.8$ and $5.5<y<9.5$.

In Figure 3.5 we can see that all the operators manage to reproduce the expected Dirichlet data with the noise level set at $4 \cdot 10^{-6}$. This suggests that they're all reasonable operators to use at this level of noise. However the biggest difference between the operators is found in the middle section of Figure 3.5. A bigger version of that zoomed in section is found in Figure 3.6.

In Figure 3.6 we can see that the CFT-T is closer to the true function in the left section and TFS is closer in the right section. In the middle they're both pretty close to eachother. The only conclusion is that PFT-T is slight worse than the other two at the noise level $4 \cdot 10^{-6}$.


Figure 3.7: This figure illustrates how the three operators predicted the data of $f(x)$ with noise level $4 \cdot 10^{-4}$.

In Figure 3.7, we can visually see how instable the PFT-T truely is. It shall be noted that the amplitudes have increased 5 times as the noise has increased by a factor 100. In order to differentiatate between the operators CFT-T and TFS we shall look at a more zoomed in version of this Figure.


Figure 3.8: A zoomed in section of Figure 3.7 adjusted to the true data.

In Figure 3.8 we can see that both operators are still functioning and are stable in the sense that the amplitudes are still reasonably close to the true data. The TFS is more stable and closer to the true data than PFT-T is at the noise level $4 \cdot 10^{-4}$.


Figure 3.9: Illustration of the prediction with noise level: $4 \cdot 10^{-2}$.

Figure 3.9 shows the instability of PFT-T. Here the amplitudes have increased by factor 100 compared to Figure 3.7. From Figure 3.9b, we can see that the CFT-T operator is far closer to the real data than the operator of TFS. This suggests that CFT-T might be the most robust operator with the given Dirichlet data, $f(x)$, on $\Lambda_{1}$.

## Chapter 4

## Conclusion

Many problems in real life are ill-posed. Our measurement equipment are not perfect either, there is always some noise lurking around. Regularization techniques are of extreme importance in these areas. Tikhonov's regularization can be found in many forms ( $L^{2}$-space was used in this paper), and therefore qualifying it as a powerful regularization technique.

In this work its demonstrated how powerful and robust the Tikhonov's regularization (in $L^{2}$ ) method really is. The implementation of this technique requires a large matrix, which can make some problems numerically expensive. The PFT-T and TFS are very cheap regularization techniques compared to it.

The conclusion from the error estimation with our given Dirichlet data $f(x)$ in 3.2 ) is: if the noise level is at the interval of $\left[0,4 \cdot 10^{-6}\right]$ then PFT-T is the best option due to how cheap it is. If the noise rises to interval of $\left[4 \cdot 10^{-6}, 4 \cdot 10^{-4}\right]$ then TFS shall be in consideration due to the accuracy and cheapness. If the noise gets any larger than that then CFT-T should be considered. Overall CFTT is the most robust operator amongst them.

Many times the noise level is unknown and therefore having an operator which is robust at high levels of noise just as it is robust at low levels of noise is essential for an accurate result. That is one reason why Tikhonov's regularization is a powerful technique, which is demonstrated in this paper.

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## Appendix A

## Proving Divergence With Maclaurin Expansion

Proof. Suppose the function $g \in \mathbb{P}$ has the following definition,

$$
\begin{equation*}
g(x)=x(1-x) \tag{A.1}
\end{equation*}
$$

this gives us the corresponding Fourier coefficients as

$$
\begin{equation*}
\left\langle g, s_{n}\right\rangle=\frac{2(1-\cos (\pi n))}{\pi^{3} n^{3}}, \quad n \in \mathbb{N}^{+} \tag{A.2}
\end{equation*}
$$

The inverse operation becomes:

$$
\begin{equation*}
T^{-1} g=\sum_{n=1}^{\infty} \frac{2(1-\cos (\pi n))}{\pi^{3} n^{3}} \sin (n \pi x) \cosh \left(b d_{n}\right) \tag{A.3}
\end{equation*}
$$

Since all the even values of $n$ becomes zero and all the odd values on $n$ becomes 4 , the summation simplifies to,

$$
\begin{equation*}
T^{-1} g=\sum_{n=1}^{\infty} \frac{4}{\pi^{3}(2 n-1)^{3}} \sin ((2 n-1) \pi x) \cosh \left(b d_{2 n-1}\right) . \tag{A.4}
\end{equation*}
$$

Now, in order to show its unboundedness, we'll use uniform norm:

$$
\begin{equation*}
\left\|T^{-1} g\right\|_{\infty}=\left\|\sum_{n=1}^{\infty} \frac{4}{\pi^{3}(2 n-1)^{3}} \sin ((2 n-1) \pi x) \cosh \left(b d_{2 n-1}\right)\right\|_{\infty} \tag{A.5}
\end{equation*}
$$

Now a divergence test seems appropriate. Let $a_{N}$ be denoted as the $n$ :th element in the series and let $x$ be such that $\sin ((2 N-1) \pi x)$ is nonzero, then we get following equations,

$$
\begin{equation*}
a_{N}=\frac{4}{\pi^{3}(2 N-1)^{3}} \sin ((2 N-1) \pi x) \cosh \left(b d_{2 N-1}\right) \tag{A.6}
\end{equation*}
$$

and applying the Maclaurin expansion to $\cosh \left(b d_{n}\right)$ :

$$
\begin{equation*}
a_{N}=\frac{4 \sin ((2 N-1)}{\pi^{3}(2 N-1)^{3}} \sum_{i=0}^{\infty} \frac{b^{2 i}\left[(\pi N)^{2}-k^{2}\right]^{i}}{(2 i)!} \tag{A.7}
\end{equation*}
$$

When $n \rightarrow \infty$, we get

$$
\begin{equation*}
\lim _{N \rightarrow \infty} a_{N}=\lim _{N \rightarrow \infty} \frac{4 \sin (2 N-1)}{\pi^{3}(2 N-1)^{3}} \sum_{i=0}^{\infty} \frac{b^{2 i}\left[(\pi N)^{2}-k^{2}\right]^{i}}{(2 i)!} \tag{A.8}
\end{equation*}
$$

By analyzing equation A.8) we understand that $\mathcal{R}(\sin ((2 N-1)) \in[-1,1]$ and if $N \rightarrow \infty$ the expression $\sin ((2 N-1) x)$ becomes undefined while the Maclaurin expansion will consume the expanding denominator and diverge. According to the Defintion (10.1) in [3] " if the limit is either infinity or does not exist, the series is defined as divergent. ". Thus $a_{N}$ can be concluded as divergent resulting in that the inverse operator can be concluded as unbounded.

In conclusion the function space $\mathbb{P}$ seems to be too large for this operator and thereby strengthening the use of $\mathcal{R}(T)$ in Theorem 3.5.1

## Appendix B

## Applying Boundary Conditions

## Condition 3.32

Applying,

$$
\begin{equation*}
X(0)=0 \tag{B.1}
\end{equation*}
$$

to the solution,

$$
\begin{equation*}
X(x)=S_{x} \sin (\lambda x)+C_{x} \cos (\lambda x) \tag{B.2}
\end{equation*}
$$

we get

$$
\begin{equation*}
X(0)=C_{x}=0 . \tag{B.3}
\end{equation*}
$$

Thus we get $X(x)$ as:

$$
\begin{equation*}
X(x)=S_{x} \sin (\lambda x) \tag{B.4}
\end{equation*}
$$

## Condition 3.33

Similarly,

$$
\begin{equation*}
X(1)=0 \tag{B.5}
\end{equation*}
$$

gives the solution,

$$
\begin{equation*}
X(x)=S_{x} \sin (\lambda x), \tag{B.6}
\end{equation*}
$$

as,

$$
\begin{align*}
& X(1)=S_{x} \sin (\lambda)=0  \tag{B.7}\\
& \Longrightarrow \lambda=\lambda_{n}=\pi n
\end{align*}
$$

where $n \in \mathbb{N}^{+}$.

Considering that $d$ is a function of $\lambda, d(\lambda)$ and $\lambda$ is a function of $n$, we need to redefine $d$ as function of $n$. From equation $3.15 d$ is expressed as:

$$
\begin{equation*}
d^{2}=k^{2}-\lambda^{2} \Longrightarrow d_{n}^{2}=k^{2}-\lambda^{2} \tag{B.8}
\end{equation*}
$$

we choose

$$
\begin{equation*}
d_{n}=\sqrt{k^{2}-\lambda_{n}^{2}} \tag{B.9}
\end{equation*}
$$

For $d_{n}$ to exist the following inequality must be true for all $n \in \mathbb{N}^{+}: k \leq \pi$. (Since all $\lambda \geq \pi$ )
Condition 3.34 Applying,

$$
\begin{equation*}
Y^{\prime}(0)=0 \tag{B.10}
\end{equation*}
$$

yields the following,

$$
\begin{equation*}
Y^{\prime}(x)=S_{y} \cosh \left(d_{n} y\right)+C_{y} \sinh \left(d_{n} y\right) \tag{B.11}
\end{equation*}
$$

Here we can see that,

$$
\begin{equation*}
Y^{\prime}(0)=S_{y}=0 \tag{B.12}
\end{equation*}
$$

Thus we get $\mathrm{Y}(\mathrm{y})$ as:

$$
\begin{equation*}
Y(x)=C_{y} \cosh \left(d_{n} y\right) \tag{B.13}
\end{equation*}
$$

Now we have simplified the expression as far as it goes and we get the following equation for $u(x, y)$ as:

$$
\begin{equation*}
u(x, y)=X(x) Y(x)=\Gamma \sin \left(\lambda_{n} x\right) \cos \left(d_{n} y\right) \tag{B.14}
\end{equation*}
$$

where $\Gamma$ is defined as $\Gamma=S_{x} C_{y}$.

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[^0]:    ${ }^{1}$ The definition involves a general field $K$, where $K$ can either be $\mathbb{R}$ or $\mathbb{C}$.

[^1]:    ${ }^{2}$ Scalar: rather than a vector is a real number and is always bigger or equal to 0 .

[^2]:    ${ }^{3}$ The Laplacian is also known as the laplace operator.
    ${ }^{4}$ Generally when talking about wave equation for monochromatic waves we're talking about wave equations in the frequency domain.

[^3]:    ${ }^{1}$ Correction: The $x$-axis in this figure is $N(\alpha)$ and not $\alpha$.

