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The Use of Landweber Algorithm in Image Reconstruction

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To Afsaneh

Abstract

Ill-posed sets of linear equations typically arise when discretizing certain types of integral transforms. A well known example is image reconstruction, which can be modelled using the Radon transform. After expanding the solution into a finite series of basis functions a large, sparse and ill-conditioned linear system arises. We consider the solution of such systems. In particular we study a new class of iteration methods named DROP (for Diagonal Relaxed Orthogonal Projections) constructed for solving both linear equations and linear inequalities. This class can also be viewed, when applied to linear equations, as a generalized Landweber iteration. The method is compared with other iteration methods using test data from a medical application and from electron microscopy. Our theoretical analysis include convergence proofs of the fully-simultaneous DROP algorithm for linear equations without consistency assumptions, and of block-iterative algorithms both for linear equations and linear inequalities, for the consistent case.

When applying an iterative solver to an ill-posed set of linear equations the error typically initially decreases but after some iterations (depending on the amount of noise in the data, and the degree of ill-posedness) it starts to increase. This phenomena is called semi-convergence. It is therefore vital to find good stopping rules for the iteration.

We describe a class of stopping rules for Landweber type iterations for solving linear inverse problems. The class includes, e.g., the well known discrepancy principle, and also the monotone error rule. We also unify the error analysis of these two methods. The stopping rules depend critically on a certain parameter whose value needs to be specified. A training procedure is therefore introduced for securing robustness. The advantages of using trained rules are demonstrated on examples taken from image reconstruction from projections.

Sammanfattning

Vi betraktar lösning av sådana linjära ekvationssystem som uppkommer vid diskretisering av inversa problem. Dessa problem karakteriseras av att den sökta informationen inte direkt kan mätas. Ett välkänt exempel utgör datortomografi. Där mäts hur mycket strålning som passerar genom ett föremål som belyses av en strålningskälla vilken intar olika vinklar i förhållande till objektet. Syftet är förstås att generera bilder av föremålets inre (i medicinska tillämpningar av det inre av kroppen). Vi studerar en klass av iterativa lösningsmetoder för lösning av ekvationssystemen. Metoderna tillämpas på testdata från bildrekonstruktion och jämförs med andra föreslagna iterationsmetoder. Vi gör även en konvergensanalys för olika val av metod-parametrar.

När man använder en iterativ metod startar man med en begynnelse approximation som sedan gradvis förbättras. Emellertid är inversa problem känsliga även för relativt små fel i uppmätta data. Detta visar sig i att iterationerna först förbättras för att senare försämras. Detta fenomen, s.k. 'semi-convergence' är väl känt och förklarat. Emellertid innebär detta att det är viktigt att konstruera goda stoppregler. Om man avbryter iterationen för tidigt fås dålig upplösning och om den avbryts för sent fås en oskarp och brusig bild.

I avhandligen studeras en klass av stoppregler. Dessa analyseras teoretiskt och testas på mätdata. Speciellt föreslås en inlärningsförfarande där stoppregeln presenteras med data där det korrekta värdet på stopp-indexet är känt. Dessa data används för att bestämma en viktig parameter i regeln. Sedan används regeln för nya okända data. En sådan tränad stoppregel visar sig fungera väl på testdata från bildrekonstruktionsområdet.

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Papers

The following papers are appended and will be referred to by their Roman numerals. The paper [I] is accepted for publication and the manuscript [II] is submitted.

- [I] Yair Censor, Tommy Elfving, Gabor T. Herman and Touraj Nikazad, On Diagonally-Relaxed Orthogonal Projection Methods. Accepted for publication in *SIAM Journal on Scientific Computing (SISC)*.
- [II] Tommy Elfving and Touraj Nikazad, Stopping Rules for Landweber Type Iteration.

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1

Introduction

A mark-point in the history of medical imaging, was the discovery by Wilhelm Röntgen in 1895 of x-rays [12, 26]. The problem of generating medical images from measurements of the radiation around the body of a patient was considered much later. Hounsfield patented the first CT-scanner in 1972 (and was awarded, together with Cormack, in 1979 the Nobel Prize for this invention). This reconstruction problem belongs to the class of inverse problem, which are characterized by the fact that the information of interest is not directly available for measurements. The imaging device (the camera) provides measurements of a transformation of this information. In practice, these measurements are both imperfect (sampling) and inexact (noise).

The mathematical basis for tomographic imaging was laid down by Johann Radon already in 1917 [23]. The word tomography means 'reconstruction from slices'. It is applied in Computerized (Computed) Tomography (CT) to obtain cross-sectional images of patients. Fundamentally, tomographic imaging deals with reconstructing an image from its projections. The relationship between the unknown distribution (or object) and the physical quantity which can be measured (the projections) is referred to as the forward problem. For several imaging techniques, such as CT, the simplest model for the forward problem involves using the Radon transform R , see [2, 19, 22]. If χ denotes the unknown distribution and β the quantity measured by the imaging device, we have

$$R\chi = \beta.$$

The discrete version, which is based on expanding χ in a finite series of basis-function, can be written as

$$Ax = b,$$

where the vector b is a sampling of β and the vector x , in the case of pixel-(2D) or voxel-(3D) basis, is a finite representation of the unknown object. The matrix A , typically large and sparse, is a discretized version of the Radon transform. An approximative solution to this linear system could therefore be computed by iterative methods, which only require matrix-vector products and hence do not alter the structure of A .

1 Semi-convergence behavior of Landweber iteration

When solving a set of linear ill-posed equations by an iterative method typically the iterates first improve, while at later stages the influence of the noise becomes more and more noticeable. This phenomenon is called semi-convergence [22]. In order to better understand the mechanism of semi-convergence, we take a closer look at the errors in the regularized solution using the following Landweber method

$$x^{k+1} = x^k + \lambda A^T M (b - Ax^k), \quad (1.1)$$

where λ is a relaxation parameter and M is a given symmetric positive definite matrix. Also the following additive noise model

$$b = \bar{b} + \delta b$$

is assumed. Here \bar{b} is the noise free right-hand side and δb in the noise-component. We also assume, without loss of generality, that $x^0 = 0$. Let

$$B = A^T M A \text{ and } c = A^T M b.$$

Then using (1.1)

$$\begin{aligned} x^k &= (I - \lambda B)x^{k-1} + \lambda c \\ &= \lambda \sum_{j=0}^{k-1} (I - \lambda B)^{k-j-1} c. \end{aligned}$$

Suppose

$$M^{\frac{1}{2}} A = U \Sigma V^T$$

is the singular value decomposition (SVD) of $M^{\frac{1}{2}} A$, where $M^{\frac{1}{2}}$ is the square root of M (a good presentation of SVD can be found in, e.g., [3]). Then

$$B = (M^{\frac{1}{2}} A)^T (M^{\frac{1}{2}} A) = V \Sigma^T \Sigma V^T = V F V^T, \quad (1.2)$$

where

$$F = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_p^2, 0, \dots, 0), \text{ and } \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p > 0,$$

and assuming that $\text{rank}(A) = p$.

By using (1.2)

$$\sum_{j=0}^{k-1} (I - \lambda B)^{k-j-1} = V E_k V^T,$$

where

$$E_k = \text{diag}\left(\frac{1 - (1 - \lambda \sigma_1^2)^k}{\lambda \sigma_1^2}, \dots, \frac{1 - (1 - \lambda \sigma_p^2)^k}{\lambda \sigma_p^2}, 0, \dots, 0\right). \quad (1.3)$$

It follows,

$$\begin{aligned} x^k &= V (\lambda E_k) V^T c = V (\lambda E_k) \Sigma^T U^T M^{\frac{1}{2}} (\bar{b} + \delta b) \\ &= \sum_{i=1}^p \{1 - (1 - \lambda \sigma_i^2)^k\} \frac{u_i^T M^{\frac{1}{2}} (\bar{b} + \delta b)}{\sigma_i} v_i. \end{aligned} \quad (1.4)$$

The functions

$$\phi_i = 1 - (1 - \lambda\sigma_i^2)^k, \quad i = 1, 2, \dots, p$$

are called filter factors, see, e.g., [4] and [15, p. 138].

Let $x^* = \operatorname{argmin} \|Ax - \bar{b}\|_M$ be the unique weighted least squares solution of minimal 2-norm. Note that \bar{b} is the noise free right hand side vector. Using the SVD one easily finds

$$x^* = VE\Sigma^T U^T M^{\frac{1}{2}} \bar{b}, \quad (1.5)$$

where

$$E = \operatorname{diag}\left(\frac{1}{\sigma_1^2}, \dots, \frac{1}{\sigma_p^2}, 0, \dots, 0\right). \quad (1.6)$$

Also note that if $|1 - \lambda\sigma_i^2| < 1$ for $i = 1, 2, \dots, p$, that is, $0 < \lambda < \frac{2}{\sigma_1^2}$, then

$$\lim_{k \rightarrow \infty} (\lambda E_k) = E.$$

It follows easily

Theorem 1.1. *Let $\lambda_k = \lambda$, $k \geq 0$. Then the iterates of (1.1) converges to a solution \hat{x} of $\min \|Ax - b\|_M$ if and only if $0 < \epsilon \leq \lambda \leq 2/\sigma_1^2 - \epsilon$ with σ_1 the largest singular value of $M^{\frac{1}{2}}A$. If in addition $x^0 \in R(A^T)$ then \hat{x} is the unique solution of minimal Euclidean norm.*

Recent extensions, including ordered subset versions, can be found in [7] and [18].

Using (1.4) and (1.5) we find

$$\begin{aligned} x^k - x^* &= V(\lambda E_k)\Sigma^T U^T M^{\frac{1}{2}}(\bar{b} + \delta b) - VE\Sigma^T U^T M^{\frac{1}{2}}\bar{b} \\ &= V\left((\lambda E_k - E)\Sigma^T U^T M^{\frac{1}{2}}\bar{b} + \lambda E_k \Sigma^T U^T M^{\frac{1}{2}}\delta b\right). \end{aligned}$$

Now using (1.3) and (1.6) we get

$$D_1 \equiv (\lambda E_k - E)\Sigma^T = -\operatorname{diag}\left(\frac{(1 - \lambda\sigma_1^2)^k}{\sigma_1}, \dots, \frac{(1 - \lambda\sigma_p^2)^k}{\sigma_p}, 0, \dots, 0\right),$$

and

$$D_2 \equiv \lambda E_k \Sigma^T = \operatorname{diag}\left(\frac{(1 - (1 - \lambda\sigma_1^2)^k)}{\sigma_1}, \dots, \frac{(1 - (1 - \lambda\sigma_p^2)^k)}{\sigma_p}, 0, \dots, 0\right).$$

Put

$$\hat{b} = U^T M^{\frac{1}{2}} \bar{b}, \quad \hat{\delta b} = U^T M^{\frac{1}{2}} \delta b.$$

Using these notations one may write, for the projected error,

$$e^{V,k} \equiv V^T(x^k - x^*) = D_1 \hat{b} + D_2 \hat{\delta b}.$$

Let

$$\Phi^\lambda(\sigma, k) = \frac{(1 - \lambda\sigma^2)^k}{\sigma}$$

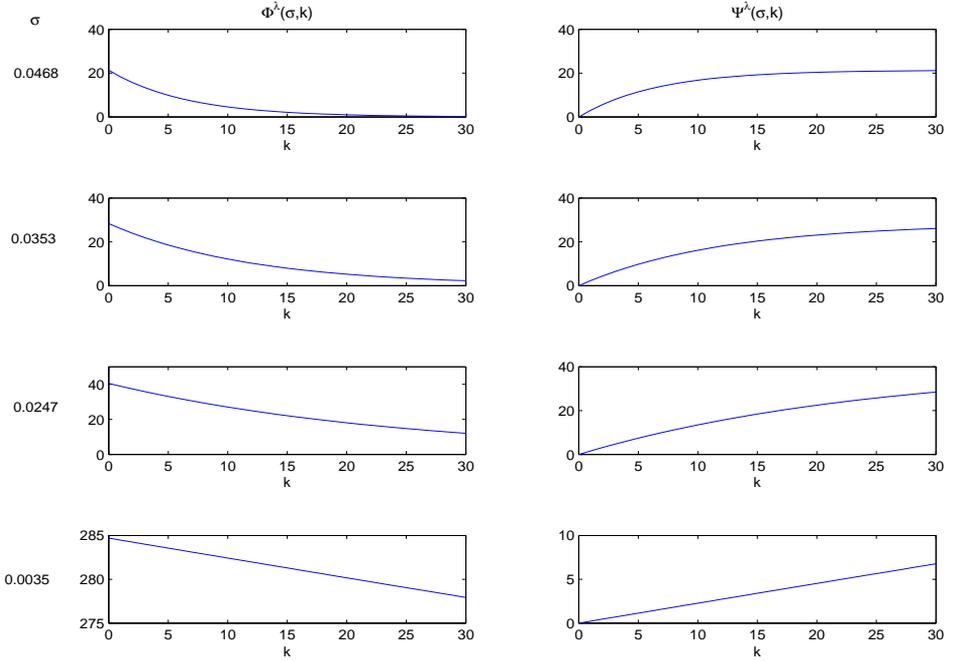


Figure 1.1: The behavior of Φ (left) and Ψ (right) using different σ - values, with $\lambda = 1.8/\sigma_1^2$.

and

$$\Psi^\lambda(\sigma, k) = \frac{1 - (1 - \lambda\sigma^2)^k}{\sigma}.$$

Then the j th component of the projected error $e^{V,k}$ is

$$e_j^{V,k} = -\Phi^\lambda(\sigma_j, k)\hat{b}_j + \Psi^\lambda(\sigma_j, k)\delta\hat{b}_j.$$

It can be observed that the projected total error $e_j^{V,k}$ has two components, an approximation error (first term) and a data error (second term). Figure 1.1 displays $\Phi^\lambda(\sigma, k)$ and $\Psi^\lambda(\sigma, k)$, for a fixed λ and various σ , as a function of iteration index k . It is seen that, for small values of k the data error is negligible and the iteration seems to converge to the exact solution. When the data error reaches the order of magnitude of the approximation error, the propagated data error is no longer hidden in the regularized solution, and the total error starts to increase. Indeed the projected error goes to a constant value as the number of iteration goes to infinity and it explains why we face with the semi-convergence problem. The typical overall error behavior is shown in Figure 1.2.

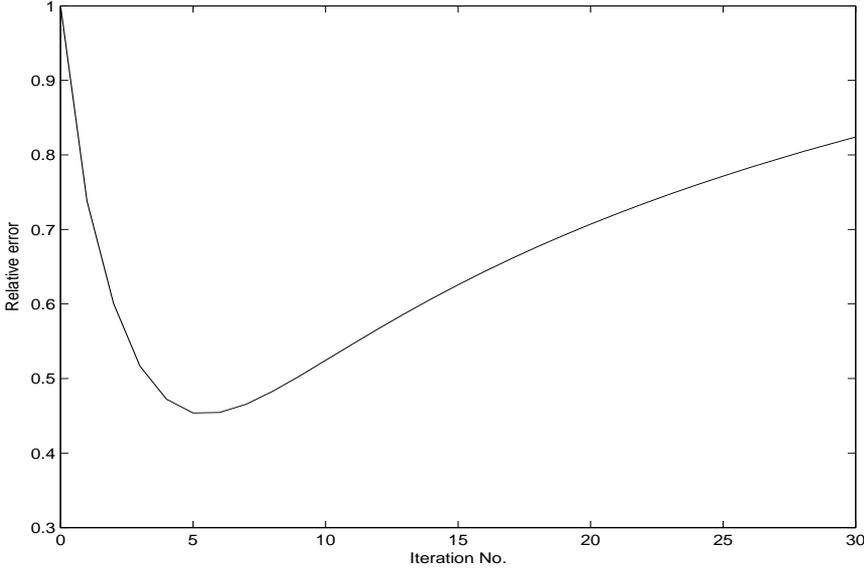


Figure 1.2: *Semi-convergence behavior.*

It follows easily

$$|e_j^{V,k}| \leq \|M^{1/2}\| (|\Phi^\lambda(\sigma_j, k)|(|b| + \delta) + |\Psi^\lambda(\sigma_j, k)|\delta),$$

here $\delta = \|\delta b\|$. We can also get the following bound for the norm of the error

$$\begin{aligned} \|x^k - x^*\| &\leq \|D_1\| * \|\hat{b}\| + \|D_2\| * \|\delta \hat{b}\| \\ &\leq \|M^{1/2}\| \max_{1 \leq j \leq p} |\Phi^\lambda(\sigma_j, k)|(|b| + \delta) + \|M^{1/2}\| \max_{1 \leq j \leq p} |\Psi^\lambda(\sigma_j, k)|\delta. \end{aligned}$$

This semi-convergence property also occurs in other iteration methods, e.g., Krylov subspace methods, see, e.g., [14].

2 Projection Algorithms

A common problem in different areas of mathematics and physical sciences consists of finding a point in the intersection of convex sets. This problem is referred to as the convex feasibility problem. Its mathematical formulation is as follows.

Suppose X is a Hilbert space and C_1, \dots, C_N are closed convex subsets with nonempty intersection C :

$$C = C_1 \cap \dots \cap C_N \neq \emptyset.$$

The convex feasibility problem is to find some point x in C . In image reconstruction using the fully discretized model each set C_i is a hyperplane or pairs of halfspaces,

so called hyperslabs, see [9, p. 269-270]. A common solution approach to such problems is to use projection algorithms, see, e.g., [1], which employ orthogonal projections (i.e., nearest point mappings) onto the individual sets C_i . Note that these projections are well defined here. These methods can have different algorithmic structures (e.g., [5, 6] and [9, Section 1.3]) some of which are particularly suitable for parallel computing, and they demonstrate nice convergence properties and/or good initial behavior patterns.

This class of algorithms has witnessed much progress in recent years and its member algorithms have been applied with success to fully-discretized models of problems in image reconstruction from projections (e.g., [16]), in image processing (e.g., [25]), and in intensity-modulated radiation therapy (IMRT) (e.g., [8]). Apart from theoretical interest, the main advantage of projection methods that makes them successful in real-world applications is computational. They commonly have the ability to handle huge-size problems of dimensions beyond which other, more sophisticated currently available, methods cease to be efficient. This is so because the building bricks of a projection algorithm are the projections onto the individual sets (that are assumed easy to perform) and the algorithmic structure is either sequential or simultaneous (or in-between). In paper I we study a new class of projection methods. This class when applied to linear equations, can also be seen as a generalized Landweber iteration. Another established class of iterations for solving linear equations is Krylov subspace methods, with CGLS (conjugate gradient applied to the normal equations) as a well known member. For low-noise and moderately ill-conditioned problems CGLS is usually very efficient. However for noisy and ill-conditioned problems (where the number of iterations is rather small before the noise component in the iterates starts to increase) projection methods become competitive.

3 Stopping rules

All regularization methods make use of a certain regularization parameter that controls the amount of stabilization imposed on the solution. In iterative methods one can use the stopping index as regularization parameter. When an iterative method is employed, the user can also study on-line adequate visualizations of the iterates as soon as they are computed, and simply halt the iteration when the approximations reach the desired quality. This may actually be the most appropriate stopping rule in many practical applications, but it requires a good intuitive imagination of what to expect. In other situations the user will need the computer's help to determine the optimal approximation, and this is the case we consider here. The stopping rule strategies naturally divide into two categories: rules which are based on knowledge of the norm of the errors, and rules which do not require such information.

If the error norm is known within reasonable accuracy, the perhaps most well known stopping rule is the discrepancy principle due to Morozov [21]. Another related rule is the monotone error rule by Hämarik and Tautenhahn [13]. Examples of the second category of methods are the L-curve criteria [15], and the generalized cross-validation criteria [11]. The performance of these parameter choice methods

depends in a complex way on both regularization method and the inverse problem at hand. E.g., the results of using the discrepancy principle for the classical Landweber method [20] are quite good. However using the discrepancy principle for the Cimmino's method [10] requires special care as is demonstrated in paper II.

2

Summary of papers

Paper I

On Diagonally-Relaxed Orthogonal Projection Methods

In the literature on reconstruction from projections, e.g., [17] and [24, Eq. (3)], researchers introduced diagonally-relaxed orthogonal projections (DROP) for heuristic reasons. However, there has been until now no mathematical study of the convergence behavior of such algorithms. Our paper makes a contribution to the convergence analysis.

We first consider a fully-simultaneous DROP algorithm for linear equations and prove its convergence without consistency assumptions. We also introduce general (block-iterative) algorithms both for linear equations and for linear inequalities and study their convergence, but only for the consistent case. Then we describe a number of iterative algorithms that we have implemented for the purpose of an experimental study. For the experiments a phantom based on a medical problem and another based on a problem from electron microscopy have been used to generate both noiseless and noisy projection data, and various algorithms have been applied to such data for the purpose of comparison. The results show that the use of DROP as an image reconstruction algorithm is not inferior to previously used methods. Those practitioners who used it without the mathematical justification offered here were indeed creating very good reconstructions. All our experiments are performed in a single processor environment. Further computational gains can be achieved by using DROP in a parallel computing environment with appropriate block choices but doing so and comparing it to other algorithms that were used in the comparisons made here calls for a separate study.

Paper II

Stopping Rules for Landweber Type Iteration

A class of stopping rules for Landweber type iterations for solving linear inverse problems is considered. The discrepancy principle (DP-rule) and the monotone error rule (ME-rule) are included in this class. Our analysis therefore unifies the DP- and ME-rule by showing that they both are special cases of a more general rule. We also unify the analysis of their error reduction properties and clarify

the role of the relaxation parameter. Also new results concerning the number of iterations needed in the DP- and ME-rule respectively are presented . We also shortly discuss possible errors in the matrix A , and show how the stopping rules can be modified to handle this case.

The DP-rule is, stop when for the first time $\|Ax^k - b\| \leq \tau\delta$ where δ , the norm of the noise, is assumed known. Using the Cauchy-Schwarz inequality we show that $\tau \in (0, 2]$ (for insuring error reduction). However the actual value of τ is critical for the performance of the stopping rules. It was found, during our experiments, that generalized Landweber methods were quite sensitive to the choice of τ . We therefore introduce a training procedure for securing a robust rule. The training is based on knowing the index where the error is minimal for certain training samples. The information gathered during the training phase is then used in the evaluation phase where unseen data is treated. We have found (experimentally) a scaling procedure, that allows using samples from a medium sized problem for predicting the stopping index for a large sized problem. The data samples all come from the field of image reconstruction from projections but differ in size and noise level.

In the last Section the advantages of using a trained rule, cf. to using fixed values like $\tau = 1, 2$ as suggested previously, are demonstrated on some examples taken from image reconstruction. In fact after training the stopping rules became quite robust and only small differences were observed between, e.g. the DP-rule and ME-rule.

Notification

The alphabetic order of authors of the two papers reflects approximatively equal inputs to the papers. It is of course natural that the adviser mostly inputs ideas and the student works out the details. Many improvements have emerged after the results of numerical experiments. All experimental work was done by the student.

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