

Linköping Studies in Science and Technology  
Dissertations, No 1136

# The Chevreton Superenergy Tensor in Einstein–Maxwell Spacetimes

Ingemar Eriksson



**Linköping University**  
**INSTITUTE OF TECHNOLOGY**

Division of Applied Mathematics  
Department of Mathematics

Linköping 2007

**The Chevreton Superenergy Tensor  
in Einstein–Maxwell Spacetimes**

Copyright © 2007 Ingemar Eriksson, unless otherwise noted.

Matematiska institutionen  
Linköpings universitet  
SE-581 83 Linköping, Sweden  
[ineri@mai.liu.se](mailto:ineri@mai.liu.se)

Linköping Studies in Science and Technology  
Dissertations, No 1136

ISBN 978-91-85895-76-2  
ISSN 0345-7524

Printed by LiU-Tryck, Linköping 2007

## Abstract

In this thesis we investigate the superenergy tensor that was introduced by Chevreton in 1964 as an electromagnetic counterpart to the Bel–Robinson tensor for the gravitational field.

We show that in Einstein–Maxwell spacetimes with a source-free electromagnetic field, the Chevreton superenergy tensor has many interesting properties. It is a completely symmetric rank-4 tensor and it gives rise to conserved currents for orthogonally transitive 1- and 2-parameter isometry groups.

The trace of this tensor is divergence-free and it is related to the Bach tensor. We investigate the implications for when the trace vanishes and we are able to determine the full set of such spacetimes. We use this to treat the problem of Einstein–Maxwell spacetimes that are conformally related to Einstein spaces and we find new exact solutions with this property.

## Acknowledgements

First of all I would like to thank my supervisor professor Göran Bergqvist for his support and encouragement and for giving me the opportunity to study General Relativity. I would also like to thank my second supervisor professor Rolf Riklund for his support and for sharing his enthusiasm for theoretical physics.

I would like to thank professor José Senovilla for many things; for much valuable input on my work, for a good time in Bilbao at the University of the Basque Country, and for being a great scientist! Many thanks to professor Brian Edgar for many interesting discussions and suggestions.

This work was carried out at the Graduate School for Interdisciplinary Mathematics at Linköping University, and I would like to thank professor Lars–Erik Andersson for supporting this project. I would also like to thank our very helpful Director of postgraduate studies Dr Bengt Ove Turesson.

I would like to thank Dr. Luuk van Dijk for revitalizing my interest in computer science.

I am indebted to the physical parameters of our Universe for allowing the evolutionary process to develop big blobs of molecules that are able to appreciate mathematics.

Many thanks to all my friends and colleagues at the Department of Mathematics. In particular I would like to mention Jens Jonasson, Martin Hessler, Magnus Herberthson, and Alfonso García–Parrado.

Finally, I would like to thank Iida for all her love and support!

Linköping, 19 October 2007

Ingemar Eriksson



## Populärvetenskaplig sammanfattning

Denna avhandling behandlar strukturer som har sitt ursprung i problemet att bestämma gravitationsfältets energiinnehåll inom allmän relativitetsteori. Vanligtvis inom fysiken, kan energin inuti ett område bestämmas genom integrering av en energidensitet över området. Denna energidensitet kan vanligtvis ges punktvis av en energi-rörelsemängdstensor. Inom allmän relativitetsteori bestäms via Einsteins ekvationer en del av rummets geometri av materiens energi-rörelsemängdstensor. Resterande del av geometrin bestäms av gravitationsfältet och på grund av ekvivalensprincipen kan dess energi inte lokaliseras punktvis.

Några olika försök att lösa detta problem har gjorts och speciellt intressant för oss är Bels konstruktion. Istället för att försöka definiera en vanlig energi-rörelsemängdstensor för gravitationsfältet, konstruerade Bel 1958 en högre ordnings tensor inspirerad av energi-rörelsemängdstensorn för elektromagnetiska fält. Denna tensor, kallad Bel–Robinson tensorn, har inte energidensitetensheter och refereras därför till som en superenergitensor. Denna tensor har många bra egenskaper, däribland att den ger upphov till konserveringslagar i avsaknad av materia. Några år senare, 1964, konstruerade Chevreton en liknande högre ordnings superenergitensor för det elektromagnetiska fältet. Denna tensor ger på motsvarande sätt konserveringslagar i avsaknad av gravitation.

Konserveringslagar är av allmänt intresse av flera skäl. De ger upphov till matematiska förenklingar av ekvationer och de kan bidra med tolkningar av fysikaliska storheter.

I denna avhandling undersöker vi Chevretons superenergitensor i allmän-relativistiska rumtider där materien enbart består av elektromagnetiska fält, så kallade Einstein–Maxwell rumtider. Vi bevisar att den har flera intressanta matematiska egenskaper, däribland att den även i närvaro av gravitation ger upphov till konserveringslagar under särskilda omständigheter. I det allmänna fallet hoppas man också att den, tillsammans med en superenergitensor för gravitationsfältet, ger konserveringslagar. Detta är ett öppet problem som vi ger ytterligare stöd till. Vi bevisar att spåret av Chevretons superenergitensor ger upphov till nya konserveringslagar under allmänna omständigheter.

Vi visar också hur denna tensor kan användas för att klassificera Einstein–Maxwell-rumtider som har vissa geometriska egenskaper. Mer specifikt undersöker vi problemet att hitta Einstein–Maxwell-rumtider som har en geometri med samma kausala struktur som så kallade Einstein-rum. Vi ger bland annat en ny lösning till Einsteins ekvationer med denna egenskap.



# Contents

<b>Introduction</b>	<b>1</b>
<b>1 Background</b>	<b>1</b>
<b>2 Preliminaries</b>	<b>3</b>
2.1 Notation and conventions . . . . .	3
2.2 Spinors . . . . .	4
2.3 Petrov classification . . . . .	5
2.4 The electromagnetic field . . . . .	6
<b>3 Superenergy tensors</b>	<b>7</b>
3.1 The Bel and Bel–Robinson tensors . . . . .	7
3.2 General construction and properties . . . . .	8
3.3 Spinor form . . . . .	10
3.4 The electromagnetic superenergy tensor . . . . .	11
3.5 Summary of paper 1 . . . . .	12
<b>4 Symmetries and conservation laws</b>	<b>13</b>
4.1 Killing vector fields . . . . .	13
4.2 Symmetry of the electromagnetic field . . . . .	16
4.3 Conservation laws . . . . .	17
4.4 Summary of papers 2 and 4 . . . . .	18
<b>5 Conformal Einstein spaces</b>	<b>19</b>
5.1 Einstein spaces and conformal transformations . . . . .	20
5.2 Necessary and sufficient conditions . . . . .	20
5.3 Summary of paper 3 . . . . .	21
<b>6 Conclusion</b>	<b>23</b>

---

Due to copyright restrictions the articles have been removed from this Ph.D. Thesis.

<b>Paper 1: New electromagnetic conservation laws</b>	<b>31</b>
<i>G Bergqvist, I Eriksson, J M M Senovilla</i>	

---

<b>Paper 2: Conserved matter superenergy currents for hyper-surface orthogonal Killing vectors</b>	<b>41</b>
<i>I Eriksson</i>	
<b>1 Introduction</b>	<b>41</b>
<b>2 Einstein–Klein–Gordon theory</b>	<b>44</b>
<b>3 Einstein–Maxwell theory</b>	<b>45</b>

4	Example for non-inherited symmetry	49
5	Conclusion	50
A	Proofs of lemmas 2, 4 and 5	51

---

<b>Paper 3: The Chevreton tensor and Einstein–Maxwell spacetimes conformal to Einstein spaces</b>		<b>61</b>
<i>G Bergqvist, I Eriksson</i>		
1	Introduction	61
1.1	Conventions . . . . .	63
2	Vanishing trace of the Chevreton tensor	64
3	Relation to the Bach tensor	69
4	Conformally Einstein spaces	71
5	Solutions	74
5.1	Non-aligned . . . . .	74
5.2	Aligned, non-null electromagnetic field . . . . .	77
5.3	Aligned, null electromagnetic field . . . . .	80
6	Conclusion	81
A	Integration of the aligned non-null solution	82

---

<b>Paper 4: Conserved matter superenergy currents for orthogonally transitive Abelian <math>G_2</math> isometry groups</b>		<b>89</b>
<i>I Eriksson</i>		
1	Introduction	89
2	Conventions and some results	91
3	Einstein–Klein–Gordon theory	93
4	Einstein–Maxwell theory	94
5	Example	99
6	Conclusion	100
A	Proofs of lemmas 2, 5, and 6	101

# Introduction

## 1 Background

The subject of this thesis has its origin in the problem of defining the energy of the gravitational field in General Relativity. For isolated systems there is a good notion of total energy, including the contributions of both the matter fields and the gravitational field, but its expression is not given in the form of a unique local energy density integrated over the whole space [28]. What one would like to have is a well-defined local energy density that can be integrated over any spacelike volume to yield the total gravitational energy inside that volume. However, such a local definition of gravitational energy has not been found, and because of the equivalence principle one may not expect to find one [34]. In any case, several different routes have been pursued in trying to deal with gravitational energy; pseudo-tensors which are quadratic in the first derivative of the metric, superenergy tensors, and quasi-local constructions.

The problem with pseudo-tensors is that they are dependent on the specific choice of coordinates and a gravitational energy-momentum pseudo-tensor based on the first derivative of the metric can always be set to zero at a point.

Bel [2, 3] instead, in 1958, introduced a rank-4 tensor, the Bel–Robinson superenergy tensor,  $T_{abcd}$ , which is quadratic in the Weyl tensor and in vacuum it has many similar properties to the energy-momentum tensor for the electromagnetic field. It is locally well defined, symmetric, trace-free, divergence-free, and its timelike component is non-negative. However, it does not have units of energy density and it lacks a clear-cut physical interpretation.

Instead of trying to find a locally defined quantity, Hawking [27] in 1968 defined a quasi-local energy inside a spacelike closed 2-surface,  $S$ , by considering an integral,  $E(S)$ , over the 2-surface. This energy vanishes when the 2-surface shrinks to a point and it agrees with the asymptotic definition of total energy when the 2-surface becomes very large. Since then, several different quasi-local energies have been proposed, but they all have some problem, e.g., unclear dependence on the shape of the 2-surface or lacking monotonicity properties [5].

We note that there is a very interesting connection between Hawking’s quasi-local energy and the Bel–Robinson tensor; if one considers small 2-

spheres  $S_r$ , with area proportional to  $r^2$ , then in vacuum [28],

$$\lim_{r \rightarrow 0} \frac{E(S_r)}{r^5} \propto T_{abcd} t^a t^b t^c t^d, \quad (1)$$

where  $t^a$  is a unit timelike vector orthogonal to  $S_r$ . So, it seems that the Bel–Robinson tensor at least has something to do with gravitational energy. When matter is present its energy density dominates  $E(S_r)$ .

Leaving aside the problem with the interpretation of the Bel–Robinson tensor, we instead focus on its similarity to the electromagnetic energy-momentum tensor. Particularly interesting is the property that its timelike component is positive, unless the Weyl tensor vanishes, in which case the Bel–Robinson tensor also vanishes. Bel soon generalized the construction to non-vacuum situations by instead using the Riemann tensor as the seed tensor, yielding the Bel tensor, and in 1964 Chevreton [15] went one step further in generalizing the superenergy construction to the electromagnetic field by constructing a rank-4 tensor based on the covariant derivative of the electromagnetic field. This tensor has the same positivity property as the Bel and the Bel–Robinson tensors and it is divergence-free for source-free electromagnetic fields in flat spacetimes.

Senovilla [38] has now given a general construction of superenergy tensors from any seed tensor. The superenergy tensors are quadratic in the seed tensor and they all have a positivity property called the dominant property, which is a generalization of the dominant energy condition for energy-momentum tensors.

The superenergy tensors have proven to be useful in many different applications, like for example estimates for solutions to partial differential equations [16, 41], proofs for causal propagation of fields [11, 12], algebraic classification of exact solutions to Einstein’s equations [6], defining intrinsic radiative states [4, 25], and Rainich theory [10].

One interesting thing to note is the fact that the Bel tensor is not divergence-free in the presence of matter need not be a problem. On the contrary, Senovilla [38] has shown that when the matter is represented by a scalar field and the spacetime contains an isometry, it is possible to construct a conserved current based on the Bel tensor and the superenergy tensor of the scalar field. This current will then govern interchange of superenergy between the gravitational field and the scalar field.

In this thesis we study the Chevreton superenergy tensor in Einstein–Maxwell spacetimes and we prove many interesting properties in relation to it; several conservation laws for the electromagnetic field at the superenergy level, implications for certain algebraic properties of the Chevreton tensor, and applications to Einstein–Maxwell spacetimes that are conformally related to Einstein spaces. Chevreton’s superenergy tensor appears to be the correct tensor to combine with the Bel tensor in the construction of conserved mixed currents for Einstein–Maxwell spacetimes. The specific problem of proving the general existence of a mixed current is still an open problem, but our results give support to the possibility of doing so.

The results of the thesis have been published in four papers [8, 9, 22, 23] and this introduction is devoted to the technical background and a summary of the papers.

## 2 Preliminaries

In this section, we specify our conventions and recall the spinor form of the curvature tensors, the Petrov classification of the Weyl spinor, and some properties of the electromagnetic field.

### 2.1 Notation and conventions

We use the abstract index notation [35, 46]. Symmetrization of indices is denoted by round brackets and antisymmetrization by square brackets.

Spacetime is given by a  $C^\infty$  manifold  $\mathcal{M}$  of dimension  $n$  and equipped with a metric  $g_{ab}$  of Lorentzian signature  $(+, -, \dots, -)$ . Unless otherwise indicated, we assume that  $n = 4$ . The connection,  $\nabla_a$ , is always the torsion-free Levi-Civita connection.

We will refer to both contravariant and covariant vectors as just vectors; the distinction is here only important for Lie derivatives. However, we only consider Lie derivatives along Killing vectors, for which the metric is constant.

The Hodge dual of a  $p$ -form  $F_{a_1 \dots a_p} = F_{[a_1 \dots a_p]}$  in  $n$  dimensions is given by

$${}^*F_{a_{p+1} \dots a_n} = \frac{1}{p!} e_{a_{p+1} \dots a_n}{}^{a_1 \dots a_p} F_{a_1 \dots a_p}, \quad (2)$$

where  $e_{a_1 \dots a_n}$  is the  $n$ -dimensional canonical volume element.

The Riemann curvature tensor is defined by

$$(\nabla_a \nabla_b - \nabla_b \nabla_a)v_c = -R_{abcd}v^d. \quad (3)$$

The trace-free part of the Riemann tensor is given by the Weyl tensor and it is denoted by  $C_{abcd}$ .

Einstein's equations are given by

$$R_{ab} - \frac{1}{2}g_{ab}R + \lambda g_{ab} = -T_{ab}, \quad (4)$$

where  $R_{ab}$  is the Ricci tensor,  $R$  is the Ricci scalar,  $\lambda$  is the cosmological constant, and  $T_{ab}$  is the energy-momentum tensor. We use geometrized units with  $8\pi G = c = 1$ .

## 2.2 Spinors

Some of the results in this thesis are derived by the use of spinor methods [35, 36] and without going into detail, we here present the spinor form of the metric and the curvature tensors. Note that this only applies to four dimensions and Lorentzian signature. We identify tensor indices ‘ $a$ ’ etc. with spinor indices ‘ $AA'$ ’ etc.

The rank-2 antisymmetric spinor  $\varepsilon_{AB} = \varepsilon_{[AB]}$  acts as an inner product on the 2-dimensional complex spin space and it is connected to the metric by

$$g_{ab} = \varepsilon_{AB}\varepsilon_{A'B'}. \quad (5)$$

The spinor form of the Riemann tensor is given by

$$\begin{aligned} R_{abcd} = & \varepsilon_{A'B'}\varepsilon_{C'D'}X_{ABCD} + \varepsilon_{AB}\varepsilon_{CD}\bar{X}_{A'B'C'D'} \\ & + \varepsilon_{A'B'}\varepsilon_{CD}\Phi_{ABC'D'} + \varepsilon_{AB}\varepsilon_{C'D'}\bar{\Phi}_{A'B'CD}, \end{aligned} \quad (6)$$

where

$$X_{ABCD} = \frac{1}{4}R_{AE'B}{}^{E'}{}_{CF'D}{}^{F'}, \quad \Phi_{ABC'D'} = \frac{1}{4}R_{AE'B}{}^{E'}{}_{FC'}{}^F{}_{D'}. \quad (7)$$

These spinors have the following symmetries:

$$\begin{aligned} X_{ABCD} &= X_{(AB)(CD)} = X_{CDAB}, \\ \Phi_{ABC'D'} &= \Phi_{(AB)(C'D')}, \\ \bar{\Phi}_{ABC'D'} &= \bar{\Phi}_{ABC'D'}. \end{aligned} \quad (8)$$

The spinor  $\Phi_{ABC'D'}$  is related to the trace-free part of the Ricci tensor by

$$-2\bar{\Phi}_{ABA'B'} = R_{ab} - \frac{1}{4}Rg_{ab}. \quad (9)$$

The spinor  $X_{ABCD}$  can be decomposed as

$$X_{ABCD} = \Psi_{ABCD} + \Lambda(\varepsilon_{AC}\varepsilon_{BD} + \varepsilon_{AD}\varepsilon_{BC}), \quad (10)$$

where  $\Psi_{ABCD} = \Psi_{(ABCD)}$  is the Weyl spinor and  $\Lambda = \frac{1}{24}R$ . The spinor form of the Weyl tensor is given by

$$C_{abcd} = \varepsilon_{A'B'}\varepsilon_{C'D'}\Psi_{ABCD} + \varepsilon_{AB}\varepsilon_{CD}\bar{\Psi}_{A'B'C'D'}. \quad (11)$$

## 2.3 Petrov classification

Any completely symmetric spinor  $S_{AB\dots C} = S_{(AB\dots C)}$  factorizes as

$$S_{AB\dots C} = \alpha_{(A}\beta_B \cdots \gamma_{C)}, \quad (12)$$

where the factorization is unique up to proportionality and reordering of the factors [35]. The spinors  $\alpha_A$  etc. are called the principal spinors of  $S_{AB\dots C}$  and the null vectors  $l_a = \alpha_A \bar{\alpha}_{A'}$  etc. are referred to as the principal null directions. Note that  $\alpha_A$  is a principal spinor of  $S_{AB\dots C}$  if and only if  $S_{AB\dots C} \alpha^A \alpha^B \cdots \alpha^C = 0$ . It is a  $k$ -fold repeated principal spinor if and only if  $S_{AB\dots C}$  contracted with  $r - k + 1$  copies of  $\alpha^A$  vanishes, where  $r$  is the rank of  $S_{AB\dots C}$ .

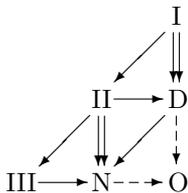


Figure 1: Petrov types

This is the basis for the Petrov classification of the Weyl spinor, which is equivalent to the Petrov classification of the Weyl tensor. The Weyl spinor factorizes as

$$\Psi_{ABCD} = \alpha_{(A}\beta_B\gamma_C\delta_{D)}, \quad (13)$$

and it is said to be of Petrov type **I** if there is no coincidence among the principal spinors. This is the algebraically general case. The other Petrov types are algebraically special and two or more principal spinors coincide. In Petrov type **II**, two principal spinors coincide, and in type **D** there are two different pairs of repeated principal spinors. In Petrov type **III**, three principal spinors coincide. Petrov type **N** is the most specialized type with all four principal spinors coinciding,

$$\Psi_{ABCD} = \alpha_A\alpha_B\alpha_C\alpha_D. \quad (14)$$

For Petrov type **O**, the Weyl spinor is zero. Figure 1 pictures the hierarchy of the Petrov types, where the arrows indicate increasing algebraic specialization.

The exact solutions appearing in paper 3 [8] are of Petrov type **N** or **O**.

## 2.4 The electromagnetic field

A source-free electromagnetic field is represented by the Faraday tensor  $F_{ab} = F_{[ab]}$ , which satisfies the source-free Maxwell equations,

$$\nabla^a F_{ab} = 0, \quad \nabla_{[a} F_{bc]} = 0. \quad (15)$$

In spinors the electromagnetic field is represented by the Maxwell spinor,

$$\varphi_{AB} = \varphi_{(AB)} = \frac{1}{2} F_{AC'B}{}^{C'}, \quad (16)$$

so that

$$F_{ab} = \varepsilon_{A'B'} \varphi_{AB} + \varepsilon_{AB} \bar{\varphi}_{A'B'}. \quad (17)$$

The spinor form of the source-free Maxwell equations simplifies to

$$\nabla^{AA'} \varphi_{AB} = 0. \quad (18)$$

The Maxwell spinor can be factorized into its principal spinors as  $\varphi_{AB} = \alpha_{(A} \beta_{B)}$ , and the electromagnetic field is said to be null if the two principal spinors coincide. It is said to be aligned if it has a principal null direction in common with the Weyl tensor.

The energy-momentum tensor for the electromagnetic field is given by<sup>1</sup>

$$T_{ab} = 2(-F_{ac} F_b{}^c + \frac{1}{4} g_{ab} F_{cd} F^{cd}) = 4\varphi_{AB} \bar{\varphi}_{A'B'}. \quad (19)$$

It is symmetric and trace-free,

$$T_{ab} = T_{(ab)}, \quad T_a{}^a = 0, \quad (20)$$

and divergence-free,

$$\nabla^a T_{ab} = 0. \quad (21)$$

In addition, it satisfies the *dominant energy condition*,

$$T_{ab} u^a v^b \geq 0, \quad (22)$$

for any future-pointing causal (i.e., timelike or null) vectors  $u^a$  and  $v^a$ .

The Einstein–Maxwell equations are given by

$$R_{ab} - \frac{1}{2} R g_{ab} + \lambda g_{ab} = -T_{ab} = 2F_{ac} F_b{}^c - \frac{1}{2} g_{ab} F_{cd} F^{cd}, \quad (23)$$

and the Ricci scalar,  $R$ , satisfies  $R = 4\lambda$ . In spinors,

$$\Phi_{ABA'B'} = 2\varphi_{AB} \bar{\varphi}_{A'B'}. \quad (24)$$

---

<sup>1</sup>The factor 2 here is also used in papers 1 and 3, where spinors are used, while it is absent in papers 2 and 4.

### 3 Superenergy tensors

In this section, we review some properties of the original superenergy tensors and the general construction of the superenergy of any tensor, both in terms of tensors and spinors. We consider the electromagnetic superenergy tensor introduced by Chevreton [15] and give a summary of paper 1 [9].

#### 3.1 The Bel and Bel–Robinson tensors

The Bel–Robinson tensor is given in  $n$  dimensions by [38]

$$\begin{aligned} T_{abcd} = & C_{aecf}C_b{}^e{}_d{}^f + C_{aedf}C_b{}^e{}_c{}^f - \frac{1}{2}g_{ab}C_{efcg}C^{ef}{}_d{}^g \\ & - \frac{1}{2}g_{cd}C_{aefg}C_b{}^{efg} + \frac{1}{8}g_{ab}g_{cd}C_{efgh}C^{efgh}. \end{aligned} \quad (25)$$

It has the following symmetries:

$$T_{abcd} = T_{(ab)(cd)} = T_{cdab}. \quad (26)$$

It also satisfies the *dominant property*,

$$T_{abcd}v_1^av_2^bv_3^cv_4^d \geq 0, \quad (27)$$

where  $v_i^a$  are future-pointing causal vectors, and it is divergence-free in vacuum,

$$\nabla^a T_{abcd} = 0. \quad (28)$$

In four dimensions it is also completely symmetric and trace-free [38],

$$T_{abcd} = T_{(abcd)}, \quad T_{abc}{}^c = 0. \quad (29)$$

The Bel–Robinson tensor thus has very similar properties to the energy-momentum tensor for the electromagnetic field. However, the Bel–Robinson tensor is a rank-4 tensor and its units are  $(\text{length})^{-4}$  as compared to  $(\text{length})^{-2}$  (or energy density in non-geometrized units) for ordinary energy-momentum tensors. The units of the Bel–Robinson tensor can be interpreted as either  $(\text{energy density})^2$  or as energy density per unit surface. Lately, the latter interpretation has been more favored, for example in the relation to Hawking’s quasi-local energy (1) [28].

When considering non-vacuum spacetimes, it seems natural to replace the Weyl tensor with the Riemann tensor in the definition of the Bel–Robinson tensor to get the Bel tensor, given in  $n$  dimensions by

$$\begin{aligned} B_{abcd} = & R_{aecf}R_b{}^e{}_d{}^f + R_{aedf}R_b{}^e{}_c{}^f - \frac{1}{2}g_{ab}R_{efcg}R^{ef}{}_d{}^g \\ & - \frac{1}{2}g_{cd}R_{aefg}R_b{}^{efg} + \frac{1}{8}g_{ab}g_{cd}R_{efgh}R^{efgh}. \end{aligned} \quad (30)$$

It has the following symmetries<sup>2</sup>:

$$B_{abcd} = B_{(ab)(cd)} = B_{cdab}, \quad (31)$$

and it also satisfies the dominant property. However, neither the Bel–Robinson tensor nor the Bel tensor are divergence-free in general. The resolution to this problem seems to be to consider not only the superenergy of the gravitational field (i.e., the Bel tensor or the Bel–Robinson tensor) by itself, but to combine it with the superenergy of the matter fields. Indeed, Senovilla [38] has shown that when the matter is represented by a massive scalar (i.e., classical Klein–Gordon) field and the spacetime contains an isometry, then the Bel tensor and the superenergy of the scalar field give rise to a combined conserved current, which then governs interchange of superenergy between the fields.

### 3.2 General construction and properties

The generalization of the superenergy tensors is due to Senovilla [38]. The construction works for any  $n$ -dimensional Lorentzian manifold. Given a seed tensor  $A$ , its superenergy is defined as follows: We group together the indices of the tensor in which it is anti-symmetric. Thus, we get an index rearranged version  $\tilde{A}_{[n_1] \dots [n_r]}$  of  $A$ , with  $r$  groups of indices,  $n_i$  denoting the number of indices in each group. This tensor can be seen as an  $r$ -fold  $(n_1, \dots, n_r)$ -form. We also need to consider (generalized Hodge) duals<sup>3</sup> of  $\tilde{A}$ . For example, the dual with respect to the  $i$ th and  $j$ th index groups is denoted by  $\tilde{A}_{[n_1] \dots [n-n_i] \dots [n-n_j] \dots [n_r]}$ . There is a total of  $2^r$  combinations of duals of  $\tilde{A}$  and without further specification they will be denoted by  $\tilde{A}_{\mathcal{P}}$ , where  $\mathcal{P} = 1, 2, \dots, 2^r$ .

We now define a product ‘ $\times$ ’ of an  $r$ -fold form with itself resulting in a rank- $2r$  tensor as<sup>4</sup>

$$\begin{aligned} (\tilde{A} \times \tilde{A})_{a_1 b_1 \dots a_r b_r} &\equiv \frac{(-1)^{n_1-1}}{(n_1-1)!} \dots \frac{(-1)^{n_r-1}}{(n_r-1)!} \\ &\cdot \tilde{A}_{a_1 c_{12} \dots c_{1n_1} \dots a_r c_{r2} \dots c_{rn_r}} \tilde{A}_{b_1 \dots b_r}^{c_{12} \dots c_{1n_1} \dots c_{r2} \dots c_{rn_r}}. \end{aligned} \quad (32)$$

The basic superenergy tensor of the tensor  $A$  is then defined as

$$T_{a_1 b_1 \dots a_r b_r} \{A\} \equiv \frac{1}{2} \sum_{\mathcal{P}=1}^{2^r} (\tilde{A}_{\mathcal{P}} \times \tilde{A}_{\mathcal{P}})_{a_1 b_1 \dots a_r b_r}. \quad (33)$$

This tensor has the following properties [38]:

<sup>2</sup>Because of these symmetries, the complete symmetrization is given by  $B_{a(bcd)}$ .

<sup>3</sup>We are excluding  $n$ -forms from the construction, since they are dual to scalar functions.

<sup>4</sup>The  $(-1)$  factors here appear because of the different sign convention here as compared to [38].

- If  $A$  is an  $r$ -fold form, then  $T\{A\}$  is a rank- $2r$  covariant tensor.
- $T\{A\} = T\{\tilde{A}_{\mathcal{P}}\}$  for any  $\mathcal{P}$ .
- $T\{A\} = 0$  if and only if  $A = 0$ .
- It is symmetric in each pair  $a_i b_i$  of indices,

$$T\{A\}_{a_1 b_1 \dots a_r b_r} = T\{A\}_{(a_1 b_1) \dots (a_r b_r)}. \quad (34)$$

- If  $\tilde{A}$  is symmetric or antisymmetric under the interchange of two groups of indices  $[n_i]$  and  $[n_j]$ , then the superenergy tensor is symmetric in the pairs  $a_i b_i$  and  $a_j b_j$ ,

$$T\{A\}_{\dots a_i b_i \dots a_j b_j \dots} = T\{A\}_{\dots a_j b_j \dots a_i b_i \dots}. \quad (35)$$

- It satisfies the *dominant property*. For any future-pointing causal vectors  $v_1^a, \dots, v_s^a$ , we have that

$$T\{A\}_{a_1 \dots a_s} v_1^{a_1} \dots v_s^{a_s} \geq 0. \quad (36)$$

The superenergy of the Riemann tensor in four dimensions is by this construction given by

$$\begin{aligned} T_{abcd}\{R_{[2][2]}\} &= \frac{1}{2}(R_{aecf}R_b{}^e{}_d{}^f + {}^*R_{aecf}{}^*R_b{}^e{}_d{}^f \\ &\quad + R^*{}_{aecf}R^*{}^e{}_b{}^f + {}^*R^*{}_{aecf}{}^*R^*{}^e{}_b{}^f), \end{aligned} \quad (37)$$

where ‘\*’ on the left side of  $R$  indicates the (Hodge) dual with respect to the first index pair and on the right side with respect to the second index pair. By expanding the duals in terms of the canonical volume element one sees that this tensor is precisely the Bel tensor (30).

A more general superenergy tensor can be constructed from the basic superenergy tensor by

$$\tilde{T}\{A\}_{a_1 \dots a_{2r}} \equiv \sum_{\sigma} c_{\sigma} T\{A\}_{a_{\sigma(1)} \dots a_{\sigma(2r)}}. \quad (38)$$

Here,  $c_{\sigma}$  is non-negative and  $a_{\sigma(1)} \dots a_{\sigma(2r)}$  denotes any possible permutation of  $1, \dots, 2r$ . In general this tensor will lose the symmetries of the basic superenergy tensor, but the dominant property still holds.

The natural way to generate higher rank superenergy tensors from a seed tensor  $A$ , seems to be to consider covariant derivatives of the tensor,  $\nabla_{[1]} \dots \nabla_{[1]} A$ . The superenergy tensor generated from  $k - 1$  covariant derivatives of the seed tensors is referred to as the (super) <sup>$k$</sup> -energy tensor [38].

Another generalization is the ‘mathematical energy-tensor’, which is obtained by ignoring the grouping into anti-symmetric indices and instead

treating a rank- $r$  tensor as an  $r$ -fold 1-form in the construction of the basic superenergy tensor. This way a rank- $r$  tensor yields a rank- $2r$  superenergy tensor, from which it is possible to recover the basic superenergy tensor [40].

Other generalizations are possible with the help of Clifford algebras or spinor methods, for example to define odd-rank superenergy tensors [10, 30].

### 3.3 Spinor form

We now turn to the general spinor form of superenergy tensors, which is due to Bergqvist [7]. Since we are now in four dimensions, we assume that any 4-forms have been removed.

We arrange the sets of indices of  $A$  such that we first have  $s$  2-forms, then  $t$  1-forms, and last  $r - s - t$  3-forms. The first term of the superenergy will be

$$\begin{aligned}
T_1 = & \frac{1}{2s0^{r-s-t}} \tilde{A}_{A_1 C'_1 B_1}{}^{C'_1} \dots A_s C'_s B_s{}^{C'_s} A_{s+1} B'_{s+1} \dots A_{s+t} B'_{s+t} \\
& A_{s+t+1} D'_{s+t+1} C_{s+t+1} B'_{s+t+1}{}^{C_{s+t+1} D'_{s+t+1}} \dots A_r D'_r C_r B'_r{}^{C_r D'_r} \\
& \cdot \tilde{A}_{C_1 A'_1}{}^{C_1} B'_1 \dots C_s A'_s{}^{C_s} B'_s B_{s+1} A'_{s+1} \dots B_{s+t} A'_{s+t} \\
& D_{s+t+1} A'_{s+t+1} B_{s+t+1} C'_{s+t+1}{}^{D_{s+t+1} C'_{s+t+1}} \dots D_r A'_r B_r C'_r{}^{D_r C'_r}, \quad (39)
\end{aligned}$$

and subsequent terms  $T_Q$  from  $T_1$ , with

$$Q = 1 + z_1 2^{r-1} + z_2 2^{r-2} + \dots + z_r, \quad (40)$$

where  $z_j = 1$  if the  $j$ -indices in  $T_Q$  have been interchanged between the two  $\tilde{A}$ s according to

$$A_j C'_j B_j{}^{C'_j} \leftrightarrow C_j A'_j{}^{C_j} B'_j, \quad 1 \leq j \leq s, \quad (41a)$$

$$A_j B'_j \leftrightarrow B_j A'_j, \quad s+1 \leq j \leq s+t, \quad (41b)$$

$$A_j D'_j C_j B'_j{}^{C_j D'_j} \leftrightarrow D_j A'_j B_j C'_j{}^{D_j C'_j}, \quad s+t+1 \leq j \leq r, \quad (41c)$$

and  $z_j = 0$  otherwise.

The spinor form of the superenergy tensor can now be written as

$$T_{a_1 b_1 \dots a_r b_r} \{A\} = \frac{1}{2} \sum_{Q=1}^{2^r} (T_Q)_{a_1 b_1 \dots a_r b_r} = \sum_{Q=1}^{2^{r-1}} (T_Q)_{a_1 b_1 \dots a_r b_r}, \quad (42)$$

since  $T_Q = T_{2^{r+1-Q}}$ . Note that each term in the superenergy is given by the product of a spinor and its conjugate. This gives a very simple proof of the dominant property [7]. Note also that there are only contractions within the spinors.

By this construction, the spinor form of the Bel tensor is given by [7]

$$\begin{aligned} B_{abcd} &= \frac{1}{4} \left( R_{AE'B}{}^{E'}{}_{CF'D}{}^{F'} R_{A'EB'}{}^E{}_{C'FD'}{}^F \right. \\ &\quad \left. + R_{AE'B}{}^{E'}{}_{FC'D'}{}^F R_{A'EB'}{}^E{}_{F'C}{}^{F'}{}^D \right) \\ &= 4 \left( X_{ABCD} \bar{X}_{A'B'C'D'} + \Phi_{ABC'D'} \bar{\Phi}_{A'B'CD} \right), \end{aligned} \quad (43)$$

and the Bel–Robinson tensor by

$$T_{abcd} = 4 \Psi_{ABCD} \bar{\Psi}_{A'B'C'D'}. \quad (44)$$

### 3.4 The electromagnetic superenergy tensor

The superenergy of the electromagnetic field tensor is

$$T_{ab}\{F_{[2]}\} = -\frac{1}{2} \left( F_{ac} F_b{}^c + F_{ac}^* F_b{}^{*c} \right) = -F_{ac} F_b{}^c + \frac{1}{4} g_{ab} F_{cd} F^{cd}, \quad (45)$$

which is just proportional to the ordinary energy-momentum tensor. The natural way to go to the next superenergy level is to consider the covariant derivative of the electromagnetic field,  $\nabla_a F_{bc}$ , which gives the superenergy tensor

$$\begin{aligned} E_{abcd} &= T_{abcd}\{\nabla_{[1]} F_{[2]}\} \\ &= -\nabla_a F_{ce} \nabla_b F_d{}^e - \nabla_b F_{ce} \nabla_a F_d{}^e + g_{ab} \nabla_f F_{ce} \nabla^f F_d{}^e \\ &\quad + \frac{1}{2} g_{cd} \nabla_a F_{ef} \nabla_b F^{ef} - \frac{1}{4} g_{ab} g_{cd} \nabla_e F_{fg} \nabla^e F^{fg}, \end{aligned} \quad (46)$$

where the duals have been expanded in terms of the canonical volume element. This tensor is referred to as the basic electromagnetic superenergy tensor and it has no other symmetries than those guaranteed by the construction,  $E_{abcd} = E_{(ab)(cd)}$ . Note that this tensor has units of (length)<sup>-4</sup> and it thus has the same units as the Bel tensor.

The Chevreton superenergy tensor, introduced in 1964, is defined as [15, 38]

$$\begin{aligned} H_{abcd} &= \frac{1}{2} (E_{abcd} + E_{cdab}) \\ &= -\frac{1}{2} (\nabla_a F_{ce} \nabla_b F_d{}^e + \nabla_b F_{ce} \nabla_a F_d{}^e + \nabla_c F_{ae} \nabla_d F_b{}^e + \nabla_d F_{ae} \nabla_c F_b{}^e) \\ &\quad + \frac{1}{2} (g_{ab} \nabla_f F_{ce} \nabla^f F_d{}^e + g_{cd} \nabla_f F_{ae} \nabla^f F_b{}^e) \\ &\quad + \frac{1}{4} (g_{ab} \nabla_c F_{ef} \nabla_d F^{ef} + g_{cd} \nabla_a F_{ef} \nabla_b F^{ef}) - \frac{1}{4} g_{ab} g_{cd} \nabla_e F_{fg} \nabla^e F^{fg}. \end{aligned} \quad (47)$$

It has the same obvious symmetries as the Bel tensor, and both this tensor and the basic electromagnetic superenergy tensor are divergence-free in flat spacetimes [38].

The spinor form of the Chevreton tensor is given by

$$H_{abcd} = \nabla_{AB'}\varphi_{CD}\nabla_{BA'}\bar{\varphi}_{C'D'} + \nabla_{BA'}\varphi_{CD}\nabla_{AB'}\bar{\varphi}_{C'D'} \\ + \nabla_{CD'}\varphi_{AB}\nabla_{DC'}\bar{\varphi}_{A'B'} + \nabla_{DC'}\varphi_{AB}\nabla_{CD'}\bar{\varphi}_{A'B'}. \quad (48)$$

### 3.5 Summary of paper 1

In paper 1 [9], we study the Chevreton superenergy tensor in four-dimensional source-free Einstein–Maxwell spacetimes. Using the spinor form of the Chevreton tensor, we prove that it is completely symmetric (theorem 1),

$$H_{abcd} = H_{(abcd)}. \quad (49)$$

This result actually holds for any source-free electromagnetic field in four dimensions, independently of Einstein’s equations. Now, in Einstein–Maxwell spacetimes, the divergence of this tensor is given by

$$\nabla^a H_{abcd} \\ = 2\Psi^{EF}{}_{(BC}\varphi_{D)F}\nabla_{E(B'}\bar{\varphi}_{C'D')} + 4\varphi^{FE}\Psi_{FE(BC}\nabla_{D)(B'}\bar{\varphi}_{C'D')} \\ + 2\bar{\Psi}^{E'F'}{}_{(B'C'}\bar{\varphi}_{D')F'}\nabla_{E'(B}\varphi_{CD)} + 4\bar{\varphi}^{F'E'}\bar{\Psi}_{F'E'(B'C'}\nabla_{D')(B}\varphi_{CD)} \\ - 18\Lambda\varphi_{(BC}\nabla_{D)(B'}\bar{\varphi}_{C'D')} - 18\Lambda\bar{\varphi}_{(B'C'}\nabla_{D')(B}\varphi_{CD)}. \quad (50)$$

Since this expression is completely symmetric with respect to the spinor indices, it follows that the divergence is trace-free. This can be restated as that the trace,

$$H_{ab} = -2\nabla_{CC'}\varphi_{AB}\nabla^{CC'}\bar{\varphi}_{A'B'}, \quad (51)$$

is divergence-free. Hence, we prove that the trace of the Chevreton tensor is a divergence-free, symmetric, and trace-free rank-2 tensor (theorem 2),

$$\nabla^a H_{ab} = 0, \quad H_{ab} = H_{(ab)}, \quad H_a{}^a = 0. \quad (52)$$

This result also holds for test fields in Einstein spaces. A tensor version of the trace is given by

$$H_{ab} = \nabla_c F_{ad}\nabla^c F_b{}^d - \frac{1}{4}g_{ab}\nabla_c F_{de}\nabla^c F^{de}. \quad (53)$$

The divergence-free property is trivially true in flat spacetimes. Deser [18] showed later that for test fields in Ricci-flat spacetimes the result follows naturally by using the Lichnerowicz operator. However, for Einstein–Maxwell spacetimes it holds only in four dimensions and one has to use the Einstein equations. A proof of this using tensor methods was later given by Edgar [19], using two different dimensionally dependent identities. In [39] the explicit expression for the trace of the Chevreton tensor was calculated

for some spacetimes and a general formula was derived for null electromagnetic fields. For the Kerr–Newman spacetime the trace of the Chevreton tensor was calculated in [21].

In paper 3 [8], we prove that the trace of the Chevreton tensor is connected to the Bach tensor [1]. (See also section 5 and equation (97).)

## 4 Symmetries and conservation laws

In this section, we give a general review of Killing vector fields and their implications for the geometry of the spacetime, the electromagnetic field, and for conservation laws. A summary of paper 2 [22] and paper 4 [23] is given.

### 4.1 Killing vector fields

To a 1-parameter group of diffeomorphisms  $\phi_t : \mathbb{R} \times \mathcal{M} \rightarrow \mathcal{M}$  we can associate a vector field  $v^a$  as follows: For a fixed  $p \in \mathcal{M}$  the orbit of  $\phi_t(p) : \mathbb{R} \rightarrow \mathcal{M}$  is a curve that passes through  $p$  at  $t = 0$ , then  $v^a|_p$  is defined as the tangent of the curve at  $t = 0$ . Likewise, a vector field  $v^a$  on  $\mathcal{M}$  generates a 1-parameter group of diffeomorphisms  $\phi_t$  [46].

With aid of the map  $(\phi_t)_*$  we can compare tensors at different points and we thus define the *Lie derivative* of a tensor  $T$  at a point  $p$  along the curve generated by  $v^a$  by [24]

$$\mathcal{L}_v T|_p = \lim_{t \rightarrow 0} \frac{(\phi_{-t})_* T|_{\phi_t(p)} - T|_p}{t}. \quad (54)$$

For a rank- $(k+l)$  tensor  $T^{a_1 \dots a_k}_{b_1 \dots b_l}$  with  $k$  contravariant and  $l$  covariant indices we have that [46]

$$\begin{aligned} \mathcal{L}_v T^{a_1 \dots a_k}_{b_1 \dots b_l} &= v^c \nabla_c T^{a_1 \dots a_k}_{b_1 \dots b_l} - \sum_{i=1}^k T^{a_1 \dots c \dots a_k}_{b_1 \dots b_l} \nabla_c v^{a_i} \\ &\quad + \sum_{j=1}^l T^{a_1 \dots a_k}_{b_1 \dots c \dots b_l} \nabla_{b_j} v^c. \end{aligned} \quad (55)$$

If the metric is invariant with respect to a vector field  $\xi_a$ , then

$$\mathcal{L}_\xi g_{ab} = \nabla_a \xi_b + \nabla_b \xi_a = 0, \quad (56)$$

or

$$\nabla_a \xi_b = \nabla_{[a} \xi_{b]}. \quad (57)$$

The vector field  $\xi_a$  is called a *Killing vector field* and it represents an *isometry* of the spacetime. An  $n$ -dimensional spacetime can have at most

$n(n+1)/2$  independent Killing vectors. By differentiating the above equation it can be shown that Killing vectors satisfy the equation [46]

$$\nabla_a \nabla_b \xi_c = R_{bcad} \xi^d. \quad (58)$$

It follows from this equation that the Lie derivative with respect to Killing vectors commutes with the covariant derivative [48],

$$\mathcal{L}_\xi \nabla_a T^{b_1 \dots b_i}_{c_1 \dots c_j} = \nabla_a \mathcal{L}_\xi T^{b_1 \dots b_i}_{c_1 \dots c_j}. \quad (59)$$

Further differentiation of (58) shows that the Riemann tensor has a vanishing Lie derivative [37],

$$\mathcal{L}_\xi R_{abcd} = 0, \quad (60)$$

and this implies that tensors derived from the Riemann tensor, the metric, and covariant derivatives also have vanishing Lie derivatives with respect to Killing vectors. For example, the Ricci tensor and the Ricci scalar are Lie derived and by Einstein's equations this also applies to the energy-momentum tensor,

$$\mathcal{L}_\xi T_{ab} = 0. \quad (61)$$

We consider the full set of  $r$  solutions to (56),  $\xi_{(i)a}$ . The Lie bracket of two Killing vectors is another vector,  $v_a$ ,<sup>5</sup>

$$\mathcal{L}_{\xi_{(i)}} \xi_{(j)b} = [\xi_{(i)}, \xi_{(j)}]_b = \xi_{(i)}^a \nabla_a \xi_{(j)b} - \xi_{(j)}^a \nabla_a \xi_{(i)b} = v_b. \quad (62)$$

Since the commutator of Lie derivatives of two vectors equals the Lie derivative of the commutator, it follows that the commutator is also a Killing vector [20],

$$\mathcal{L}_v g_{ab} = \mathcal{L}_{[\xi_{(i)}, \xi_{(j)}]} g_{ab} = (\mathcal{L}_{\xi_{(i)}} \mathcal{L}_{\xi_{(j)}} - \mathcal{L}_{\xi_{(j)}} \mathcal{L}_{\xi_{(i)}}) g_{ab} = 0. \quad (63)$$

Hence, the Lie bracket of two Killing vectors must be a linear combination of the Killing vectors of the spacetime [20],

$$[\xi_{(i)}, \xi_{(j)}]_b = C_{(i)(j)}^{(k)} \xi_{(k)b}. \quad (64)$$

The full set of solutions to (56) thus defines a Lie algebra and the  $r$  Killing vectors are generators of an  $r$ -parameter Lie group  $G_r$  —the isometry group of the spacetime. The group is Abelian if all the structure constants  $C_{(i)(j)}^{(k)}$  are zero. The orbit of  $G_r$  through a point  $p \in \mathcal{M}$  is the set of points  $\mathcal{O}_p$  that can be reached by applying the group transformations to  $p$ . The group is said to be transitive on its orbits, and also to be transitive if  $\mathcal{O}_p = \mathcal{M}$ , or intransitive if  $\mathcal{O}_p \neq \mathcal{M}$ .

---

<sup>5</sup>Strictly speaking, the Lie bracket is defined for contravariant vectors, but we don't need to make that distinction here, since for Killing vectors, the Lie derivative commutes with metric contractions.

If the (sub-) group  $G_r$  has  $r \leq n - 1$  parameters, then it is said to be orthogonally transitive if the  $r$ -surfaces generated by the orbits of the group are orthogonal to a family of  $(n - r)$ -surfaces. By Frobenius's theorem, the surface element  $w_{a_1 \dots a_r} = \xi_{(1)[a_1} \cdots \xi_{(r)a_r]}$  then satisfies [37]

$$w_{a_1 \dots a_{r-1}[a_r} \nabla_b w_{c_1 \dots c_r]} = 0. \quad (65)$$

In papers 2 [22] and 4 [23], we consider orthogonally transitive groups  $G_1$  and  $G_2$ , respectively. In the case of an orthogonally transitive  $G_1$  we say that the Killing vector  $\xi_a$  is hypersurface orthogonal and it satisfies

$$\xi_{[a} \nabla_b \xi_{c]} = 0. \quad (66)$$

If the Killing vector is also timelike, then the spacetime is said to be static.

In the case of an orthogonally transitive  $G_2$ , the two Killing vectors  $\xi_a$  and  $\eta_a$  satisfy

$$\xi_{[a} \eta_b \nabla_c \xi_{d]} = 0, \quad \xi_{[a} \eta_b \nabla_c \eta_{d]} = 0. \quad (67)$$

We will further assume that this group is Abelian and that the surfaces of transitivity are non-null. This is satisfied by a large number of cylindrically symmetric and stationary axisymmetric spacetimes. By the first assumption, the Killing vectors commute and it is possible to introduce coordinates of which the metric components are independent. The second assumption implies that  $\xi_{[a} \eta_b] \xi^a \eta^b \neq 0$ , so it is possible to split the identity  $\delta_b^a$  into projections onto the surfaces of transitivity and onto the orthogonal surfaces,  $\delta_b^a = P_{\parallel b}^a + P_{\perp b}^a$  [31].

By taking divergences of equations (66) and (67), and using (58), we see that the Ricci tensor satisfies

$$\xi_{[a} R_{b]c} \xi^c = 0, \quad (68)$$

for a hypersurface orthogonal Killing vector, and

$$\xi_{[a} \eta_b R_{c]d} \xi^d = 0, \quad \xi_{[a} \eta_b R_{c]d} \eta^d = 0, \quad (69)$$

for two Killing vectors that generate an orthogonally transitive Abelian isometry group [14]. By Einstein's equations this also holds for the energy-momentum tensor  $T_{ab}$ , that is, the matter currents generated from the Killing vectors lie in the orbits of the groups,

$$T_{ab} \xi^b = \omega \xi_a, \quad (70)$$

and

$$T_{ab} \xi^b = \alpha \xi_a + \beta \eta_a, \quad T_{ab} \eta^b = \gamma \xi_a + \delta \eta_a, \quad (71)$$

respectively.

Inspired by the properties of the Ricci tensor under these symmetries, Lazkoz, Senovilla, and Vera [31] showed that the Bel tensor generates currents with similar properties. In the first case, we have the current

$$B_{abcd}\xi^b\xi^c\xi^d = \omega\xi_a, \quad (72)$$

and in the second case, the four currents

$$B_{a(bcd)}\xi^{Ib}\xi^{Jc}\xi^{Kd} = \omega_{IJK}\xi_a + \Omega_{IJK}\eta_a, \quad (73)$$

where  $\xi_a^1 = \xi_a$  and  $\xi_a^2 = \eta_a$ .

## 4.2 Symmetry of the electromagnetic field

Even though the energy-momentum tensor in Einstein–Maxwell spacetimes has a vanishing Lie derivative with respect to a Killing vector, it does not follow that this also applies to the electromagnetic field itself. In the general situation we have that [33, 45]

$$\mathcal{L}_\xi F_{ab} = \xi^c\nabla_c F_{ab} + F_{cb}\nabla_a\xi^c + F_{ac}\nabla_b\xi^c = \Psi^* F_{ab}, \quad (74)$$

where  $\Psi$  is a constant in the case of a non-null electromagnetic field and for a null electromagnetic field it satisfies  $l_{[a}\nabla_{b]}\Psi = 0$ , where  $l_a$  is the repeated principal null direction of the field. If  $\Psi = 0$ , the electromagnetic field is said to inherit the symmetry of the spacetime. Generally, there is a different  $\Psi$  for each Killing vector.

If the spacetime has a 2-parameter Abelian isometry group  $G_2$  that acts orthogonally transitive on non-null surfaces, then if the electromagnetic field is non-null, it will inherit the symmetries of this group. If we assume that the electromagnetic field is source-free, and that null fields also inherit the symmetries, then the two scalars  $F_{ab}\xi^a\eta^b$  and  ${}^*F_{ab}\xi^a\eta^b$ , where  $\xi_a$  and  $\eta_b$  are the generators of the group, will be constants [14]. If, for example, one of the Killing vectors vanishes at a point, as is the case of a generator of axisymmetry, these scalars will vanish and we have that

$$F_{ab}\xi^a\eta^b = 0 = {}^*F_{ab}\xi^a\eta^b. \quad (75)$$

For a hypersurface orthogonal Killing vector  $\Psi$  can be nonzero. If the electromagnetic field is non-null and the spacetime admits two hypersurface orthogonal Killing vectors, then the electromagnetic field inherits all symmetries of the spacetime [33].

### 4.3 Conservation laws

If  $\mathcal{N}$  is a compact  $n$ -dimensional submanifold of  $\mathcal{M}$  (also  $n$ -dimensional) and if the vector field  $J_a$  is defined on all of  $\mathcal{N}$ , then by Gauss's theorem [46],

$$\int_{\partial\mathcal{N}} J_a n^a d^{(n-1)}\sigma = \int_{\mathcal{N}} \nabla^a J_a d^n\sigma, \quad (76)$$

where  $d^{(n-1)}\sigma$  and  $d^n\sigma$  are the natural volume forms on  $\partial\mathcal{N}$  and  $\mathcal{N}$ , respectively, and  $n^a$  is the unit normal (with proper orientation) to  $\partial\mathcal{N}$ . Hence, if  $J_a$  is divergence-free, then the total flux over  $\partial\mathcal{N}$  is zero. One may then talk about conserved quantities related to the vector in the sense that the flow over one spacelike hypersurface is equal to the flow over another spacelike hypersurface.

This of course happens when a region, in which  $J_a$  is defined, is bounded by a compact spacelike hypersurface  $\mathcal{S}$  which is divided into two parts,  $\mathcal{S}_1$  and  $\mathcal{S}_2$ , so that the interior of  $\mathcal{S}_1$  is in the past of  $\mathcal{S}_2$ . Then the flow over  $\mathcal{S}_1$  is equal to that over  $\mathcal{S}_2$ . This can also happen in more general situations when the bounding surface  $\mathcal{S}$  has components which are timelike or null, and the flows over these components vanish.

If we have a divergence-free tensor, then we can, in general, derive no conserved quantities from it. However, if the spacetime contains a Killing vector we can construct a conserved current from the tensor. Suppose that we have a symmetric rank-2 tensor,  $T_{ab} = T_{(ab)}$ , then if either the tensor is divergence-free,  $\nabla^a T_{ab} = 0$ , or its divergence is orthogonal to the Killing vector, then the current  $J_a = T_{ab}\xi^b$  is divergence-free,

$$\nabla^a J_a = \nabla^a(T_{ab}\xi^b) = \xi^b\nabla^a T_{ab} + T_{ab}\nabla^a\xi^a = 0, \quad (77)$$

and  $J_a$  can thus be used to create conserved quantities related to the tensor and the particular symmetry of the spacetime. Specifically, the energy-momentum tensor of a spacetime will give currents that are related to conserved physical quantities. Similarly, if we have a completely symmetric rank- $r$  tensor  $T_{a_1\dots a_r}$ , whose divergence contracted with  $r-1$  (possibly different) Killing vectors vanishes, then we can again construct an associated conserved current.

The Bel currents (72) and (73) are easily seen to be divergence-free: The Bel tensor is quadratic in the Riemann tensor and thus has a vanishing Lie derivative with respect to a Killing vector. Since the isometry groups under consideration are Abelian, the Lie derivative of the currents with respect to a member of the group vanish, e.g.,

$$\mathcal{L}_\eta B_{a(bcd)}\xi^b\xi^c\eta^d = 0 = \xi_a\mathcal{L}_\eta\omega + \eta_a\mathcal{L}_\eta\Omega. \quad (78)$$

We see that the proportionality factors in the currents are Lie derived and the currents are thus divergence-free [31],

$$\nabla^a(B_{abcd}\xi^b\xi^c\xi^d) = 0, \quad (79)$$

and

$$\nabla^a (B_{a(bcd)} \xi^{Ib} \xi^{Jc} \xi^{Kd}) = 0, \quad (80)$$

respectively.

#### 4.4 Summary of papers 2 and 4

We prove in papers 2 [22] and 4 [23], that the Chevreton tensor in Einstein–Maxwell theory generates currents with similar properties to those of the Bel tensor when the spacetimes admit a hypersurface orthogonal Killing vector or an Abelian 2-parameter group that acts orthogonally transitive on non-null surfaces. This holds for a source-free electromagnetic field that inherits the symmetries of the spacetime and it seems to be restricted to four dimensions.

For a hypersurface orthogonal Killing vector, we have that (theorem 6, paper 2)

$$H_{abcd} \xi^b \xi^c \xi^d = \omega \xi_a, \quad \nabla^a (H_{abcd} \xi^b \xi^c \xi^d) = 0, \quad (81)$$

and for two commutative Killing vectors,  $\xi_a$  and  $\eta_a$ , that act orthogonally transitive on non-null surfaces, we have that (theorem 7, paper 4)

$$\begin{aligned} H_{abcd} \xi^b \xi^c \xi^d &= \omega_1 \xi_a + \Omega_1 \eta_a, & \nabla^a (H_{abcd} \xi^b \xi^c \xi^d) &= 0, \\ H_{abcd} \xi^b \xi^c \eta^d &= \omega_2 \xi_a + \Omega_2 \eta_a, & \nabla^a (H_{abcd} \xi^b \xi^c \eta^d) &= 0, \\ H_{abcd} \xi^b \eta^c \eta^d &= \omega_3 \xi_a + \Omega_3 \eta_a, & \nabla^a (H_{abcd} \xi^b \eta^c \eta^d) &= 0, \\ H_{abcd} \eta^b \eta^c \eta^d &= \omega_4 \xi_a + \Omega_4 \eta_a, & \nabla^a (H_{abcd} \eta^b \eta^c \eta^d) &= 0. \end{aligned} \quad (82)$$

We also prove that the trace of the Chevreton tensor has currents that lie in the orbits of the groups (theorem 3, paper 2)

$$H_{ab} \xi^b = \omega \xi_a, \quad (83)$$

and (theorem 3, paper 4)

$$H_{ab} \xi^b = \omega_5 \xi_a + \Omega_5 \eta_a, \quad H_{ab} \eta^b = \omega_6 \xi_a + \Omega_6 \eta_a, \quad (84)$$

respectively. These currents are trivially conserved, since the trace is divergence-free. We show in [8] that the Bach tensor [1] is related to the trace of the Chevreton tensor and the energy-momentum tensor (see also next section and equation (97)), so as a corollary we also have that (corollary 4, paper 4)

$$B_{ab} \xi^b = \omega \xi_a, \quad (85)$$

and

$$B_{ab} \xi^b = \omega_7 \xi_a + \Omega_7 \eta_a, \quad B_{ab} \eta^b = \omega_8 \xi_a + \Omega_8 \eta_a, \quad (86)$$

respectively.

Senovilla [38] has shown that for Einstein–Klein–Gordon spacetimes, the Bel tensor together with the superenergy tensor of the scalar field gives a conserved current when there is a Killing vector present. Hence, since the Bel tensor is independently conserved for the two isometry groups under consideration, it follows that the corresponding superenergy currents of the scalar field are also conserved. We prove, for completeness, in [22] and [23], that these currents also lie in the orbits of the groups (theorem 1, paper 2)

$$S_{abcd}\xi^b\xi^c\xi^d = \omega\xi_a, \quad (87)$$

and (theorem 1, paper 4)

$$S_{abcd}\xi^{Ib}\xi^{Jc}\xi^{Kd} = \omega_{IJK}\xi_a + \Omega_{IJK}\eta_a, \quad (88)$$

respectively, where  $S_{abcd}$  is the superenergy tensor of the scalar field.

In paper 4 [23], we give an example of an Einstein–Maxwell spacetime with a 3-parameter isometry group which contains an Abelian 2-parameter subgroup that acts orthogonally transitive on non-null surfaces. The Chevreton currents constructed from this subgroup are of course independently conserved, but this is not true for any of the currents involving the third Killing vector. However, in this example, by combining the Chevreton tensor with the symmetrized Bel tensor we get conserved currents for the full 3-parameter group,

$$\nabla^a \left( \left( B_{a(bcd)} + \frac{1}{3}H_{abcd} \right) \xi^{Ib}\xi^{Jc}\xi^{Kd} \right) = 0. \quad (89)$$

This example, and the fact that the Bel tensor and the Chevreton tensor give independently conserved currents for the  $G_1$  and  $G_2$  isometry groups treated here, give support to the possibility of proving a general mixed conservation law for the gravitational and electromagnetic fields.

## 5 Conformal Einstein spaces

In this section, we consider the problem of finding spacetimes that are conformally related to Einstein spaces. We review some results on necessary and sufficient conditions and we give a summary of paper 3 [8], which uses the Chevreton tensor as a tool in finding Einstein–Maxwell spacetimes that are conformally related to Einstein spaces.

## 5.1 Einstein spaces and conformal transformations

A spacetime is said to be *flat* if  $R_{abcd} = 0$ , it is *Ricci-flat* if  $R_{ab} = 0$ , it is said to be an *Einstein space* if  $R_{ab} - \frac{1}{n}Rg_{ab} = 0$ , and a *C-space* if  $\nabla^a C_{abcd} = 0$ . We have the following hierarchy [29]:

$$\text{flat} \subset \text{Ricci-flat} \subset \text{Einstein space} \subset C\text{-space.}$$

A conformal transformation of the spacetime gives a new spacetime with metric  $\hat{g}_{ab}$  by

$$g_{ab} \mapsto \hat{g}_{ab} = e^{2\Omega} g_{ab}, \quad (90)$$

where  $\Omega$  is called the conformal factor. The curvature tensors transform as [46]

$$\begin{aligned} \hat{R}_{ab} &= R_{ab} + (n-2)\nabla_a\nabla_b\Omega + g_{ab}\nabla^c\nabla_c\Omega \\ &\quad - (n-2)\nabla_a\Omega\nabla_b\Omega + (n-2)g_{ab}\nabla^c\Omega\nabla_c\Omega, \\ \hat{R} &= e^{-2\Omega}(R + 2(n-1)\nabla^a\nabla_a\Omega + (n-2)(n-1)\nabla^a\Omega\nabla_a\Omega), \\ \hat{C}_{abcd} &= e^{2\Omega}C_{abcd}. \end{aligned} \quad (91)$$

We say that a tensor that transforms as  $\hat{T} = e^{k\Omega}T$  is *conformally invariant of weight  $k$* . Note that the Weyl tensor is conformally well-behaved with conformal weight 2.

A spacetime is said to be a conformal Einstein space if it can be conformally transformed spacetime into an Einstein space, i.e.,

$$\hat{R}_{ab} - \frac{1}{n}\hat{R}\hat{g}_{ab} = 0. \quad (92)$$

Likewise a spacetime is said to be conformally flat, conformally Ricci-flat, or a conformal *C-space*.

An interesting property of conformal Einstein spaces is that their null cone structures are the same as that of their corresponding Einstein spaces.

## 5.2 Necessary and sufficient conditions

The study of spacetimes conformally related to Einstein spaces dates back to Brinkmann's work in 1924 [13]. In four dimensions a necessary condition for a spacetime to be conformal to an Einstein space is the vanishing of the conformally invariant Bach tensor [1, 29]

$$B_{ab} = \nabla^c\nabla^d C_{acbd} - \frac{1}{2}R^{cd}C_{acbd}. \quad (93)$$

The problem of finding conditions that are sufficient and useful (i.e., not involving solvability of differential equations) has only quite recently

been solved. Kozameh, Newman, and Tod [29] solved the problem in spacetimes for which the complex invariant  $J = \Psi_{ABCD}\Psi^{CD}{}_{EF}\Psi^{EFAB}$  is non-zero. This excludes spacetimes of Petrov type **N**, **III**, and some particular cases of type **I**. They showed that necessary and sufficient conditions for a spacetime with  $J \neq 0$  to be conformally related to an Einstein space are given by

$$\begin{aligned} B_{ab} &= 0, \\ \nabla^d C_{abcd} + K^d C_{abcd} &= 0, \end{aligned} \tag{94}$$

where  $K^e = -4C^{abce}\nabla^d C_{abcd}/(C^{ijkl}C_{ijkl})$ . The second condition here is a necessary and sufficient condition for the spacetime to be conformal to a  $C$ -space. Hence, for  $C$ -spaces with  $J \neq 0$  the vanishing of the Bach tensor implies that the spacetime is conformal to an Einstein space. For Petrov type **I** spacetimes Wünsch has shown that there are no  $C$ -spaces with  $J = 0$  [47].

For Petrov type **III** spacetimes the problem was solved by Wünsch. The conditions for a conformal  $C$ -space are more involved, and for a  $C$ -space to be conformal to an Einstein space the vanishing of the Bach tensor has to be supplemented by properties of a certain conformally invariant rank-4 tensor,  $N_{abcd}$  [47].

The problem of finding the necessary and sufficient conditions for a Petrov type **N** spacetime to be conformal to an Einstein space has not yet been solved. Czapor, McLenaghan, and Wünsch [17] have solved the problem for conformally Ricci-flat spacetimes. The conditions for conformal  $C$ -spaces are more complicated than those for Petrov type **III**, and for a  $C$ -space to be conformally Ricci-flat the vanishing of the Bach tensor,  $N_{abcd}$ , and a certain conformally invariant rank-6 tensor is necessary and sufficient [17].

One of the problems with Petrov type **N** seems to be that the equation for conformal  $C$ -spaces does not guarantee a unique solution for the conformal factor [29, 42].

For the generic  $n$ -dimensional case with non-degenerate Weyl tensor sufficient conditions have been given by Listing [32] and Gover and Nurowski [26].

### 5.3 Summary of paper 3

In paper 3 [8], we study the source-free Einstein–Maxwell spacetimes for which the Chevreton tensor is trace-free. This gives an opportunity to study the open problem of source-free Einstein–Maxwell spacetimes that are conformal to Einstein spaces. The trace-free condition is given by the equation

$$\nabla_{CC'}\varphi_{AB}\nabla^{CC'}\bar{\varphi}_{A'B'} = 0, \tag{95}$$

with the simple solution

$$\nabla_{CC'}\varphi_{AB} = \eta o_C o_{C'} o_A o_B. \quad (96)$$

From this expression we prove that (theorems 1, 2, and 3):

- The null vector  $l_a = o_A o_{A'}$  generates a shear-free geodetic null congruence.
- The Chevreton tensor is of pure-radiation type,  $H_{abcd} = 4\eta\bar{\eta}l_a l_b l_c l_d$ .
- The spacetime is of Petrov type **N** or **O**, and in the former case  $l_a$  coincides with the principal null direction of the Weyl tensor.
- The cosmological constant,  $\lambda$ , is zero.

In addition, we get a number of equations in terms of spin-coefficients that characterize the spacetimes with a trace-free Chevreton tensor.

We prove that the Bach tensor and the trace of the Chevreton tensor are connected (theorem 4),

$$B_{ab} = 2H_{ab} + \frac{2}{3}\lambda T_{ab}, \quad (97)$$

where  $T_{ab}$  is the energy-momentum tensor and  $\lambda$  is the cosmological constant. Hence, for source-free Einstein–Maxwell spacetimes with a zero cosmological constant, the vanishing of the Bach tensor is equivalent to the Chevreton tensor being trace-free. We determine all Einstein–Maxwell spacetimes with a vanishing Bach tensor and search for conformal Einstein spaces among them. There are five classes of exact solutions to consider.

There are two Petrov type **O** solutions. In the first case we have the Bertotti–Robinson solution which has a covariantly constant electromagnetic field,  $\nabla_a F_{bc} = 0$ , and in the second case we have a null electromagnetic field. Both solutions are conformally flat, as are all Petrov type **O** solutions [37].

The Petrov type **N** solutions are divided into three different cases depending on the principal null directions of the electromagnetic field and the Weyl tensor.

In the first case, the electromagnetic field is non-null and there are no coincidences of the principal null directions with that of the Weyl tensor. These solutions are given in [43] and all of them are conformal  $C$ -spaces with a zero Bach tensor but none of them are conformal to an Einstein space.

In the second case, one of the principal null directions of the non-null electromagnetic field coincides with the one of the Weyl tensor. The general solutions are given in [43]. The set of solutions with a vanishing Bach tensor constitutes a subset of this class of solutions and corresponds to the case when both of the two principal null directions are shear-free.

These spaces are generally not even conformal  $C$ -spaces, but they contain a subset of solutions which are conformal to Einstein spaces. We are not able to determine the whole set of conformal Einstein spaces, but we do find an explicit example. The metric is given by

$$ds^2 = 2(\Phi_{11}r^2 - h(u)x)du^2 + 2dudr - \frac{1}{2P(y)^2}(dx^2 + dy^2), \quad (98)$$

with

$$P(y) = \frac{\Phi_{11} + e^{2C_1y+2C_2}}{2C_1e^{C_1y+C_2}}, \quad (99)$$

where  $h(u)$  is an arbitrary real function and  $C_1$  and  $C_2$  are constants. The conformal factor

$$\Omega(y) = \ln \left( \frac{\Phi_{11} + e^{2C_1y+2C_2}}{\Phi_{11} - e^{2C_1y+2C_2}} \right), \quad (100)$$

transforms the spacetime into an Einstein space. This solution is a proper conformal Einstein space, i.e., it is not conformally Ricci-flat. To the best of our knowledge, this is the first example of such an Einstein–Maxwell spacetime.

In the last case, we have a null electromagnetic field whose principal null direction coincides with that of the Weyl tensor. The solutions with a vanishing Bach tensor belong to a subset of the family of  $pp$  waves. They were found by Van den Bergh [44] and they are all conformally Ricci-flat.

Finally, we note that in the case when the Bach tensor vanishes but the cosmological constant is non-zero, the equation (97) is much harder to analyze.

## 6 Conclusion

We have seen that Chevreton’s superenergy tensor has many interesting properties in Einstein–Maxwell spacetimes. The general results on conservation laws in this thesis have all been generated from the Chevreton tensor alone. The first question that comes to mind is: What exactly are those conserved quantities that are derived from the conserved currents, i.e., what are their physical interpretations? This is of course related to the problem of the physical interpretation of the Chevreton tensor itself. The second question concerns mixed conserved quantities. Is there always a conservation of superenergy between the gravitational field and the matter fields? It is desirable to have a proof of a general conserved mixed current between the superenergy of the gravitational field and the superenergy electromagnetic field, as in the case of the Einstein–Klein–Gordon spacetimes.

The physical interpretation could also be strengthened by performing an orthogonal splitting of the Chevreton tensor to see if one could get

results similar to the Poynting theorem for electromagnetism or the characterization of intrinsic superenergy radiative states for the gravitational field [4, 25].

In [39], for the case of null electromagnetic fields a simple expression for the trace of the Chevreton tensor was derived that contained only the principal null direction of the field and its covariant derivatives. Finding such an expression for the Chevreton tensor itself might prove useful.

Related to this is the subject of classification of spacetimes. We have seen that a trace-free Chevreton tensor assumes a particularly simple form and that it also restricts the spacetime to the class with a vanishing Bach tensor. It would be interesting to see in what other ways the Chevreton tensor can be used in classifications.

Finding all Einstein–Maxwell spacetimes that are conformal to Einstein spaces is another interesting and challenging task.

Hence, there is still more to do!

## References

- [1] Bach R, *Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs*, Mathematische Zeitschrift **9** (1921), 110–135.
- [2] Bel L, *Sur la radiation gravitationnelle*, Les Comptes Rendus de l’Académie des Sciences de Paris **247** (1958), 1094–1096.
- [3] Bel L, *Introduction d’un tenseur de quatrième ordre*, Les Comptes Rendus de l’Académie des Sciences de Paris **248** (1959), 1297–1300.
- [4] Bel L, *Les états de radiation et le problème de l’énergie en relativité générale*, Cahiers de Physique **16** (1962), 59–80 (English translation: *General Relativity and Gravitation* **32** (2000), 2047–2078).
- [5] Bergqvist G, *Quasilocal mass for event horizons*, Classical and Quantum Gravity **9** (1992), 1753–1768.
- [6] Bergqvist G, *Positivity properties of the Bel–Robinson tensor*, Journal of Mathematical Physics **39** (1998), 2141–2147.
- [7] Bergqvist G, *Positivity of general superenergy tensors*, Communications in Mathematical Physics **207** (1999), 467–479.
- [8] Bergqvist G and Eriksson I, *The Chevreton tensor and Einstein–Maxwell spacetimes conformal to Einstein spaces*, Classical and Quantum Gravity **24** (2007), 3437–3455.
- [9] Bergqvist G, Eriksson I, and Senovilla J M M, *New electromagnetic conservation laws*, Classical and Quantum Gravity **20** (2003), 2663–2668.

- [10] Bergqvist G and Lankinen P, *Algebraic and differential Rainich conditions for symmetric trace-free tensors of higher rank*, Proceedings of the Royal Society A **461** (2005), 2181–2195.
- [11] Bergqvist G and Senovilla J M M, *On the causal propagation of fields*, Classical and Quantum Gravity **16** (1999), L55–L61.
- [12] Bonilla M Á G and Senovilla J M M, *Very simple proof of the causal propagation of gravity in vacuum*, Physical Review Letters **78** (1997), 783–786.
- [13] Brinkmann H W, *Riemann spaces conformal to Einstein spaces*, Mathematische Annalen **91** (1924), 269–278.
- [14] Carter B, *Killing horizons and orthogonally transitive groups in space-time*, Journal of Mathematical Physics **10** (1969), 70–81.
- [15] Chevreton M, *Sur le tenseur de superénergie du champ électromagnétique*, Il Nuovo Cimento **34** (1964), 901–913.
- [16] Christodoulou D and Klainerman S, *The global nonlinear stability of the Minkowski space*, Princeton University Press, 1993.
- [17] Czapor S R, McLenaghan R G, and Wunsch V, *Conformal C and empty spaces of Petrov type N*, General Relativity and Gravitation **34** (2002), 385–402.
- [18] Deser S, *A note on matter superenergy tensors*, Classical and Quantum Gravity **20** (2003), L213–L215.
- [19] Edgar S B, *On the structure of the new electromagnetic conservation laws*, Classical and Quantum Gravity **21** (2004), L21–L25.
- [20] Eisenhart L P, *Continuous groups of transformations*, Dover Publications, 1961.
- [21] Eriksson I, *The Chevreton tensor and its trace*, Licentiate Thesis, Linköping University, 2005.
- [22] Eriksson I, *Conserved matter superenergy currents for hypersurface orthogonal Killing vectors*, Classical and Quantum Gravity **23** (2006), 2279–2290.
- [23] Eriksson I, *Conserved matter superenergy currents for orthogonally transitive Abelian  $G_2$  isometry groups*, Classical and Quantum Gravity **24** (2007), 4955–4968.
- [24] Frankel T, *The geometry of physics*, Cambridge University Press, 1997.
- [25] García-Parrado A, *Dynamical laws of superenergy in General Relativity*, arXiv:0707.1475 (2007).

- [26] Gover A R and Nurowski P, *Obstructions to conformally Einstein metrics in  $n$  dimensions*, Journal of Geometry and Physics **56** (2006), 450–484.
- [27] Hawking S W, *Gravitational radiation in an expanding universe*, Journal of Mathematical Physics **9** (1968), 598–604.
- [28] Horowitz G T and Schmidt B G, *Note on gravitational energy*, Proceedings of the Royal Society of London A **381** (1982), 215–224.
- [29] Kozameh C N, Newman E T, and Tod K P, *Conformal Einstein spaces*, General Relativity and Gravitation **17** (1985), 343–352.
- [30] Lankinen P, *Spinors, Clifford algebras and superenergy tensors*, PhD thesis, Mälardalen University, 2004.
- [31] Lazkoz R, Senovilla J M M, and Vera R, *Conserved superenergy currents*, Classical and Quantum Gravity, **20** (2003), 4135–4152.
- [32] Listing M, *Conformal Einstein spaces in  $N$ -dimensions*, Annals of Global Analysis and Geometry **20** (2001), 183–197.
- [33] Michalski H and Wainwright J, *Killing vector fields and the Einstein–Maxwell field equations in General Relativity*, General Relativity and Gravitation **6** (1975), 289–318.
- [34] Misner C W, Thorne K S, and Wheeler J A, *Gravitation*, W H Freeman and Company, 1973.
- [35] Penrose R and Rindler W, *Spinors and spacetime vol 1*, Cambridge University Press, 1984.
- [36] Penrose R and Rindler W, *Spinors and spacetime vol 2*, Cambridge University Press, 1986.
- [37] Schouten J A, *Ricci-calculus*, Springer-Verlag, 1954.
- [38] Senovilla J M M, *Superenergy tensors*, Classical and Quantum Gravity **17** (2000), 2799–2841.
- [39] Senovilla J M M, *New conservation laws for electromagnetic fields in gravity*, Symmetries and gravity in field theory, Workshop in honour of Prof. J. A. de Azcárraga, June 9–11, 2003, Salamanca (Spain) (V Aldaya and J M Cerveró, eds.), Ediciones Universidad de Salamanca, Salamanca, 2004. gr-qc/0311033.
- [40] Senovilla J M M, *The universal ‘energy’ operator*, Classical and Quantum Gravity **23** (2006), 7143–7147.

- [41] Senovilla J M M, *Symmetric hyperbolic systems for a large class of fields in arbitrary dimension*, General Relativity and Gravitation **39** (2007), 361–386.
- [42] Szekeres P, *Spaces conformal to a class of spaces in general relativity*, Proceedings of the Royal Society of London A **274** (1963), 206–211.
- [43] Szekeres P, *On the propagation of gravitational fields in matter*, Journal of Mathematical Physics **7** (1966), 751–761.
- [44] Van den Bergh N, *Conformally Ricci flat Einstein-Maxwell solutions with a null electromagnetic field*, General Relativity and Gravitation **18** (1986), 1105–1110.
- [45] Wainwright J and Yaremowicz P A E, *Symmetries and the Einstein-Maxwell field equations — The null field case*, General Relativity and Gravitation **7** (1976), 595–608.
- [46] Wald R M, *General Relativity*, University of Chicago Press, 1984.
- [47] Wunsch V, *Conformal C- and Einstein spaces*, Mathematische Nachrichten **146** (1990), 237–245.
- [48] Yano K, *The theory of Lie derivatives and its applications*, North-Holland Publishing Company, 1955.