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*Computational Models of
Adhesively Bonded Joints*

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I dedicate this thesis to my daughter Nathalie

Preface

The work presented herein constitutes a thesis for the degree of Teknologie Doktor (PhD) at the Division of Mechanics, Department of Management and Engineering, Linköpings universitet. I would like to thank my supervisor Professor Anders Klarbring for introducing me into the field of computational mechanics. I am also very grateful to Dr Ulf Edlund for his guidance and encouragement during this work. They both possesses an enormous insight into the field and have inspired me to keep on and finishing the work. The author is also grateful for the financial support from Swedish National Aeronautical Research Programme (NFFP). Thanks also to Dr Jan Åslund and Dr Carl-Gustaf Aronsson for keeping the spirit up. Last but not least, I would like to thank my parents, my friends and colleagues, and my family Panom and Nathalie for their help and support.

Linköping in January 2007,

Peter Schmidt

Abstract

Simulations using the Finite Element Method (FEM) play an increasingly important role in the design process of joints and fasteners in the aerospace industry. In order to utilize the potential of such adhesive bonding, there is an increasing need for effective and accurate computational methods. The geometry and the nature of an adhesive joint are, however, not so simple to describe effectively using standard FEM-codes. To overcome this difficulty, special FEM-elements can be developed that provide a material surface treatment of the adhesive and the joined parts. In order to create a model that reflects the above features, one may introduce proper scalings on the geometry and on the material properties in terms of a perturbation parameter. Within the framework of three-dimensional elasticity, together with an asymptotic expansion method, a material surface model is obtained through a systematic procedure. In such a derivation, no *a priori* assumptions for the displacements or stress fields are needed. The final result is a variational equation posed over a single reference surface which forms the basis of a structural element for the compound joint.

Through the usage of continuum damage mechanics and the framework of a generalized standard material, the linear elastic model is extended to include an elastic-plastic material model with damage for the adhesive. The model is FE-discretized and an important implication is that the (quasi-static) propagation of the local failure zone in the adhesive layer can be simulated. The failure load is obtained as a computational result and consequently *no postulated failure criterion is needed*. The derived FE-method opens up the possibility to efficiently model and analyze the mechanical behavior of large bonded structures.

Dissertation

This dissertation consists of an introduction and the following four papers:

- Paper I : P. Schmidt, Modelling of Adhesively Bonded Joints–I. Asymptotic Analysis, *submitted*.
- Paper II : P. Schmidt, Modelling of Adhesively Bonded Joints–II. Numerical Results and a Proposed Structural Model, *submitted*.
- Paper III : P. Schmidt and U. Edlund, Analysis of Adhesively Bonded Joints; A Finite Element Method and a Material Model with Damage, *International Journal for Numerical Methods in Engineering 2006*; **66**:1271–1308.
- Paper IV : P. Schmidt and U. Edlund, Analysis of Adhesively Bonded Structures; A Finite Element Method and Failure Simulations using a Material Model with Damage, *submitted*.

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Appended papers

- Paper I : Modelling of Adhesively Bonded Joints–I.
Asymptotic Analysis.
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Numerical Results and a Proposed Structural Model.
- Paper III : Analysis of Adhesively Bonded Joints;
A Finite Element Method and a Material Model
with Damage.
- Paper IV : Analysis of Adhesively Bonded Structures;
A Finite Element Method and Failure Simulations using
a Material Model with Damage.

1 Introduction

When constructing machines and vehicles of any kind there is always a need to join different parts together. Besides the function itself, the strength, or the ability to resist loads, is then a factor of vital importance. These structures usually need to satisfy a number of criteria such as lightness, strength and stiffness. These key features are often present in the automobile and aerospace industry, usually together with safety aspects. In these applications, there are several examples of structures where a load is transferred from one structural detail to another through an adhesive layer. In an aircraft structure, for instance, stiffener and stringer fastenings are well established examples where adhesive bonding is frequently used. In such lightweight structures, adhesive bonding is a good complement to traditional bonding techniques (bolting, riveting), and can in many cases replace these traditional techniques and the complications due to drilled holes can therefore be avoided. Adhesive bonding is used in metal to metal joints (single and multi-layered structures, lap-joints, T-joints and L-shaped joints etc.), but also to join metal to fibre reinforced plastics or composites and sandwich panels [1, 2]. As modern technology moves forward to lighter, stiffer and larger structures, which are to be used in new and more demanding environments, there is an increasing need to quantify the strength of an adhesively bonded structure. This means that both efficient and reliable computational methods need to be developed that can predict initiation and growth of cracks as well as the global failure-load of such structures.

Pioneering work on simple geometries, such as single lap-joints, was performed by Volkersen [3], where closed form solutions were obtained for linear elastic materials using a shear-lag analysis. Goland and Reissner [4], analyzed single- and double lap-joints using a plain strain assumption where bending of the joined parts was incorporated. Although the validity of these models is restricted to linear elastic materials, specific geometries and loading situations, they are important benchmarks in the development of more advanced models. Generalizations of the above models that provide closed form solutions has been developed by Ojalvo and Eidinoff [5] (varying shear stress through the thickness), Hart-Smith [6] (elastic-perfectly plastic adhesive), Delale and Erdogan [7] (viscoelastic adhesive), Ottosen and Olsson [8] and Gustafsson [9] (material softening), see also Tsai and Morton [10], Oplinger [11] and Bigwood and Crocombe [12]. In order to analyze more complex geometries, loading conditions and material behaviors, the usage of numerical methods, such as the Finite Element Method (FEM), is to be applied. However, due to the fact that the adhesive layer is thin, regular FE-techniques are not efficient to use, since they require a large number of 3-D solid elements through the thickness of the adhesive and the adherends (the joined parts) to achieve convergence of the three-dimensional state of stress. In practice, this means that to analyze structures, special finite elements need to be adopted which are more rational to use, but still incorporate a three-dimensional state of stress and strain. Different types of finite elements have been proposed where simplifications have been introduced due to the thinness of the adhesive (Carpenter [13, 14, 15], Kuo [16], Groth [17], Stigh [18],

Reddy and Roy [19], Edlund and Klarbring [20, 21] among others). For further references see the bibliography by Mackerle [22]. In most cases, these models are based on several simplifying *ad hoc* assumptions concerning the displacement field and the stress field in the constituents. Typically, a specific form of the displacement field through the thickness is assumed, and one or more assumptions of the stress/strain components are introduced. Recent FE-models that can be sorted into this category can be found in Andruett [23, 24] and Mortensen [25]. In references [23, 24] it was assumed that the adherends behave as shells and the adhesive was modelled as a 3-D solid using brick elements. To obtain continuity of displacements at the adherend–adhesive interface special interpolation functions were used. In reference [25] the adherends were modelled as orthotropic plates using assumptions as in classical laminate theory. Another approach, which is more systematic, is to use an asymptotic expansion technique. This approach provides a derivation of a model for an adhesive interacting with its joined parts which is consistent in an asymptotic sense as the thicknesses of the constituents tends to zero. Here the displacement field (and/or stress field) is expanded into an asymptotic series in terms of a small perturbation parameter. Through this parameter, scalings are introduced that reflect the thinness of the adhesive and/or the adherends, and the fact that Young’s modulus for the adhesive is a small number. No *a priori* assumptions as regards the displacements and the stress/strain field are needed. As a result, the distributions through the thickness are derived, and the dominating terms can be identified. In practice, this means that the transition from an original 3-D problem to a simpler lower dimensional surface description is found. Depending on the geometrical scalings made on the adherends, the usage of asymptotic techniques to analyze adhesive joints can be sorted into two groups. In the first group, the adherends are considered to be 3D–solids of arbitrary shape, joined together by a thin adhesive with a low Young’s modulus. In this case, scalings are introduced that reflect the thinness and the low modulus of the adhesive. This type of scaling for linear elastic materials was first considered by Klarbring [26], Destuynder et al. [27], Geymonat et al. [28, 29] where a consistent surface description was derived for the adhesive in conjunction with a 3-D solid description for the adherends. Krasucki et al. [30, 31] used the asymptotic technique to derive an FE-method for the above problem in the case of large displacement incorporating Griffith’s fracture energy, and Mishouri [32] studied the influence of a nearly incompressible adhesive material. See also the works by Bigoni et al. [33] and Benveniste and Miloh [34]. The next group consists of problems where not only the adhesive but also the adherends can be regarded as thin solids. In this case, the adherends are considered to be thin planar solids, joined together by an even thinner adhesive with a low Young’s modulus. In this class of problems, both the thickness of the adhesive and the adherends are scaled (as well as the adhesive modulus) in terms of the perturbation parameter. This type of scaling has been studied in a strong form formulation by Klarbring and Movchan [35, 36], Avila-Pozos [37], and through a variational formulation by Schmidt [38], and Åslund [39] (reference [39] considers non-linear elastic plates). The introduction of rotational degrees of freedom, compatible with a three-dimensional state

of stress in the adhesive, and the dominating work conjugate stress and strain measures that are associated with a surface treatment are identified.

If a systematic derivation of a lower dimensional surface model in the linear elastic case concretizes the first foundation toward an efficient computational model, then the next step is to formulate a constitutive law that reflects some of the major material non-linearities present in the adhesive. This can be accomplished by phenomenological considerations using the concepts of Continuum Damage Mechanics (CDM), [89, 90, 91, 95, 96, 97], where the continuous progression of debonding is described by gradual decrease in local strength. The derived continuum model is next transformed to the surface description above replacing the linear elastic material law. This means that the adhesive is modelled as a third part in the joint, with a finite thickness, supporting a three-dimensional state of stress and strain. As in the works by Edlund and Klarbring [43], Edlund [44], Luccioni et al. [45], Ganghoffer et al. [46], Imanaka et al. [47], Edlund and Volgers [48], Sampaio et al. [49], the introduction of internal variables, such as an irreversible strain and a damage variable, the initiation and evolution of a damage zone can be traced. A theory for the adhesive (for both loading and unloading situations), incorporating the complete stress-strain curve, including the softening branch, is derived. As a result, no postulated failure criterion is needed. This differs from the classical *global* approach of fracture mechanics, where it is postulated that fracture occurs when the stress-intensity factor K , or the J -integral (Rice-integral), or Griffith's fracture energy G (energy release rate) reaches a critical value for an existing crack with a given length and location. The usage of CDM to model the adhesive in a continuum context, transformed to a surface description obtained through an asymptotic analysis, also differs in some sense from the approach using interface models. Interface models usually start from a kinematic displacement discontinuity between the bonded parts. In these models, there is no distinct bond line thickness between the adherends, and a material model (cohesive law) for the interface is constructed describing a softening behavior of the traction vector in terms of the displacement jump. The latter is frequently used in the study of delamination of composite materials (e.g. Allix and Ladeveze [50], Corigliano [51], Schellekens and De Borst [52], Crisfield et al. [53], Mi et al. [54], Alfano and Crisfield [55], Corigliano and Ricci [56], Larsson and Jansson [57], Borg [58]). However, both these approaches coincide in the sense that they result in a surface description of the bonding material, and can be sorted into the category of *local* approaches.

2 The adhesive joint problem

As a starting point in this dissertation we consider the three constituents of the joint as three solid bodies, shown in Figure 1, where the adherends occupy the domains Ω^1 and Ω^2 , and the adhesive the domain Ω^0 . The two interfaces are denoted $S^1 := \partial\Omega^1 \cap \partial\Omega^0$ and $S^2 := \partial\Omega^2 \cap \partial\Omega^0$ respectively. The domain for the compound joint is defined as $\Omega^1 \cup \Omega^0 \cup \Omega^2 \cup S^1 \cup S^2$. In the following (if

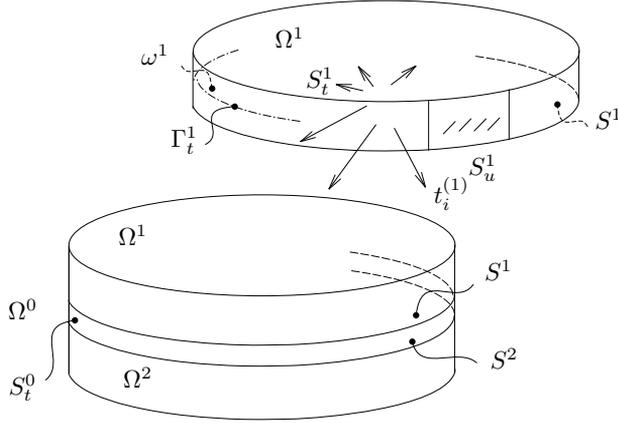


Figure 1: The adhesively bonded joint. Notations associated with the upper adherend are shown separately.

not stated otherwise), the standard summation convention over repeated indices is used, with Latin letters $i, j, k \in \{1, 2, 3\}$ and Greek letters $\alpha, \theta, \gamma \in \{1, 2\}$, the numbers $r \in \{1, 2\}$ and $p \in \{0, 1, 2\}$. The displacements \mathbf{u} are prescribed on the surface S_u^r , and the tractions \mathbf{t} on the surface S_t^r and $\mathbf{t} = \mathbf{0}$ on S_t^0 . The thicknesses for the adherends and adhesive are denoted h_1 , h_2 and h_0 , respectively, and an orthonormal reference frame x_i is used, together with local thickness coordinates $\zeta_p \in [-\frac{h_p}{2}, \frac{h_p}{2}]$, measured from each middle surface ω^p , see Figure 2.

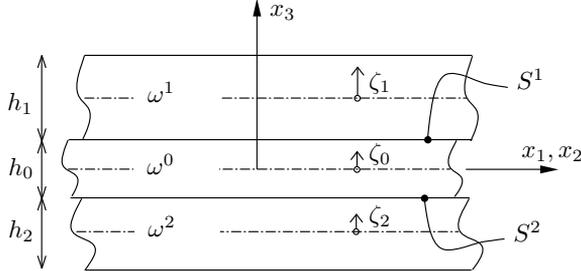


Figure 2: The geometry of the adhesive joint and definitions of local thickness coordinates ζ_0 , ζ_1 , ζ_2 .

If we deal with isotropic linear elastic materials, the constitutive law within each domain reads

$$\sigma_{ij} = 2\mu_p \varepsilon_{ij} + \lambda_p \varepsilon_{kk} \delta_{ij} \quad \text{in } \Omega^p \quad (1)$$

where σ_{ij} is the Cauchy stress, μ_p , λ_p are Lamé parameters, and ε_{ij} are the linear strain components

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$$

where $u_{i,j} := \partial u_i / \partial x_j$ denotes partial differentiation of u_i with respect to the coordinate x_j .

2.1 Formulation in strong form

Besides the constitutive law (1), the governing equations for a compound joint involve equilibrium equations in the interior of each domain, continuity conditions on the interfaces and boundary conditions. In the absence of body forces, the following is a formulation of our problem in strong form (see Appendix A):

Problem (\mathcal{P}^s): Given the applied traction $\mathbf{t} = (t_i)$, find $\mathbf{u} = (u_i)$ such that equilibrium is satisfied

$$\sigma_{ij,j} = 0 \quad \text{in } \Omega^p$$

where

$$\sigma_{ij} = 2\mu_p \varepsilon_{ij} + \lambda_p \varepsilon_{kk} \delta_{ij}, \quad \varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

the continuity conditions on the interfaces¹

$$\begin{aligned} (u_i)_+ &= (u_i)_- & \text{on } S^1 \cup S^2 \\ (\sigma_{3i})_+ &= (\sigma_{3i})_- & \text{on } S^1 \cup S^2 \end{aligned}$$

and the boundary conditions hold

$$\begin{aligned} u_i &= 0 & \text{on } S_u^1 \cup S_u^2 \\ \sigma_{ij} n_j &= t_i & \text{on } S_t^1 \cup S_t^2 \\ \sigma_{ij} n_j &= 0 & \text{on } S_t^0 \end{aligned}$$

2.2 Variational formulation

A formulation that is more suitable for a finite element discretization is the equivalent variational form of problem (\mathcal{P}^s) stated in the following: Let V denote the set of kinematically admissible displacements which are assumed continuous and differentiable as many times as necessary on $\Omega^1 \cup \Omega^0 \cup \Omega^2 \cup S^1 \cup S^2$:

$$V := \{\mathbf{u} = (u_i) \mid \mathbf{u} = \mathbf{0} \text{ on } S_u^1 \cup S_u^2\} \quad (2)$$

Problem (\mathcal{P}^v): Given the applied traction $\mathbf{t} = (t_i)$, find $\mathbf{u} \in V$ such that

$$\int_{\Omega^0 \cup \Omega^1 \cup \Omega^2} \sigma_{ij} \varepsilon_{ij}(\mathbf{v}) d\Omega - \int_{S_t^1 \cup S_t^2} t_i v_i dS = 0 \quad \forall \mathbf{v} \in V \quad (3)$$

and such that the constitutive law (1) is satisfied.

¹The brackets $(\cdot)_+$ and $(\cdot)_-$ mean approaching the interfaces from above and from below respectively.

3 The asymptotic expansion technique

3.1 Perturbation methods

Many problems in applied science and engineering involve an algebraic or a differential equation where one or more coefficients can be regarded as small quantities as compared to other coefficients in the equation. This can be seen by writing the equation in a non-dimensional form. For instance, in slender bodies, such as rods and beams, the thickness to length ratio is a small number as compared to unity. For this class of problems, this ratio can be used as a perturbation parameter (a scalar ϵ) in the formulation of the three-dimensional equations. Through a systematic asymptotic expansion technique, a simpler equation of lower dimension can be obtained by letting the perturbation parameter tend to zero. Typically, we consider equations in the form:

$$L(\mathbf{u}, \epsilon) = 0$$

with the boundary condition

$$B(\mathbf{u}, \epsilon) = 0$$

The solution to this equation is denoted $\mathbf{u}(\mathbf{x}, \epsilon)$, where \mathbf{x} is the independent variable and ϵ is the perturbation parameter (i.e. $\epsilon \ll 1$). The fact that ϵ is small can be used to obtain an approximative solution to the original problem. More readily, one seeks to find a solution for small ϵ in, say, a power series of ϵ , i.e. $\mathbf{u}(\mathbf{x}, \epsilon) = \mathbf{u}^0(\mathbf{x}) + \epsilon\mathbf{u}^1(\mathbf{x}) + \epsilon^2\mathbf{u}^2(\mathbf{x}) + \dots$, where each \mathbf{u}^m is independent of ϵ , and the m^{th} term is a small correction to the $(m-1)^{\text{th}}$ term. If this expansion is substituted into $L(\mathbf{u}, \epsilon) = 0$ and $B(\mathbf{u}, \epsilon) = 0$, and coefficients of each power of ϵ are collected, one finds that each coefficient must vanish independently, since this equation must hold for all values of ϵ . This results in a set of new equations governing \mathbf{u}^m , which are usually simpler, and can be solved successively (recursively). In more general terms, asymptotic expansions can be constructed by using a sequence of functions $\delta_m(\epsilon)$, $m \in \{0, 1, 2, \dots\}$, which are such that $\delta_m(\epsilon) = o(\delta_{m-1}(\epsilon))$ as $\epsilon \rightarrow 0$, see for instance Nayfeh [59] and Hinch [60]. If the components of a scalar or vector field \mathbf{a}^m are independent of ϵ and $\delta_m(\epsilon)$ is an asymptotic sequence, then

$$\mathbf{a} = \sum_{m=0}^{\infty} \delta_m(\epsilon) \mathbf{a}^m(\mathbf{x}) \tag{4}$$

is an asymptotic expansion of the field \mathbf{a} , if and only if

$$\mathbf{a} = \sum_{m=0}^{n-1} \delta_m(\epsilon) \mathbf{a}^m(\mathbf{x}) + O(\delta_n(\epsilon)) \quad \text{as } \epsilon \rightarrow 0$$

for all given $n \in \{1, 2, \dots\}$. The straightforward expansion in (4) is said to be uniformly valid if the term $\delta_m(\epsilon) \mathbf{a}^m(\mathbf{x})$ is small compared to the preceding term $\delta_{m-1}(\epsilon) \mathbf{a}^{m-1}(\mathbf{x})$, for each m , and for all values of \mathbf{x} . Otherwise, the expansion

is said to be nonuniformly valid, see [59] for a more detailed discussion of sources of nonuniformities and how they can be resolved. Since one of the main goals in the current investigation is to obtain an efficient computational model for adhesively bonded structures, the usage of perturbation methods facilitates a systematic way to accomplish such a model. The transition from the original three-dimensional problem to a two-dimensional surface description of the equations is shown schematically in Figure 3. The main steps in the derivation are

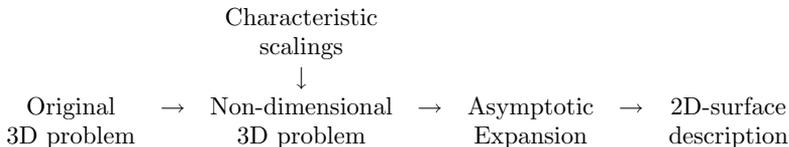


Figure 3: Transition scheme

as follows: As a starting point, the three-dimensional continuum equations are formulated. This is followed by a non-dimensionalization procedure and a perturbation parameter ϵ is chosen. Certain geometrical and material scalings are introduced in terms of ϵ that characterize the current problem. Finally, a solution is sought as an asymptotic expansion of the displacement field. As a result, the expansion is determined by a set of simpler equations that can approximately replace the original three-dimensional problem for small values of the parameter ϵ . By letting the perturbation parameter tend to zero, a limit problem governing the leading order terms is obtained. The effect of higher order terms in the expansion, and the presence of nonuniformities such as boundary layers, can also be identified as a consequence of the analysis.

3.2 Non-dimensionalization and introduction of scalings

As described in Section 1, the specific class of problems considered here is characterized by the thinness of the constituents in comparison with the inplane dimension, and that Young’s modulus for the adhesive is small as compared to the adherends modulus. This means that if the governing equations in problem (\mathcal{P}^s) or (\mathcal{P}^v) are non-dimensionalized with respect to a set of representative reference properties, these features will appear explicitly in the equations in terms of small non-dimensional ratios. For the present purpose, the chosen reference properties are given in Table 1, where L is a length measure of the same order of magnitude as the inplane dimension of the joint and E and ν are representative material parameters for the adherends of the same order of magnitude as E_1 , ν_1 and E_2 , ν_2 , respectively. The Lamé parameters corresponding to E and ν are obtained from

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{5}$$

The non-dimensionalization procedure of the variational problem (\mathcal{P}^v) reads:

Reference properties	
L	Characteristic inplane length of the joint
E, ν	Characteristic Young's modulus and Poisson's ratio for the adherends

Table 1: The introduced reference properties

First, we let the small ratio $\epsilon = h/L$ serve as the perturbation parameter, where h is a thickness of the same order of magnitude as the adherend thicknesses h_1 and h_2 , respectively. The introduction of non-dimensional displacements, stresses, tractions and Lamé parameters according to

$$\bar{u}_i = u_i/L, \quad (\bar{\sigma}_{ij}, \bar{t}_i, \bar{\lambda}_p, \bar{\mu}_p) = (\sigma_{ij}, t_i, \lambda_p, \mu_p)/(2\mu + \lambda) \quad (6)$$

together with a change of coordinates to a non-dimensional domain $\bar{\Omega}^p$, close to unity $\bar{\mathbf{x}} = (\bar{x}_1, \bar{x}_2, \bar{\zeta}_p)$ where $\bar{\zeta}_p \in [-\bar{h}_p/2, \bar{h}_p/2]$, leads to a variational equation involving the ratios h_1/L , h_2/L , h_0/L and λ_0 , μ_0 explicitly. Scalings on these ratios that reveal the geometrical and material features considered here are shown in Table 2. Here it is assumed that the thickness to length ratio for the adhesive is an order ϵ lower than for the adherends, and the parameter m is a positive integer reflecting the order of magnitude of the adhesive modulus relative to the adherends (Poisson's ratio is of the same order of magnitude as for the adherends). These material scalings in terms of Lamé parameters are

$$\bar{\lambda}_0 = \epsilon^m \bar{\lambda}, \quad \bar{\mu}_0 = \epsilon^m \bar{\mu} \quad (7)$$

where $(\bar{\lambda}, \bar{\mu}) = (\lambda, \mu)/(2\mu + \lambda)$ denotes non-dimensional Lamé parameters of order $O(1)$. The result is a non-dimensional problem ($\bar{\mathcal{P}}^v$) in the form below, suitable for an analysis using perturbation methods.

$$\bar{V} := \{\bar{\mathbf{u}} = (\bar{u}_i) \mid \bar{\mathbf{u}} = \mathbf{0} \text{ on } \bar{S}_u^1 \cup \bar{S}_u^2\}$$

Problem ($\bar{\mathcal{P}}^v$): Given $\bar{\mathbf{t}}$, find $\bar{\mathbf{u}} \in \bar{V}$ such that

$$\int_{\bar{\Omega}^0 \cup \bar{\Omega}^1 \cup \bar{\Omega}^2} \bar{\sigma}_{ij} \bar{\epsilon}_{ij}(\bar{\mathbf{v}}) \det(\mathbf{J}) d\bar{\Omega} - \int_{\bar{S}_t^1 \cup \bar{S}_t^2} \bar{t}_i \bar{v}_i \frac{\det(\mathbf{J})}{\sqrt{\mathbf{n} \cdot (\mathbf{J}^T \mathbf{J}) \cdot \mathbf{n}}} d\bar{S} = 0 \quad \forall \bar{\mathbf{v}} \in \bar{V} \quad (8)$$

where \mathbf{n} is the outward unit normal to \bar{S}_t^1 and \bar{S}_t^2 respectively and

$$\bar{\sigma}_{ij} = \epsilon^m \left(\bar{\lambda} \bar{\epsilon}_{kk}(\bar{\mathbf{u}}) \delta_{ij} + 2\bar{\mu} \bar{\epsilon}_{ij}(\bar{\mathbf{u}}) \right), \quad \mathbf{J} = (J_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon^2 \end{pmatrix} \text{ for } \bar{\mathbf{x}} \in \bar{\Omega}^0 \cup \partial\bar{\Omega}^0$$

$$\bar{\sigma}_{ij} = \bar{\lambda}_r \bar{\epsilon}_{kk}(\bar{\mathbf{u}}) \delta_{ij} + 2\bar{\mu}_r \bar{\epsilon}_{ij}(\bar{\mathbf{u}}), \quad \mathbf{J} = (J_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \epsilon \end{pmatrix} \text{ for } \bar{\mathbf{x}} \in \bar{\Omega}^r \cup \partial\bar{\Omega}^r$$

and

$$\bar{\varepsilon}_{ij}(\bar{\mathbf{u}}) = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial \bar{x}_k} (J_{jk}^{-1}) + \frac{\partial \bar{u}_j}{\partial \bar{x}_k} (J_{ik}^{-1}) \right), \quad \mathbf{J}^{-1} = (J_{ij}^{-1})$$

Scalings on the geometrical and material properties

$$\frac{h_1}{L} = \epsilon \bar{h}_1, \quad \frac{h_2}{L} = \epsilon \bar{h}_2, \quad \frac{h_0}{L} = \epsilon^2 \bar{h}_0$$

$$\frac{E_0}{E} = \epsilon^m, \quad \frac{\nu_0}{\nu} = 1, \quad m \in \{1, 2, 3, 4, \dots\}$$

Table 2: Introduced scalings, where $(\bar{h}_1, \bar{h}_2, \bar{h}_0)$ are non-dimensional thicknesses of order $O(1)$.

3.3 Asymptotic expansion

In Maz'ya et al. [61], Klarbring and Movchan [35], it has been shown that in the case of a *single beam*, transversally loaded with a traction component of order ϵ (i.e. $\epsilon \bar{t}_3$), an expansion in the form of a general power series $\bar{\mathbf{u}} = \dots + \epsilon^{-2} \bar{\mathbf{u}}^{-2} + \epsilon^{-1} \bar{\mathbf{u}}^{-1} + \bar{\mathbf{u}}^0 + \epsilon \bar{\mathbf{u}}^1 + \epsilon^2 \bar{\mathbf{u}}^2 + \dots$ can be constructed that satisfies the equilibrium equations in the interior, and the stress boundary condition on the horizontal boundary surfaces. This expansion starts with the term $\epsilon^{-2} \bar{\mathbf{u}}^{-2}$, where $\bar{\mathbf{u}}^{-2}$ only contains the transversal displacement component \bar{u}_3 . Since the problem is linear it can be re-scaled, without loss of generality, by multiplying the applied stress vector by ϵ^2 , and consequently the expansion starts with the term $\bar{\mathbf{u}}^0$. In Klarbring and Movchan [35, 36], Avila-Pozos et al. [37], Schmidt [38] and Åslund [39], a solution $\bar{\mathbf{u}} = \bar{\mathbf{u}}(\bar{\mathbf{x}}, \epsilon)$ to the adhesive joint problem corresponding to the scalings in Table 2, was sought as an asymptotic expansion in ϵ according to

$$\bar{\mathbf{u}} = \bar{\mathbf{u}}^0 + \epsilon \bar{\mathbf{u}}^1 + \epsilon^2 \bar{\mathbf{u}}^2 + \epsilon^3 \bar{\mathbf{u}}^3 + \epsilon^4 \bar{\mathbf{u}}^4 + \dots \quad (9)$$

In these works, a set of scalings on the applied traction vector $\bar{\mathbf{t}}$ were introduced compatible with (9). By assuming (9), no *a priori* assumptions are made concerning the displacements or stress fields distributed through the thickness of the three layers. These distributions are determined by the systematic asymptotic procedure. In reference [36], the strong formulation stated in problem (\mathcal{P}^s) was analyzed in the case of bonded beams, and in [37] orthotropic materials were considered. In [38] the variational problem (\mathcal{P}^v) was analyzed using the expansion (9), and in [39] an extension to non-linear elastic adherends was performed. Substitution of the series (9) into the variational problem $(\bar{\mathcal{P}}^v)$ results in that the expansion (9) is determined by a set of recursive variational equations posed over a surface. The behavior of these equations for different scalings on the adhesive modulus can be analyzed by altering the parameter m in Table 2 (see Avila-Pozos [40], Benveniste [34], Schmidt [38]). The result is

a family of limit problems which are spread from the case where the adhesive imposes a kinematic constraint (i.e. $u_i^+ = u_i^-$), to the case where the interfaces starts to uncouples and a deformation in the adhesive is present. These limit problems form the basis of a proposed surface model for the compound adhesive joint [41, 42]. The approach where the displacement field is expanded into a series in the perturbation parameter is usually referred to as the displacement approach, cf. Ciarlet [62, 63]. An alternative approach is to formulate the original problem in terms of a Hellinger–Reissner variational equation. In this case, both the displacement vector $\mathbf{u} = \mathbf{u}(\mathbf{x}, \epsilon)$ and the stress tensor $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x}, \epsilon)$ are expanded into asymptotic series. This formulation was used in the paper by Klarbring [26]. See also Ciarlet and Destuynder [64] who used this approach to justify plate theories.

3.4 A surface description of the compound adhesive joint

As a result of the asymptotic analysis, work-conjugate generalized stress and strain measures, associated with a geometrical two-dimensional surface description, can be identified. The order of magnitude of these measures, in terms of the power of the parameter ϵ in the strain energy function, can be found. It turns out that the kinematic variables are the displacement $f_i^{(r)}$ on the adherends middle surface, and the generalized stress and strain measures to be used are summarized in Table 3. If we let the inplane displacement $f_\alpha^{(r)}$ be prescribed to zero on Γ_u^r , and $f_3^{(r)}$ and the rotation $f_{3,\alpha}^{(r)}$ be prescribed to zero on $\Gamma_{u_3}^r$ and define the spaces for the admissible displacements as: $V := \{f_\alpha^{(r)} \mid f_\alpha^{(r)} = 0 \text{ on } \Gamma_u^r\}$ and $V_3 := \{f_3^{(r)} \mid f_3^{(r)} = 0, f_{3,\alpha}^{(r)} = 0 \text{ on } \Gamma_{u_3}^r\}$, the following approximations of the variational equation (3) in problem (\mathcal{P}^v) is in effect made:

$$\begin{aligned} \int_{\Omega^r} \sigma_{ij} \varepsilon_{ij}(\mathbf{v}) d\Omega &\approx \int_{\omega^r} \left[N_{\theta\alpha}^{(r)} n_{\theta\alpha}^{(r)}(v_\gamma^{(r)}) + M_{\theta\alpha}^{(r)} \kappa_{\theta\alpha}^{(r)}(v_3^{(r)}) \right] d\omega \\ \int_{\Omega^0} \sigma_{ij} \varepsilon_{ij}(\mathbf{v}) d\Omega &\approx \int_{\omega^0} p_i w_i(v_j^{(r)}) d\omega \\ \int_{S_t^r} t_i v_i dS &\approx \int_{\omega^r} \left[t_i^{(r)} v_i^{(r)} + (-1)^r \frac{h_r}{2} t_\alpha^{(r)} v_{3,\alpha}^{(r)} \right] d\omega \\ &+ \int_{\Gamma_t^r} \left[T_i^{(r)} v_i^{(r)} - M_\alpha^{(r)} v_{3,\alpha}^{(r)} \right] d\Gamma \end{aligned}$$

where the stress and couple (moment) resultants on the boundary Γ_t^r due to a traction $\mathbf{t} = (t_i^{(r)})$ applied on $\Gamma_t^r \times [-\frac{h_r}{2}, \frac{h_r}{2}]$ are:

$$T_i^{(r)} := \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} t_i^{(r)} d\zeta_r, \quad M_\alpha^{(r)} := \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} t_\alpha^{(r)} \zeta_r d\zeta_r \quad \text{on } \Gamma_t^r \quad (10)$$

For the adherend middle surface ω^r	
Generalized stress	Generalized strain
$N_{\theta\alpha}^{(r)} := \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} \sigma_{\theta\alpha} d\zeta_r,$	$n_{\theta\alpha}^{(r)} = \frac{1}{2}(f_{\theta,\alpha}^{(r)} + f_{\alpha,\theta}^{(r)})$
$M_{\theta\alpha}^{(r)} := - \int_{-\frac{h_r}{2}}^{\frac{h_r}{2}} \sigma_{\theta\alpha} \zeta_r d\zeta_r,$	$\kappa_{\theta\alpha}^{(r)} = f_{3,\theta\alpha}^{(r)}$
For the adhesive middle surface ω^0	
Generalized stress	Generalized strain
$p_\alpha := \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \sigma_{3\alpha} d\zeta_0,$	$w_\alpha = \frac{1}{h_0} \left(f_\alpha^{(1)} - f_\alpha^{(2)} + \frac{h_1}{2} f_{3,\alpha}^{(1)} + \frac{h_2}{2} f_{3,\alpha}^{(2)} \right)$
$p_3 := \int_{-\frac{h_0}{2}}^{\frac{h_0}{2}} \sigma_{33} d\zeta_0,$	$w_3 = \frac{1}{h_0} \left(f_3^{(1)} - f_3^{(2)} \right)$

Table 3: Generalized stress and strain measures.

The expressions for the constitutive equations within the three layers are given next. For the material surfaces defined on ω^r representing the adherends, the following holds with $G_r = \frac{E_r}{2(1+\nu_r)}$:

$$N_{\theta\alpha}^{(r)} = 2G_r h_r \left(\frac{\nu_r}{1-\nu_r} n_{\gamma\gamma}^{(r)} \delta_{\theta\alpha} + n_{\theta\alpha}^{(r)} \right) \quad (11)$$

$$M_{\theta\alpha}^{(r)} = \frac{h_r^3}{12} 2G_r \left(\frac{\nu_r}{1-\nu_r} \kappa_{\gamma\gamma}^{(r)} \delta_{\theta\alpha} + \kappa_{\theta\alpha}^{(r)} \right) \quad (12)$$

For the adhesive material surface defined on ω^0 the following relation is obtained:

$$p_i = C_{ij} w_j \quad (13)$$

where

$$[C_{ij}] = \frac{E_0 h_0}{1 + \nu_0} \begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & \frac{1-\nu_0}{1-2\nu_0} \end{bmatrix}$$

An overview of the model, illustrating the mapping between generalized stresses and strains and the corresponding 3-D continuum counterpart is given in Figure 4. The approximations of displacements, stresses and strains are summarized in Appendix B.

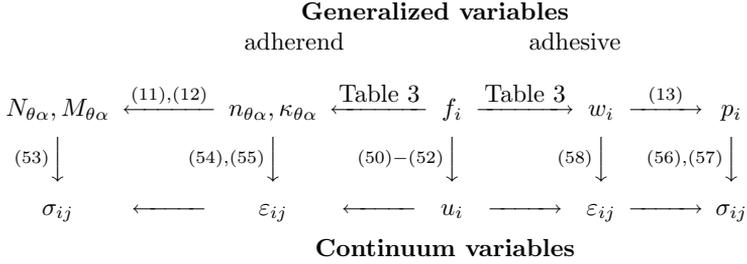


Figure 4: Overview of the mapping between generalized stresses and strains and the 3-D continuum counterpart. The superscript (r) has been omitted for simplicity and equations (50)–(58) are explicitly given in Appendix B.

4 Modelling the material behavior of the adhesive

The type of adhesive considered in this work is a high strength rubber-modified (toughened) epoxy adhesive. Even though this particular type of adhesives is under consideration here, the theory presented can be applied to a wider group of polymeric materials. From a micromechanical point of view, the fracture process takes place through initiation and growth of microcracks and microcavities (Figure 5), which leads to formation of a macrocrack and a softening behavior of the adhesive material. The early study of fracture surfaces indicates the growth of cavities (Kinloch et al. [66], Yee and Pearson [67], Bascom and Hunston [68]), and from the existing experiments one can also conclude that growth of cracks takes place under extensive plastic flow ahead of the crack tip (Kinloch and Young [65]). The above mechanisms of the fracture process were identified and fracture energies determined by Chai [69, 70, 71, 72], Chai and Chiang [73], and Hunston et al. [74].

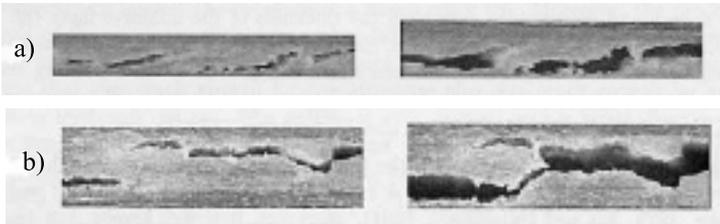


Figure 5: Fracture process. Initiation (left) and coalescence (right) of microcracks and microcavities in a high strength adhesive loaded in a) shear and b) peel. Experimental results from Leffler [84] and Anderson and Biel [88].

From phenomenological observations concerning the macro-mechanical be-

havior of adhesives, one can conclude that: (i) They possess a linear elastic branch, and as for many polymers, the yield limit is dependent on the hydrostatic state of stress (Sultan and McGarry [75], Bowden and Jukes [76], Raghava, Caddell and Yeh [77]); (ii) This is followed by an elastic-plastic branch with irreversible strains both with a hardening and a softening behavior, where plastic flow is essentially a deviatoric process (Kinloch et al. [66], Yee and Pearson [67]); (iii) Polymeric adhesives are usually viscoelastic and viscoplastic (Perets and Weitsman [78]), and (iv) the response is highly temperature dependent. In many cases, initial defects are also induced during the manufacturing process (e.g. porosity and cavities caused by entrained air, voids and microcracks due to poor curing, delaminated areas because of poor surface preparation, see Adams et al. [1]). It is always desirable to formulate a computational model governed by simplicity where as few material parameters as possible need to be determined from experiments, still reflecting some of the material behavior described above. Such a model can be derived from phenomenological considerations using the concepts of Continuum Damage Mechanics (CDM). Through the introduction of internal variables, such as an irreversible (plastic) strain tensor, a scalar hardening variable, and a scalar damage variable, a consistent theory can be constructed that reflects both a hardening and a softening behavior in the multi-axial case (see for instance the phenomenological considerations by Edlund and Klarbring [43]). This means that a material model that reproduces the complete stress-strain relation in Figure 6 is derived. Even though adhesive bonding has been used in several engineering applications during the last decades, there is still a lack of experimental data describing the softening branch for high-strength adhesives. From regular standard tests (cf. Adams [1], Thick-Adherent test [80]) the yield limit, the hardening slope and the maximum stress can usually be identified. However, from recent measurement techniques, that use an inverse method, the complete stress-strain curve, including the softening part, can be detected, in the case of shear (Alfredsson [81, 82], Alfredsson et al. [83], Leffler [84]), and peel (Stigh [85], Stigh and Andersson [86], Andersson and Stigh [87], Andersson and Biel [88]).

4.1 Continuum damage mechanics

Damage is the progressive process by which a material breaks. Continuum damage mechanics deals with continuous fields on a conceptual level, where the material degradation is described by irreversible internal variables defined on a scale of a representative volume element². From a physical point of view, these variables at a given point represent averages on this volume. The evolution of damage within this volume is considered to be the growth of surface and volume discontinuities (i.e. microcracks and microcavities), which initiate a crack (a broken volume). The advantage of using internal variables defined on this scale, is that the deterioration may be studied without detailed knowledge of the

²This volume is considered to be small enough to capture high gradients and local variations at the macroscale, but large enough to represent an average of the micro-processes taking place (cf. Lemaitre and Chaboche [92], Lemaitre [93]).

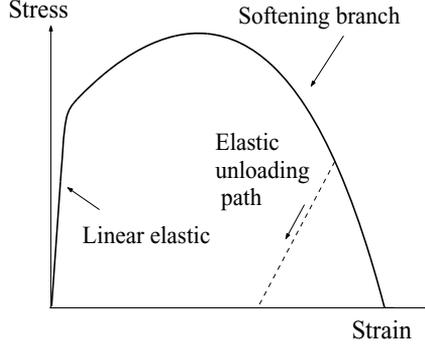


Figure 6: The complete stress–strain curve for an adhesive. The elastic unloading path is also indicated.

complex microstructure, and a constitutive law can be written for this volume such that the mechanics of continuous media can be applied. The evolution of the internal variables is finally described by a kinetic law. In this dissertation, this evolution law is constructed within the framework of thermodynamics of irreversible processes using the formal structure of a Generalized Standard Material (GSM), summarized in Sections 4.2 and 4.3.

4.1.1 Representation of isotropic damage

The trace of microcracks and microcavities on an intersected plane with a representative volume element, may be approximated by a total area of defects ΔS_d , as in Figure 7. This area incorporates the intersections of all present defects, accounting for micro-stress concentrations due to discontinuities and their interactions. The remaining area that effectively resists load is then reduced to $\Delta \tilde{S} = \Delta S - \Delta S_d$, where ΔS is the intersection area with no defects.

L.M. Kachanov [89] proposed a one-dimensional continuous variable Φ to represent the damage, where $\Phi = 1$ corresponds to a continuous and completely undamaged material (no discrete defects), and $\Phi = 0$ for a completely broken (destroyed) material. Another widely used damage measure is the variable $d = 1 - \Phi$ (Odqvist and Hult [90]), which is related to the concept of effective stress Rabotnov [91], see also Lemaitre and Chaboche [92]. In order to define the damage measure d , we consider a damaged body and a representative volume element at a point O intersected by a plane at x with a normal \mathbf{n} (Figure 7). In Lemaitre and Chaboche [92], and Lemaitre [93] the damage is defined as the maximum of the ratio between the total area of defects $\Delta S_d(x)$ and the area ΔS . Thus, the damage at point O in the direction \mathbf{n} is

$$d_n := \max_x \left(\frac{\Delta S_d(x)}{\Delta S} \right)_n = \left(\frac{\Delta S_d}{\Delta S} \right)_n, \quad 0 \leq d_n \leq 1 \quad (14)$$

where ΔS_d denotes the maximum of $\Delta S_d(x)$ along the abscissa x . Through this definition $d_n = 0$ corresponds to an undamaged material, and $d_n = 1$ corresponds to a volume element completely broken into two parts.³ If the damage variable is independent of the orientation \mathbf{n} , the damage is said to be isotropic and is represented by

$$d = \frac{\Delta S_d}{\Delta S}, \quad \forall \mathbf{n} \quad (15)$$

Due to the physical nature of damage, it is in many case anisotropic in character. Several definitions of anisotropic damage exist where damage is defined as a tensor (second order or as a fourth order tensor) [93]. See also the review by Olsson [94]. However, an isotropic damage measure in terms of a scalar variable has benefits due to its simplicity.

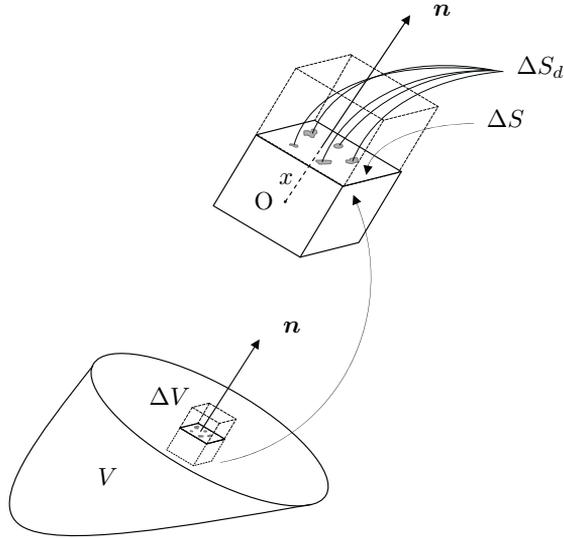


Figure 7: A representative volume element ΔV and the total area of defects denoted ΔS_d .

4.1.2 The concept of effective stress

Y.N. Rabotnov [91] introduced the concept of effective stress by considering the stress $\bar{\sigma}$ related to the surface $\Delta \tilde{S} = \Delta S - \Delta S_d$ that effectively resists the load.

³The failure usually occurs for $d < 1$ due to a process of instability. In many cases one writes $0 \leq d \leq d_c$, where $d_c < 1$ denotes the critical value, see [92, 93].

In the uniaxial case, one obtain (see Figure 8)

$$\Delta F = \sigma \Delta S = \tilde{\sigma} \Delta \tilde{S} \quad (16)$$

and since $d = \Delta S_d / \Delta S$, the effective stress $\tilde{\sigma}$ can be identified as:

$$\tilde{\sigma} = \frac{\sigma}{1-d} \quad (17)$$

A generalization to the multi-axial case of isotropic damage (Chaboche [95]) reads:

$$\tilde{\boldsymbol{\sigma}} := \frac{\boldsymbol{\sigma}}{1-d} \quad (18)$$

In this definition, $\boldsymbol{\sigma}$ is referred to as the homogenized continuum stress.

4.1.3 Strain equivalence principle

In order to avoid modelling each specific type of defect in a damaged body in a micromechanical sense, the concept of strain equivalence can be used to obtain a homogenized constitutive law. This can be accomplished by postulating a principle on the scale of a representative volume element. After J. Lemaitre [96] and Sidoroff [97], the postulated Principle of Strain Equivalence reads:

”Any strain constitutive equation for a damaged material may be derived in the same way as for a virgin material except that the usual stress is replaced by the effective stress ”.

This means that the strain constitutive equations (i.e. the state laws for the damaged state) can be derived for a given tensor valued strain function F . Letting $\boldsymbol{\varepsilon}^e$ denote the elastic strain tensor,

- Undamaged virgin state $d = 0$

$$\boldsymbol{\varepsilon}^e(\boldsymbol{\sigma}, 0) = F(\boldsymbol{\sigma}) \quad (19)$$

- Damaged state $0 < d < 1$

$$\boldsymbol{\varepsilon}^e(\boldsymbol{\sigma}, d) = F(\tilde{\boldsymbol{\sigma}}) \quad (20)$$

This can be expressed equivalently as:

$$\boldsymbol{\varepsilon}^e(\boldsymbol{\sigma}, d) = \boldsymbol{\varepsilon}^e(\tilde{\boldsymbol{\sigma}}, 0) \quad (21)$$

When applied to the uniaxial linear elastic case, where $F(\sigma) = \sigma/E$, one finds

$$\varepsilon^e(\sigma, d) = \varepsilon^e(\tilde{\sigma}, 0) = \frac{\tilde{\sigma}}{E} = \frac{\sigma}{(1-d)E}$$

See Figure 9 for an interpretation of the above equations. Note that the last expression can be interpreted as $\tilde{E} = (1-d)E$ being the elastic modulus of the damaged material, and since \tilde{E} is a measurable quantity, the damage variable d can be determined from $d = 1 - \tilde{E}/E$, cf. [92, 93].

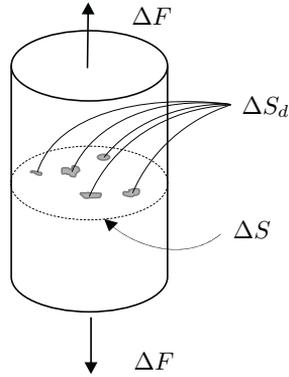


Figure 8: The effective stress $\tilde{\sigma}$ associates to the surface $\Delta\tilde{S} = \Delta S - \Delta S_d$ that effectively resists the load ΔF , and is defined through the relation $\sigma\Delta S = \tilde{\sigma}\Delta\tilde{S}$.

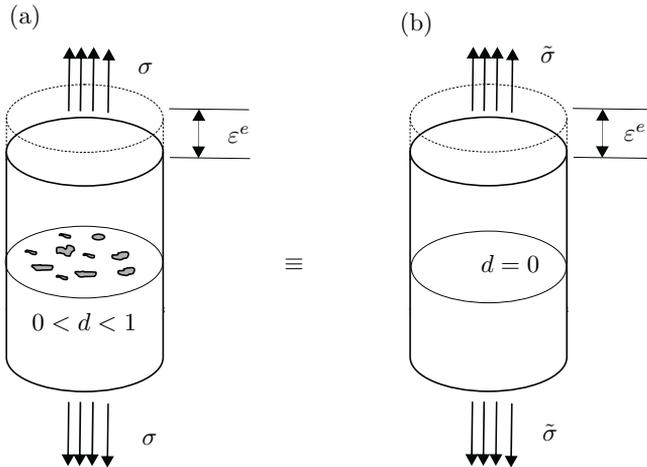


Figure 9: Strain equivalence; (a) damaged state where $\varepsilon^e = \sigma/(1 - d)E$, and (b) a fictitious state where $\varepsilon^e = \tilde{\sigma}/E$. In both these cases the strain ε^e is equal.

4.2 Thermodynamic relations

Any constitutive law intended to describe the relation between stress and strain in a material body must fulfil the fundamental principles of thermodynamics. These are summarized in Appendix A. The first law of thermodynamics (48) can be rewritten in a local form using the divergence theorem, Cauchy's postulate (42), conservation of mass (43), and the equations of motion (44)–(45):

$$\rho \dot{e} = \boldsymbol{\sigma} : \mathbf{D} + \rho r_q - \operatorname{div} \mathbf{q} \quad (22)$$

where $\mathbf{D} := \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ is the rate of deformation tensor and \mathbf{v} the velocity. The combination of the second law of thermodynamics (49) and equation (22) results in Clausius–Duhems inequality after localization:

$$\rho T \dot{s} + \boldsymbol{\sigma} : \mathbf{D} - \rho \dot{e} - \frac{1}{T} \mathbf{q} \cdot \nabla T \geq 0 \quad (23)$$

This inequality must be satisfied for any process taking place in a material body. In the case of small strains, one may approximate the rate of deformation tensor \mathbf{D} with the time derivative of the linear strain tensor, i.e. $\mathbf{D} \approx \dot{\boldsymbol{\varepsilon}}$. Further, the introduction of Helmholtz⁴ free energy $\psi = e - sT$, results in

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \rho(\dot{\psi} + s\dot{T}) - \frac{1}{T} \mathbf{q} \cdot \nabla T \geq 0 \quad (24)$$

If we assume adiabatic processes at constant temperature, we let $\mathbf{q} = \mathbf{0}$ and $\dot{T} = 0$, and introduce Helmholtz energy per unit volume $\Psi := \rho\psi$, letting ρ be constant. Under these circumstances, the inequality (24) can be further simplified to

$$\boldsymbol{\sigma} : \dot{\boldsymbol{\varepsilon}} - \dot{\Psi} \geq 0 \quad (25)$$

In this dissertation it will be assumed that the free energy Ψ and the Cauchy stress $\boldsymbol{\sigma}$ are functions of the strain $\boldsymbol{\varepsilon}$ and a set of n tensor or scalar valued internal variables $\boldsymbol{\alpha}_k$, where $k = 1, 2, \dots, n$

$$\Psi = \Psi(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_k), \quad \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}, \boldsymbol{\alpha}_k) \quad (26)$$

If the functions (26) are inserted into (25), one finds that the following inequality must hold

$$\left(\boldsymbol{\sigma} - \frac{\partial \Psi}{\partial \boldsymbol{\varepsilon}} \right) : \dot{\boldsymbol{\varepsilon}} - \frac{\partial \Psi}{\partial \boldsymbol{\alpha}_k} : \dot{\boldsymbol{\alpha}}_k \geq 0 \quad (27)$$

⁴Helmholtz energy can be identified as a Legendre transformation of the convex function $e = e(\boldsymbol{\varepsilon}, s)$ obtained from the fundamental functions $s = s(\boldsymbol{\varepsilon}, e)$ and $T = T(\boldsymbol{\varepsilon}, e)$. Any convex function $y(x)$ can be described by the slope $p = dy/dx$ and ordinate intersection ξ of the tangent line. It follows that $p = dy/dx = (y - \xi)/(x - 0)$ or $\xi = y - px$. This means that $y(x)$ and its Legendre transformation $\xi(p)$ contain the same information.

Since Ψ and σ are independent of the strain rate $\dot{\varepsilon}$, a Coleman–Noll⁵ argument can be used to obtain the state laws

$$\sigma = \frac{\partial \Psi}{\partial \varepsilon} \tag{28}$$

$$\mathbf{A}_k := -\frac{\partial \Psi}{\partial \alpha_k} \tag{29}$$

and the inequality (27) takes the reduced form

$$\mathbf{A}_k : \dot{\alpha}_k \geq 0 \tag{30}$$

This is the reduced dissipation inequality, which states that the power developed by the thermodynamical forces \mathbf{A}_k and their associated fluxes $\dot{\alpha}_k$, is always greater than or equal to zero.

4.3 The generalized standard material approach

The framework referred to as the Generalized Standard Material approach (Halphen, Nguyen, [98]) brings a systematic structure to the procedure of deriving a material model. This approach mainly consists of two steps: First a set of internal variables α_k is chosen, and the function Ψ in (26) is specified. The thermodynamic forces are then given by the state laws (28)–(29). Next, an evolution law for the internal variables is constructed which ensures that the dissipation inequality (30) is satisfied. This is accomplished by postulating the existence of a dissipation potential⁶ $\phi(\mathbf{A}_k)$, (Moreau [100]). If ϕ has the following properties:

- $\phi(\mathbf{A}_k)$ is convex
- $0 = \phi(\mathbf{0})$
- $\mathbf{0} \in \partial\phi(\mathbf{0})$

it follows that the dissipation inequality (30) is satisfied if $\dot{\alpha}_k$ belongs to the set

$$\dot{\alpha}_k \in \partial\phi(\mathbf{A}_k) \tag{31}$$

where $\partial\phi$ denotes the sub-differential defined in Box 2. This is a sufficient but not a necessary condition for (30) to be satisfied. In the case where ϕ is a differentiable function, the subdifferential is unique and equals the gradient (i.e. $\partial\phi = \{\nabla\phi\} = \{\partial\phi/\partial\mathbf{A}_k\}$). If the dissipation potential is chosen as the non-differentiable indicator function I_C (Box 1) of a closed set C containing m convex functions g_i

$$\phi := I_C, \quad C := \{\mathbf{A}_k \mid g_i(\mathbf{A}_k) \leq 0, \quad i = 1, 2, \dots, m\} \tag{32}$$

⁵This argument reads: If $xy + z \geq 0 \forall y$, where x and z are independent of y , then $x = 0$ and $z \geq 0$.

⁶This is actually a Legendre transformation of the dual dissipation potential $\phi^*(\alpha_k)$.

the following evolution law follows:

$$\dot{\alpha}_k = \sum_{i=1}^m \dot{\lambda}_i \frac{\partial g_i}{\partial \mathbf{A}_k}$$

$$g_i \leq 0, \quad \dot{\lambda}_i \geq 0, \quad g_i \dot{\lambda}_i = 0 \quad (\text{no sum on } i)$$

Box 1: Indicator function

The indicator function I_C of the set $C \subset \mathbb{R}^n$ is defined by:

$$I_C(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in C \\ +\infty & \text{if } \mathbf{x} \notin C \end{cases}$$

If C is a convex set, then I_C is a convex function.

Box 2: Sub-differential

The sub-differential $\partial f(\mathbf{x})$ of a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ at a point \mathbf{x} is the set

$$\partial f(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} \cdot (\mathbf{z} - \mathbf{x}) \leq f(\mathbf{z}) - f(\mathbf{x}) \quad \forall \mathbf{z} \in \mathbb{R}^n\}$$

The elements $\mathbf{y} \in \partial f(\mathbf{x})$ are called sub-gradients. If f is smooth at \mathbf{x} , then

$$\partial f(\mathbf{x}) = \{\nabla f(\mathbf{x})\}$$

where ∇ is the gradient operator.

Of particular interest is the sub-differential of the indicator function:

let $f = I_C$ where the convex set C is defined by m convex functions g_i according to:

$$C = \{\mathbf{x} \in \mathbb{R}^n \mid g_i(\mathbf{x}) \leq 0, i = 1, 2, \dots, m\}$$

Then,

$$\partial I_C(\mathbf{x}) = \left\{ \mathbf{y} \in \mathbb{R}^n \mid \mathbf{y} = \sum_{i=1}^m \lambda_i \nabla g_i(\mathbf{x}), g_i(\mathbf{x}) \leq 0, \lambda_i \geq 0, g_i(\mathbf{x}) \lambda_i = 0 \right\}$$

(no sum on i unless a summation symbol is given.)

The particular choices of internal variables in this dissertation are as follows: an irreversible strain tensor $\boldsymbol{\varepsilon}^p$, a scalar hardening variable r , and a scalar damage variable d , that is $(\alpha_1, \alpha_2, \alpha_3) := (\boldsymbol{\varepsilon}^p, r, d)$. The thermodynamic forces associated with these internal variables are denoted $(\mathbf{A}_1, \mathbf{A}_2, \mathbf{A}_3) := (\boldsymbol{\sigma}, R, Y)$. In the work by Lemaitre and Chaboche [92], the choice of the free energy Ψ is based on a decoupling between elasticity (associated with damage) and the hardening effect. In the case of isotropic hardening the decomposition reads:

$$\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, r, d) = \Psi^e(\boldsymbol{\varepsilon}^e, d) + \Psi^p(r) \tag{33}$$

where

$$\Psi^e = (1 - d) \frac{1}{2} \boldsymbol{\varepsilon}^e : \mathbf{E} : \boldsymbol{\varepsilon}^e \tag{34}$$

Here Ψ^e represents the elastic potential of the damaged material, where $\boldsymbol{\varepsilon}^e := \boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}^p$ is the elastic strain, and \mathbf{E} is the fourth order elasticity tensor. $\Psi^p(r)$ is introduced as a plastic hardening function. A different form was used by Edlund and Klarbring [43], where the function was chosen as

$$\Psi(\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}^p, r, d) = (1 - d)\Psi^0(\boldsymbol{\varepsilon}^e, r) \quad (35)$$

where Ψ^0 is the energy of the undamaged material decomposed into an elastic and plastic part

$$\Psi^0(\boldsymbol{\varepsilon}^e, r) = \frac{1}{2}\boldsymbol{\varepsilon}^e : \mathbf{E} : \boldsymbol{\varepsilon}^e + \Psi^p(r) \quad (36)$$

In both these references, the evolution laws were derived using an extension of the principle of strain equivalence (section 4.1.3). This means that not only the state laws, but also the evolution laws are expressed in terms of effective quantities. In reference [92], the stress tensor $\boldsymbol{\sigma}$ was replaced by the effective stress in the two yield functions g_i ($i = 1, 2$) defining the set C in (32) according to: $g_1 = g_1(\frac{\boldsymbol{\sigma}}{1-d}, R)$ and $g_2 = g_2(Y)$ where d is treated as a parameter. In reference [43], the choice of Ψ in equation (35) implies that all the thermodynamical forces $(\boldsymbol{\sigma}, R, Y)$ should be treated equal, and be replaced by their effective counterparts.⁷ In [43], three yield functions g_i ($i = 1, 2, 3$) were used to define the set C according to: $g_1 = g_1(\frac{\boldsymbol{\sigma}}{1-d}, \frac{R}{1-d})$, $g_2 = g_2(Y)$ and $g_3 = g_3(\frac{\boldsymbol{\sigma}}{1-d}, Y)$. See also the paper by Olsson and Ristinmaa [99], where the usage of different concepts, such as (complementary) energy equivalence, was discussed.

In the material model presented in this dissertation, a *single* yield function g_0 is used to enclose the set C of the dissipation potential $\phi = I_C$. This differs from the above formulations in [92] and [43]. Furthermore, the function g_0 is decomposed into a function $f_1(\tilde{\boldsymbol{\sigma}}, \tilde{R})$ reflecting isotropic hardening behavior, and a function $f_2(Y; d)$ that involves the thermodynamic force associated with isotropic damage. Terms appearing after the semicolon are treated as parameters, and the form of the energy function Ψ in (35) is used:

$$\phi := I_C \quad (37)$$

$$C := \{(\tilde{\boldsymbol{\sigma}}, \tilde{R}, Y) \mid g_0 = f_1(\tilde{\boldsymbol{\sigma}}, \tilde{R}) + f_2(Y; d) \leq 0\} \quad (38)$$

and the evolution law reads

$$(\dot{\boldsymbol{\varepsilon}}^p, \dot{r}, \dot{d}) \in \partial\phi(\tilde{\boldsymbol{\sigma}}, \tilde{R}, Y; d) \quad (39)$$

This formulation provides simplicity in the implementation and requires few material parameters, still reflecting some of the material non-linearities stated at the beginning of this section (an extension to a rate-dependent visco-plastic model is straightforward). Note that the gradients in (39) is taken with respect to the effective thermodynamical forces, and as a result, the evolution laws are written completely in effective quantities without violating (30). This means that the evolution of the internal variables are determined with respect to the state of the undamaged material. As a consequence, a single yield surface model with an accelerated damage growth ($\partial^2 d / \partial \varepsilon^2 > 0$) can be retained.

⁷ $Y = \Psi^0$ is already effective in character

5 Summary of appended papers

In Paper I, the formal and systematic asymptotic expansion technique, used to derive a surface model for the compound adhesive joint is described in detail. Through the derivation procedure, certain scalings are introduced in the three-dimensional linear elastic problem in terms of a perturbation parameter. These scalings reflect the thinness of the constituents and the low Young's modulus for the adhesive material. A set of compatible scalings on the components of the applied traction vector are also introduced. As a result of the analysis, the distribution of displacements, stresses and strains through the thickness are derived, and the order of magnitude of these properties is found in terms of a power series in the perturbation parameter. The leading order terms in the expansion are determined by solving a geometrical two-dimensional variational equation. Work-conjugate, generalized stress and strain measures associated with the two-dimensional surface treatment are identified.

In Paper II, a justification of the introduced scalings on the applied traction components is presented. Second, the behavior of the governing equations for different scalings on Young's modulus for the adhesive material are investigated. The result is a family of limit (leading order) problems, in terms of variational equations posed over a surface. The numerical behavior of these equations near the boundary of the joint is investigated and a final structural model is proposed. This model approximately replaces the original three-dimensional problem for arbitrarily (small) values of the perturbation parameter.

Paper III concerns the derivation of an elastic-plastic adhesive material model with a softening behavior due to damage. This model is first derived for a continuum and then transformed to the surface description proposed, replacing the linear elastic material model for the adhesive. The resulting variational equation is FE-discretized and numerical results are presented and compared with experiments.

The aim of Paper IV is to summarize the above derived FE-method in a concise way and to illustrate the ability of the model to follow the damage evolution in a structure during the descending part of the global load-displacement curve. This is demonstrated through a number of FE-simulations of adhesively bonded structures of varying complexity. The matrices needed for an implementation are also derived.

6 Appendix A

Balance laws:

- Linear momentum

$$\frac{d}{dt} \int_{\Omega} \rho \dot{\mathbf{r}} d\Omega = \int_{\Omega} \rho \mathbf{b} d\Omega + \int_{\partial\Omega} \mathbf{t} dS \quad (40)$$

- Angular momentum

$$\frac{d}{dt} \int_{\Omega} \mathbf{r} \times \rho \dot{\mathbf{r}} d\Omega = \int_{\Omega} \mathbf{r} \times \rho \mathbf{b} d\Omega + \int_{\partial\Omega} \mathbf{r} \times \mathbf{t} dS \quad (41)$$

where \mathbf{r} is the position vector and ρ the density, \mathbf{b} the applied body force (per unit mass) and \mathbf{t} the surface traction. The Cauchy stress tensor $\boldsymbol{\sigma}$ is related to the surface traction according

$$\mathbf{t} = \boldsymbol{\sigma} \cdot \mathbf{n} \quad (42)$$

where \mathbf{n} is the outward unit normal to the boundary $\partial\Omega$. Further, conservation of mass can be expressed as

- Conservation of mass

$$\frac{d}{dt} \int_{\Omega} \rho d\Omega = 0 \quad (43)$$

The balance of linear and angular momentum, conservation of mass together with Cauchy's postulate (42), results in the following equations after using the divergence theorem and localization

$$\operatorname{div} \boldsymbol{\sigma} + \rho \mathbf{b} = \rho \ddot{\mathbf{r}} \quad \text{in } \Omega \quad (44)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad (45)$$

In the absence of body forces \mathbf{b} , the laws of motion (44)–(45) simplify to the following equilibrium laws in the (quasi) static case

$$\operatorname{div} \boldsymbol{\sigma} = \mathbf{0} \quad \text{in } \Omega \quad (46)$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T \quad (47)$$

Laws of thermodynamics:

- First law of thermodynamics

The first law of thermodynamics states that the rate of change of the internal energy \mathcal{E} and kinetic energy K of a body occupying Ω , equals the power of external forces L_e and the power Q supplied by heat:

$$\frac{d}{dt} (\mathcal{E} + K) = L_e + Q \quad (48)$$

where

$$\mathcal{E} = \int_{\Omega} \rho e d\Omega, \quad K = \int_{\Omega} \frac{1}{2} \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \rho d\Omega$$

$$L_e = \int_{\Omega} \mathbf{b} \cdot \dot{\mathbf{r}} \rho d\Omega + \int_{\partial\Omega} \mathbf{t} \cdot \dot{\mathbf{r}} dS, \quad Q = \int_{\Omega} r_q \rho d\Omega - \int_{\partial\Omega} \mathbf{q} \cdot \mathbf{n} dS$$

and e is the specific internal energy per unit mass, r_q the internal heat production per unit mass and \mathbf{q} the heat flux vector.

- Second law of thermodynamics

The second law of thermodynamics states that the change of entropy of a volume Ω is always greater than or equal to the supply of entropy through heat

$$\frac{d}{dt}\mathcal{N} \geq \int_{\Omega} \frac{r_q}{T} \rho d\Omega - \int_{\partial\Omega} \frac{\mathbf{q}}{T} \cdot \mathbf{n} dS \quad (49)$$

where

$$\mathcal{N} = \int_{\Omega} \rho s d\Omega$$

is the entropy, and s is the specific entropy per unit mass.

7 Appendix B

Displacements

The approximations of the displacement fields in the compound joint are expressed in terms of the displacements of the middle surfaces.

In the adherend domain $\Omega^r = \omega^r \times [-\frac{h_r}{2}, \frac{h_r}{2}]$:

$$u_{\alpha}^{(r)}(x_1, x_2, \zeta_r) \approx f_{\alpha}^{(r)}(x_1, x_2) - f_{3,\alpha}^{(r)}(x_1, x_2)\zeta_r \quad (50)$$

$$u_3^{(r)}(x_1, x_2, \zeta_r) \approx f_3^{(r)}(x_1, x_2) \quad (51)$$

where $\zeta_r \in [-\frac{h_r}{2}, \frac{h_r}{2}]$.

In the adhesive domain $\Omega^0 = \omega^0 \times [-\frac{h_0}{2}, \frac{h_0}{2}]$:

$$u_i^{(0)}(x_1, x_2, \zeta_0) \approx \frac{1}{2}(u_i^+ + u_i^-) + (u_i^+ - u_i^-)\frac{\zeta_0}{h_0} \quad (52)$$

where $\zeta_0 \in [-\frac{h_0}{2}, \frac{h_0}{2}]$, and $u_i^+ := u_i^{(1)}(x_1, x_2, \zeta_1 = -\frac{h_1}{2})$ and $u_i^- := u_i^{(2)}(x_1, x_2, \zeta_2 = \frac{h_2}{2})$ are the displacements on the upper and lower interface surfaces S^1 and S^2 respectively.

Stress and strain components⁸

In the adherend domain $\Omega^r = \omega^r \times [-\frac{h_r}{2}, \frac{h_r}{2}]$:

$$\sigma_{\theta\alpha} = \frac{N_{\theta\alpha}^{(r)}}{h_r} - M_{\theta\alpha}^{(r)} \frac{12}{h_r^3} \zeta_r, \quad \sigma_{3i} \ll \sigma_{\theta\alpha} \quad (53)$$

$$\varepsilon_{\theta\alpha} = n_{\theta\alpha}^{(r)} - \kappa_{\theta\alpha}^{(r)} \zeta_r, \quad \varepsilon_{3\alpha} \ll \varepsilon_{\theta\alpha}, \varepsilon_{33} \quad (54)$$

$$\varepsilon_{33} = \frac{\nu_r}{1 - \nu_r} (\kappa_{\gamma\gamma}^{(r)} \zeta_r - n_{\gamma\gamma}^{(r)}) \quad (55)$$

⁸The adherend stress and strain components σ_{3i} , $\varepsilon_{3\alpha}$, and the adhesive stress and strain components σ_{12} , $\varepsilon_{\theta\alpha}$ are neglected higher order terms, see [38, 41, 42].

In the adhesive domain $\Omega^0 = \omega^0 \times [-\frac{h_0}{2}, \frac{h_0}{2}]$:

$$\sigma_{3\alpha} = \frac{p_\alpha}{h_0}, \quad \sigma_{33} = \frac{p_3}{h_0}, \quad \sigma_{12} \ll \sigma_{3i}, \sigma_{11}, \sigma_{22} \quad (56)$$

$$\sigma_{\theta\theta} = \frac{\nu_0}{1-\nu_0} \frac{p_3}{h_0}, \quad \text{no sum on } \theta \quad (57)$$

$$\varepsilon_{3\alpha} = \frac{1}{2} w_\alpha, \quad \varepsilon_{33} = w_3, \quad \varepsilon_{\theta\alpha} \ll \varepsilon_{3i} \quad (58)$$

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