Automatica 149 (2023) 110801

Contents lists available at ScienceDirect

Automatica

journal homepage: www.elsevier.com/locate/automatica

Brief paper

Investigating the effect of edge modifications on networked control systems: Stability analysis ☆

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ARTICLE INFO

Article history:
Received 27 July 2020
Received in revised form 8 July 2022
Accepted 22 November 2022
Available online 30 December 2022

Keywords:
Complex networks
Network topology design
Robustness
Positive systems

ABSTRACT

This paper investigates the impact of addition/removal/reweighting of edges in a complex networked linear control system. For networks of positive edge weights, we show that when adding edges leads to the creation of new cycles, these in turn may lead to instabilities. Dynamically, these cycles correspond to positive feedback loops. Conditions are provided under which the modified network is guaranteed to be stable. These conditions are related to the steady state value of the transfer function matrix of the newly created positive feedbacks. The tools we develop in the paper can be used to investigate the fragility of a network, i.e., its robustness to structured perturbations.

1. Introduction

In a networked control system, a possible degree of freedom that can be leveraged upon for perturbation analysis and/or performance improvement is based on rewiring the network by addition/ removal/reweighting of edges. This approach is promising, given the significant impact that network topology has on system performances (see for instance (Bianchin, Pasqualetti, & Zampieri, 2015)). Edge modifications are often feasible in applications and correspond to changes in e.g. connectivity of smart grids or traffic routing. There are a few studies using this approach to investigate controllability: for a given budget of edges and weights that can be added, Chanekar, Nozari, and Cortés (2019) applies differential analysis for maximization of the trace of the Gramian control energy metric. In Becker, Pequito, Pappas, and Preciado (2017), re-weighting of existing edges is applied in order to reduce the worst case control energy, as measured by the minimal eigenvalue of the Gramian. Edge addition in consensus networks has received more attention, the focus being often on network robustness to external disturbances (Hassan-Moghaddam, Wu, & Jovanović, 2017; Siami & Motee, 2017; Zhang, Chen, & Mo, 2017). A natural problem that emerges in this context is to investigate the fragility of a network, i.e., its sensitivity to perturbations (Pasqualetti, Favaretto, Zhao, & Zampieri, 2018). If for general linear systems the problem can be reformulated in terms of stability radius (Hinrichsen & Pritchard, 1986), by treating edge modifications as structured perturbations, a more network-tailored fragility analysis can be set up. A similar problem is investigated e.g. in Hara, Tanaka, and Iwasaki (2014) using transfer function representations of interconnected systems and LMIs. Other studies instead focus on investigating controllability and robustness in correspondence of edge failures (Gundeti, Moothedath, & Chaporkar, 2021; Rahimian & Aghdam, 2013).

In this paper we consider edge modifications in discrete-time linear networks with input/output nodes. The focus is on stable networks with positive edge weights. For this important class (appearing often in applications) we rely on the theory of positive systems in order to derive several new results. We show that the addition of edges or the increase of the weight of existing edges may render a stable network unstable if the weights are large and if new cycles appear in the network. For a single edge modification, the stability margin associated with the edge addition/increase coincides with the maximal weight by which the edge can be modified without causing instability, and can be computed explicitly. Necessary and sufficient conditions for stability are given also for multiple, simultaneous edge additions/increases. These conditions can be used to obtain bounds on the admissible values of the edge weights modifications. A transfer function formulation for the changes in output caused by edge modifications is also derived. Such transfer function sheds light into the origin of instability. Namely, it shows that when the addition of one or more edges creates new cycles in the graph, then these new cycles correspond dynamically to positive

☆ Work supported in part by a grant from the Swedish Research Council (grant no. 2020-03701 to C.A.). The material in this paper was partially presented at the 21st IFAC World Congress (IFAC 2020), July 12–17, 2020, Berlin, Germany (Lindmark and Altafini, 2020). This paper was recommended for publication in revised form by Associate Editor Julien M. Hendrickx under the direction of Editor Christos G. Cassandras.

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https://doi.org/10.1016/j.automatica.2022.110801

feedback loops, which can trigger instability. In particular, we show that the stability margins can be expressed in terms of the steady state value of the transfer function matrix of these new feedback loops.

2. Preliminaries

2.1. Notation

$\mathbb{R}^+$ is the set of non-negative real numbers, $\mathbb{N}$ the set of natural numbers and $\mathbb{N}_0$ the set of natural numbers including zero. Given a matrix $P \in \mathbb{R}^{n\times m}$, let P_{ij} denote the element on row i and column j. For P and Q two matrices of the same dimension, $P \geqslant Q$ should be interpreted element-vise, i.e. $P_{ij} \geqslant Q_{ij}$ $\forall i, j$. The spectral radius of the square matrix $P \in \mathbb{R}^{n\times n}$ is denoted by $\rho(P)$, while $\bar{\sigma}(P)$ denotes its maximal singular value. The jth vector of the canonical basis of $\mathbb{R}^n$ is denoted e_j, $j = 1, \ldots, n$, while a collection of canonical vectors for the set of indices $\mathcal{J} = \{j_1, \ldots, j_{\mathcal{J}}\}$ (of cardinality $n_{\mathcal{J}} = |\mathcal{J}|$ and with $j_k \in \{1, \ldots, n\}$) is denoted $E_{\mathcal{J}} = \begin{bmatrix} e_{j_1} & \ldots & e_{j_{\mathcal{J}}} \end{bmatrix} \in \mathbb{R}^{n\times n_{\mathcal{J}}}$. Notice that indices may be repeated in $\mathcal{J}$. Given $\mathcal{J}$ and $\mathcal{K} = \{k_1, \ldots, k_{\mathcal{K}}\}$ of the same cardinality $n_{\mathcal{J}}$ and $Q \in \mathbb{R}^{n\times n}$, the matrix $P = E_{\mathcal{J}}^\top Q E_{\mathcal{K}} \in \mathbb{R}^{n_{\mathcal{J}}\times n_{\mathcal{J}}}$ is just a selection of the entries of Q:

$$P = E_{\mathcal{J}}^\top Q E_{\mathcal{K}} = \begin{bmatrix} Q_{j_1 k_1} & \ldots Q_{j_1 k_{\mathcal{K}}} \\ \vdots & \\ Q_{j_{\mathcal{J}} k_1} & \ldots Q_{j_{\mathcal{J}} k_{\mathcal{K}}} \end{bmatrix} =: Q_{\mathcal{J}\mathcal{K}}. \tag{1}$$

The canonical matrices $E_{\mathcal{J}}$ and $E_{\mathcal{K}}$ will also be used to form $n \times n$ matrices from $n_{\mathcal{J}} \times n_{\mathcal{J}}$ ones: $Q = E_{\mathcal{K}} P E_{\mathcal{J}}^\top$.

The identity matrix is I (or I_m when it is useful to emphasize the dimension m). For matrices of compatible dimensions, the Sherman–Morrison–Woodbury (aka matrix inversion) formula is

$$(F + PHQ)^{-1} = F^{-1} - F^{-1}P(I + HQF^{-1}P)^{-1}HQF^{-1}. \tag{2}$$

A matrix P is said a *Z-matrix* if $P_{ij} \leqslant 0$ $\forall$ $i \neq j$. A Z-matrix can always be written as $P = \alpha I - Q$ where $Q \geqslant 0$ and $\alpha \in \mathbb{R}$. A Z-matrix $P = \alpha I - Q$ is said an M-matrix if $\alpha > \rho(Q)$. A property of M-matrices is that they are nonsingular and their inverses are nonnegative.

Proposition 1 (*Horn and Johnson (1994), Thm 2.5.3.17*)**.** *A Z-matrix P is an M-matrix if and only if $P^{-1} \geqslant 0$.*

Furthermore, M-matrices have all eigenvalues with positive real part: $\mathrm{Re}[\lambda_i(P)] > 0$, $i = 1, \ldots, n$. The following is a special case of Theorem 6 of Altafini and Lini (2015), for Z-matrices.

Lemma 1. *Consider a matrix $P = D - Q$ where $Q \geqslant 0$ irreducible and $D = \mathrm{diag}(d_1, \ldots, d_n)$, $d_i > 0$ $i = 1, \ldots, n$.*

(1) If $d_i \geqslant \rho(Q)$ $\forall i$ and $d_i > \rho(Q)$ for some i, then $\mathrm{Re}[\lambda_i(P)] > 0$, $i = 1, \ldots, n$;
(2) If $d_i = \rho(Q)$ $\forall i$, then $\mathrm{Re}[\lambda_i(P)] \geqslant 0$, $i = 1, \ldots, n$, and $\mathrm{Re}[\lambda_i(P)] = 0$ for some i;
(3) If $d_i \leqslant \rho(Q)$ $\forall i$ and $d_i < \rho(Q)$ for some i, then $\mathrm{Re}[\lambda_i(P)] < 0$, for some i.

A graph is indicated by the triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes, $\mathcal{E} \subseteq \{(i, j),\ i, j \in \mathcal{V}\}$ is the set of edges (directed from i to j) and $\mathcal{W} = \{w_{ji} \in \mathbb{R},\ i, j \text{ s.t. } (i, j) \in \mathcal{E}\}$ the set of edge weights. The weighted adjacency matrix $A \in \mathbb{R}^{n\times n}$ is defined in such a way that $A_{ji} = w_{ji}$ if $(i, j) \in \mathcal{E}$ and $A_{ji} = 0$ otherwise. A path in $\mathcal{G}$ is a subgraph of nodes $\mathcal{V}^* = \{i_1, \ldots, i_j\}$ and edges $\mathcal{E}^* = \{(i_1, i_2), \ldots, (i_{j-1}, i_j)\}$. The path is directed from i_1 to i_j and it is denoted $i_1 \to i_j$.

2.2. Network model

In this work we consider a linear networked system represented by the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{W})$. Each external input is assumed to act only on one node which is then called an *input node*. The set of input nodes is $\mathcal{I} \subseteq \mathcal{V}$, $|\mathcal{I}| = n_{\mathcal{I}}$, and it is represented by the input matrix $E_{\mathcal{I}} \in \mathbb{R}^{n\times n_{\mathcal{I}}}$. Similarly, the *output nodes* are given by the set $\mathcal{O} \subseteq \mathcal{V}$, $|\mathcal{O}| = n_{\mathcal{O}}$, and represented by the output matrix $E_{\mathcal{O}}^\top \in \mathbb{R}^{n_{\mathcal{O}}\times n}$. If we choose as state update matrix the adjacency matrix A, then our model of discrete-time, linear, time-invariant (LTI) networked system is the following

$$\begin{aligned} x(t + 1) &= Ax(t) + E_{\mathcal{I}}u(t), \\ y(t) &= E_{\mathcal{O}}^\top x(t), \end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state of the network at time $t \in \mathbb{N}_0$, $u(t) \in \mathbb{R}^{n_{\mathcal{I}}}$ is the input vector, and $y(t) \in \mathbb{R}^{n_{\mathcal{O}}}$ the output vector.

When $E_{\mathcal{I}} = E_{\mathcal{O}} = I$, the transfer function of (3), $G^{(A)}(z) = (zI - A)^{-1}$, and the impulse response,

$$g^{(A)}(t) = \begin{cases} 0 & \text{for } t = 0, \\ A^{t-1} & \text{for } t \in \mathbb{N}, \end{cases}$$

are only functions of A (superindex in the notation). For generic sets $\mathcal{I}$, $\mathcal{O}$, using the notation (1), the transfer function becomes

$$G^{(A)}_{\mathcal{O}\mathcal{I}}(z) = E_{\mathcal{O}}^\top (zI - A)^{-1} E_{\mathcal{I}} = \left[\begin{array}{c|c} A & E_{\mathcal{I}} \\ \hline E_{\mathcal{O}}^\top & 0 \end{array}\right](z)$$

where the rightmost block-matrix expression is a standard shorthand representation for the realization $(A, E_{\mathcal{I}}, E_{\mathcal{O}}^\top)$, used e.g. in the robust control literature, see Zhou and Doyle (1998). The z argument in the transfer function will be omitted from now on. The associated impulse response is

$$g^{(A)}_{\mathcal{O}\mathcal{I}}(t) = E_{\mathcal{O}}^\top g^{(A)}(t) E_{\mathcal{I}} = \begin{cases} 0 & \text{for } t = 0, \\ E_{\mathcal{O}}^\top A^{t-1} E_{\mathcal{I}} & \text{for } t \in \mathbb{N}. \end{cases} \tag{4}$$

Since all the networks considered in this paper have nonnegative edge weights, i.e., $A \geqslant 0$, the system (3) is a case of positive system.

Definition 1. The linear system $(A,\ E_{\mathcal{I}},\ E_{\mathcal{O}}^\top)$ is said to be *externally positive* if its forced output is non-negative for every non-negative input function and 0 initial state. It is said to be *positive* if for every non-negative initial state and for every non-negative input, both its state and outputs are non-negative.

Clearly, positivity implies external positivity but not viceversa. A necessary and sufficient condition for $(A,\ E_{\mathcal{I}},\ E_{\mathcal{O}}^\top)$ to be externally positive is that the impulse response is non-negative. For our choice of inputs and outputs (corresponding to $E_{\mathcal{I}} \geqslant 0$ and $E_{\mathcal{O}} \geqslant 0$), $(A,\ E_{\mathcal{I}},\ E_{\mathcal{O}}^\top)$ is positive if and only if $A \geqslant 0$ (Farina & Rinaldi, 2011).

Internal stability of a positive system of the form (3) holds true if $\rho(A) < 1$. The networks considered in this paper are always assumed internally stable. The following results will be useful when investigating the internal stability of positive systems.

Proposition 2 (*Farina & Rinaldi, 2011*)**.** *For $A \geqslant 0$, $(I - A)^{-1}$ exist and is non-negative if and only if $\rho(A) < 1$.*

Proposition 3 (*Horn and Johnson (1985), Cor. 8.1.19*)**.** *For A, $B \in \mathbb{R}^{n\times n}$ such that $0 \leqslant B \leqslant A$ it is $0 \leqslant \rho(B) \leqslant \rho(A)$.*

For the network model (3), with $A \geqslant 0$, $\rho(A) < 1$ and input/output sets $\mathcal{I}$, $\mathcal{O}$, the steady state transfer function is

$$\begin{aligned} G^{(A)}_{\mathcal{O}\mathcal{I}}(e^{i\omega})\Big|_{\omega=0} &= G^{(A)}_{\mathcal{O}\mathcal{I}}(1) = E_{\mathcal{O}}^\top \left(I + A + A^2 + \cdots\right) E_{\mathcal{I}} \\ &= E_{\mathcal{O}}^\top (I - A)^{-1} E_{\mathcal{I}}. \end{aligned} \tag{5}$$

In this paper we sometimes consider input/output relations between other sets of nodes than $\mathcal{I}$ and $\mathcal{O}$. For instance we will consider the transfer function from a set of "source" nodes $\mathcal{S}$ to a set of "target" nodes $\mathcal{T}$ (as well as from $\mathcal{T}$ to $\mathcal{S}$). All expressions above can be easily rewritten with $\mathcal{S}, \mathcal{T}$ in place of $\mathcal{I}, \mathcal{O}$.

3. Main result

Consider a network given by the state update matrix $A \geqslant 0$ and the input/output sets $\mathcal{I}$ and $\mathcal{O}$. Assume that ν edges, corresponding to the node pairs $(\mathfrak{s}_1, \mathfrak{t}_1), \dots (\mathfrak{s}_\nu, \mathfrak{t}_\nu)$, are modified with the weights $w_1, \dots, w_\nu$. In this notation, $\mathfrak{s}_i$ is the source node and $\mathfrak{t}_i$ is the target node of the edge being modified. Denote $\mathcal{S} = \{\mathfrak{s}_1, \dots, \mathfrak{s}_\nu\}$, $\mathcal{T} = \{\mathfrak{t}_1, \dots, \mathfrak{t}_\nu\}$ and $E_\mathcal{S}$, $E_\mathcal{T}$ the corresponding collection of elementary vectors. Let further $W = \operatorname{diag}(w_1, \dots, w_\nu)$ be the diagonal matrix of weights. When the modifications $\{(\mathcal{S}, \mathcal{T}), W\}$ are applied simultaneously, the modified adjacency matrix is

$$\bar{A} = A + \sum_{i=1}^{\nu} e_{\mathfrak{t}_i} w_i e_{\mathfrak{s}_i}^\top = A + E_\mathcal{T} W E_\mathcal{S}^\top. \tag{6}$$

We assume further that the weights w_i in the modifications are either positive (i.e., correspond to edge addition or edge weight increment) or negative but lower bounded by $w_i \geqslant -A_{\mathfrak{t}_i\mathfrak{s}_i}$ (i.e., edge weight reduction, or edge elimination when $w_i = -A_{\mathfrak{t}_i\mathfrak{s}_i}$). In this way the modifications preserve positivity of the system, i.e., both $A \geqslant 0$ and $\bar{A} \geqslant 0$. In the following, we use the triplet $\{(\mathcal{S}, \mathcal{T}), W\}$ to identify the edge modifications.

The main problem of interest in this paper is the following.

Problem 1. Given a network (3) with adjacency matrix $A \geqslant 0$ and input/output sets $\mathcal{I}$, $\mathcal{O}$, understand the impact of the edge modifications $\{(\mathcal{S}, \mathcal{T}), W\}$ on the dynamics, and provide (analytical) conditions under which the edge modifications preserve stability.

Denote $y = G^{(A)}_{\mathcal{O}\mathcal{I}} u$ and $\bar{y} = G^{(\bar{A})}_{\mathcal{O}\mathcal{I}} u$ the outputs of the networks associated to A and $\bar{A}$. For a given input u, the difference

$$y^\delta = \bar{y} - y = \left(G^{(\bar{A})}_{\mathcal{O}\mathcal{I}} - G^{(A)}_{\mathcal{O}\mathcal{I}} \right) u \tag{7}$$

is the change in the states of the output nodes due to the edge modifications (6). The corresponding transfer function,

$$G^\delta = G^{(\bar{A})}_{\mathcal{O}\mathcal{I}} - G^{(A)}_{\mathcal{O}\mathcal{I}}, \tag{8}$$

is from now on referred to as the *delta system*.

We seek an expression for the transfer function of the delta system that depends explicitly on $\mathcal{S}, \mathcal{T}$ and W, but not on $\bar{A}$.

Proposition 4. *Consider a network (3) with adjacency matrix $A \geqslant 0$, input/output sets $\mathcal{I}$, $\mathcal{O}$, and the edge modifications $\{(\mathcal{S}, \mathcal{T}), W\}$. The transfer function of the delta system is*

$$\begin{aligned} G^\delta &= G^{(A)}_{\mathcal{O}\mathcal{T}} (I_\nu - W G^{(A)}_{\mathcal{S}\mathcal{T}})^{-1} W G^{(A)}_{\mathcal{S}\mathcal{I}} \\ &= \begin{bmatrix} G^{(A)}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(A)}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} \left(I_\nu - W G^{(A)}_{\mathcal{S}\mathcal{T}} \right)^{-1} W \begin{bmatrix} G^{(A)}_{\mathfrak{s}_1\mathcal{I}} \\ \vdots \\ G^{(A)}_{\mathfrak{s}_\nu\mathcal{I}} \end{bmatrix} \end{aligned} \tag{9}$$

Proof. Writing the difference $G^\delta = G^{(\bar{A})}_{\mathcal{O}\mathcal{I}} - G^{(A)}_{\mathcal{O}\mathcal{I}}$ in terms of block-matrices for the corresponding realizations, and performing elementary operations ((7) is a parallel interconnection of systems) we get

$$G^\delta = \left[\begin{array}{c|c} \bar{A} & E_\mathcal{I} \\ \hline E_\mathcal{O}^\top & 0 \end{array} \right] - \left[\begin{array}{c|c} A & E_\mathcal{I} \\ \hline E_\mathcal{O}^\top & 0 \end{array} \right] = \left[\begin{array}{cc|c} \bar{A} & 0 & E_\mathcal{I} \\ 0 & A & E_\mathcal{I} \\ \hline E_\mathcal{O}^\top & -E_\mathcal{O}^\top & 0 \end{array} \right].$$

The rightmost formulation corresponds to the state vector $[\bar{x}^\top x^\top]^\top$, where $\bar{x}$ is the state of the modified network and x that of the original network. Define the state transformation

$$\begin{bmatrix} \tilde{x} \\ x \end{bmatrix} = \begin{bmatrix} \bar{x} - x \\ x \end{bmatrix} = \begin{bmatrix} I & -I \\ 0 & I \end{bmatrix} \begin{bmatrix} \bar{x} \\ x \end{bmatrix}, \text{ with inverse}$$

$$\begin{bmatrix} \bar{x} \\ x \end{bmatrix} = \begin{bmatrix} I & I \\ 0 & I \end{bmatrix} \begin{bmatrix} \tilde{x} \\ x \end{bmatrix}.$$

Changing basis,

$$\left[\begin{array}{cc|c} I & -I & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right] \left[\begin{array}{cc|c} \bar{A} & 0 & E_\mathcal{I} \\ 0 & A & E_\mathcal{I} \\ \hline E_\mathcal{O}^\top & -E_\mathcal{O}^\top & 0 \end{array} \right] \left[\begin{array}{cc|c} I & I & 0 \\ 0 & I & 0 \\ \hline 0 & 0 & I \end{array} \right] = \left[\begin{array}{cc|c} \bar{A} & \bar{A} - A & 0 \\ 0 & A & E_\mathcal{I} \\ \hline E_\mathcal{O}^\top & 0 & 0 \end{array} \right]$$

from which

$$\begin{aligned} G^\delta &= \left[\begin{array}{cc|c} \bar{A} & E_\mathcal{T} W E_\mathcal{S}^\top & 0 \\ 0 & A & E_\mathcal{I} \\ \hline E_\mathcal{O}^\top & 0 & 0 \end{array} \right] = \sum_{i=1}^{\nu} \left[\begin{array}{cc|c} \bar{A} & e_{\mathfrak{t}_i} w_i e_{\mathfrak{s}_i}^\top & 0 \\ 0 & A & E_\mathcal{I} \\ \hline E_\mathcal{O}^\top & 0 & 0 \end{array} \right] \\ &= \sum_{i=1}^{\nu} \left[\begin{array}{c|c} \bar{A} & e_{\mathfrak{t}_i} \\ \hline E_\mathcal{O}^\top & 0 \end{array} \right] w_i \left[\begin{array}{c|c} A & E_\mathcal{I} \\ \hline e_{\mathfrak{s}_i}^\top & 0 \end{array} \right] = \sum_{i=1}^{\nu} G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_i} w_i G^{(A)}_{\mathfrak{s}_i\mathcal{I}} \\ &= \begin{bmatrix} G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} W \begin{bmatrix} G^{(A)}_{\mathfrak{s}_1\mathcal{I}} \\ \vdots \\ G^{(A)}_{\mathfrak{s}_\nu\mathcal{I}} \end{bmatrix} = G^{(\bar{A})}_{\mathcal{O}\mathcal{T}} W G^{(A)}_{\mathcal{S}\mathcal{I}}. \end{aligned} \tag{10}$$

Analogous calculations lead, for $i = 1, \dots, \nu$, to

$$G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_i} - G^{(A)}_{\mathcal{O}\mathfrak{t}_i} = \begin{bmatrix} G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} W \begin{bmatrix} G^{(A)}_{\mathfrak{s}_i\mathfrak{t}_1} \\ \vdots \\ G^{(A)}_{\mathfrak{s}_i\mathfrak{t}_\nu} \end{bmatrix}$$

and hence to

$$\begin{bmatrix} G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} = \begin{bmatrix} G^{(A)}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(A)}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} + \begin{bmatrix} G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} W \underbrace{\begin{bmatrix} G^{(A)}_{\mathfrak{s}_1\mathfrak{t}_1} & \dots & G^{(A)}_{\mathfrak{s}_1\mathfrak{t}_\nu} \\ \vdots & & \vdots \\ G^{(A)}_{\mathfrak{s}_\nu\mathfrak{t}_1} & \dots & G^{(A)}_{\mathfrak{s}_\nu\mathfrak{t}_\nu} \end{bmatrix}}_{= G^{(A)}_{\mathcal{S}\mathcal{T}}}.$$

Gathering all terms in $\bar{A}$ on the left hand side:

$$\begin{bmatrix} G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} \left(I - W G^{(A)}_{\mathcal{S}\mathcal{T}} \right) = \begin{bmatrix} G^{(A)}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(A)}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix}$$

i.e.,

$$\begin{bmatrix} G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(\bar{A})}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} = \begin{bmatrix} G^{(A)}_{\mathcal{O}\mathfrak{t}_1} & \dots & G^{(A)}_{\mathcal{O}\mathfrak{t}_\nu} \end{bmatrix} \left(I - W G^{(A)}_{\mathcal{S}\mathcal{T}} \right)^{-1}$$

Substituting this expression into (10), (9) follows. ■

3.1. Interpretation: feedback along reverse paths

In order to understand the meaning of the expression (9) it is convenient to look first at the single edge modification case $\{(\mathfrak{s}, \mathfrak{t}), w\}$, illustrated in block diagram form in Fig. 1. For the single edge case the equivalent of (9) is

$$G^\delta = G^{(A)}_{\mathcal{O}\mathfrak{t}} \left(I - w G^{(A)}_{\mathfrak{s}\mathfrak{t}} \right)^{-1} w G^{(A)}_{\mathfrak{s}\mathcal{I}}. \tag{11}$$

We can view G^δ in (11) as composed of three parts through which the edge modification $\{(\mathfrak{s}, \mathfrak{t}), w\}$ perturbs the network

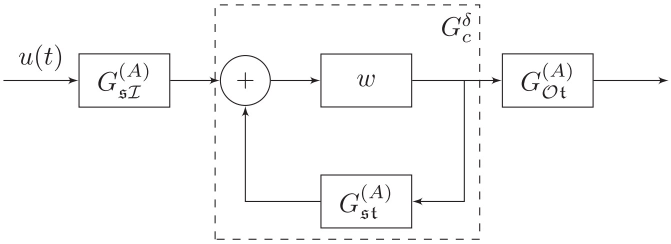

Fig. 1. Block diagram of the delta system G^{δ} of (11).

transfer function. These three parts reflect the structure of the underlying graph $\mathcal{G}$. The first and last parts are the transfer functions $G^{(A)}_{\mathfrak{s}\mathcal{I}}$ (from the nodes $\mathcal{I}$ to node $\mathfrak{s}$) and $G^{(A)}_{\mathcal{O}\mathfrak{t}}$ (from the node $\mathfrak{t}$ to the nodes $\mathcal{O}$) respectively. The middle part of G^{δ},

$$G^{\delta}_{c} = \left(1 - wG^{(A)}_{\mathfrak{s}\mathfrak{t}}\right)^{-1} w, \tag{12}$$

expresses the overall transfer function for the new edge $(\mathfrak{s}, \mathfrak{t})$ (i.e., from $\mathfrak{s}$ to $\mathfrak{t}$).

Interestingly, (12) shows that G^{δ}_{c} depends on the transfer function $G^{(A)}_{\mathfrak{s}\mathfrak{t}}$ of the "reverse path" $\mathfrak{t} \to \mathfrak{s}$. In particular, when $w > 0$, G^{δ}_{c} in (12) corresponds to a positive feedback loop containing $G^{(A)}_{\mathfrak{s}\mathfrak{t}}$. The fact that it has the structure of a positive feedback anticipates that it might be a source of instabilities, as we will see in detail in the next subsection. If instead the edge modification corresponds to a negative edge weight $-A_{\mathfrak{t}\mathfrak{s}} \leqslant w < 0$, then G^{δ}_{c} in (12) becomes a negative feedback, which does not lead to instabilities when concatenated in a series connection as in Fig. 1.

For multiple edge modifications, the interpretation of (9) is similar to that of the single edge modification case. In particular, when $w_i > 0$ for all i

$$G^{\delta}_{c} = \left(I_{\nu} - WG^{(A)}_{\mathcal{S}\mathcal{T}}\right)^{-1} W \tag{13}$$

still has the meaning of ($\nu \times \nu$, i.e., MIMO) positive feedback as in Fig. 1. In this case, however, all possible paths $\mathfrak{t}_i \to \mathfrak{s}_j \; \forall \, \mathfrak{t}_i \in \mathcal{T}$ and $\mathfrak{s}_j \in \mathcal{S}$ enter into the transfer function G^{δ}_{c}, i.e., not only the "reverse paths" $\mathfrak{t}_i \to \mathfrak{s}_i$ connecting the pairs $(\mathfrak{s}_i, \mathfrak{t}_i)$ for which an edge is being modified, but also all possible "cross-paths" $\mathfrak{t}_i \to \mathfrak{s}_j$, $j \neq i$, including those that do not correspond to any modified edge.

Remark 1. Edge modification/addition for control of large scale networks is studied with differential analysis in Chanekar et al. (2019). Such analysis is only valid when the weights W are small, corresponding to the approximation $G^{\delta} \approx G^{(A)}_{\mathcal{O}\mathcal{T}} W G^{(A)}_{\mathcal{S}\mathcal{I}}$. However, as the w_i increase, the effect of the feedback loop quickly becomes significant, and its effect on stability cannot be neglected.

3.2. Stability

If we consider the single edge modification $\{(\mathfrak{s}, \mathfrak{t}), w\}$ described in Section 3.1 and illustrated in Fig. 1, then we can interpret G^{δ} in (11) also in terms of cycles in the network graph $\mathcal{G}$: $G^{(A)}_{\mathfrak{s}\mathfrak{t}} > 0$ if there is a path $\mathfrak{t} \to \mathfrak{s}$ in $\mathcal{G}$; this path forms a cycle with the modified edge $\{(\mathfrak{s}, \mathfrak{t}), w\}$. As mentioned in Section 3.1, only edge additions that create new cycles, or edge modifications that increase the weight of an existing edge that is part of a cycle, may cause instability because they are associated to positive feedback loops G^{δ}_{c}. Edge removal or reduction of the weight of an existing edge in a positive network will on the other hand never cause instability (Farina & Rinaldi, 2011, p. 43), see also Proposition 3.

For multiple edge modifications the situation is similar, and it is shown next that stability of the modified system $\bar{A}$ depends only on the edge additions/increments and not on the edge reductions/eliminations. To do so, it is convenient to split $\{(\mathcal{S}, \mathcal{T}), W\}$ into two subsets according to the sign of the corresponding w_i: $\mathcal{S} = \mathcal{S}^{+} \cup \mathcal{S}^{-}$, $\mathcal{T} = \mathcal{T}^{+} \cup \mathcal{T}^{-}$, where $w_i > 0 \; \forall \, (\mathfrak{s}_i, \mathfrak{t}_i) \in \mathcal{S}^{+} \times \mathcal{T}^{+}$, and $-A_{\mathfrak{t}_i\mathfrak{s}_i} \leqslant w_i < 0 \; \forall \, (\mathfrak{s}_i, \mathfrak{t}_i) \in \mathcal{S}^{-} \times \mathcal{T}^{-}$. Denote W^{+} and W^{-} the corresponding diagonal submatrices, of size $\nu^{+} = |\mathcal{S}^{+}| = |\mathcal{T}^{+}|$ and $\nu^{-} = |\mathcal{S}^{-}| = |\mathcal{T}^{-}|$ respectively. Denote further $\bar{A}^{-} = A + E_{\mathcal{T}^{-}} W^{-} E^{\top}_{\mathcal{S}^{-}}$ the matrix obtained by considering only the negative edge modifications.

Next theorem provides necessary and sufficient conditions for internal stability in presence of edge modifications for two different cases.

Theorem 1. *Consider a network (3) with adjacency matrix $A \geqslant 0$, $\rho(A) < 1$, and the (simultaneous) edge modifications $\{(\mathcal{S}, \mathcal{T}), W\}$. The modified network (6) is internally stable if and only if*
Case 1 *(All positive edge modifications: $w_i > 0 \; \forall \, i = 1, \ldots, \nu$):*

$$I_{\nu} - WE^{\top}_{\mathcal{S}}(I - A)^{-1}E_{\mathcal{T}} \tag{14}$$

is an M-matrix.
Case 2 *(Edge modifications of mixed signs: $w_i \geqslant -A_{\mathfrak{t}_i\mathfrak{s}_i} \forall \, i = 1, \ldots, \nu$):*

$$I_{\nu^{+}} - W^{+}E^{\top}_{\mathcal{S}^{+}}(I - \bar{A}^{-})^{-1}E_{\mathcal{T}^{+}} \tag{15}$$

is an M-matrix.

Proof.
Case 1. *Sufficiency*: Applying the Sherman–Morrison–Woodbury formula:

$$\begin{aligned}(I - \bar{A})^{-1} &= (I - A - E_{\mathcal{T}} W E^{\top}_{\mathcal{S}})^{-1} \\ &= (I - A)^{-1} + (I - A)^{-1}E_{\mathcal{T}} \\ &\quad \cdot \left(I_{\nu} - WE^{\top}_{\mathcal{S}}(I - A)^{-1}E_{\mathcal{T}}\right)^{-1} WE^{\top}_{\mathcal{S}}(I - A)^{-1}.\end{aligned} \tag{16}$$

Since $(I - A)^{-1} \geqslant 0$, if $I_{\nu} - WE^{\top}_{\mathcal{S}}(I - A)^{-1}E_{\mathcal{T}}$ is an M-matrix, then, from Proposition 1, its inverse is a nonnegative matrix. Hence the entire expression for $(I - \bar{A})^{-1}$ is nonnegative and the result follows from Proposition 2.

Necessity: When $w_i > 0 \; \forall \, i = 1, \ldots, \nu$, then by construction (14) is a Z-matrix. In fact, from Proposition 2, A stable $\Longrightarrow$ $(I - A)^{-1} \geqslant 0 \Longrightarrow E^{\top}_{\mathcal{S}}(I - A)^{-1}E_{\mathcal{T}} \geqslant 0$, hence (14) has all off-diagonal elements $\leqslant 0$. If we apply again the Sherman–Morrison–Woodbury formula to it:

$$\begin{aligned}&\left(I_{\nu} - WE^{\top}_{\mathcal{S}}(I - A)^{-1}E_{\mathcal{T}}\right)^{-1} = \\ &\quad = I_{\nu} + WE^{\top}_{\mathcal{S}}(I - A - E_{\mathcal{T}}WE^{\top}_{\mathcal{S}})^{-1}E_{\mathcal{T}} \\ &\quad = I_{\nu} + WE^{\top}_{\mathcal{S}}(I - \bar{A})^{-1}E_{\mathcal{T}}.\end{aligned}$$

Since $\bar{A}$ is stable, it is $(I - \bar{A})^{-1} \geqslant 0$, hence

$$\left(I_{\nu} - WE^{\top}_{\mathcal{S}}(I - A)^{-1}E_{\mathcal{T}}\right)^{-1} \geqslant 0.$$

Now, $I_{\nu} - WE^{\top}_{\mathcal{S}}(I - A)^{-1}E_{\mathcal{T}}$ is a Z-matrix characterized by a nonnegative inverse. It follows from Proposition 1 that it must be an M-matrix.

Case 2. Since $\bar{A}^{-}$ is obtained subtracting the negative edge modifications from A, by construction it is $0 \leqslant \bar{A}^{-} \leqslant A$, hence from Proposition 3 it is $0 \leqslant \rho(\bar{A}^{-}) \leqslant \rho(A)$, i.e., $\bar{A}^{-}$ is always internally stable. Since

$$\bar{A} = \underbrace{A + E_{\mathcal{T}^{-}} W^{-} E^{\top}_{\mathcal{S}^{-}}}_{\bar{A}^{-}} + E_{\mathcal{T}^{+}} W^{+} E^{\top}_{\mathcal{S}^{+}},$$

we can now repeat the arguments used in Case 1, replacing A with $\bar{A}^-$. For instance, (16) becomes

$$(I-\bar{A})^{-1} = (I-\bar{A}^- - E_{\mathcal{T}^+}W^+E_{\mathcal{S}^+}^\top)^{-1}$$
$$= (I-\bar{A}^-)^{-1} + (I-\bar{A}^-)^{-1}E_{\mathcal{T}^+}$$
$$\cdot \left(I_{\nu^+} - W^+E_{\mathcal{S}^+}^\top(I-\bar{A}^-)^{-1}E_{\mathcal{T}^+}\right)^{-1} W^+E_{\mathcal{S}^+}^\top(I-\bar{A}^-)^{-1}.$$

where $(I-\bar{A}^-)^{-1} \geqslant 0$. ■

Remark 2. Notice that the matrix (15) being an M-matrix is not equivalent to the following matrix

$$I_{\nu^+} - W^+E_{\mathcal{S}^+}^\top(I-A)^{-1}E_{\mathcal{T}^+} \tag{17}$$

being an M-matrix. In words: checking whether the addition of the positive edge modifications preserve stability in $\bar{A}$ without first subtracting the negative edge modifications may lead to a wrong result. See Example 2 in Section 4.2 for a counterexample. Similar problems occur when checking the M-matrix property directly on (14) (and negative edge modifications are present).

When $\nu = 1$ (single edge modification) we obtain an exact upper bound on the value of w that preserves internal stability, regardless of the sign of w.

Corollary 1. *Under the same assumptions as in Theorem 1, consider the single edge modification $\{(\mathfrak{s}, \mathfrak{t}), w\}$. If the original network has no path $\mathfrak{t} \to \mathfrak{s}$, then the modified network is internally stable (and positive) for any $w \geqslant -A_{\mathfrak{ts}}$. On the other hand, if there is a path $\mathfrak{t} \to \mathfrak{s}$, then the modified network is internally stable (and positive) if and only if $-A_{\mathfrak{ts}} \leqslant w < 1/\left((I-A)^{-1}\right)_{\mathfrak{st}}$.*

The proof follows from Theorem 1 and Corollary 3. See also (Lindmark & Altafini, 2020) for a self-contained proof.

3.3. Multiple edge additions/increments that preserve stability: a constructive condition

While in the single edge modification case Corollary 1 readily provides an upper bound on the stability-preserving edge modification (i.e., $w < 1/((I-A)^{-1})_{\mathfrak{st}}$), in the multiple edge case the values w_i only enter implicitly into the M-matrix condition of (14). From it, it is however possible to provide explicit upper bounds for w_i.

Since it follows from Theorem 1 that negative edge modifications do not jeopardize stability, in the rest of this section it is enough to concentrate on positive edge modifications only: $w_i > 0 \; \forall i = 1, \ldots, \nu$ (Case 1 in Theorem 1).

Theorem 2. *Consider a network (3) with adjacency matrix $A \geqslant 0$, $\rho(A) < 1$, and the positive edge modifications $\{(\mathcal{S}, \mathcal{T}), W\}$. Assume that the $\nu \times \nu$ matrix $E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}$ is irreducible. Then the modified network (6) is internally stable if*

$$w_i \leqslant \frac{1}{\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right)} \quad \forall\, i = 1, \ldots, \nu \tag{18}$$

and

$$w_i < \frac{1}{\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right)} \quad \text{for at least one } i. \tag{19}$$

Proof. Notice that (14) is an M-matrix if and only if

$$W^{-1} - E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T} \tag{20}$$

is an M-matrix, since both are Z-matrices and have nonnegative inverses simultaneously. If $E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}$ is irreducible, then (20) obeys Lemma 1, hence if

$$\frac{1}{w_i} \geqslant \rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right) \quad \forall\, i$$

and

$$\frac{1}{w_i} > \rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right) \quad \text{for at least one } i$$

(i.e., (18) and (19) hold) then (20) is an M-matrix, hence so is (14). It follows from Theorem 1 that the modified network is internally stable. ■

Corollary 2. *Under the same assumptions as in Theorem 2, if*

$$w_i \geqslant \frac{1}{\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right)} \quad \forall\, i = 1, \ldots, \nu$$

and

$$w_i > \frac{1}{\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right)} \quad \text{for at least one } i$$

then the modified network is unstable.

Proof. Just apply item (3) of Lemma 1. ■

In between the two cases discussed in Theorem 2 and Corollary 2, there is a gap, in which internal stability cannot be established just by looking at the edge weights. For multiedge addition, the only case in which stability of the augmented network can be formulated as a necessary and sufficient condition in terms of the w_i is when all weights are identical.

Corollary 3. *Under the same assumptions as in Theorem 2, if $w_i = w_j = w > 0 \; \forall\, i = 1, \ldots, \nu$, then the modified network is internally stable if and only if*

$$w < \frac{1}{\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right)}.$$

Proof. In this case (14) can be written as

$$I_\nu - wE_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}$$

which is an M-matrix if and only if $w\rho(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}) < 1$. ■

3.4. Stability and steady state delta system

We now show that (14) (an hence (18) and (19)) are related to the feedback transfer function matrix G_c^δ at steady state. For that it is enough to express $G_c^\delta(1)$ at steady state.

Lemma 2.

$$G_c^\delta(1) = (I_\nu - WE_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T})^{-1}W \tag{21}$$
$$= W + WE_\mathcal{S}^\top(I-A-E_\mathcal{T}WE_\mathcal{S}^\top)^{-1}E_\mathcal{T}W. \tag{22}$$

Proof. Eq. (21) is straightforward: from (5) and (13)

$$G_c^\delta(1) = (I_\nu - WG_{\mathcal{ST}}^{(A)}(1))^{-1}W$$
$$= (I_\nu - WE_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T})^{-1}W.$$

As for (22), just apply (2):

$$G_c^\delta(1) = (I_\nu + WE_\mathcal{S}^\top(I-A-E_\mathcal{T}WE_\mathcal{S}^\top)^{-1}E_\mathcal{T})W,$$

and the expression follows. ■

Eq. (21) tells us that the for positive edge modifications the matrix investigated in Theorem 1 is basically $G_c^\delta(1)$.

Remark 3. It is well known that for positive systems several dynamical properties are expressible in terms of the static "DC-gain" (e.g. $\mathcal{H}_\infty$ norm, see Lindmark and Altafini (2020), Rantzer (2011), Tanaka and Langbort (2011)). Theorem 2 and Lemma 2 provide a novel one: internal stability upon positive edge modifications can be checked simply by looking at the steady state $G_c^\delta(1)$.

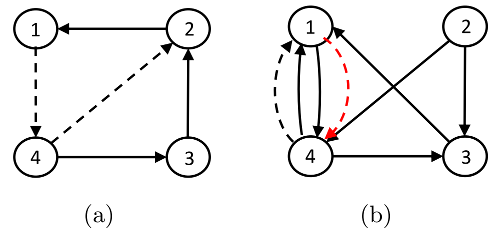

Fig. 2. (a): Example 1. (b): Example 2. For both, edged in A are solid lines. Black (resp. red) dashed lines are positive (resp. negative) edge modifications. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

Remark 4. Notice that from (6), Eq. (22) can be rewritten as

$$G_c^\delta(1) = W + WE_\mathcal{S}^\top(I-\bar{A})^{-1}E_\mathcal{T}W = W + WG_{\mathcal{ST}}^{(\bar{A})}(1)W$$

i.e., the steady state feedback transfer function $G_c^\delta(1)$ is expressing how the steady state transfer function of the modified network appears along the "reverse paths" induced by the new edges, weighted by the weights W.

4. Applications and examples

4.1. Network fragility under structured perturbations

Fragility of internally stable networks can be defined in many different ways. In Pasqualetti et al. (2018), fragility refers to the sensitivity of a network to variations in the edge weights and it is quantified by the stability radius,

$$r(A) = \min\{\bar{\sigma}(\Delta) \text{ s.t. } \rho(A+\Delta) \geqslant 1\},$$

i.e. the spectral norm of the smallest change in the network weights that renders it unstable (Hinrichsen & Pritchard, 1986). This definition of fragility assumes no particular structure on the matrix Δ. However, if we restrict Δ to the set of real matrices with non-zero entries only in correspondence of the $(\mathcal{S},\mathcal{T})$ entries, then it represents edge modifications as studied in this paper. In this case, rather that $r(A)$, a more suitable concept for fragility appears to be the maximal amplitude $w_i^{\max}$ tolerable on the edge weights w_i. The bound $w_i^{\max}$ can be computed from Theorem 2. It is exact in the single edge case (Corollary 1).

Proposition 5. *Consider a network (3) with adjacency matrix A, $A \geqslant 0$, $\rho(A) < 1$ and positive edge modifications $\{(\mathcal{S},\mathcal{T}), W\}$.*

(1) Single edge case: If $\Delta = e_t w e_s^\top$ then $w^{\max} = 1/((I-A)^{-1})_{st} = r(A)$;
(2) Multiple edge case: If $\Delta = E_\mathcal{T} W E_\mathcal{S}^\top$ (and $E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}$ is irreducible) then $w_i^{\max} = 1/\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right)$, $i = 1,\dots,\nu$.

Proof. The proof follows straightforwardly from Corollary 1 and Theorem 2. ■

4.2. Numerical examples

Example 1.
Consider the network of Fig. 2(a), with the following adjacency matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0.4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and the two edge additions (4, 2) and (1, 4). We consider first 2 cases in which each edge is added singularly in sequence, and then the case of simultaneous multiedge addition.

- If the first edge to be added is (4, 2), then $((I-A)^{-1})_{42} = 0$, i.e., no new cycle is created, hence, according to Corollary 1, w_1 can be chosen arbitrarily high, say for instance $w_1 = 100$. The augmented matrix is now $\bar{A}_1 = A + 100e_2e_4^\top$. When we add the second edge (1, 4), we have to use $\bar{A}_1$ as current adjacency matrix, and for it the new edge creates cycles (e.g. $4 \to 2 \to 1 \to 4$). Therefore w_2 is upper bounded. In particular it is $w_2 < 1/((I-\bar{A}_1)^{-1})_{14} = 0.01$ i.e., w_2 must be very small in order to preserve stability (because of the large w_1), in spite of $\bar{A}_1$ being nilpotent.
- If instead we switch the order of the two edge additions, adding first (1, 4) to A gives a cycle $4 \to 3 \to 2 \to 1 \to 4$, hence $w_1 < 1.25$. If we choose $w_1 = 1$ then for the second edge (4, 2) we have $w_2 < 0.2$.
- When adding both edges simultaneously, then the condition that the weights w_1 and w_2 have to respect to have stability in the augmented system is described in Theorem 2: we must compute the spectral radius

$$\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right) = \rho\left(\begin{bmatrix} 0.8 & 1 \\ 1 & 0 \end{bmatrix}\right) = 1.477$$

from which $w^{\max} = 1/\rho\left(E_\mathcal{S}^\top(I-A)^{-1}E_\mathcal{T}\right) = 0.677$. Notice that by choosing $w_1 = w_2 = w^{\max}$ the resulting $\bar{A}$ has exactly $\rho(\bar{A}) = 1$, as predicted by Corollary 3. Notice further that both choices of w_i made above violate Theorem 2 but are nevertheless leading to a stable $\bar{A}$ (they satisfy Theorem 1), confirming that the condition of Theorem 2 is sufficient but not necessary (and so is the instability condition of Corollary 2).

Example 2. For the network in Fig. 2(b), it is

$$A = \begin{bmatrix} 0 & 0 & 0.25 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0.25 & 0 & 0.25 \\ 0.75 & 0.25 & 0 & 0 \end{bmatrix}$$

and the edge modifications (4, 1) and (1, 4) have weights 0.85 and −0.1 respectively. In this case it can be checked straightforwardly that $\rho(\bar{A}) < 1$ i.e., the double edge modification preserves stability. However, because of the negative weight, both matrices (14) and (17) are not M-matrices, while instead (15) is.

5. Conclusions

In large scale networks with stable dynamics, the particular structure of the transfer function G^δ that we derive for the changes in network output due to edge modifications enables us to quantify the impact of all possible edge modifications on the network. The impact from modifying the edges $(\mathcal{S},\mathcal{T})$ depends on three network properties: (i) the strength of the connections from the input nodes to $\mathcal{S}$, (ii) that from $\mathcal{T}$ to the output nodes, and (iii) the feedback connections from $\mathcal{T}$ to $\mathcal{S}$. In particular, the third factor appears not to have been observed before in the context of networked control systems. In the case of stable positive dynamics, it provides a stability margin which leads to an upper bound on the admissible edge weights. For single edge modifications the bound is sharp and the stability margin exact.

It is at the moment unclear if any of the conditions we have found can be extended beyond positive systems, for instance to the case of steady state transfer function (5) which is still nonnegative and stable, even tough the adjacency matrix A is not nonnegative. Natural candidates for such an extension would be eventually positive matrices which we have already studied in Altafini and Lini (2015) and Altafini (2016).

References

Altafini, C. (2016). Minimal eventually positive realizations of externally positive systems. *Automatica*, *68*, 140–147.

Altafini, C., & Lini, G. (2015). Predictable dynamics of opinion forming for networks with antagonistic interactions. *IEEE Transactions on Automatic Control*, *60*(2), 342–357.

Becker, C. O., Pequito, S., Pappas, G. J., & Preciado, V. M. (2017). Network design for controllability metrics. In *2017 IEEE 56th conference on decision and control* (pp. 4193–4198). IEEE.

Bianchin, G., Pasqualetti, F., & Zampieri, S. (2015). The role of diameter in the controllability of complex networks. In *2015 54th IEEE conference on decision and control* (pp. 980–985). IEEE.

Chanekar, Nozari, & Cortés (2019). Network modification using a novel gramian-based edge centrality. In *2019 58th IEEE conference on decision and control*. IEEE.

Farina, L., & Rinaldi, S. (2011). *Positive linear systems: theory and applications, vol. 50*. John Wiley & Sons.

Gundeti, R., Moothedath, S., & Chaporkar, P. (2021). Feedback robustness in structured closed-loop system. *European Journal of Control*, *57*, 95–108.

Hara, S., Tanaka, H., & Iwasaki, T. (2014). Stability analysis of systems with generalized frequency variables. *IEEE Transactions on Automatic Control*, *59*(2), 313–326.

Hassan-Moghaddam, S., Wu, X., & Jovanović, M. R. (2017). Edge addition in directed consensus networks. In *2017 American control conference* (pp. 5592–5597). IEEE.

Hinrichsen, D., & Pritchard, A. (1986). Stability radii of linear systems. *Systems & Control Letters*, *7*(1), 1–10.

Horn, R., & Johnson, C. R. (1985). *Matrix analysis*. Cambridge University Press.

Horn, R., & Johnson, C. (1994). *Topics in matrix analysis*. Cambridge University Press.

Lindmark, G., & Altafini, C. (2020). On the impact of edge modifications for networked control systems. In *Proceedings of the 2020 IFAC world congress*.

Pasqualetti, F., Favaretto, C., Zhao, S., & Zampieri, S. (2018). Fragility and controllability tradeoff in complex networks. In *2018 American control conference* (pp. 216–221). IEEE.

Rahimian, M. A., & Aghdam, A. G. (2013). Structural controllability of multi-agent networks: Robustness against simultaneous failures. *Automatica*, *49*(11), 3149–3157.

Rantzer, A. (2011). Distributed control of positive systems. In *2011 50th IEEE conference on decision and control and European control conference* (pp. 6608–6611).

Siami, M., & Motee, N. (2017). Growing linear dynamical networks endowed by spectral systemic performance measures. *IEEE Transactions on Automatic Control*, *63*(7), 2091–2106.

Tanaka, T., & Langbort, C. (2011). The bounded real lemma for internally positive systems and H-infinity structured static state feedback. *IEEE Transactions on Automatic Control*, *56*(9), 2218–2223.

Zhang, H.-T., Chen, Z., & Mo, X. (2017). Effect of adding edges to consensus networks with directed acyclic graphs. *IEEE Transactions on Automatic Control*, *62*(9), 4891–4897.

Zhou, K., & Doyle, J. (1998). *Prentice hall modular series for eng., Essentials of robust control*. Prentice Hall.

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