Reciprocal properties of random fields on undirected graphs

Torkel Erhardsson

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RECIPROCAL PROPERTIES OF RANDOM FIELDS ON UNDIRECTED GRAPHS

TORKEL ERHARDSSON,* Linköping University

Abstract

We clarify and refine the definition of a reciprocal random field on an undirected graph, with the reciprocal chain as a special case, by introducing four new properties: the factorizing, global, local, and pairwise reciprocal properties, in decreasing order of strength, with respect to a set of nodes \( \delta \). They reduce to the better known Markov properties if \( \delta \) is the empty set, or, with the exception of the local property, if \( \delta \) is a complete set. Conditions for each reciprocal property to imply the next stronger property are derived, and it is shown that conditionally on the values at a set of nodes \( \delta_0 \), all four properties are preserved for the subgraph induced by the remaining nodes, with respect to the node set \( \delta \setminus \delta_0 \). We note that many of the above results are new even for reciprocal chains.

Keywords: Conditional independence; Markov property; Random field; Reciprocal chain; Reciprocal property; Undirected graph.

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Secondary 60J99; 62H22

1. Introduction

In this paper we will be concerned with reciprocal random fields on undirected graphs. They contain the reciprocal chains a special case, but have up to now received comparably little attention, and have suffered from a lingering ambiguity in the definition. We will here lay a solid foundation for the study of reciprocal random fields on undirected graphs, by defining four properties: the factorizing, global, local, and pairwise reciprocal properties, in decreasing order of strength, with respect to a
set of nodes $\delta$. The relations of the reciprocal properties to each other, and to the better known Markov properties, will be described in detail, as will be the reciprocal properties of a reciprocal random field conditioned on the values at a set of nodes $\delta_0$. We first give a background.

Continuous time reciprocal processes, also called Bernstein processes, were introduced in Bernstein [1], following an acclaimed paper by Schrödinger [24] on the relation between certain problems in classical and quantum dynamics. Jamison [11], [12] provided a rigorous definition of a reciprocal process, and proofs of their basic properties. According to this definition, a real valued process $\{X_t; t \in [0, 1]\}$ is reciprocal if

$$P(X_u \in \cdot | X_{s_1}, \ldots, X_{s_m}, X_{t_n}, \ldots, X_{t_1}) = P(X_u \in \cdot | X_{s_m}, X_{t_n})$$

$$\forall 0 \leq s_1 < \ldots < s_m < u < t_n < \ldots < t_1 \leq 1.$$ 

In particular, a Markov process is always a reciprocal process, but a reciprocal process is not always Markov. Reciprocal chains, i.e., discrete time reciprocal processes, were introduced in Chay [4], and studied by Levy et al. [17], and others. A random sequence $\{X_t; t = 1, \ldots, n\}$, where $n \geq 3$, is said to be a reciprocal chain if

$$P(X_k \in \cdot | X_1, \ldots, X_j, X_l, \ldots, X_n) = P(X_k \in \cdot | X_j, X_l) \quad \forall 0 \leq j < k < l \leq n.$$ 

In much of the work on reciprocal processes, attention has been restricted to the stationary and/or Gaussian cases. In [11], [4], and Carmichael et al. [2], characterizations of a stationary Gaussian reciprocal process in terms of its autocovariance function were given, and in [4] conditions for stationary Gaussian reciprocal processes to be Markov. Later work has to a large extent dealt with the construction of representations of a reciprocal process by a second order nearest-neighbour model, driven by a locally correlated noise process; see Krener [14], Levy et al. [17], Sand [23], Carravetta [3], and White and Carravetta [29]. Such representations are then used to construct smoothing algorithms for hidden reciprocal processes. Examples of engineering applications of reciprocal chains include image simulation, in Picci and Carli [21], and track extraction, in Stamatescu et al. [27].

It should also be noted that in [3], [29], [27] and other papers, a different definition of a reciprocal chain was used: a random sequence $\{X_t; t = 1, \ldots, n\}$, where $n \geq 3$, 

was said to be a reciprocal chain if
\[ P(X_k \in \cdot | X_1, \ldots, X_{k-1}, X_{k+1}, \ldots, X_n) = P(X_k \in \cdot | X_{k-1}, X_{k+1}) \quad \forall 0 < k < n. \]

It was pointed out in Erhardsson et al. [9] that if the latter definition is used, some of the properties that a reciprocal chain is known to satisfy need not hold; for example that, conditionally on \( X_n \), a reciprocal chain is a Markov chain.

In the present paper, we take a different approach to reciprocal chains, assuming neither stationarity nor Gaussianity. Since any Markov chain is a reciprocal chain, and given the existence of a well established theory for Markov random fields on undirected graphs, it is natural to attempt to define in a proper way a class of reciprocal random fields on undirected graphs, which should contain both Markov random fields and reciprocal chains as special cases. The theory of Markov random fields on undirected graphs was developed by Dobrushin [7], [8], Spitzer [26], Preston [22], and others, originally as an attempt to extend the Ising model for ferromagnetic materials to a wider class of probabilistic models. Markov random fields have shown themselves useful in statistics (e.g., graphical models) and in probabilistic expert systems; see Darroch et al. [6], Pearl [20], Lauritzen [15], and Whittaker [30].

In the early literature, a random field \( \{X_t; t \in V\} \) is typically said to satisfy the Markov property if
\[ P(\{X_t; t \in T\} \in \cdot | \{X_s; s \notin T\}) = P(\{X_t; t \in T\} \in \cdot | \{X_s; s \in N_T\}) \quad \forall T \subset V, \]
where \( N_T \) is the set of nodes not in \( T \) that are neighbours of \( T \) in the graph. This property holds e.g. if the random field has a Gibbs distribution; see Kindermann and Snell [13]. Later, it became standard to distinguish between four different Markov properties for random fields: the factorizing, global, local, and pairwise Markov properties, in order of decreasing strength. They are all satisfied if the random field has a Gibbs distribution. The relations between these properties have been studied by several authors, including Speed [25], Studeny [28], and Matúš [18]. As for reciprocal properties of random fields, an early attempt to define such a property was the quasi-Markov, or \( L \)-Markov, property for random fields of the type \( \{X_t; t \in \mathbb{Z}^d\} \), defined and investigated in [4].

The main contribution of the present paper is to define four so-called reciprocal
properties for random fields on an undirected graph, all of them with respect to an arbitrary set of nodes $\delta \subset V$: the factorizing, global, local, and pairwise reciprocal properties, in order of decreasing strength. It will be seen that these properties reduce to the corresponding Markov properties when $\delta = \emptyset$; in fact, the factorizing, global, and pairwise reciprocal properties reduce to the corresponding Markov properties whenever $\delta$ is a complete set. Moreover, necessary and sufficient conditions on the graph for the global and local properties to be equivalent, as well as for the local and pairwise properties to be equivalent, and a sufficient condition for the pairwise property to imply the factorizing property, will be given.

We also consider the conditional distributions of a random field on an undirected graph given the values at a subset of nodes $\delta_0 \subset V$, and show that for the subgraph induced by $V \setminus \delta_0$, under such a conditioning, all reciprocal properties are preserved, with respect to the set of nodes $\delta \setminus \delta_0$. The converse statement is wrong: even if a random field does not satisfy any reciprocal property with respect to $\delta$, the subgraph induced by $V \setminus \delta_0$ may still, conditionally on the values at $\delta_0$, satisfy any reciprocal property with respect to $\delta \setminus \delta_0$.

Specializing to reciprocal chains, we show that a random sequence is a reciprocal chain according to the definition in [4] and [17] if and only if, when seen as a random field on an undirected graph, it has the global reciprocal property with respect to the node set $\delta = \{0, n\}$. Similarly, it is a “reciprocal chain” according to the definition in [3] if and only if it has the (weaker) local reciprocal property with respect to $\delta$. More importantly, a random sequence has the factorizing reciprocal property with respect to $\delta$ if and only if the joint distribution has a density $f_X$ with respect to a product of $\sigma$-finite measures $\mu$, of the form:

$$f_X(x) = \phi_n(x_0, x_n) \prod_{i=0}^{n-1} \phi_i(x_i, x_{i+1}) \quad \mu\text{-a.e.}$$

for some measurable functions $\{\phi_i : \mathcal{X}_{\{i,i+1\}} \to \mathbb{R}_+; i = 0, 1, \ldots, n-1\}, \phi_n : \mathcal{X}_{\{0,n\}} \to \mathbb{R}_+$. The factorizing reciprocal property does not hold for reciprocal chains in general, but it does hold if the density $f_X$ is positive; a result which, to the best of our knowledge, is new.

The rest of the paper is organised as follows. Section 2 contains some preliminary material on graphs, random fields, and conditional independence. In Section 3, defini-
tions of the four reciprocal properties of a random field are given, and several results
concerning the relations between these properties, and their relation to the Markov
properties, are derived. In Section 4, conditional distributions of reciprocal random
fields given the values at a set of nodes $\delta_0$ are considered. Lastly, in Section 5, the
results obtained are applied to the special case of reciprocal chains.

2. Preliminaries

This section contains some basic definitions and notation pertaining to undirected
graphs and random fields on graphs, and some basic results on conditional indepen-
dence. For more information, see [15].

2.1. Undirected graphs

Let $G = (V,E)$ be a graph, where $V$ is an ordered finite set of nodes, and $E$ is a set
of edges. The edges are always assumed to be undirected. An undirected edge between
two nodes $\alpha, \beta \in V$ is denoted by $\langle \alpha, \beta \rangle$, or equivalently $\langle \beta, \alpha \rangle$. $G$ is called simple, if
there is at most one edge between any pair of nodes, and if there are no edges of type
$\langle \alpha, \alpha \rangle$ for any $\alpha \in V$ (these are called loops). In this paper, we only consider simple
graphs.

A graph $G_A = (A,E')$ is called a subgraph of $G$, if $A \subset V$ and $E' \subset E_A = \{\langle \alpha, \beta \rangle \in E; \alpha, \beta \in A\}$. The particular subgraph $G_A = (A,E_A)$ is called the subgraph induced by $A$. A graph $G$ is said to be complete, if $E = \{\langle \alpha, \beta \rangle; \alpha \neq \beta, \alpha, \beta \in V\}$. A subset $A \subset V$ is said to be complete, if the induced subgraph $G_A$ is complete. The collection of complete subsets of $V$ is denoted $\mathbb{K}$. A complete subset which is maximal with respect to set inclusion is called a clique.

For any $\alpha, \beta \in V$, a sequence $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ of elements in $V$ is called a path
between $\alpha$ and $\beta$ of length $n \geq 1$, if $\alpha_0 = \alpha$, $\alpha_n = \beta$, and $\langle \alpha_i, \alpha_{i+1} \rangle \in E$ for each
$i = 0,1,\ldots,n-1$. $\alpha$ and $\beta$ are said to be connected if either $\alpha = \beta$, or $\alpha \neq \beta$ and
there exists a path between $\alpha$ and $\beta$. Two subsets $A,B \subset V$ are said to be connected if
either $A \cap B \neq \emptyset$, or $A \cap B = \emptyset$ and there exists a path in $G$ between $A$ and $B$, by which
we mean a path between a node $\alpha \in A$ and a node $\beta \in B$. Clearly, connectedness
is an equivalence relation on $V$. The subgraphs of $G$ induced by the corresponding
equivalence classes are called the connected components of $\mathcal{G}$.

For each triple $(A, B, S)$ of disjoint subsets of $V$, $S$ is said to separate $A$ from $B$ in $\mathcal{G}$, if there is no path between $A$ and $B$ of length $n = 1$, and, for each path $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ between $A$ and $B$ of length $n \geq 2$, there is an $i \in \{1, \ldots, n-1\}$ such that $\alpha_i \in S$.

For each $\alpha \in V$, the boundary of $\alpha$ is defined by: $\text{bd}(\alpha) = \{\beta \in V; \langle \alpha, \beta \rangle \in E\}$, and the closure of $\alpha$ is defined by: $\text{cl}(\alpha) = \{\alpha\} \cup \text{bd}(\alpha)$.

2.2. Random fields

By a random field on an undirected graph $\mathcal{G} = (V, E)$, we mean a collection $\{X_i; t \in V\}$ of random variables defined on the same probability space and indexed by the node set $V$. For each $t \in V$, $X_t$ takes values in a measurable space $(\mathcal{X}_t, \mathcal{B}_{\mathcal{X}_t})$, which is assumed to be either $(\mathbb{R}^d, \mathcal{B}^d)$, i.e., $\mathbb{R}^d$ equipped with its Borel $\sigma$-algebra, or a finite or countably infinite set equipped with its power $\sigma$-algebra (= the collection of all its subsets). For each $A \subset V$, we denote by $X_A$ the random variable $\{X_i; t \in A\}$, taking values in the product space $\mathcal{X}_A = \prod_{i \in A} \mathcal{X}_i$, equipped with the product $\sigma$-algebra $\mathcal{B}_{\mathcal{X}_A}$. The random variable $X_V$ is denoted by $X$, the product space $\mathcal{X}_V$ by $\mathcal{X}$, and the $\sigma$-algebra $\mathcal{B}_{\mathcal{X}_V}$ by $\mathcal{B}_X$. Also, for each $x \in \mathcal{X}$ and each $A \subset V$, we denote by $x_A$ the projection of $x$ onto $\mathcal{X}_A$.

We denote by $\mathcal{L}(\cdot)$ the probability distribution of a random variable. Throughout the paper we will assume that $\mathcal{L}(X)$ has a density $f_X$ with respect to a product measure $\mu = \prod_{i \in V} \mu_i$ on $(\mathcal{X}, \mathcal{B}_X)$, where, for each $i \in V$, $\mu_i$ is a $\sigma$-finite (nonnegative) measure on $(\mathcal{X}_i, \mathcal{B}_{\mathcal{X}_i})$. For each $A \subset V$, we denote by $\mu_A$ the product measure $\mu_A = \prod_{i \in A} \mu_i$ on $(\mathcal{X}_A, \mathcal{B}_{\mathcal{X}_A})$, and by $f_{X_A}$ the marginal density of $\mathcal{L}(X_A)$ with respect to $\mu_A$, defined by

$$f_{X_A}(x_A) = \int_{\mathcal{X}_{V \setminus A}} f_X(x_A, x_{V \setminus A}) d\mu_{V \setminus A}(x_{V \setminus A}) \quad \forall x_A \in \mathcal{X}_A.$$ 

We will use the fact that, for each $A \subset B \subset V$, if $N_A \in \mathcal{B}_{\mathcal{X}_A}$ is such that $\mu_A(N_A) = 0$, then also $\mu_B(N_A \times \mathcal{X}_{B \setminus A}) = 0$, since $\mu_{B \setminus A}$ is $\sigma$-finite. Furthermore, it will be used that, for each $B \subset V$,

$$f_{X_A}(x_A) = 0 \Rightarrow f_{X_B}(x_A, x_{B \setminus A}) = 0 \quad \forall A \subset B \quad \mu_B\text{-a.e.} \quad (2.1)$$

To prove (2.1), define the sets $\{N_A \in \mathcal{B}_{\mathcal{X}_A}; A \subset V\}$ and $\{N_{A,B} \in \mathcal{B}_{\mathcal{X}_B}; A \subset B \subset V\}$ by: $N_A = \{x_A \in \mathcal{X}_A; f_{X_A}(x_A) = 0\}$ and $N_{A,B} = \{x_B \in \mathcal{X}_B; f_{X_A}(x_A) = 0\} =$
Reciprocal properties of random fields on undirected graphs

\[ N_A \times X_{B \setminus A}, \text{ respectively. Then, it holds that} \]

\[ 0 \leq \int_{N_A \cap N_B} f_{X_B}(x_B)d\mu_B(x_B) \]
\[ = \int_{N_A \cap N_B} \int_{X \setminus B} f(x_B, x_{V \setminus B})d\mu_{V \setminus B}(x_{V \setminus B})d\mu_B(x_B) = \int_{N_A \cap N_{B,V}} f(x)d\mu(x) \]
\[ \leq \int_{N_A} f(x)d\mu(x) = \int_{N_A} \int_{X \setminus A} f(x_A, x_{V \setminus A})d\mu_{V \setminus A}(x_{V \setminus A})d\mu_A(x_A) \]
\[ = \int_{N_A} f_{X_A}(x_A)d\mu_A(x_A) = 0 \quad \forall A \subset B \subset V. \]

In order for the first integral to be 0, it is necessary that \( \mu_B(N_{A,B} \cap N_B) = 0\). Since \( V \) has finitely many subsets, we obtain that
\[ \mu_B \left( \bigcup_{A \subset B} (N_{A,B} \cap N_B) \right) \leq \sum_{A \subset B} \mu(N_{A,B} \cap N_B) = 0 \quad \forall B \subset V. \]

2.3. Conditional independence

For each pair \((B, C)\) of disjoint subsets of \( V \), there exists a regular conditional distribution of \( X_B \) given \( X_C \), which has a density \( f_{X_B|X_C} \) with respect to \( \mu_B \). For each \( x_C \in X_C \) such that \( f_{X_C}(x_C) > 0 \), \( f_{X_B|X_C}(\cdot|x_C) \) can (and will in this paper) be chosen as:
\[ f_{X_B|X_C}(x_B|x_C) = \frac{f_{X_B\cap C}(x_B, x_C)}{f_{X_C}(x_C)} \quad \forall x_B \in X_B. \]

For all \( x_C \in X_C \) such that \( f_{X_C}(x_C) = 0 \), \( f_{X_B|X_C}(\cdot|x_C) \) can be chosen as an arbitrary fixed density. Using (2.1), this gives:
\[ f_{X_B\cap C}(x_B, x_C) = f_{X_B\cap C}(x_B|x_C)f_{X_C}(x_C) \quad \mu_{B\cup C}\text{-a.e.} \quad (2.2) \]

For each triple \((A, B, C)\) of disjoint subsets of \( V \), we say that \( X_A \) and \( X_B \) are conditionally independent given \( X_C \), denoted \( X_A \perp X_B|X_C \), if there exists a version of \( f_{X_{A\cup B\cup C}} \) such that, for each \( x_C \in X_C \) such that \( f_{X_C}(x_C) > 0 \),
\[ f_{X_{A\cup B\cup C}}(x_A, x_B, x_C) = \frac{f_{X_{A\cup C}}(x_A, x_C)}{f_{X_C}(x_C)}f_{X_{B\cup C}}(x_B, x_C) \quad \forall (x_A, x_B, x_C) \in X_{A\cup B}. \quad (2.3) \]

Using (2.1), \( X_A \perp X_B|X_C \) implies that
\[ f_{X_{A\cup B\cup C}}(x_A, x_B, x_C) = f_{X_A\cap C}(x_A|x_C)f_{X_{B\cup C}}(x_B, x_C) \quad \mu_{A\cup B\cup C}\text{-a.e.} \quad (2.4) \]

Note that (2.4) remains valid if \( f_{X_{B\cup C}} \) is replaced on the right hand side by any \( \mu_{B\cup C}\)-version of \( f_{X_{B\cup C}} \).
A sufficient condition for $X_A \perp X_B|X_C$ to hold is that there exists a version of $f_{X_{A\cup B\cup C}}$ and measurable functions $h : \mathcal{X}_{A\cup C} \to \mathbb{R}_+$ and $k : \mathcal{X}_{B\cup C} \to \mathbb{R}_+$, such that

$$f_{X_{A\cup B\cup C}}(x_A, x_B, x_C) = h(x_A, x_C)k(x_B, x_C) \quad \forall (x_A, x_B, x_C) \in \mathcal{X}_{A\cup B\cup C}. \quad (2.5)$$

To see this, note that the version of $f_{X_{A\cup B\cup C}}$ given in (2.5) satisfies (2.3), since:

$$f_{X_C}(x_C) = \int_{\mathcal{X}_A} h(x_A, x_C)d\mu_A(x_A) \int_{\mathcal{X}_B} k(x_B, x_C)d\mu_B(x_B) \quad \forall x_C \in \mathcal{X}_C;$$

$$f_{X_{A\cup C}}(x_A, x_C) = h(x_A, x_C) \int_{\mathcal{X}_B} k(x_B, x_C)d\mu_B(x_B) \quad \forall (x_A, x_C) \in \mathcal{X}_{A\cup C};$$

$$f_{X_{B\cup C}}(x_B, x_C) = k(x_B, x_C) \int_{\mathcal{X}_A} h(x_A, x_C)d\mu_A(x_A) \quad \forall (x_B, x_C) \in \mathcal{X}_{B\cup C}.$$ 

3. Reciprocal properties for random fields

**Definition 3.1.** A random field $X$ on an undirected graph $\mathcal{G} = (V, E)$ is said to satisfy the factorizing reciprocal property with respect to $\delta \subset V$, abbreviated $F[\delta]$, if $\mathcal{L}(X)$ has a density $f_X$ with respect to a product of $\sigma$-finite measures $\mu$, of the following form:

$$f_X(x) = \prod_{C \in \mathbb{K}^\delta} \phi_C(x) \quad \mu\text{-a.e.,} \quad (3.1)$$

for some measurable functions $\{\phi_C : \mathcal{X}_C \to \mathbb{R}_+ ; C \in \mathbb{K}^\delta\}$, where $\mathbb{K}^\delta$ is the collection of subsets of $V$ defined as follows:

$$\mathbb{K}^\delta = \{C \subset V; (C \setminus \delta) \in \mathbb{K}, (C \setminus \delta) \cup \{\alpha\} \in \mathbb{K} \quad \forall \alpha \in C \cap \delta\}.$$ 

**Example 3.1.** Consider an undirected graph $\mathcal{G} = (V, E)$, for which the subgraph $\mathcal{G}_A$ induced by $A = \{1, 2, 3, 4, 5, 6, 7\} \subset V$ is the following:

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1
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5
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6
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7
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If $\delta \cap A = \{1, 2, 6, 7\}$, then the sets $\{1, 3, 4, 6\}$ and $\{2, 4, 5, 7\}$ both belong to $\mathbb{K}^\delta$, even though neither of them belongs to $\mathbb{K}$. However, the set $\{1, 2, 3, 4\}$, for example, does not belong to $\mathbb{K}^\delta$, since the edge $\langle 2, 3 \rangle$ is not present.
**Definition 3.2.** A random field $X$ on an undirected graph $\mathcal{G} = (V, E)$ is said to satisfy the *global reciprocal property* with respect to $\delta \subset V$, abbreviated $G[\delta]$, if, for each triple $(A, B, S)$ of disjoint subsets of $V$ such that $S$ separates $A$ from $B \cup (\delta \setminus S)$, it holds that $X_A \perp X_B | X_S$.

**Definition 3.3.** A random field $X$ on an undirected graph $\mathcal{G} = (V, E)$ is said to satisfy the *local reciprocal property* with respect to $\delta \subset V$, abbreviated $L[\delta]$, if, for each $\alpha \in V \setminus \delta$, it holds that $X_\alpha \perp X_{V \setminus cl(\alpha)} | X_{bd(\alpha)}$.

**Definition 3.4.** A random field $X$ on an undirected graph $\mathcal{G} = (V, E)$ is said to satisfy the *pairwise reciprocal property* with respect to $\delta \subset V$, abbreviated $P[\delta]$, if, for each $\alpha, \beta \in V$ such that $\alpha \neq \beta$, $\alpha \in V \setminus \delta$ and $\langle \alpha, \beta \rangle \notin E$, it holds that $X_\alpha \perp X_\beta | X_{V \setminus \{\alpha, \beta\}}$.

**Remark 3.1.** All four reciprocal properties reduce to the corresponding Markov properties in the case when $\delta = \emptyset$; cf. the definitions in [15], Section 3.2.

**Remark 3.2.** It is easily seen that if $\delta \subset \delta_1 \subset V$, then the following implications hold: $F[\delta] \Rightarrow F[\delta_1]$, $G[\delta] \Rightarrow G[\delta_1]$, $L[\delta] \Rightarrow L[\delta_1]$, and $P[\delta] \Rightarrow P[\delta_1]$. In particular, each of the four Markov properties implies the corresponding reciprocal property, with respect to any set $\delta \subset V$.

**Theorem 3.1.** Let $X$ be a random field on an undirected graph $\mathcal{G} = (V, E)$, and let $\delta \subset V$. Then, $X$ satisfies $F[\delta]$, $G[\delta]$, $L[\delta]$, or $P[\delta]$ in $\mathcal{G}$, if and only if $X$ satisfies the same property in the undirected graph $\mathcal{G}_\delta = (V, E_\delta^+)$, where $E_\delta^+ = E \cup \{\langle \alpha, \beta \rangle ; \alpha \neq \beta, \alpha, \beta \in \delta\}$.

**Proof.** The claims for $F[\delta]$ and $L[\delta]$ follow from the easily checked facts that neither the collection $K^\delta$, nor, for any $\alpha \in V \setminus \delta$, the set $bd(\alpha)$, depends on which edges in $\{\langle \alpha, \beta \rangle ; \alpha \neq \beta, \alpha, \beta \in \delta\}$ belong to $E$. Similarly, the claim for $P[\delta]$ follows since, for any $\alpha, \beta \in V$ such that $\alpha \neq \beta$ and $\alpha \in V \setminus \delta$, it holds that $\langle \alpha, \beta \rangle \in E_\delta^+$ if and only if $\langle \alpha, \beta \rangle \in E$.

The claim for $G[\delta]$ follows from the fact that for any triple $(A, B, S)$ of disjoint subsets of $V$, it holds that $S$ separates $A$ from $B \cup (\delta \setminus S)$ in $\mathcal{G}$ if and only if $S$ separates $A$ from $B \cup (\delta \setminus S)$ in $\mathcal{G}_\delta$. To see this, we first observe that any path in $\mathcal{G}$ between $A$ and $B \cup (\delta \setminus S)$ which does not intersect $S$ must also be such a path in $\mathcal{G}_\delta$. Conversely, for $n \geq 1$, let $\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$ be a path in $\mathcal{G}_\delta$ but not in $\mathcal{G}$, between $\alpha_0 \in A$ and
\[ \alpha_n \in B \cup (\delta \setminus S), \text{ which does not intersect } S. \] Then, there must exist \( 0 < k < n \) such that \( \alpha_k \in \delta \setminus S \), and \( \alpha_i \notin \delta \) for each \( i = 1, \ldots, k - 1 \). Hence, \( \{\alpha_0, \alpha_1, \ldots, \alpha_k\} \) is a path in \( \mathcal{G} \) between \( A \) and \( \delta \setminus S \) which does not intersect \( S \). \( \square \)

**Theorem 3.2.** For a random field \( X \) on an undirected graph \( \mathcal{G} = (V, E) \), the following implications hold: \( F[\delta] \Rightarrow G[\delta] \Rightarrow L[\delta] \Rightarrow P[\delta] \).

Proof. \( F[\delta] \Rightarrow G[\delta] \): Let \( (A, B, S) \) be a triple of disjoint subsets of \( V \) such that \( S \) separates \( A \) from \( B \cup (\delta \setminus S) \). Note that we implicitly assume that \( A \cap \delta = \emptyset \). We will also assume, without loss of generality, that \( A \cup B \cup S = V \), and that \( \delta \subset B \cup S \). That this can be done is seen as follows: let \( \tilde{A} \subset V \setminus S \) be the set of nodes in \( V \setminus S \) which are connected to \( A \) in \( \mathcal{G}_{V \setminus S} \). By construction, \( \tilde{A} \cap (B \cup \delta) = \emptyset \), and \( S \) separates \( \tilde{A} \) from \( B \cup (\delta \setminus S) \) in \( \mathcal{G} \). Define \( \tilde{B} = V \setminus (\tilde{A} \cup S) \). Then, \( \tilde{A} \cup \tilde{B} \cup S = V \), and \( S \) separates \( \tilde{A} \) from \( \tilde{B} \).

Let \( C \in \mathbb{K}^d \), and assume first that \( C \cap \delta = \emptyset \), so that \( C \in K \). Then, we must have either \( C \subset A \cup S \) or \( C \subset B \cup S \); both statements can hold only if \( C \subset S \). Next, assume that \( C \setminus \delta \neq \emptyset \). By assumption, \( C \setminus \delta \subset B \cup S \). If \( C \setminus \delta \subset S \), then again we must have either \( C \subset A \cup S \) or \( C \subset B \cup S \), or both if \( C \subset S \). On the other hand, if there is a node \( \beta \in C \) such that \( \beta \in B \), then, since \( (C \setminus \delta) \cup \{\beta\} \in \mathbb{K} \), it must hold that \( C \setminus \delta \subset B \cup S \), implying that \( C \subset B \cup S \).

Define \( \mathbb{K}^d_A = \{C \in \mathbb{K}^d; C \subset A \cup S\} \) and \( \mathbb{K}^d_B = \{C \in \mathbb{K}^d; C \subset B \cup S\} \); if \( C \subset S \), we arbitrarily assign \( C \) to one of \( \mathbb{K}^d_A \) or \( \mathbb{K}^d_B \). By (3.1), the joint pdf \( f_X \) satisfies:

\[
 f_X(x) = \prod_{C \in \mathbb{K}^d} \phi_C(x_C) = \prod_{C \in \mathbb{K}^d_A} \phi_C(x_C) \prod_{C' \in \mathbb{K}^d_B} \phi_{C'}(x_{C'}) \quad \mu\text{-a.e.},
\]

where the first product depends only on \( (x_A, x_S) \in \mathcal{X}_{A \cup S} \), and the second product depends only on \( (x_B, x_S) \in \mathcal{X}_{B \cup S} \). It follows from (2.5) that \( X_A \perp X_B|X_S \), which implies the claim.

\( G[\delta] \Rightarrow L[\delta] \): For any \( \alpha \in V \setminus \delta \), let \( A = \{\alpha\}, B = V \setminus \text{cl}(\alpha) \), and \( S = \text{bd}(\alpha) \).

\( L[\delta] \Rightarrow P[\delta] \): For any \( \alpha, \beta \in V \) such that \( \alpha \neq \beta, \alpha \in V \setminus \delta \) and \( \langle \alpha, \beta \rangle \notin E \), it holds that \( X_\alpha \perp X_{V \setminus \text{cl}(\alpha)}|X_{\text{bd}(\alpha)} \). Therefore,

\[
 f_X(x) = f_{X_\alpha|x_{\text{bd}(\alpha)}}(x_\alpha|x_{\text{bd}(\alpha)}) f_{X_{V \setminus \{\alpha\}}}(x_{V \setminus \{\alpha\}}) \quad \mu\text{-a.e.}
\]

Since \( \langle \alpha, \beta \rangle \notin E \), it holds that \( \beta \notin \text{cl}(\alpha) \), so the first function in the product on the right...
Reciprocal properties of random fields on undirected graphs

hand side depends only on \( x_{V \setminus \{\beta\}} \in \mathcal{X}_{V \setminus \{\beta\}} \), while the second function in the product depends only on \( x_{V \setminus \{\alpha\}} \in \mathcal{X}_{V \setminus \{\alpha\}} \). It follows from (2.5) that \( X_\alpha \perp X_\beta | X_{V \setminus \{\alpha, \beta\}} \).

**Theorem 3.3.** For an undirected graph \( G = (V, E) \) and a set \( \delta \subset V \), the following conditions are equivalent:

(i) For any random field \( X \) on \( G \), it holds that \( L[\delta] \Rightarrow G[\delta] \).

(ii) No subset \( C \subset V \) exists of any of the following two types: \( C = \{\alpha_1, \alpha_2, \beta_1, \beta_2\} \subset V \setminus \delta \), with induced subgraph \( G_C = (C, \{\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle \}) \), or: \( C = \{\alpha_1, \alpha_2, \beta\} \subset V \), where \( \{\alpha_1, \alpha_2\} \subset V \setminus \delta \) and \( \beta \in \delta \), with induced subgraph \( G_C = (C, \{\langle \alpha_1, \alpha_2 \rangle \}) \).

**Proof.** (i) \( \Rightarrow \) (ii). Assume, in order to derive a contradiction, that there exists a subset \( C = \{\alpha_1, \alpha_2, \beta_1, \beta_2\} \subset V \setminus \delta \), with induced subgraph \( G_C = (C, \{\langle \alpha_1, \alpha_2 \rangle, \langle \beta_1, \beta_2 \rangle \}) \). Define a random field \( X \) on \( G \) by: \( X_{\alpha_1} = X_{\alpha_2} = X_{\beta_1} = X_{\beta_2} = Y \), where \( Y \) is a non-degenerate random variable, and \( X_v = \gamma \) for each \( v \in V \setminus C \), where \( \gamma \in \mathbb{R} \) is a constant. \( X \) trivially satisfies \( L[\delta] \). It is also clear that \( S = V \setminus C \) separates \( \{\alpha_1\} \) from \( \{\beta_1\} \cup (\delta \setminus S) \). However, since \( X_{\alpha_1} \) and \( X_{\beta_1} \) are not conditionally independent given \( X_S \), \( X \) does not satisfy \( G[\delta] \). The case when \( C \) is of the second type is handled similarly.

(ii) \( \Rightarrow \) (i). Assume that the random field \( X \) on \( G \) satisfies \( L[\delta] \). Let \( (A, B, S) \) be a triple of disjoint subsets of \( V \) such that \( S \) separates \( A \) from \( B \cup (\delta \setminus S) \). As in the proof of Theorem 3.2, we assume without loss of generality that \( A \cup B \cup S = V \), and that \( \delta \subset B \cup S \). By (ii), one of two (possibly overlapping) cases must hold: in case 1, there exists no nodes \( \alpha_1, \alpha_2 \in A \) such that \( \langle \alpha_1, \alpha_2 \rangle \in E \); in case 2, \( B \cap \delta = \emptyset \), and there exists no nodes \( \beta_1, \beta_2 \in B \) such that \( \langle \beta_1, \beta_2 \rangle \in E \). In the first of these cases, it holds that \( \text{bd}(\alpha) \subset S \) for each \( \alpha \in A \). Let \( A = \{\alpha_i; i = 1, \ldots, m\} \). Using \( L[\delta] \), we get:

\[
\begin{align*}
  f_X(x) &= f_{X_{\alpha_1} | X_{\text{bd}(\alpha_1)}}(x_{\alpha_1} | x_{\text{bd}(\alpha_1)}) f_{X_{V \setminus \{\alpha_1\}}}(x_{V \setminus \{\alpha_1\}}) \\
  &= f_{X_{\alpha_1} | X_{\text{bd}(\alpha_1)}}(x_{\alpha_1} | x_{\text{bd}(\alpha_1)}) f_{X_{\alpha_2} | X_{\text{bd}(\alpha_2)}}(x_{\alpha_2} | x_{\text{bd}(\alpha_2)}) f_{X_{V \setminus \{\alpha_1, \alpha_2\}}}(x_{V \setminus \{\alpha_1, \alpha_2\}}) \\
  &= \ldots = \prod_{i=1}^{m} f_{X_{\alpha_i} | X_{\text{bd}(\alpha_i)}}(x_{\alpha_i} | x_{\text{bd}(\alpha_i)}) f_{X_{V \setminus A}}(x_{V \setminus A}) \quad \mu\text{-a.e.}
\end{align*}
\]

By (2.5), it holds that \( X_A \perp X_{B | X_S} \). Case 2 is handled analogously, with the roles of \( A \) and \( B \) interchanged. Hence, \( X \) satisfies \( G[\delta] \).

**Example 3.2.** Consider an undirected graph \( G = (V, E) \), for which the subgraph \( G_C \) induced by \( C = \{1, 2, 3, 4\} \subset V \) is the following:
If $\delta \cap C = \emptyset$, then the implication $L[\delta] \Rightarrow G[\delta]$ does not hold in $\mathcal{G}$. Suppose instead that the subgraph $\mathcal{G}_D$ induced by $D = \{1,2,3\} \subset V$ is the following:

![Diagram 1](1-2-3-4)

If $\delta \cap D = \{3\}$, then again the implication $L[\delta] \Rightarrow G[\delta]$ does not hold in $\mathcal{G}$.

**Theorem 3.4.** For an undirected graph $\mathcal{G} = (V, E)$ and a set $\delta \subset V$, the following conditions are equivalent:

(i) For any random field $X$ on $\mathcal{G}$, it holds that $P[\delta] \Rightarrow L[\delta]$.

(ii) No subset $C \subset V$ exists of the following type: $C = \{\alpha, \beta_1, \beta_2\} \subset V$, where $\alpha \in V \setminus \delta$, with induced subgraph $\mathcal{G}_C = (C, \{\langle \beta_1, \beta_2 \rangle\})$ or $\mathcal{G}_C = (C, \emptyset)$.

**Proof.** (i) $\Rightarrow$ (ii). Assume that there exists a subset $C = \{\alpha, \beta_1, \beta_2\} \subset V$, where $\alpha \in V \setminus \delta$, with induced subgraph $\mathcal{G}_C = (C, \{\langle \beta_1, \beta_2 \rangle\})$ or $\mathcal{G}_C = (C, \emptyset)$. Define a random field $X$ on $\mathcal{G}$ by: $X_\alpha = X_{\beta_1} = X_{\beta_2} = Y$, where $Y$ is a non-degenerate random variable, and $X_v = \gamma$ for each $v \in V \setminus C$, where $\gamma \in \mathbb{R}$ is a constant. $X$ clearly satisfies $P[\delta]$. Since $\text{bd}(\alpha) \subset V \setminus \{\beta_1, \beta_2\}$, but $X_\alpha$ and $X_{\beta_1}$ are not conditionally independent given $X_{\text{bd}(\alpha)}$, $X$ does not satisfy $L[\delta]$.

(ii) $\Rightarrow$ (i). Assume that the random field $X$ on $\mathcal{G}$ satisfies $P[\delta]$. Let $\alpha \in V \setminus \delta$. By (ii), $V \setminus \text{cl}(\alpha)$ can contain at most one node. Assume that $\beta \in V \setminus \text{cl}(\alpha)$. Using $P[\delta]$, we get:

$$f_X(x) = f_{X_\alpha | X_{\text{bd}(\alpha)}}(x_{\alpha} | x_{\text{bd}(\alpha)}) f_{X_{V \setminus \{\alpha\}}}(x_{V \setminus \{\alpha\}}) \quad \mu\text{-a.e.}$$

By (2.5), it holds that $X_\alpha \perp X_\beta | X_{\text{bd}(\alpha)}$. Hence, $X$ satisfies $L[\delta]$.\qed

**Example 3.3.** Consider an undirected graph $\mathcal{G} = (V, E)$, for which the subgraph $\mathcal{G}_D$ induced by $D = \{1,2,3\} \subset V$ is one of the following: either

![Diagram 2](1-2-3)

or:

![Diagram 3](1-2-3)

If $\{3\} \subset V \setminus \delta$, then the implication $P[\delta] \Rightarrow P[\delta]$ does not hold in $\mathcal{G}$. 
Theorem 3.5. Let $X$ be a random field on an undirected graph $\mathcal{G} = (V,E)$. Assume that $X$ satisfies the following condition: for any four disjoint subsets $A, B, C, D \subset V$ such that either $(A \cup C) \cap \delta = \emptyset$ or $B \cap \delta = \emptyset$, it holds that

$$X_A \perp X_B \mid X_{A \cup D} \quad \Rightarrow \quad X_{A \cup C} \perp X_B \mid X_D. \quad (3.2)$$

Then, the following implications hold: $G[\delta] \Leftrightarrow L[\delta] \Leftrightarrow P[\delta]$.

Proof. By Theorem 3.2, we need only prove that $P[\delta] \Rightarrow G[\delta]$. Let $(A, B, S)$ be a triple of disjoint subsets of $V$ such that $S$ separates $A$ from $B \cup (\delta \setminus S)$. As in the proof of Theorem 3.2, we assume without loss of generality that $A \cup B \cup S = V$ and that $\delta \subset B \cup S$. We also assume, again without loss of generality, that both $A$ and $B$ are non-empty. The assertion is proved using backwards induction in the number of nodes of $S$.

Assume first that $|S| = |V| - 2$, so that $A$ and $B$ each contain one node. Since $A \cap \delta = \emptyset$, $P[\delta]$ implies that $X_A \perp X_B \mid X_S$. Next, assume that the claim holds when $|S| = n - 1$. Since $|S| < n$, at least one of $A$ or $B$ contains more than one node. If $A$ contains more than one node, choose any $\alpha \in A$. By the induction assumption, both $X_{A \setminus \{\alpha\}} \perp X_B \mid X_{S \cup \{\alpha\}}$ and $X_\alpha \perp X_B \mid X_{S \cup (A \setminus \{\alpha\})}$ hold, so by (3.2), it holds that $X_A \perp X_B \mid X_S$. If $B$ contains more than one node, choose any $\beta \in B$. As before, both $X_A \perp X_B \setminus \{\beta\} \mid X_{S \cup \{\beta\}}$ and $X_A \perp X_\beta \mid X_S \setminus B \cup \{\beta\}$ hold, so again by (3.2), it holds that $X_A \perp X_B \mid X_S$. □

Theorem 3.6. Let $X$ be a random field on an undirected graph $\mathcal{G} = (V,E)$ such that $\mathcal{L}(X)$ has a positive density $f_X$ with respect to a product of $\sigma$-finite measures $\mu$. Then, the following implications hold: $F[\delta] \Leftrightarrow G[\delta] \Leftrightarrow L[\delta] \Leftrightarrow P[\delta]$.

Proof. By Theorem 3.2, we need only prove that $P[\delta] \Rightarrow F[\delta]$. In the Markov case, Theorem 3.6 is known as the Clifford-Hammersley theorem, a version of which appears as Theorem 3.9 in [15]. We shall use the proof of the latter, with appropriate modifications. Fix $x^* \in \mathcal{X}$, and define, for all subsets $C \subset V$,

$$H_C(x) = \ln f_X(x_C, x^*_V \setminus C); \quad \psi_C(x) = \sum_{A \subset C} (-1)^{|C\setminus A|} H_A(x), \quad \forall x \in \mathcal{X}.$$ 

By definition, $H_C$ and $\psi_C$ both depend on $x$ through $x_C \in \mathcal{X}_C$. By Möbius inversion,
cf. [15], Lemma A.2,
\[
\ln f_X(x) = H_V(x) = \sum_{C \subset V} \psi_C(x) \quad \forall x \in \mathcal{X},
\]
so we have proven the claim if we can show that \( \psi_C \equiv 0 \) whenever \( C \notin \mathbb{K}^\delta \); cf. Definition 3.1. If \( C \notin \mathbb{K}^\delta \), then there exists \( \alpha \in C \setminus \delta \) and \( \beta \in C \) such that \( \alpha \neq \beta \) and \( \langle \alpha, \beta \rangle \notin E \). Let \( C_0 = C \setminus \{ \alpha, \beta \} \) and \( D = V \setminus \{ \alpha, \beta \} \). Then, as in the proof of Theorem 3.9 in [15],
\[
\psi_C(x) = \sum_{B \subset C_0} (-1)^{|C_0 \setminus B|} (H_B(x) - H_{B \cup \{ \alpha \}}(x) - H_{B \cup \{ \beta \}}(x) + H_{B \cup \{ \alpha, \beta \}}(x)) \quad \forall x \in \mathcal{X},
\]
so using property \( P[\delta] \) and (2.3), we get:
\[
H_{B \cup \{ \alpha, \beta \}}(x) - H_{B \cup \{ \beta \}}(x) = \ln \frac{\sum_{x \in \mathcal{X}} x^*_{\alpha, \beta, x_{D \setminus B}}}{\sum_{x \in \mathcal{X}} x^*_{\alpha, \beta, x_{D \setminus B}}}
\]
\[
= \ln \frac{\sum_{x \in \mathcal{X}} x^*_{\alpha, \beta, x_{D \setminus B}}}{\sum_{x \in \mathcal{X}} x^*_{\alpha, \beta, x_{D \setminus B}}}
\]
\[
= \ln \frac{\sum_{x \in \mathcal{X}} x^*_{\alpha, \beta, x_{D \setminus B}}}{\sum_{x \in \mathcal{X}} x^*_{\alpha, \beta, x_{D \setminus B}}} = H_{B \cup \{ \alpha \}}(x) - H_B(x) \quad \forall x \in \mathcal{X}.
\]

\[\square\]

**Theorem 3.7.** Let \( X \) be a random field on an undirected graph \( \mathcal{G} = (V, E) \), and let \( \delta \subset V \). Then, \( X \) satisfies \( F[\delta] \), \( G[\delta] \), or \( P[\delta] \) if and only if \( X \) satisfies \( F[\emptyset] \), \( G[\emptyset] \), or \( P[\emptyset] \) in the undirected graph \( \mathcal{G}_\delta = (V, E^\delta_+) \), where \( E^\delta_+ = E \cup \{ \langle \alpha, \beta \rangle ; \alpha \neq \beta, \alpha, \beta \in \delta \} \).

**Proof.** If \( X \) satisfies \( F[\emptyset] \), \( G[\emptyset] \), or \( P[\emptyset] \) in \( \mathcal{G}_\delta \), then, by Remark 3.2, \( X \) satisfies \( F[\delta] \), \( G[\delta] \), or \( P[\delta] \) in \( \mathcal{G}_\delta \), and by Theorem 3.1, \( X \) also satisfies \( F[\delta] \), \( G[\delta] \), or \( P[\delta] \) in \( \mathcal{G} \). It remains to prove the reverse implications.

\( F[\delta] \) in \( \mathcal{G} \Rightarrow F[\emptyset] \) in \( \mathcal{G}_\delta \): By Theorem 3.1, \( X \) satisfies \( F[\delta] \) in \( \mathcal{G}_\delta \). Since \( \delta \) is a complete set in \( \mathcal{G}_\delta \), it is easy to see that \( \mathbb{K}^\delta \) is equal to the collection of complete sets in \( \mathcal{G}_\delta \). Hence, \( X \) satisfies \( F[\emptyset] \) in \( \mathcal{G}_\delta \).

\( G[\delta] \) in \( \mathcal{G} \Rightarrow G[\emptyset] \) in \( \mathcal{G}_\delta \): By Theorem 3.1, \( X \) satisfies \( G[\delta] \) in \( \mathcal{G}_\delta \). Let \( (A, B, S) \) be a triple of disjoint subsets of \( V \) such that \( S \) separates \( A \) from \( B \) in \( \mathcal{G}_\delta \). As in the proof of Theorem 3.2, we assume without loss of generality that \( A \cup B \cup S = V \). Since \( \delta \) is a complete set in \( \mathcal{G}_\delta \), either \( A \cap \delta = \emptyset \), meaning that \( S \) separates \( A \) from \( B \cup (\delta \setminus S) \) in
Reciprocal properties of random fields on undirected graphs

Let \( G \), or \( B \cap \delta = \emptyset \), meaning that \( S \) separates \( B \) from \( A \cup (\delta \setminus S) \) in \( G_{\delta} \). Either way, it follows that \( X_A \perp X_B | X_S \). Hence, \( X \) satisfies \( G[\emptyset] \) in \( G_{\delta} \).

\[ P[\delta] \in G \Rightarrow P[\emptyset] \in G_{\delta} \]: By Theorem 3.1, \( X \) satisfies \( P[\delta] \) in \( G_{\delta} \). Since \( \delta \) is a complete set in \( G_{\delta} \), for any \( \alpha, \beta \in V \) such that \( \alpha \neq \beta \) and \( \langle \alpha, \beta \rangle \notin E_{\delta}^+ \), at least one of \( \alpha \) or \( \beta \) belongs to \( V \setminus \delta \). It follows that \( X_{\alpha} \perp X_{\beta} | X_{V \setminus \{\alpha, \beta\}} \). Hence, \( X \) satisfies \( P[\emptyset] \) in \( G_{\delta} \). \( \square \)

An immediate consequence of the preceding theorem is that if a random field \( X \) on an undirected graph \( G = (V, E) \) satisfies \( F[\delta], G[\delta], \) or \( P[\delta] \), where \( \delta \subset V \) is a complete set, then \( X \) also satisfies \( F[\emptyset], G[\emptyset], \) or \( P[\emptyset] \) in \( G \). However, the corresponding statement for \( L[\delta] \) is false, as the final example of this section shows.

**Example 3.4.** Consider \( G = (V, E) \), where \( V = \{0, 1, 2, 3, 4\} \) and \( E = \{(i, i+1); i = 0, 1, 2, 3\} \), and let

\[
X_0 = X, \quad X_1 = X, \quad X_2 = X + Y, \quad X_3 = Y, \quad X_4 = X,
\]

where \( X \) and \( Y \) are two independent random variables having the common distribution \( P(X = 0) = P(X = 1) = \frac{1}{2} \). It can be seen that this random field has the local reciprocal property with respect to \( \delta = \{4\} \) (which is a complete subset of \( V \)), but not the local Markov property.

### 4. Conditioned reciprocal random fields

Let \( X \) be a random field on an undirected graph \( G = (V, E) \), such that \( \mathcal{L}(X) \) has a density \( f_X \) with respect to a product of \( \sigma \)-finite measures \( \mu \). Recall that for each \( \delta_0 \subset V \) there exists a regular conditional distribution of \( X_{V \setminus \delta_0} \) given \( X_{\delta_0} \), which has a density \( f_{X_{V \setminus \delta_0}|X_{\delta_0}} \) with respect to \( \mu_{V \setminus \delta_0} \). For all \( x_{\delta_0} \in X_{\delta_0} \) such that \( f_{X_{\delta_0}}(x_{\delta_0}) > 0 \), \( f_{X_{V \setminus \delta_0}|X_{\delta_0}}(\cdot|x_{\delta_0}) \) can be chosen as:

\[
f_{X_{V \setminus \delta_0}|X_{\delta_0}}(x_{V \setminus \delta_0}|x_{\delta_0}) = \frac{f_X(x_{V \setminus \delta_0}, x_{\delta_0})}{f_{X_{\delta_0}}(x_{\delta_0})} \quad \forall x_{V \setminus \delta_0} \in X_{V \setminus \delta_0}.
\]

For all \( x_{\delta_0} \in X_{\delta_0} \) such that \( f_{X_{\delta_0}}(x_{\delta_0}) = 0 \), \( f_{X_{V \setminus \delta_0}|X_{\delta_0}}(\cdot|x_{\delta_0}) \) can be chosen as an arbitrary fixed density.

**Theorem 4.1.** Let \( X \) be a random field on an undirected graph \( G = (V, E) \), such that \( \mathcal{L}(X) \) has a density \( f_X \) with respect to a product of \( \sigma \)-finite measures \( \mu \). Assume that
Define the conditional density (4.1) satisfies:

\[ f_{X_0}(x_{\delta_0}) > 0, \] under the conditional distribution of \( X_V \setminus \delta_0 \) given \( X_{\delta_0} = x_{\delta_0}, X_V \setminus \delta_0 \) satisfies \( F[\delta \setminus \delta_0], G[\delta \setminus \delta_0], L[\delta \setminus \delta_0], \) or \( P[\delta \setminus \delta_0], \) respectively.

Proof. \( F[\delta] \Rightarrow F[\delta \setminus \delta_0] \): Since \( X \) satisfies \( F[\delta], f_X \) has the form (3.1). Consider any function \( \phi_C : \mathcal{X}_C \to \mathbb{R}_+ \), where \( C \in \mathbb{K}^\delta \), and fix \( x^*_{\delta_0} \in \mathcal{X}_{\delta_0} \) such that \( f_{X_{\delta_0}}(x_{\delta_0}) > 0 \). Define \( \phi^*_C : \mathcal{X}_{C \setminus \delta_0} \to \mathbb{R}_+ \) by

\[
\phi^*_C(x_{C \setminus \delta_0}) = \phi_C(x_{C \setminus \delta_0}, x^*_{C \setminus \delta_0}) \quad \forall x_{C \setminus \delta_0} \in \mathcal{X}_{C \setminus \delta_0}.
\]

The conditional density (4.1) satisfies:

\[
fx_{V \setminus \delta_0}|x_{\delta_0}(x_{V \setminus \delta_0}|x^*_{\delta_0}) = \frac{1}{fx_{\delta_0}(x^*_{\delta_0})} \prod_{C \in \mathbb{K}^\delta} \phi_C(x_{C \setminus \delta_0}, x^*_{C \setminus \delta_0})
\]

\[
= \frac{1}{fx_{\delta_0}(x^*_{\delta_0})} \prod_{C \in \mathbb{K}^\delta} \phi^*_C(x_{C \setminus \delta_0}) \quad \forall x_{V \setminus \delta_0} \in \mathcal{X}_{V \setminus \delta_0},
\]

and it is easy to see that \( \{C \setminus \delta_0; C \in \mathbb{K}^\delta\} \subset \mathbb{K}^{\delta \setminus \delta_0} \).

\( G[\delta] \Rightarrow G[\delta \setminus \delta_0] \): Let \( (A, B, S) \) be a triple of disjoint subsets of \( V \setminus \delta_0 \) such that \( S \) separates \( A \) from \( B \cup ((\delta \setminus \delta_0) \setminus S) = B \cup (\delta \setminus (S \cup \delta_0)) \) in \( \mathcal{G}_{V \setminus \delta_0} \). As in the proof of Theorem 3.2, we assume without loss of generality that \( A \cup B \cup S = V \setminus \delta_0 \), and that \( \delta \setminus \delta_0 \subset B \cup S \). This implies that \( S \cup \delta_0 \) separates \( A \) from \( B \cup (\delta \setminus (S \cup \delta_0)) \) in \( \mathcal{G} \). By (2.4), a \( \mu \)-version of \( f_X \) is given by:

\[
f_X(x) = fx_{A|S,\delta_0}(x_A|x_S, x_{\delta_0})fx_{B|S,\delta_0}(x_B|x_S, x_{\delta_0}) \quad \forall x \in \mathcal{X}.
\]

Using this \( \mu \)-version of \( f_X \), for each fixed \( x^*_{\delta_0} \in \mathcal{X}_{\delta_0} \) such that \( f_{X_{\delta_0}}(x^*_{\delta_0}) > 0 \), the conditional density (4.1) can be written:

\[
fx_{V \setminus \delta_0}|x_{\delta_0}(x_{V \setminus \delta_0}|x^*_{\delta_0}) = fx_{A|S,\delta_0}(x_A|x_S, x^*_{\delta_0})
\]

\[
\times \frac{fx_{B|S,\delta_0}(x_B|x_S, x^*_{\delta_0})}{fx_{\delta_0}(x^*_{\delta_0})} \quad \forall x_{V \setminus \delta_0} \in \mathcal{X}_{V \setminus \delta_0}.
\]

By (2.5), under the conditional distribution of \( X_{V \setminus \delta_0} \) given \( X_{\delta_0} = x^*_{\delta_0} \), it holds that \( X_A \perp X_B|X_S \) in \( \mathcal{G}_{V \setminus \delta_0} \).

\( L[\delta] \Rightarrow L[\delta \setminus \delta_0] \): Let \( \alpha \in V \setminus (\delta \cup \delta_0) \). Then, \( X_\alpha \perp X_{V \setminus cl(\alpha)}|X_{bd(\alpha)} \) in \( \mathcal{G} \), so a \( \mu \)-version of \( f_X \) is given by:

\[
f_X(x) = fx_{\alpha|X_{bd(\alpha)}}(x_\alpha|x_{bd(\alpha)})fx_{V \setminus \{\alpha\}}(x_{V \setminus \{\alpha\}}) \quad \forall x \in \mathcal{X}.
\]
Using this μ-version of $f_X$, for each fixed $x^*_{\delta_0} \in X_{\delta_0}$ such that $f_{X_{\delta_0}}(x^*_{\delta_0}) > 0$, the conditional density (4.1) can be written:

$$f_{X_{V \setminus \delta_0}|X_{\delta_0}}(x_{V \setminus \delta_0}|x^*_{\delta_0}) = f_{X_{\alpha}|X_{bd(\alpha)}\setminus \delta_0}(x_{\alpha}|x_{bd(\alpha)\setminus \delta_0}, x^*_{\delta_0}) \times \frac{f_{X_{V \setminus \{\alpha\}}}(x_{V \setminus \{\alpha\}}\setminus \delta_0), x^*_{\delta_0})}{f_{X_{\delta_0}}(x^*_{\delta_0})} \quad \forall x_{V \setminus \delta_0} \in X_{V \setminus \delta_0}.$$  

By (2.5), under the conditional distribution of $X_{V \setminus \delta_0}$ given $X_{\delta_0} = x^*_{\delta_0}$, it holds that $X_{\alpha} \perp X_{V \setminus (cl(\alpha)\cup \delta_0)}|X_{bd(\alpha)\setminus \delta_0}$ in $\mathcal{G}_{V \setminus \delta_0}$.

$P[\delta] = P[\delta \setminus \delta_0]$; Let $\alpha \in V \setminus (\delta \cup \delta_0)$ and $\beta \in V \setminus \delta_0$ be such that $\alpha \neq \beta$ and $\langle \alpha, \beta \rangle \notin E$. Then, $X_{\alpha} \perp X_{\beta}|X_{V \setminus \{\alpha, \beta\}}$ in $\mathcal{G}$, so a μ-version of $f_X$ is given by:

$$f_X(x) = f_{X_{\alpha}|X_{V \setminus \{\alpha, \beta\}}}(x_{\alpha}|x_{V \setminus \{\alpha, \beta\}}) f_{X_{V \setminus \{\alpha\}}}(x_{V \setminus \{\alpha\}}) \quad \forall x \in \mathcal{X}.$$  

Using this μ-version of $f_X$, for each fixed $x^*_{\delta_0} \in X_{\delta_0}$ such that $f_{X_{\delta_0}}(x^*_{\delta_0}) > 0$, the conditional density (4.1) can be written:

$$f_{X_{V \setminus \delta_0}|X_{\delta_0}}(x_{V \setminus \delta_0}|x^*_{\delta_0}) = f_{X_{\alpha}|X_{V \setminus \{\alpha, \beta\}}}(x_{\alpha}|x_{V \setminus \{\alpha, \beta\} \setminus \delta_0}, x^*_{\delta_0}) \times \frac{f_{X_{V \setminus \{\alpha\}}}(x_{V \setminus \{\alpha\}}\setminus \delta_0), x^*_{\delta_0})}{f_{X_{\delta_0}}(x^*_{\delta_0})} \quad \forall x_{V \setminus \delta_0} \in X_{V \setminus \delta_0}.$$  

By (2.5), under the conditional distribution of $X_{V \setminus \delta_0}$ given $X_{\delta_0} = x^*_{\delta_0}$, it holds that $X_{\alpha} \perp X_{\beta}|X_{V \setminus (\{\alpha, \beta\} \cup \delta_0)}$ in $\mathcal{G}_{V \setminus \delta_0}$. \qed

The next example shows that the converse of Theorem 4.1 is false, in the sense that even if a random field $X$ on an undirected graph $\mathcal{G} = (V, E)$ does not satisfy $P[\delta]$, the subgraph induced by $V \setminus \delta_0$ may still satisfy $F[\delta \setminus \delta_0]$ conditionally on $X_{\delta_0}$.

**Example 4.1.** Consider the undirected graph $\mathcal{G} = (V, E)$, where $V = \{0, 1, 2, 3, 4\}$ and $E = \{(i, i+1); i = 0, 1, 2, 3\}$. Let $\delta = \delta_0 = \{0, 4\}$. Let

$$X_0 = Y, \quad X_1 = Y + U, \quad X_2 = Y, \quad X_3 = Y + V, \quad X_4 = Z,$$

where $Y, Z, U$ and $V$ are independent random variables having the common distribution $P(Y = 0) = P(Y = 1) = 1/2$. Clearly, $X$ does not satisfy $P[\delta]$, since $X_0$ and $X_2$ are not conditionally independent given $X_{V \setminus \{0, 2\}}$. However, conditionally on $X_{\delta_0} = (x^*_0, x^*_4)$ for any fixed $(x^*_0, x^*_4) \in \{0, 1\}^2$, $X_1$ and $X_3$ are conditionally independent given $X_2$. For any fixed $(x^*_0, x^*_4) \in \{0, 1\}^2$, denoting

$$f^*(x_1, x_2, x_3) = f_{X_1, X_2, X_3|X_0, X_4}(x_1, x_2, x_3|x^*_0, x^*_4) \quad \forall (x_1, x_2, x_3) \in \{0, 1\}^3,$$
we see that, by (2.4), $f^*$ has the factorization
\[
f^*(x_1, x_2, x_3) = f_{X_1|X_2}(x_1|x_2)f_{X_2,X_3}(x_2, x_3) \quad \forall (x_1, x_2, x_3) \in \{0, 1\}^3.
\]
Hence, conditionally on $X_{\delta_0}$, $X_{V \setminus \delta_0}$ satisfies $F[\emptyset]$.

**Theorem 4.2.** Let $X$ and $Y$ be random fields on an undirected graph $G$, such that $\mathcal{L}(X)$ and $\mathcal{L}(Y)$ have densities $f_X$ and $f_Y$ with respect to a product of $\sigma$-finite measures $\mu$. Let $\delta_0 \subset V$. Assume that for each $x \in X$ such that $f_{Y,0}(x_{\delta_0}) > 0$, it holds that
\[
f_{X,\delta_0}(x_{\delta_0}) > 0 \quad \text{and} \quad \frac{f_X(x_{V \setminus \delta_0}, x_{\delta_0})}{f_{X,\delta_0}(x_{\delta_0})} = \frac{f_Y(x_{V \setminus \delta_0}, x_{\delta_0})}{f_{Y,\delta_0}(x_{\delta_0})}.
\]
If $X$ satisfies $F[\delta], G[\delta], L[\delta], \text{ or } P[\delta]$, then $Y$ satisfies $F[\delta \cup \delta_0], G[\delta \cup \delta_0], L[\delta \cup \delta_0]$, or $P[\delta \cup \delta_0]$. If in addition the function $\phi_{\delta_0} : \mathcal{X}_{\delta_0} \to \mathbb{R}_+$, defined by
\[
\phi_{\delta_0}(x_{\delta_0}) = \begin{cases} \frac{f_{Y,\delta_0}(x_{\delta_0})}{f_{X,\delta_0}(x_{\delta_0})} & \text{if } f_{Y,\delta_0}(x_{\delta_0}) > 0; \\ 0, & \text{if } f_{Y,\delta_0}(x_{\delta_0}) = 0,
\end{cases}
\]
has the form
\[
\phi_{\delta_0}(x_{\delta_0}) = \prod_{C \in \mathbb{K}^\delta} \psi_C(x_{C \cap \delta_0}) \quad \mu_{\delta_0}\text{-a.e. (4.2)}
\]
for some measurable functions $\{\psi_C : \mathcal{X}_{C \cap \delta_0} \to \mathbb{R}_+ ; C \in \mathbb{K}^\delta\}$, then $Y$ satisfies $F[\delta]$, $G[\delta], L[\delta], \text{ or } P[\delta]$.

**Proof.** By assumption, and using (2.2), it holds that
\[
f_Y(x) = f_{Y_{\setminus \delta_0}|Y_{\delta_0}}(x_{V \setminus \delta_0}|x_{\delta_0})f_{Y,\delta_0}(x_{\delta_0})
= f_X_{\setminus \delta_0}|X_{\delta_0}(x_{V \setminus \delta_0}|x_{\delta_0})f_{Y,\delta_0}(x_{\delta_0}) = f_X(x)\phi_{\delta_0}(x_{\delta_0}) \quad \mu\text{-a.e. (4.3)}
\]
$X$ satisfies $F[\delta]$: Since $f_X$ has the form (3.1), we conclude that $f_Y$ has the form (3.1) with $\delta$ replaced by $\delta \cup \delta_0$, implying that $Y$ satisfies $F[\delta \cup \delta_0]$. If $\phi_{\delta_0}$ has the form (4.2), then $f_Y$ has the form (3.1), implying that $Y$ satisfies $F[\delta]$.

$X$ satisfies $G[\delta]$: Let $(A, B, S)$ be a triple of disjoint subsets of $V$ such that $S$ separates $A$ from $B \cup ((\delta \cup \delta_0) \setminus S)$. As in the proof of Theorem 3.2, we assume without loss of generality that $A \cup B \cup S = V$, and that $\delta \cup \delta_0 \subset B \cup S$. From (4.3) and (2.4),
\[
f_Y(x) = f_{X_A|X_S}(x_A|x_S)f_{X_B \cup S}(x_B, x_S)\phi_{\delta_0}(x_{\delta_0}) \quad \mu\text{-a.e. (4.4)}
\]
Since $\delta \cup \delta_0 \subset B \cup S$, it follows from (2.5) that $Y_A \perp Y_B \mid Y_S$, implying that $Y$ satisfies $G[\delta \cup \delta_0]$. If $\phi_{\delta_0}$ has the form (4.2), then we let $(A, B, S)$ be disjoint subsets of $V$ such that $S$ separates $A$ from $B \cup (\delta \setminus S)$, and assume that $A \cup B \cup S = V$, and that $\delta \subset B \cup S$. It can be shown, as in the proof of Theorem 3.2, that for each $C \in \mathbb{K}^\delta$, either $C \in A \cup S$ or $C \in B \cup S$. Therefore, it follows from (4.4) and (2.5) that $Y_A \perp Y_B \mid Y_S$, implying that $Y$ satisfies $G[\delta]$.

$x$ satisfies $L[\delta]$: For each $\alpha \in V \setminus (\delta \cup \delta_0)$, replace $A, B$ and $S$ in (4.4) by $\{\alpha\}$, $V \setminus \text{cl}(\alpha)$ and $\text{bd}(\alpha)$, respectively, and conclude that $Y$ satisfies $L[\delta \cup \delta_0]$. If $\phi_{\delta_0}$ has the form (4.2), then, for each $\alpha \in V \setminus \delta$, replace $A, B$ and $S$ in (4.4) by $\{\alpha\}$, $V \setminus \text{cl}(\alpha)$ and $\text{bd}(\alpha)$, and conclude that $Y$ satisfies $L[\delta]$.

$x$ satisfies $P[\delta]$: For each $\alpha \in V \setminus (\delta \cup \delta_0)$ and $\beta \in V$ such that $(\alpha, \beta) \notin E$, replace $A, B$ and $S$ in (4.4) by $\{\alpha\}$, $\{\beta\}$, and $V \setminus \{\alpha, \beta\}$, and conclude that $Y$ satisfies $P[\delta \cup \delta_0]$. If $\phi_{\delta_0}$ has the form (4.2), then, for each $\alpha \in V \setminus \delta$ and $\beta \in V$ such that $(\alpha, \beta) \notin E$, replace $A, B$ and $S$ in (4.4) by $\{\alpha\}$, $\{\beta\}$, and $V \setminus \{\alpha, \beta\}$, and conclude that $Y$ satisfies $P[\delta]$. \hfill \square

Remark 4.1. (Schrödinger problems.) As an application of Theorem 4.2, we mention Schrödinger problems for random fields on undirected graphs; for more details, see Csiszár [5], Föllmer [10], Section 1.3, or Léonard et al. [16], Section 3. Let $X$ be a random field on an undirected graph $G = (V, E)$, and let $\pi_X = \mathcal{L}(X)$. $\pi_X$ is assumed to have a density $f_X$ with respect to a product of $\sigma$-finite measures $\mu$. For each $A \subset V$, define $\mathcal{P}_A$ as the set of all probability distributions on $(\mathcal{X}_A, \mathcal{B}_{\mathcal{X}_A})$, and let $\mathcal{P} = \mathcal{P}_V$. Let $\delta_0 \subset V$, and let $\mathcal{P}_{\delta_0}^a$ be a fixed convex subset of $\mathcal{P}_{\delta_0}$. Denote by $D(\cdot \| \cdot)$ the relative entropy, also known as the Kullback-Leibler divergence. By the static and dynamic Schrödinger problems, we mean the following optimization problems.

\[ S_{\text{stat}}: \text{minimize } D(\pi_{\delta_0} \| \pi_{X_{\delta_0}}) \text{ over all } \pi_{\delta_0} \in \mathcal{P}_{\delta_0}^a. \]

\[ S_{\text{dyn}}: \text{minimize } D(\pi_Y \| \pi_X) \text{ over all } \pi_Y \in \mathcal{P} \text{ such that } \pi_{Y_{\delta_0}} \in \mathcal{P}_{\delta_0}^a. \]

By the strict convexity of the relative entropy, both solutions are unique if they exist. If the solution $\pi_{\delta_0}$ to $S_{\text{stat}}$ exists, it follows from the definition of relative entropy that $\pi_{\delta_0}$ must have a density $f_{\delta_0}$ with respect to $\mu_{\delta_0}$, which can be chosen so that $f_{X_{\delta_0}}(x_{\delta_0}) = 0 \Rightarrow f_{\delta_0}(x_{\delta_0}) = 0$. Moreover, from the chain rule of relative entropy, see
[10], Section 1.3, a solution $\pi_Y$ to $S_{\text{dyn}}$ exists, which has a density $f_Y$ with respect to $\mu_Y$. $f_Y$ can be chosen so that $f_{Y_{\delta_0}} = f_{\delta_0}$, and so that, for each $x \in \mathcal{X}$ so that $f_{Y_{\delta_0}}(x_{\delta_0}) > 0$,
\[
\frac{f_X(x_{V \setminus \delta_0}, x_{\delta_0})}{f_{X_{\delta_0}}(x_{\delta_0})} = \frac{f_Y(x_{V \setminus \delta_0}, x_{\delta_0})}{f_{Y_{\delta_0}}(x_{\delta_0})}.
\]

5. Reciprocal chains

In this section we apply the results of the previous sections to discrete time reciprocal processes, better known as reciprocal chains.

Definition 5.1. A sequence of random variables $\{X_t; t = 0, 1, \ldots, n\}$, where $n \geq 2$, is called a reciprocal chain if
\[
X_k \perp \{X_1, \ldots, X_{j-1}, X_{j+1}, \ldots, X_n\} | \{X_j, X_l\} \quad \forall 0 \leq j < k < l \leq n. \tag{5.1}
\]

As before, we will assume that $\mathcal{L}(X)$ has a density $f_X$ with respect to a product of $\sigma$-finite measures $\mu$. We observe that any random sequence $X = \{X_t; t = 0, 1, \ldots, n\}$ can be seen as a random field on the undirected graph $\mathcal{G} = (V, E)$, where $V = \{0, 1, \ldots, n\}$ and $E = \{(i, i+1); i = 0, 1, \ldots, n-1\}$. We will identify $X$ with this random field, since there is no risk for confusion.

Theorem 5.1. A random sequence $X = \{X_t; t = 0, 1, \ldots, n\}$ is a reciprocal chain if and only if it satisfies $G[\delta]$ with respect to $\delta = \{0, n\}$.

Proof. Assume that $X$ satisfies $G[\delta]$ with respect to $\delta = \{0, n\}$. For each fixed $0 \leq j < k < l \leq n$, let $A = \{k\}$, $S = \{j, l\}$, and $B = \{0, 1, \ldots, j-1\} \cup \{l+1, \ldots, n\}$.

Then, $S$ separates $A$ from $B \cup (\delta \setminus S)$, so by property $G[\delta]$, $X$ satisfies (5.1).

Assume instead that $X$ satisfies (5.1). Let $(A, B, S)$ be a triple of disjoint subsets of $V = \{0, 1, \ldots, n\}$ such that $S$ separates $A$ from $B \cup (\delta \setminus S)$. As in the proof of Theorem 3.2, we assume without loss of generality that $A \cup B \cup S = V$, and that $\delta \subset B \cup S$. We also assume without loss of generality that $A \neq \emptyset$. It must then hold that $A = \cup_{i=1}^m A_i$, where $m$ is a positive integer, and $A_i = \{\ell_i, \ell_i+1, \ldots, u_i\}$ for $i = 1, \ldots, m$, where $\{(\ell_i, u_i); i = 1, \ldots, m\}$ are pairs of integers such that $0 < \ell_1 \leq u_1 < \ell_2 - 1 < \ell_2 \leq u_2 < \ell_3 - 1 < \cdots < \ell_m \leq u_m < n$. Note also that, for each $i = 1, \ldots, m$, $\{\ell_i-1, u_i+1\} \subset S$. Applying (5.1) and (2.4) to $f_X$ for each $k \in A_i$ in
increasing order, we get:

\[ f_X(x) = f_{X_{t_1}}\big|_{X_{t_1-1}, X_{t_1+1}}(X_{t_1} | X_{t_1-1}, X_{t_1+1}) \]

\[ \times f_{X_0, \ldots, X_{t_1-1}, X_{t_1+1}, \ldots, X_n}(x_0, \ldots, x_{t_1-1}, x_{t_1+1}, \ldots, x_n) \]

\[ = f_{X_{t_1}}\big|_{X_{t_1-1}, X_{t_1+1}}(X_{t_1} | X_{t_1-1}, X_{t_1+1})f_{X_{t_1+1}}\big|_{X_{t_1-1}, X_{t_1+2}}(X_{t_1+1} | X_{t_1-1}, X_{t_1+2}) \]

\[ \times f_{X_0, \ldots, X_{t_1-1}, X_{t_1+1}, \ldots, X_n}(x_0, \ldots, x_{t_1-1}, x_{t_1+2}, \ldots, x_n) \]

\[ = \ldots = \prod_{r=0}^{u_1-\ell_1} f_{X_{t_1+r}}\big|_{X_{t_1-1}, X_{t_1+r+1}}(X_{t_1+r} | X_{t_1-1}, X_{t_1+r+1}) \]

\[ \times f_{X_0, \ldots, X_{t_1-1}, X_{u_1+1}, \ldots, X_n}(x_0, \ldots, x_{t_1-1}, x_{u_1+1}, \ldots, x_n) \quad \mu\text{-a.e.} \]

Proceeding in the same fashion for each \( k \in A \setminus A_1 \) in increasing order, we end up with:

\[ f_X(x) = \prod_{i=1}^{m} \prod_{r=0}^{u_i-\ell_i} f_{X_{t_i+r}}\big|_{X_{t_i-1}, X_{t_i+r+1}}(X_{t_i+r} | X_{t_i-1}, X_{t_i+r+1})f_{X_{V \setminus A}}(x_{V \setminus A}) \quad \mu\text{-a.e.} \]

The expression on the right hand side is a product of two functions, the first of which depends only on \( X_{A \cup S} \), while the second one depends only on \( X_{B \cup S} \). By (2.5), this implies that \( X_A \perp X_B | X_S \), so \( X \) satisfies \( G[\delta] \).

**Theorem 5.2.** A random sequence \( X = \{X_t; t = 0, 1, \ldots, n\} \) satisfies \( F[\delta] \) with respect to \( \delta = \{0, n\} \) if and only if \( f_X \) has the form:

\[ f_X(x) = \phi_n(x_0, x_n) \prod_{i=0}^{n-1} \phi_i(x_i, x_{i+1}) \quad \mu\text{-a.e.} \]

for some measurable functions \( \phi_i : X_{\{i,i+1\}} \to \mathbb{R}_+ \); \( i = 0, 1, \ldots, n-1 \), \( \phi_n : X_{\{0,n\}} \to \mathbb{R}_+ \). In particular, if \( X \) is a reciprocal chain and \( f_X \) is positive, then \( X \) satisfies \( F[\delta] \).

**Proof.** The first claim follows from Definition 3.1, and the second one follows from Theorem 5.1 and Theorem 3.6. \( \square \)

**Example 5.1.** Let \( X = \{X_t; t = 0, 1, \ldots, n\} \) be a random sequence with a centered, nonsingular Gaussian distribution. Then, by Theorem 5.2, \( X \) satisfies \( F[\delta] \) with respect to \( \delta = \{0, n\} \) if and only if the inverse covariance matrix \( C^{-1} \) has a cyclic tridiagonal structure, meaning that all its elements are 0 except possibly \( C_{i,j}^{-1}; |i - j| \leq 1 \) and \( C_{0,n}^{-1} = C_{n,0}^{-1} \). This result was previously obtained in [17] (their Theorem 3.2) by a completely different argument.
In the general case a reciprocal chain need not satisfy $F[\delta]$ with respect to $\delta = \{0, n\}$, as the following two examples show.

**Example 5.2.** Let $X = \{X_0, X_1, X_2\}$ be a random sequence, where each of the random variables $\{X_0, X_1, X_2\}$ takes values in $\mathcal{X}_0 = \{0, 1\}$ with a probability mass function

$$f_X(x_0, x_1, x_2) = P(\bigcap_{i=0}^{2}\{X_i = x_i\}) \quad \forall x \in \{0, 1\}^3$$

such that $f_X(0, 0, 0) = 0$, $f_X(0, 0, 1) > 0$, $f_X(0, 1, 0) > 0$, and $f_X(1, 0, 0) > 0$. Define $V = \{0, 1, 2\}$ and $\delta = \{0, 2\}$. $X$ is (trivially) a reciprocal chain. Assume that $X$ satisfies $F[\delta]$. Then, it must hold that

$$f_X(x) = \phi_0(x_0, x_1)\phi_1(x_1, x_2)\phi_2(x_0, x_2) \quad \forall (x_0, x_1, x_2) \in \{0, 1\}^3,$$

for some functions $\{\phi_i : \{0, 1\}^2 \to \mathbb{R} ; i = 0, 1, 2\}$. However, the condition $f_X(0, 0, 0) = 0$ implies that at least one of the factors $\phi_0(0, 0)$, $\phi_1(0, 0)$ and $\phi_2(0, 0)$ must be 0, while the conditions $f_X(0, 0, 1) > 0$, $f_X(0, 1, 0) > 0$, and $f_X(1, 0, 0) > 0$ imply that $\phi_0(0, 0)$, $\phi_1(0, 0)$ and $\phi_2(0, 0)$ must all be positive, which is a contradiction.

**Example 5.3.** Let $X = \{X_0, X_1, X_2, X_3\}$ be a random sequence, where each of the random variables $\{X_0, X_1, X_2, X_3\}$ takes values in $\mathcal{X}_0 = \{0, 1\}$, with a probability mass function

$$f_X(x_0, x_1, x_2, x_3) = P(\bigcap_{i=0}^{3}\{X_i = x_i\}) \quad \forall x \in \{0, 1\}^4,$$

defined by: $f_X(0, 0, 0, 0) = f_X(1, 0, 0, 0) = f_X(1, 1, 0, 0) = f_X(1, 1, 1, 0) = f_X(0, 0, 0, 1) = f_X(0, 0, 1, 1) = f_X(0, 1, 1, 1) = f_X(1, 1, 1, 1) = \frac{1}{8}$. Define $V = \{0, 1, 2, 3\}$ and $\delta = \{0, 3\}$. $X$ can be considered as a random field on $\mathcal{G} = (V, E)$, where $E = \{(0, 1), (1, 2), (2, 3), \}$, but also as a random field on $\mathcal{G}_{\delta} = (V, E^+_\delta)$, where $E^+_\delta = \{(0, 1), (1, 2), (2, 3), (0, 3)\}$. It was shown in Moussouris [19] that $X$ satisfies $G[0]$, but not $F[0]$, in $\mathcal{G}_{\delta}$. Therefore, by Theorem 3.7, $X$ satisfies $G[\delta]$, but not $F[\delta]$, in $\mathcal{G}$.

**Remark 5.1.** As mentioned in Section 1, in a number of papers starting with [3] a different definition of a reciprocal chain was used: a sequence of random variables $\{X_t ; t = 0, 1, \ldots, n\}$, where $n \geq 2$, is said to be a reciprocal chain if

$$X_k \perp \{X_1, \ldots, X_{k-2}, X_{k+2}, \ldots, X_n\} | \{X_{k-1}, X_{k+1}\} \quad \forall 0 < k < n. \quad (5.2)$$
It follows from Definition 3.3 that $X = \{X_t; t = 0, 1, \ldots, n\}$ satisfies (5.2) if and only if it satisfies the local reciprocal property $L[\delta]$ with respect to $\delta = \{0, n\}$. We propose to call such a process a local reciprocal chain.

**Remark 5.2.** (Markov chains.) Let $X = \{X_t; t = 0, 1, \ldots, n\}$ be a random sequence such that $\mathcal{L}(X)$ has a density $f_X$ with respect to a product of $\sigma$-finite measures $\mu$. $X$ is called a Markov chain, if

$$X_k \perp \{X_0, \ldots, X_{k-2}\} | X_{k-1} \quad \forall 0 < k \leq n.$$ 

It is well-known that $f_X$ has the factorization

$$f_X(x) = f_{X_0}(x_0) \prod_{i=0}^{n-1} f_{X_{i+1}|X_i}(x_{i+1}|x_i) \mu\text{-a.e.}$$

From this and Theorem 3.2 it follows that $X$ is a Markov chain if and only if $X$ has the factorizing Markov property, $F[\emptyset]$. As we have seen, this is not true for reciprocal chains in general.

**Remark 5.3.** Let $X = \{X_t; t = 0, 1, \ldots, n\}$, where $n \geq 2$, be a reciprocal chain, i.e., a random sequence satisfying $G[\delta]$, where $\delta = \{0, n\}$, and let $\delta_0 = \{n\}$. By Theorem 4.1, under the conditional distribution of $X_{V \setminus \{n\}}$ given $X_n = x_n$ for any $x_n \in \mathcal{X}_n$ such that $f_{X_n}(x_n) > 0$, $X_{V \setminus \{n\}}$ satisfies $G[\{0\}]$. Moreover, by Theorem 3.7 it holds that $G[\{0\}] \Rightarrow G[\emptyset]$, so, conditionally on $X_n = x_n$ for any $x_n \in \mathcal{X}_n$ such that $f_{X_n}(x_n) > 0$, $X_{V \setminus \{n\}}$ is a Markov chain. In contrast, if $X$ satisfies only $L[\delta]$, then, conditional on $X_n = x_n$, where $x_n \in \mathcal{X}_n$ is such that $f_{X_n}(x_n) > 0$, $X_{V \setminus \{n\}}$ need not satisfy $L[\emptyset]$; cf. Example 3.4.

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