Continuous primitives with infinite derivatives

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Abstract

In calculus the concept of an infinite derivative – i.e. $DF(x) = \pm\infty$ – is seldom studied due to a plethora of complications that arise from this definition. For instance, in this extended sense, algebraic expressions involving derivatives are generally undefined; and two continuous functions possessing identical derivatives at every point of an interval generally differ by a non-constant function. These problems are fundamentally irremediable insofar as calculus is concerned and must therefore be addressed in a more general setting. This is quite difficult since the literature on infinite derivatives is rather sparse and seldom accessible to non-specialists. Therefore we supply a self-contained thesis on continuous functions with infinite derivatives aimed at graduate students with a background in real analysis and measure theory.

Predominately we study continuous primitives which satisfy the Luzin condition (N) by establishing a deep connection with the strong Luzin condition – a weak form of absolute continuity which has its origins in the Henstock–Kurzweil theory of integration. The main result states that a function satisfies the strong Luzin condition if and only if it can be expressed as a sum of two such primitives. Furthermore, we establish some pathological properties of continuous primitives which fail to satisfy the Luzin condition (N).

Keywords:

Infinite derivative, Henstock–Kurzweil integral, primitive function, strong Luzin condition, variational measure.

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Contents

Introduction

This thesis concerns continuous real-valued functions of one variable possessing a derivative (finite or infinite) at every point of an open set $\Omega \subset \mathbb{R}$. We shall refer to these functions as *continuous primitives* and partition them into two classes: *regular primitives* which satisfy the Luzin condition (N); and *irregular primitives* which fail to satisfy the Luzin condition (N). We are primarily interested in the former class since its members are particularly reminiscent of the primitives with finite derivatives studied in calculus. Nevertheless some pathological properties of the latter class are established as well. Let us begin with a sample of the more accessible results.

Theorem 3.1. *Consider two regular primitives $F, G\colon (a,b) \to \mathbb{R}$ possessing the property that $DF = DG$ a.e. on (a,b). Then $F - G$ is constant.*

This theorem is obtained by first establishing an analogous theorem for functions which satisfy the strong Luzin condition – a weak form of absolute continuity proposed by Lee [12] – and subsequently verifying that regular primitives satisfy this condition. In addition we prove the well-known fact that the strong Luzin condition implies continuity, the Luzin condition (N) and differentiability a.e.; whereas the converse generally fails.

The first example of two continuous functions possessing identical derivatives (finite or infinite) at every point of an interval, while differing by a non-constant function, was published in 1905 by Hahn [11]. Therefore the possession of just one continuous primitive is generally insufficient for the purpose of obtaining all continuous primitives which possess the same derivative. By manipulating the Cantor ternary set we obtain a rather simple construction of this type.

Theorem 3.2. *There exists a regular primitive $F\colon (0,1) \to \mathbb{R}$ and an irregular primitive $G\colon (0,1) \to \mathbb{R}$ such that $DF = DG$ everywhere on $(0,1)$ and $G - F = T|_{(0,1)}$, where T denotes the Cantor ternary function. In particular $G - F$ is non-constant.*

A regular primitive possessing certain desirable properties can be obtained by first constructing an integrable candidate for its derivative and subsequently verifying that this candidate constitutes the derivative of its indefinite integral. Of course, it is generally insufficient to merely integrate a function in order to conclude that it constitutes the derivative of its indefinite integral. For instance, the presence of a single jump discontinuity violates the intermediate value property and therefore inhibits the classification of a derivative, i.e. derivatives are *Darboux functions*. Moreover, even for bounded derivatives the points of continuity may form a set of measure zero, see Bruckner [3, pp. 33–34]. Nevertheless this set is dense and of type G_δ (and therefore uncountable) because derivatives are pointwise limits of continuous functions, i.e. *Baire*-1 *functions*. Thus the process of constructing the derivative before the primitive can be quite challenging since one must ensure that a multitude of necessary and collectively sufficient properties of derivatives are satisfied. Indeed, the proof of the following theorem reflects this sentiment.

Theorem 4.1 (Zahorski–Choquet). *Let $Z \subset \Omega$ be a G_δ set with $m(Z) = 0$. Then there exists a bounded, non-decreasing, regular primitive $G\colon \Omega \to \mathbb{R}$ such that $DG(x) = \infty$ if and only if $x \in Z$.*

The first construction of this kind was published in 1941 by Zahorski [17] and then six years later another one appeared in the doctoral dissertation of Choquet [5, pp. 216–220]. Although these works of Zahorski and Choquet – which are written in German and French, respectively – have merely been glazed over in the writing of this thesis, both authors appear to supply constructions based on the Luzin–Menshov theorem. By the aid of Bruckner [3, pp. 20–23, 86] we supply a similar construction.

The results discussed thus far can more or less be deduced from the existing literature. Henceforth the discussion concerns results that we have not been able to locate elsewhere. We shall briefly outline some consequences of these results in relation to the *Henstock–Kurzweil integral* – an extension of the Riemann integral. This integral admits a fundamental theorem of calculus based on the strong Luzin condition. In particular the class of regular primitives is subsumed by this theory of integration. That is, for every regular primitive $F\colon \Omega \to \mathbb{R}$ and real-valued function $f\colon [a,b] \to \mathbb{R}$ satisfying $DF = f$ a.e. on $[a,b] \subset \Omega$, we have

$$F(x) = F(a) + \int_a^x f(t)\,dt \quad \text{for every } x \in [a,b].$$

Note that by Ward's Theorem 2.1 the points at which an arbitrary function possesses an infinite derivative form a set of measure zero. Thus for every regular primitive F there exists such a function f. Technically the Henstock–Kurzweil integral is not capable of integrating functions taking infinite values which necessitates this minor comprise.

Now, consider an irregular primitive $G\colon \Omega \to \mathbb{R}$ and a subinterval $[a,b] \subset \Omega$. If $g\colon [a,b] \to \mathbb{R}$ satisfies $DG = g$ a.e. on $[a,b]$, is it necessarily the case that g is Henstock–Kurzweil integrable? If such was the case, then the difference between the restricted primitive $G|_{[a,b]}$ and the indefinite integral of g would constitute a singular[1] function $H\colon [a,b] \to \mathbb{R}$ such that

$$G(x) = H(x) + \int_a^x g(t)\,dt \quad \text{for every } x \in [a,b].$$

Therefore we pose a more general question: is it necessarily the case that G can be locally expressed, at every point of Ω, as the sum of a singular function and a function which satisfies the strong Luzin condition? The following theorem disproves this conjecture.[2]

Theorem 5.1. *There exists an irregular primitive $G\colon (0,1) \to \mathbb{R}$ possessing the property that for every open interval $U \subset (0,1)$ which contains a point of the Cantor ternary set, there does not exist any function $F\colon U \to \mathbb{R}$ which satisfies both the strong Luzin condition and $DF = DG$ a.e. on U.*

As indicated by the preceding remarks, regular primitives are intimately connected with the strong Luzin condition, and thus with the Henstock–Kurzweil integral. The following theorem strengthens this claim and extends an analogous result for the Lebesgue integral due to Zahorski [18, p. 50].

Theorem 5.2. *A function $F\colon \Omega \to \mathbb{R}$ satisfies the strong Luzin condition if and only if it can be expressed as the sum of two regular primitives.*

The forward direction in particular appears to be new and is established by developing an assortment of lemmas concerning the Henstock variational measure and its connection with the more accessible Jordan variation. In contrast, the reverse direction − which remains valid for products of regular primitives − is comparatively simple to prove and is therefore obtained as a consequence of our initial investigation of the strong Luzin condition.

[1] A function is said to be *singular* provided that it is continuous, has a vanishing derivative a.e. and is non-constant.

[2] Theorem 5.1 originates from an e-mail correspondence with Valentin Skvortsov in the summer of 2021, to whom I posed the question of whether a real-valued function which coincides a.e. with the derivative of a continuous VBG_* function is necessarily Henstock–Kurzweil integrable. Skvortsov mentioned this question to his former student Piotr Sworowski who proposed a simple counterexample based on the Cantor ternary set. Although our construction is substantially different, it makes use of Sworowski's idea to consider a function which oscillates badly on the Cantor ternary set and subsequently diminish its amplitude on the complementary intervals by the addition of singular functions.

Chapter 1

Preliminaries

This chapter contains definitions, notations and some elementary results from real analysis and measure theory. Throughout the entire thesis $\Omega \subset \mathbb{R}$ will denote an open set, and throughout this particular chapter we shall fix a subset $E \subset \Omega$ and consider a function $F\colon \Omega \to \mathbb{R}$. We make an exception in Lemma 1.6 by allowing the function F to assume values in $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$.

1.1 The Lebesgue measure

The Lebesgue outer measure and the Lebesgue measure are denoted by m^* and m, respectively. Note that the latter is a *regular Borel measure*, i.e. every Borel set is measurable and if E is measurable then

$$m(E) = \inf\{m(U)\colon U \supset E \text{ is open}\} = \sup\{m(K)\colon K \subset E \text{ is compact}\}.$$

We say that F satisfies the *Luzin condition* (N) provided that $m(F(N)) = 0$ for every $N \subset \Omega$ with $m(N) = 0$.

Suppose $E \subset [0,1]$ is measurable. Then for every $x \in E$ we define the *density* of E at x by

$$d(x,E) := \liminf_{m(I)\to 0} \frac{m(E \cap I)}{m(I)},$$

where I denotes a subinterval of $[0,1]$ which contains x. Moreover, for every set $A \subset [0,1]$ we shall use the notation $A \subsetdot E$ to mean that $A \subset E$ and $d(x,E) = 1$ for every $x \in A$. Note that the points $x \in E$ for which $d(x,E) < 1$ form a set of measure zero.[1]

[1] This is called Lebesgue's density theorem, see e.g. Faure [8] for a simple proof. Note that $d(x,E) = 1$ if and only if $m(E \cap (x-\varepsilon, x+\varepsilon))/2\varepsilon \to 1$ as $\varepsilon \to 0+$.

1.2 Concepts of variation

We denote by E^o the interior of E and by $\partial E := \overline{E} \setminus E^o$ the boundary of E.

Let Q be a collection of intervals. The intervals of Q are *non-overlapping* if $I_1^o \cap I_2^o = \varnothing$ for all distinct $I_1, I_2 \in Q$; and *strictly non-overlapping* if $I_1 \cap I_2 = \varnothing$ for all distinct $I_1, I_2 \in Q$.

A finite (possibly empty) collection Q of non-overlapping compact intervals satisfying $\bigcup_{I\in Q} I \subset E$ is called a *subpartition* of E. In case $\bigcup_{I\in Q} I = E$ then we refer to Q as a *partition* of E.

The *mesh* of Q is defined by

$$\operatorname{mesh}(Q) := \max_{I\in Q} m(I).$$

We refer to a positive function $\delta\colon \Omega \to (0,\infty)$ as a *gauge* on Ω. A pair (x, I), where I is a compact interval with $x \in I$, is called a *point-interval pair.* A finite (possibly empty) collection P of point-interval pairs whose interval components are non-overlapping and contained in Ω, is called a *tagged subpartition* of Ω. If $x \in E$ for every $(x, I) \in P$, then P is *anchored* in E. If $I \subset (x - \delta(x), x + \delta(x))$ for every $(x, I) \in P$, then P is *subordinate* to δ.

For every compact interval $I \subset \Omega$ we define

$$\Delta(F, I) := \begin{cases} F(\max I) - F(\min I), & \text{if } I \neq \varnothing; \\ 0, & \text{if } I = \varnothing. \end{cases}$$

The *oscillation* of F on E is defined by

$$\operatorname{osc}(F, E) := \begin{cases} \displaystyle\sup_{u,v\in E} |F(v) - F(u)|, & \text{if } E \neq \varnothing; \\ 0, & \text{if } E = \varnothing. \end{cases}$$

Let $J \subset \Omega$ be a compact interval. The *Jordan variation* of F on J is defined by

$$\nu_0(F, J) := \sup_Q \sum_{I\in Q} |\Delta(F, I)|,$$

where the supremum is taken over all partitions Q of J. Note that sums over the empty set are interpreted as zero.

For a gauge δ on Ω we define

$$\nu(F, E, \delta) := \sup_P \sum_{(x,I)\in P} |\Delta(F, I)|,$$

where the supremum is taken over all tagged subpartitions P of Ω which are anchored in E and subordinate to δ. The *Henstock variation* of F on E is defined by

$$\nu(F, E) := \inf_{\delta} \nu(F, E, \delta),$$

where the infimum is taken over all gauges δ on Ω. Moreover, the *total variation function* of F is the non-negative (and possibly infinite) function $V_F : \Omega \to [0, \infty]$ defined by

$$V_F(x) := \nu(F, (-\infty, x] \cap \Omega).$$

We say that F satisfies the *strong Luzin condition* provided that $\nu(F, N) = 0$ for every $N \subset \Omega$ with $m(N) = 0$.

1.3 Differential operators

The *ordinary differential operator* is defined by

$$DF(x) := \lim_{\substack{m(I)\to 0:\\ x\in\partial I}} \frac{\Delta(F, I)}{m(I)},$$

where I denotes a non-degenerate compact subinterval of Ω. Note that we allow infinite limits and the condition $x \in \partial I$ can without loss of generality be replaced by $x \in I$. We shall say that F is *differentiable* on E whenever $DF(x)$ exists (as a finite or infinite value) for every $x \in E$. In case $E = \Omega$ then we simply say that F is differentiable.

Suppose that F is both continuous and differentiable. Then we refer to F as a *regular primitive* of DF provided that F satisfies the Luzin condition (N). Otherwise we refer to F as an *irregular primitive* of DF. In both cases F is a *continuous primitive* of DF.

The *extreme differential operators* $\underline{D}$ and $\overline{D}$ are defined by

$$\underline{D}F(x) := \liminf_{\substack{m(I)\to 0:\\ x\in\partial I}} \frac{\Delta(F, I)}{m(I)} \quad \text{and} \quad \overline{D}F(x) := \limsup_{\substack{m(I)\to 0:\\ x\in\partial I}} \frac{\Delta(F, I)}{m(I)},$$

where I denotes a non-degenerate compact subinterval of Ω. Both operators are allowed to assume infinite values and once again the condition $x \in \partial I$ can without loss of generality be replaced by $x \in I$.

An extended real number $d \in \overline{\mathbb{R}}$ is called a *derived number* of F at a point $x \in \Omega$ provided that there exists a collection $\{I_j\}_{j=1}^{\infty}$ of non-degenerate compact intervals, with $x \in \partial I_j$ and $I_j \subset \Omega$, such that

$$\lim_{j\to\infty} m(I_j) = 0 \quad \text{and} \quad \lim_{j\to\infty} \frac{\Delta(F, I_j)}{m(I_j)} = d.$$

Note that if F is continuous, then the derived numbers of F at a point $x \in \Omega$ form the interval $[\underline{D}F(x), \overline{D}F(x)]$.

A function which possesses the intermediate value property is said to be a *Darboux function.* As the following theorem states, every infinite derivative possessing a continuous primitive is a Darboux function.

Theorem 1.1 (Darboux). *Suppose that F is a continuous primitive. Consider a compact subinterval $[a, b] \subset \Omega$ such that $DF(a) \neq DF(b)$ and let y be a real number which lies strictly between $DF(a)$ and $DF(b)$. Then there exists a point $x \in (a, b)$ such that $DF(x) = y$.*

Proof. It will evidently suffice to consider the case when $DF(a) < DF(b)$. Define the function $G \colon \Omega \to \mathbb{R}$ by

$$G(x) := F(x) - yx.$$

Since G is continuous on the compact interval $[a, b]$, it attains a minimum m somewhere in $[a, b]$. Let $x \in [a, b]$ satisfy $G(x) = m$. Since $DG(a) < 0 < DG(b)$ we have $x \in (a, b)$, and therefore $DG(x) = 0$. This gives the desired result. □

1.4 Basic topology

The non-empty subintervals of Ω which have both their endpoints in $\overline{\mathbb{R}} \setminus \Omega$ are called the *components* of Ω. These subintervals are countably many and pairwise disjoint. If $E \subset \mathbb{R} \setminus \Omega$ is closed, then we refer to the components of Ω which have both their endpoints in E as being *contiguous* to E.

We refer to E as an F_σ set if it can be expressed as a countable union of closed sets; and as a G_δ set if it can be expressed as a countable intersection of open sets. By De Morgan's laws the complement of an F_σ set is a G_δ set, and vice versa.

Lemma 1.2. *Let A denote the set of points which are isolated in E. Then A is countable.*

Proof. For each positive integer n, let A_n be the set of points $x \in A$ such that $|x| < n$ and $\{x\} = E \cap (x - 1/n, x + 1/n)$. Then A_n contains no more than n^2 points. Since $A = \bigcup_{n=1}^{\infty} A_n$ we conclude that A is countable. □

Lemma 1.3. *Let A denote the set of points $x \in \Omega$ for which F is constant in some neighbourhood of x. Then $F(A)$ is countable.*

Proof. For each positive integer n, let B_n be the collection of open intervals $I \subset (-n,n) \cap \Omega$ such that F is constant on I and $m(I) > 1/n$. Moreover, let A_n be the subcollection of maximal intervals $I \in B_n$, i.e. for every interval $J \in B_n$ with $I \subset J$ we have $I = J$. Note that B_n contains no more than $2n^2$ intervals. Since $A = \bigcup_{n=1}^{\infty} \bigcup_{I \in A_n} I$ we conclude that $F(A)$ is countable. $\square$

Lemma 1.4 (Cousin). *Consider a compact subinterval $[a,b] \subset \Omega$ and let δ be a gauge on Ω. Then there exists a tagged subpartition P of Ω which is subordinate to δ and whose interval components form a partition of $[a,b]$.*

Proof. The case $a = b$ is trivial, hence we assume that $a < b$. Let A be the set of points $x \in (a,b]$ for which there exists a tagged subpartition P_x of Ω which is subordinate to δ and whose interval components form a partition of $[a,x]$. Since $a < b$ and $\delta(a) > 0$ it is clear that $A \neq \varnothing$, so that $\xi := \sup A \in (a,b]$. Assume towards a contradiction that $\xi < b$. Since $\xi \in (a,b)$ there exist points

$$u \in A \cap (\xi - \delta(\xi), \xi] \quad \text{and} \quad v \in (\xi, b] \cap (\xi, \xi + \delta(\xi)).$$

Then $P_u \cup \{(\xi, [u,v])\}$ is a tagged subpartition of Ω which is subordinate to δ. From this we obtain the contradiction $\sup A = \xi < v \in A$. Therefore $\xi = b$. $\square$

Consider two sets $P, Q \subset \mathbb{R}$. We refer to Q as being *dense* in P provided that $Q \subset P$ and $\overline{Q} = P$. Alternatively, we require that $Q \subset P$ and for every open interval J with $P \cap J \neq \varnothing$ we have $Q \cap J \neq \varnothing$.

We refer to Q as being *nowhere dense* in P provided that $Q \subset P$ and for every open interval J with $P \cap J \neq \varnothing$ we have $\overline{Q \cap J} \neq \overline{P \cap J}$. Alternatively, we require that $Q \subset P$ and for every open interval J with $P \cap J \neq \varnothing$ there exists an open subinterval $I \subset J$ such that $P \cap I \neq \varnothing$ and $Q \cap I = \varnothing$.

If Q can be expressed as a countable union of sets which are nowhere dense in P, then we say that Q is of the *first category* relative to P. Otherwise we say that Q is of the *second category* relative to P.

Theorem 1.5 (Baire category theorem). *Let P be a non-empty closed set. Then P is of the second category relative to itself.*

Proof. Assume towards a contradiction that P is of the first category relative to itself. Then we can write $P = \bigcup_{k=1}^{\infty} Q_k$, where $\{Q_k\}_{k=1}^{\infty}$ is a countable collection of sets which are nowhere dense in P. Since P is non-empty and Q_1 is nowhere dense in P, there exists an open interval $I_1 \subset Q_1^c$ which contains a point $x_1 \in P$ and satisfies $m(I_1) < 1$.

We proceed inductively. Suppose that we have already determined an open interval $I_k \subset Q_k^c$ which contains a point $x_k \in P$ and satisfies $m(I_k) < 1/k$. Then since $P \cap I_k \neq \varnothing$ and Q_{k+1} is nowhere dense in P, there exists an open interval

$I_{k+1} \subset I_k \cap Q_{k+1}^c$ which contains a point $x_{k+1} \in P$ and satisfies $m(I_{k+1}) < 1/(k+1)$. Without loss of generality we shall assume that there exists a compact interval J_k such that $I_{k+1} \subset J_k \subset I_k$.

Now, by the nested interval property of $\mathbb{R}$ there exists a point

$$x \in \bigcap_{k=1}^{\infty} J_k \subset \bigcap_{k=1}^{\infty} I_k \subset \bigcap_{k=1}^{\infty} Q_k^c = \left(\bigcup_{k=1}^{\infty} Q_k \right)^c = P^c.$$

However, since P is closed, $\{x_k\}_{k=1}^{\infty} \subset P$ and $x_k \to x$ as $k \to \infty$, we have $x \in P$. Thus we have obtained a contradiction. $\square$

Hereafter we allow the function F which we have considered throughout this chapter to assume values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, as opposed to just in $\mathbb{R}$. We say that F belongs to the *first class of Baire* or is a *Baire*-1 *function* if it is the pointwise limit of a sequence of finite continuous functions. It is easily verified that every infinite derivative of a continuous primitive constitutes a Baire-1 function.

Lemma 1.6. *Suppose that $F\colon \Omega \to \overline{\mathbb{R}}$ is a Baire-1 function and let $P \subset \Omega$ be a subset for which $F|_P$ is discontinuous at every point of P. Then P is of the first category relative to itself.*

Proof. First we consider the case when F is $\mathbb{R}$-valued (as opposed to $\overline{\mathbb{R}}$-valued). For each positive integer n, we denote by P_n the subset of P possessing the property that $\operatorname{osc}(F, P \cap U) > 1/n$ for every open subset $U \subset \Omega$ with $P_n \cap U \neq \varnothing$. Note that $P = \bigcup_{n=1}^{\infty} P_n$ since $F|_P$ is discontinuous at every point of P. Assume towards a contradiction that P is of the second category relative to itself. Then there exists a positive integer n_0 and an open interval J such that $P \cap J \neq \varnothing$ and $P_{n_0} \cap J$ is dense in $P \cap J$. From this, and the fact that P_{n_0} is closed relative to P, we infer that $P \cap J = P_{n_0} \cap J$.

Without loss of generality we shall assume that $\overline{J} \subset \Omega$ and each endpoint of J is either excluded from P or else constitutes an accumulation point of $P \cap J$. Thus $Q := \overline{P \cap J} \subset \Omega$ and $\operatorname{osc}(F, Q \cap U) > 1/n_0$ for every open subset $U \subset \Omega$ with $Q \cap U \neq \varnothing$. Moreover, Q is perfect because it is closed by definition and each of its isolated points is an isolated point of P, but P does not possess any isolated points as a consequence of $F|_P$ being discontinuous at every point of P. We aim to show that there exists an open subinterval $I \subset J$ such that $Q \cap I \neq \varnothing$ and $\operatorname{osc}(F, Q \cap I) \leq 1/n_0$, thereby obtaining the desired contradiction.

By hypothesis F is a Baire-1 function and so there exists a sequence $\{F_i\}_{i=1}^{\infty}$ of continuous functions $F_i\colon \Omega \to \mathbb{R}$ which converges pointwise to F. Define for each positive integer k the set

$$Q_k := \bigcap_{i,j \geq k} \left\{ x \in Q \colon |F_i(x) - F_j(x)| \leq \frac{1}{3n_0} \right\}.$$

Since Q is perfect and each $|F_i - F_j|$ is continuous it follows that Q_k is defined as an intersection of closed sets and is therefore closed itself. By Cauchy's criterion for convergence we have $Q = \bigcup_{k=1}^{\infty} Q_k$, and so the Baire category theorem yields a positive integer k_0 and an open interval U such that $Q \cap U \neq \varnothing$ and $Q_{k_0} \cap U$ is dense in $Q \cap U$. Since Q_{k_0} is closed it follows that $Q \cap U = Q_{k_0} \cap U$. That is, $|F_i(x) - F_j(x)| \leq 1/3n_0$ for all $x \in Q \cap U$ and $i, j \geq k_0$. By fixing $j := k_0$ and letting $i \to \infty$ we obtain

$$|F(x) - F_{k_0}(x)| \leq \frac{1}{3n_0} \quad \text{for all } x \in Q \cap U.$$

Since $Q \subset \overline{J}$ is perfect, $Q \cap U \neq \varnothing$ and F_{k_0} is continuous there exists an open subinterval $I \subset J \cap U$ such that $Q \cap I \neq \varnothing$, and

$$|F_{k_0}(x) - F_{k_0}(y)| \leq \frac{1}{3n_0} \quad \text{for all } x, y \in I.$$

Thus for all $x, y \in Q \cap I$ we use the two inequalities above to infer that

$$|F(x) - F(y)| \leq |F(x) - F_{k_0}(x)| + |F_{k_0}(x) - F_{k_0}(y)| + |F_{k_0}(y) - F(y)| \leq \frac{1}{n_0}.$$

That is, we have shown that $\operatorname{osc}(F, Q \cap I) \leq 1/n_0$. This completes the proof of the case when F is $\mathbb{R}$-valued.

Finally we consider the general case when F is $\overline{\mathbb{R}}$-valued. Let $G \colon \overline{\mathbb{R}} \to \mathbb{R}$ be a continuous injection, e.g. the arctangent function continuously extended to $\overline{\mathbb{R}}$. Then by applying the bounded case to the composition $G \circ F$ we conclude that P is of the first category relative to itself. $\square$

1.5 The Cantor ternary set

The *Cantor ternary set* can be defined as

$$C := \left\{ \sum_{k=1}^{\infty} \frac{2a_k}{3^k} : a_k \in \{0, 1\} \right\}.$$

In order to proceed we require the following notation:

$$C_1 := \{0\} \quad \text{and} \quad C_i := \left\{ \sum_{j=1}^{i-1} \frac{2a_j}{3^j} : a_1, \ldots, a_{i-1} \in \{0, 1\} \right\} \quad \text{for } i \geq 2.$$

Then we have

$$C = \bigcap_{i=1}^{\infty} \bigcup_{x \in C_i} \left(\left[x, x + \frac{1}{3^i} \right] \cup \left[x + \frac{2}{3^i}, x + \frac{3}{3^i} \right] \right),$$

from which it is readily inferred that C is perfect, nowhere dense in $[0,1]$ and $m(C)=0$. That is, C is closed because it is expressed as an intersection of closed sets. In addition the common length of the two compact intervals that correspond to each positive integer i and $x \in C_i$, as well as the distance between these intervals, tend to zero as $i \to \infty$. This ensures that C possesses no isolated points. Thus C is perfect. Furthermore, each C_i contains 2^{i-1} points, so that

$$m(C) \leq \sum_{x \in C_i} \frac{2}{3^i} = \left(\frac{2}{3}\right)^i.$$

From this inequality we infer that $m(C)=0$. Finally, we have

$$[0,1] \setminus C = \bigcup_{i=1}^{\infty} \bigcup_{x \in C_i} \left(x + \frac{1}{3^i}, x + \frac{2}{3^i}\right),$$

from which it follows that C is nowhere dense in $[0,1]$ and each C_i corresponds to 2^{i-1} intervals of length $1/3^i$ which are contiguous to C.

The *Cantor ternary function* $T\colon [0,1] \to [0,1]$ is defined as

$$T(x) := \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{a_k}{2^k}, & \text{if } x = \displaystyle\sum_{k=1}^{\infty} \frac{2a_k}{3^k}, a_k \in \{0,1\}, \\ \displaystyle\sup_{y \in C \cap [0,x)} T(y), & \text{if } x \in [0,1] \setminus C. \end{cases}$$

It can easily be shown that T is continuous, non-decreasing and constant on the components of $[0,1] \setminus C$. In particular T is *singular*, i.e. T is continuous, it has a vanishing derivative a.e. on $(0,1)$ and it is non-constant.

Chapter 2

Differentiation theorems

On numerous occasions throughout this thesis we shall employ the Vitali covering theorem which constitutes the first result of this chapter. Subsequently we prove two well-known differentiation theorems of Lebesgue and Ward. The former treats monotone functions whereas the latter treats the broader class of functions for which at least one extreme derivative is finite.

Theorem 2.1 (Ward). *Consider a function $F\colon \Omega \to \mathbb{R}$ and define the set*

$$E := \{x \in \Omega\colon \underline{D}F(x) > -\infty \text{ or } \overline{D}F(x) < \infty\}.$$

Then F is finitely differentiable a.e. on E.

An immediate consequence of Ward's theorem is that the points $x \in \Omega$ for which $DF(x) = \infty$ ($DF(x) = -\infty$) form a set of measure zero since these points satisfy the inequality $\underline{D}F(x) > -\infty$ ($\overline{D}F(x) < \infty$). Therefore we may use the terminology 'differentiable a.e.' unambiguously upon completing the proof of Ward's theorem. With Lebesgue's theorem at our disposal, this proof boils down to a rather simple 'squeeze argument'.

Finally, using the Baire category theorem in addition to the aforementioned theorems of Vitali and Ward we prove the following result:

Theorem 2.2 (Bongiorno–Skvortsov–Piazza). *Let $F\colon \Omega \to \mathbb{R}$ be a function which satisfies the strong Luzin condition. Then F is differentiable a.e. on Ω.*

Our proof closely resembles the original proof due to Bongiorno, Skvortsov and Di Piazza [2]. However, the original proof depends on the Denjoy–Luzin–Saks differentiability theorem for VBG_* functions [14, p. 230]; whereas our proof is based on Ward's theorem and Lemma 2.5.

2.1 Theorems of Vitali, Lebesgue and Ward

In order to state the Vitali covering theorem we require the following definition: A collection $\mathcal{V}$ of compact intervals is a *Vitali cover* of E if for every $x \in E$ and $\varepsilon > 0$ there exists an interval $J \in \mathcal{V}$ such that $x \in J$ and $0 < m(J) < \varepsilon$.

Theorem 2.3 (Vitali covering theorem). *Consider a set $E \subset \mathbb{R}$ and suppose $\mathcal{V}$ is a Vitali cover of E. Then there exists a countable subcollection $\{J_k\}_k \subset \mathcal{V}$ of strictly non-overlapping intervals such that*

$$m\Big(E \setminus \bigcup_k J_k\Big) = 0.$$

Proof. First we consider the case when E is bounded. Pick an open set U which is bounded and contains E. If the subcollection $\{J_k\}_k$ can be chosen to be finite (and possibly empty) then nothing remains to be shown. Hence we assume that this is not the case. The intervals J_k will be constructed recursively. Pick an interval $J_1 \in \mathcal{V}$ with $J_1 \subset U$.

We proceed inductively. Suppose that we have already determined a finite subcollection $\{J_k\}_{k=1}^n \subset \mathcal{V}$. Define

$$U_n := U \setminus \bigcup_{k=1}^n J_k \quad \text{and} \quad \delta_n := \sup\{m(J) \colon J \in \mathcal{V}, J \subset U_n\}.$$

Since U_n is open by construction and $E \cap U_n$ is non-empty by assumption we have $\delta_n > 0$. Moreover, since U is bounded we have $\delta_n < m(U) < \infty$. Thus there exists an interval $J_{n+1} \in \mathcal{V}$ such that $J_{n+1} \subset U_n$ and $m(J_{n+1}) > \delta_n/2$. We add this interval to our subcollection to get $\{J_k\}_{k=1}^{n+1}$.

The above procedure results in the subcollection $\{J_k\}_{k=1}^\infty$. It remains to be shown that the intervals J_k cover E up to a set of measure zero. Since

$$\sum_{k=1}^\infty m(J_k) = m\Big(\bigcup_{k=1}^\infty J_k\Big) \le m(U) < \infty, \tag{2.1}$$

we infer that

$$\lim_{M\to\infty} \sum_{k=M}^\infty m(J_k) = 0. \tag{2.2}$$

Pick a point $x \in E \setminus \bigcup_{k=1}^\infty J_k$ and a positive integer M. Then $x \in U_M$, and since $\delta_M > 0$, there exists an interval $J \in \mathcal{V}$ such that $x \in J \subset U_M$. By the inequality $2m(J_{k+1}) > \delta_k$ and the convergence of the series in (2.1) it follows that $\delta_k \to 0$ as $k \to \infty$. Therefore there exists a smallest positive integer N such that $J \not\subset U_N$. Note that $M < N$ since otherwise $J \subset U_M \subset U_N$. Moreover, we have $J \subset U_{N-1}$ and $J \not\subset U_N$, so that

$$J \cap J_N \neq \varnothing \quad \text{and} \quad m(J) \leq \delta_{N-1} < 2m(J_N).$$

That is, if the length of J_N is expanded by a factor of 5 relative to its midpoint, then the resulting interval will contain J. Since $x \in E \setminus \bigcup_{k=1}^{\infty} J_k$ was arbitrary and $N > M$, we conclude that

$$m^*\left(E \setminus \bigcup_{k=1}^{\infty} J_k\right) \leq 5 \sum_{k=M}^{\infty} m(J_k).$$

Since M was arbitrary as well, the above inequality and (2.2) yield the desired result

$$m\left(E \setminus \bigcup_{k=1}^{\infty} J_k\right) = 0.$$

This completes the proof of the case when E is bounded.

Next we consider the case when E is unbounded. Define for each integer i the set $E_i := E \cap (i, i+1)$ and let $\{J_{i,j}\}_j \subset \mathcal{V}$ be a countable subcollection of strictly non-overlapping intervals such that

$$m\left(E_i \setminus \bigcup_j J_{i,j}\right) = 0.$$

By the proof of the bounded case we may assume that $J_{i,j} \subset (i, i+1)$ for all i, j. We have

$$m^*\left(E \setminus \bigcup_{i,j} J_{i,j}\right) \leq \sum_i m\left(E_i \setminus \bigcup_j J_{i,j}\right) + m(\mathbb{Z}) = 0,$$

and therefore

$$m\left(E \setminus \bigcup_{i,j} J_{i,j}\right) = 0.$$

Since the subcollection $\{J_{i,j}\}_{i,j}$ is countable, the proof is complete. $\square$

Theorem 2.4 (Lebesgue). *Let $F\colon \Omega \to \mathbb{R}$ be a monotone function. Then F is finitely differentiable a.e. on Ω.*

Proof. It will evidently suffice to consider the case when F is non-decreasing and Ω is bounded, in addition to fixing two rational numbers p, q with $0 < p < q$ and showing that the following set has measure zero:

$$E_{p,q} := \{x \in \Omega\colon \underline{D}F(x) < p < q < \overline{D}F(x)\}.$$

Let $\varepsilon > 0$ and let U be an open set such that

$$E_{p,q} \subset U \subset \Omega \quad \text{and} \quad m(U) < m^*(E_{p,q}) + \varepsilon.$$

Define the collection of intervals

$$\mathcal{V} := \{[u,v] \subset U\colon E_{p,q} \cap [u,v] \neq \varnothing \text{ and } F(v) - F(u) < p(v-u)\}.$$

Note that $\mathcal{V}$ is a Vitali cover of $E_{p,q}$. Thus by the Vitali covering theorem there exists a countable subcollection $\{J_k\}_k \subset \mathcal{V}$ of non-overlapping intervals such that

$$m\left(E_{p,q} \setminus \bigcup_k J_k\right) = 0.$$

Furthermore, define for each k the collection of intervals

$$\mathcal{V}_k := \{[u,v] \subset J_k\colon E_{p,q} \cap [u,v] \neq \varnothing \text{ and } F(v) - F(u) > q(v-u)\}.$$

Note that $\mathcal{V}_k$ is a Vitali cover of $E_{p,q} \cap J_k^o$ (but not necessarily of $E_{p,q} \cap J_k$). We shall therefore use the Vitali covering theorem once again to obtain a countable subcollection $\{J_{k,l}\}_l \subset \mathcal{V}$ of non-overlapping intervals such that

$$m\left((E_{p,q} \cap J_k) \setminus \bigcup_l J_{k,l}\right) = 0.$$

Using all of the above we obtain the following:

$$\begin{aligned}
m^*(E_{p,q}) &\leq m^*\left(E_{p,q} \cap \bigcup_k J_k\right) \leq \sum_k \sum_l m(J_{k,l}) \\
&\leq \frac{1}{q} \sum_k \sum_l |\Delta(F, J_{k,l})| \leq \frac{1}{q} \sum_k |\Delta(F, J_k)| \\
&\leq \frac{p}{q} \sum_k m(J_k) \leq \frac{p}{q} m(U) < \frac{p}{q}(m^*(E_{p,q}) + \varepsilon).
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we infer that $qm^*(E_{p,q}) \leq pm^*(E_{p,q})$, and since $p < q$ we obtain the desired result $m(E_{p,q}) = 0$. $\square$

Proof of Theorem 2.1. Without loss of generality we shall assume the existence of a positive integer k such that $|x| < k$, $|F(x)| < k$ and $\underline{D}F(x) > -k$ for every $x \in E$. Define the function $G\colon \Omega \to \mathbb{R}$ by

$$G(x) := F(x) + kx.$$

Note that $|G(x)| < k+k^2$ and $\underline{D}G(x) > 0$ for every $x \in E$. It will suffice to show that G is finitely differentiable a.e. on E. For each open interval $(p,q) \subset \Omega$ with rational endpoints we define the set

$$E_{p,q} := \left\{ x \in E \cap (p,q) \colon \sup_{t \in (p,x)} G(t) \le G(x) \le \inf_{t \in (x,q)} G(t) \right\}.$$

Then $E = \bigcup_{p,q} E_{p,q}$ and G is non-decreasing relative to each set $E_{p,q}$. It will further suffice to fix a pair p, q for which $E_{p,q} \neq \varnothing$ and subsequently show that G is finitely differentiable a.e. on $E^*_{p,q} := E_{p,q} \setminus \{u, v\}$, where $u := \inf E_{p,q}$ and $v := \sup E_{p,q}$. Define the two functions $m, M \colon (u,v) \to \mathbb{R}$ by

$$\begin{aligned} M(x) &:= \inf\{G(t) \colon t \in E_{p,q} \cap [x,v]\}; \\ m(x) &:= \sup\{G(t) \colon t \in E_{p,q} \cap [u,x]\}. \end{aligned}$$

Note that m and M are non-decreasing in addition to satisfying $m \le G \le M$ on (u,v) and $m = G = M$ on $E^*_{p,q}$. Therefore Lebesgue's theorem guarantees that m and M are finitely differentiable a.e. on (u,v); and for every point $x \in E^*_{p,q}$ at which m and M are both differentiable we have

$$Dm(x) \le \underline{D}G(x) \le \overline{D}G(x) \le DM(x). \tag{2.3}$$

Next we show that for every point $x \in E^*_{p,q}$ which is not isolated in $E^*_{p,q}$, and at which m and M are both differentiable, we have $Dm(x) = DM(x)$. To see this, pick a sequence $\{x_k\}_{k=1}^{\infty} \subset E^*_{p,q} \setminus \{x\}$ which converges to x. Since $m = G = M$ on $E^*_{p,q}$, the difference quotients of m and M defined by each pair of points x_k, x are identical, so that $Dm(x) = DM(x)$. Thus by (2.3) we must have $Dm(x) = DG(x) = DM(x)$. Now, recall that m and M are finitely differentiable a.e. on (u,v) and by Lemma 1.2 the isolated points of $E^*_{p,q}$ form a countable set. Therefore we conclude that G is finitely differentiable a.e. on $E^*_{p,q}$. □

2.2 Theorem of Bongiorno–Skvortsov–Piazza

Lemma 2.5. *Consider a function $F \colon \Omega \to \mathbb{R}$ and define the set*

$$S := \{x \in \Omega \colon \underline{D}F(x) = -\infty \text{ or } \overline{D}F(x) = \infty\}.$$

Let $U \subset \Omega$ be an open subset with $m^*(S \cap U) > 0$ and let $\varepsilon, M > 0$. Then there exists a finite collection $\{I_j\}_j$ of non-overlapping compact intervals $I_j \subset U$ with $m^*(S \cap I_j) > 0$, such that

$$\sum_j m(I_j) < \varepsilon \quad \text{and} \quad \sum_j |\Delta(F, I_j)| > M.$$

Proof. It will suffice to consider the case when $m(U) < \varepsilon$. For every $x \in S \cap U$ and $\eta > 0$ there exists a compact interval $I_{x,\eta}$ such that

$$x \in I_{x,\eta} \subset U, \quad 0 < m(I_{x,\eta}) < \eta \quad \text{and} \quad |\Delta(F, I_{x,\eta})| > \frac{M}{m^*(S \cap U)} m(I_{x,\eta}).$$

Define the collection of intervals

$$\mathcal{V} := \{I_{x,\eta} \colon x \in S \cap U \text{ and } \eta > 0\}.$$

Note that $\mathcal{V}$ is a Vitali cover of $S \cap U$. Thus by the Vitali covering theorem there exists a countable subcollection $\{I_j\}_j \subset \mathcal{V}$ of non-overlapping intervals such that

$$m\biggl((S \cap U) \setminus \bigcup_j I_j\biggr) = 0.$$

Moreover, we have

$$\sum_j m(I_j) = m\biggl(\bigcup_j I_j\biggr) \leq m(U) < \varepsilon$$

and

$$\begin{aligned}
\sum_j |\Delta(F, I_j)| &> \frac{M}{m^*(S \cap U)} \sum_j m(I_j) \\
&= \frac{M}{m^*(S \cap U)} m\biggl(\bigcup_j I_j\biggr) \\
&\geq \frac{M}{m^*(S \cap U)} m^*\biggl(S \cap U \cap \bigcup_j I_j\biggr) \\
&= \frac{M}{m^*(S \cap U)} m^*(S \cap U) = M.
\end{aligned}$$

Consequently $\{I_j\}_j$ contains a finite subcollection which possesses the desired properties. □

Proof of Theorem 2.2. Let the set S be defined as in Lemma 2.5. If $m(S) = 0$ then by Ward's theorem it follows that F is differentiable a.e. on Ω. Hence we assume towards a contradiction that $m^*(S) > 0$. Then by Lemma 2.5 we obtain a finite collection $\{I_j^{(1)}\}_j$ of non-overlapping compact intervals $I_j^{(1)} \subset \Omega$ with $m^*(S \cap I_j^{(1)}) > 0$, such that

$$\sum_j m(I_j^{(1)}) < \frac{1}{2} \quad \text{and} \quad \sum_j |\Delta(F, I_j^{(1)})| > 1.$$

We proceed inductively. Suppose that for some positive integer k we have already determined a finite collection $\{I_i^{(k)}\}_i$ of non-overlapping compact intervals $I_i^{(k)} \subset \Omega$ with $m^*(S \cap I_i^{(k)}) > 0$. By Lemma 2.5 we obtain a finite collection $\{I_j^{(k+1)}\}_j$ of non-overlapping compact intervals for which the following conditions are met:

(i) $m^*(S \cap I_j^{(k+1)}) > 0$ for each j;

(ii) $\displaystyle\bigcup_j I_j^{(k+1)} \subset \bigcup_i (I_i^{(k)})^o$;

(iii) $\displaystyle\sum_j m(I_j^{(k+1)}) < \frac{1}{2^{k+1}}$;

(iv) $\displaystyle\sum_{j\colon I_j^{(k+1)} \subset I_i^{(k)}} |\Delta(F, I_j^{(k+1)})| > 1$ for each i.

Now, define the set

$$Z := \bigcap_k \bigcup_j I_j^{(k)}.$$

The above conditions in conjunction with the nested interval property of $\mathbb{R}$ ensure that Z is a non-empty closed set with measure zero and that each interval $I_j^{(k)}$ satisfies $Z \cap (I_j^{(k)})^o \neq \varnothing$. Let δ be an arbitrary gauge on Ω and define for each positive integer n the set

$$Z_n := \{x \in Z \colon \delta(x) > 1/n\}.$$

Note that $Z = \bigcup_{n=1}^{\infty} Z_n$. By the Baire category theorem we obtain a positive integer n_0 and an open interval I such that $Z \cap I \neq \varnothing$ and $Z_{n_0} \cap I$ is dense in $Z \cap I$. Pick an interval $I_i^{(k)} \subset I$ with $m(I_i^{(k)}) < 1/n_0$ and associate each interval $I_j^{(k+1)} \subset I_i^{(k)}$ with a point $x_j \in Z_{n_0} \cap I_j^{(k+1)}$. The resulting collection of pairs $(x_j, I_j^{(k+1)})$, which we denote by P, constitutes a tagged subpartition of Ω which is anchored in Z, subordinate to δ, and for which

$$\sum_{(x,I)\in P} |\Delta(F, I)| = \sum_{j\colon I_j^{(k+1)} \subset I_i^{(k)}} |\Delta(F, I_j^{(k+1)})| > 1.$$

Since δ was arbitrary it follows that $\nu(F, Z) \geq 1$, which contradicts the fact that F satisfies the strong Luzin condition. $\square$

Chapter 3

Ambiguity of primitives

A well-known property of finitely differentiable functions is that on open intervals they are uniquely determined up to a constant by their derivatives. In this chapter we endeavour to prove the following theorem which extends this result to regular primitives:

Theorem 3.1. *Consider two regular primitives $F, G\colon (a,b) \to \mathbb{R}$ possessing the property that $DF = DG$ a.e. on (a,b). Then $F - G$ is constant.*

This theorem is obtained by first establishing an analogous result for functions satisfying the strong condition, and subsequently verifying that regular primitives satisfy this condition, see Theorems 3.9 and 3.10, respectively. However, Theorem 3.1 cannot be extended to irregular primitives because of the following result:

Theorem 3.2. *There exists a regular primitive $F\colon (0,1) \to \mathbb{R}$ and an irregular primitive $G\colon (0,1) \to \mathbb{R}$ such that $DF = DG$ everywhere on $(0,1)$ and $G - F = T|_{(0,1)}$, where T denotes the Cantor ternary function. In particular $G - F$ is non-constant.*

We shall begin by studying various properties of the Henstock variation ν. For instance, Theorem 3.5 states that the strong Luzin condition implies continuity and the Luzin condition (N). From Theorem 2.2 we already know that differentiability a.e. is implied as well. However, as we shall see in Theorem 3.6, there exists a function which is continuous, satisfies the Luzin condition (N), is differentiable a.e. and yet has infinite Henstock variation on a set of measure zero. That is, the joint converse of Theorems 2.2 and 3.5 fails disastrously

3.1 Continuity and the Luzin condition (N)

Lemma 3.3. *Consider a function $F\colon \Omega \to \mathbb{R}$ and let $E \subset \Omega$ satisfy $\nu(F, E) = 0$. Then for every $\varepsilon > 0$ there exists a gauge δ on Ω such that*

$$\sum_{(x,I)\in P} \operatorname{osc}(F, I) < \varepsilon$$

for every tagged subpartition P of Ω which is anchored in E and subordinate to δ.

Proof. Let $\varepsilon > 0$. Since $\nu(F, E) = 0$ there exists a gauge δ on Ω such that

$$\sum_{(x,I)\in P} |\Delta(F, I)| < \frac{\varepsilon}{4}$$

for every tagged subpartition P of Ω which is anchored in E and subordinate to δ. If $P = \varnothing$ then nothing remains to be shown, hence we assume that this is not the case. Let (x_j, I_j), $j = 1, \dots, N$, denote the elements of P. Clearly we must have $\operatorname{osc}(F, I_j) < \infty$ and thus there exist points $u_j, v_j \in I_j$ such that

$$\operatorname{osc}(F, I_j) - |F(v_j) - F(u_j)| < \frac{\varepsilon}{2N}.$$

Let $I_j^{(1)}$ and $I_j^{(2)}$ denote the compact intervals for which x_j is a common endpoint with u_j and v_j, respectively, being the other endpoint. Then

$$\begin{aligned}\sum_{(x,I)\in P} \operatorname{osc}(F, I) &< N\frac{\varepsilon}{2N} + \sum_{j=1}^{N} |\Delta(F, I_j^{(1)})| + \sum_{j=1}^{N} |\Delta(F, I_j^{(2)})| \\ &< N\frac{\varepsilon}{2N} + 2\frac{\varepsilon}{4} = \varepsilon.\end{aligned}$$ □

Lemma 3.4. *Consider a function $F\colon \Omega \to \mathbb{R}$ and let $E \subset \Omega$ satisfy $\nu(F, E) = 0$. Then F is continuous on E and $m(F(E)) = 0$.*

Proof. Clearly F is continuous on E since for every $x \in E$ we have $\nu(F, \{x\}) = 0$. Thus it remains to be show that $m(F(E)) = 0$. By Lemma 1.3 we may assume that for every $x \in E$ the function F is non-constant in all neighbourhoods of x. Let $\varepsilon > 0$. By Lemma 3.3 there exists a gauge δ on Ω such that

$$\sum_{(x,I)\in P} \operatorname{osc}(F, I) < \varepsilon$$

for every tagged subpartition P of Ω which is anchored in E and subordinate to δ. For every $x \in E$ and $\eta > 0$ there exists a compact interval $I_{x,\eta}$ such that $x \in I_{x,\eta} \subset (x - \delta(x), x + \delta(x)) \cap \Omega$ and $0 < \operatorname{osc}(F, I_{x,\eta}) < \eta$. Define the set

$$\mathcal{V} := \{[\inf F(I_{x,\eta}), \sup F(I_{x,\eta})] \colon x \in E \text{ and } \eta > 0\}.$$

Note that $\mathcal{V}$ is a Vitali cover of $F(E)$. Thus by the Vitali covering theorem there exists a countable subcollection $\{J_k\}_k \subset \mathcal{V}$ of strictly non-overlapping intervals, with positive integer indices k, such that

$$m\left(F(E) \setminus \bigcup_k J_k\right) = 0.$$

For each k, let $I_k := I_{x_k,\eta_k}$ denote an interval by which J_k was defined above. Then, if J_{k_1} and J_{k_2} are distinct, we must have

$$I_{k_1} \cap I_{k_2} = \varnothing \quad \text{since} \quad F(I_{k_1}) \cap F(I_{k_2}) \subset J_{k_1} \cap J_{k_2} = \varnothing.$$

Thus every finite subcollection of pairs (x_k, I_k) constitutes a tagged subpartition of Ω which is anchored in E and subordinate to δ. Consequently

$$m\left(\bigcup_k J_k\right) = \lim_{n\to\infty} \sum_{k\le n} m(J_k) = \lim_{n\to\infty} \sum_{k\le n} \operatorname{osc}(F, I_k) \le \varepsilon,$$

so that

$$m^*(F(E)) \le m\left(F(E) \setminus \bigcup_k J_k\right) + m\left(\bigcup_k J_k\right) \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $m(F(E)) = 0$. $\square$

Theorem 3.5. *Consider a function $F \colon \Omega \to \mathbb{R}$ which satisfies the strong Luzin condition. Then F is continuous and satisfies the Luzin condition (N).*

Proof. This follows directly from Lemma 3.4. That is, for every $E \subset \Omega$ with $m(E) = 0$ we have $\nu(F, E) = 0$ by hypothesis. Therefore Lemma 3.4 guarantees that F is continuous on E and $m(F(E)) = 0$. $\square$

Theorem 3.6. There exists a function $F \colon (0, 1) \to \mathbb{R}$ which is continuous, satisfies the Luzin condition (N), is differentiable a.e. and fails to satisfy the strong Luzin condition.

Proof. Let C denote the Cantor ternary set and define $C_* := C \setminus \{0, 1\}$. For each positive integer i we denote by $I_{i,1}, \ldots, I_{i,2^{i-1}}$ the open intervals of length $1/3^i$

contiguous to C, indexed in ascending order. Define the function $F\colon (0,1) \to \mathbb{R}$ by

$$F(x) := \begin{cases} \dfrac{(x - \inf I_{i,j})(x - \sup I_{i,j})}{i(m(I_{i,j})/2)^2}, & \text{if } x \in I_{i,j}; \\ 0, & \text{if } x \in C_*. \end{cases}$$

Since $-1/i \leq F(x) < 0$ whenever $x \in I_{i,j}$ it is easily seen that F is continuous. In addition $DF(x)$ exists and is finite for every $x \in (0,1) \setminus C_*$.

Next we show that F satisfies the Luzin condition (N). Since $F(C_*) = \{0\}$ it will suffice to fix $I_{i,j}$ and show that F satisfies the Luzin condition (N) on $I_{i,j}$. Let $Z \subset I_{i,j}$ with $m(Z) = 0$. By Lemma 3.4 it will further suffice to show that $\nu(F, Z) = 0$. Note that for every $x \in I_{i,j}$ we have

$$\begin{aligned} |DF(x)| &= \frac{|2x - \inf I_{i,j} - \sup I_{i,j}|}{i(m(I_{i,j})/2)^2} = \frac{2|x - \operatorname{mid} I_{i,j}|}{i(m(I_{i,j})/2)^2} \\ &< \frac{m(I_{i,j})}{i(m(I_{i,j})/2)^2} = \frac{4}{im(I_{i,j})} = \frac{3^i \cdot 4}{i}. \end{aligned}$$

Thus by the mean value theorem we infer that $L := 3^i \cdot 4/i$ is a Lipschitz constant for F on $I_{i,j}$. Let $\varepsilon > 0$ and U be an open set such that

$$Z \subset U \subset I_{i,j} \quad \text{and} \quad m(U) < \frac{\varepsilon}{L}.$$

Furthermore, let δ be a gauge on $(0,1)$ such that $(x - \delta(x), x + \delta(x)) \subset U$ for every $x \in Z$, and let P be a tagged subpartiton of $(0,1)$ which is anchored in Z and subordinate to δ. Then

$$\sum_{(x,I)\in P} |\Delta(F, I)| \leq \sum_{(x,I)\in P} Lm(I) \leq Lm(U) < \varepsilon.$$

This proves that $\nu(F, Z) = 0$. Thus F satisfies the Luzin condition (N).

Finally, we endeavour to prove that $\nu(F, C_*) = \infty$. Let δ be a gauge on $(0,1)$. For each positive integer n we define the set

$$C_n := \{x \in C_*\colon \delta(x) > 1/n\}.$$

Note that C is perfect, $C = \{0,1\} \cup \bigcup_{n=1}^{\infty} C_n$ and $\{0,1\}$ is nowhere dense in C. Thus by the Baire category theorem there exists a positive integer n_0 and an open interval I such that $C \cap I \neq \varnothing$ and $C_{n_0} \cap I$ is dense in $C \cap I$. Let I_{i_0,j_0} be such that

$$m(I_{i_0,j_0}) < 1/n_0 \quad \text{and} \quad I_{i_0,j_0} + m(I_{i_0,j_0}) \subset I.$$

For each positive integer k we define $i_k := k + i_0$ and $j_k := 2^k j_0$, so that

$$I_{i_k,j_k} \subset I_{i_0,j_0} + m(I_{i_0,j_0}) \subset I.$$

Fix an arbitrary number $M > 0$. By the divergence of the harmonic series there exists a positive integer N such that

$$\sum_{k=1}^{N} \frac{1}{i_k} > M.$$

Furthermore, let $\{J_k\}_{k=1}^N$ be a collection of non-overlapping compact intervals $J_k \subset I_{i_0,j_0} + m(I_{i_0,j_0})$ with the endpoints x_k, y_k, where $x_k \in C_{n_0}$ and y_k is the midpoint of I_{i_k,j_k}. Then the collection of all pairs (x_k, J_k), which we denote by P, constitutes a tagged subpartition of $(0,1)$ which is anchored in C_*, subordinate to δ, and for which

$$\sum_{(x,J)\in P} |\Delta(F,J)| = \sum_{k=1}^{N} \frac{1}{i_k} > M.$$

Since both δ and $M > 0$ were arbitrary, we conclude that $\nu(F, C_*) = \infty$. Thus F does not satisfy the strong Luzin condition. □

Lemma 3.7. *Consider a function $F\colon \Omega \to \mathbb{R}$ and define the set*

$$E := \{x \in \Omega\colon DF(x) = \pm\infty\}.$$

Then $\nu(F,E) = 0$ if and only if F is continuous on E and $m(F(E)) = 0$.

Proof. The forward direction follows from Lemma 3.4, we shall therefore prove the reverse direction. It will suffice to consider the case when $DF(x) = \infty$ for every $x \in E$. For each open interval $(p,q) \subset \Omega$ with rational endpoints we define the set

$$E^*_{p,q} := \left\{x \in E \cap (p,q)\colon \sup_{t\in(p,x)} F(t) \le F(x) \le \inf_{t\in(x,q)} F(t)\right\}. \tag{3.1}$$

Then $E = \bigcup_{p,q} E^*_{p,q}$. Assign to each (p,q) a unique positive integer n and define $p_n := p$, $q_n := q$ and $E^*_n := E^*_{p,q}$. We shall assume that no positive integer n has been excluded since we may otherwise define $E^*_n := \varnothing$.

In order to acquire a collection of mutually disjoint sets we define

$$E_n := E^*_n \setminus \bigcup_{k<n} E^*_k.$$

Let $\varepsilon > 0$. Henceforth we aim to show that each n corresponds to a gauge δ_n on Ω such that

$$\sum_{(x,I)\in P_n} |\Delta(F,I)| < \frac{\varepsilon}{2^n}$$

for every tagged subpartition P_n of Ω which is anchored in E_n and subordinate to δ_n. Let n be fixed but arbitrary. If $E_n = \varnothing$ then δ_n can be chosen arbitrarily, we shall therefore assume that $E_n \neq \varnothing$. Since $m(F(E_n)) = 0$ there exists an open set U_n with $F(E_n) \subset U_n$ and $m(U_n) < \varepsilon/2^{n+1}$. Furthermore, since F is continuous on E_n there exists a gauge δ_n on Ω possessing the property that if $x \in E_n$, then with the notation $J_x := (x - \delta_n(x), x + \delta_n(x)) \cap \Omega$ we have

$$J_x \subset (p_n, q_n) \quad \text{and} \quad F(J_x) \subset U_n. \tag{3.2}$$

Let P_n be a tagged subpartition of Ω which is anchored in E_n and subordinate to δ_n. If $P_n = \varnothing$ then nothing remains to be shown, hence we assume that this is not the case. Denote the pairs of P_n by (x_j, I_j), $j = 1, \ldots, N$. For each j we partition I_j into two compact subintervals $I_j^{(1)}$ and $I_j^{(2)}$ stated in ascending order and possessing the common endpoint x_j. From (3.1) and (3.2) it follows that $F(I_i^{(k)}) \cap F(I_j^{(k)})$ contains at most a single point whenever $i \neq j$ and $k \in \{1, 2\}$. Therefore

$$\sum_{(x,I)\in P_n} |\Delta(F,I)| \leq \sum_{j=1}^{N} |\Delta(F, I_j^{(1)})| + \sum_{j=1}^{N} |\Delta(F, I_j^{(2)})| \leq 2m(U_n) < \frac{\varepsilon}{2^n}.$$

Now, let δ be a gauge on Ω such that $\delta(x) = \delta_n(x)$ whenever $x \in E_n$, and let P be a tagged subpartition of Ω which is anchored in E and subordinate to δ. Then

$$\sum_{(x,I)\in P} |\Delta(F,I)| < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon,$$

which proves that $\nu(F, E) = 0$. □

Lemma 3.8. *Consider a function $F\colon \Omega \to \mathbb{R}$. Suppose there exists a collection of functions $\{F_k\}_{k=1}^{\infty}$, all of which are non-decreasing, continuous and satisfy the Luzin condition (N), such that*

$$F = \sum_{k=1}^{\infty} F_k \quad \textit{and} \quad \sum_{k=1}^{\infty} m(F_k(\Omega)) < \infty.$$

Then F is non-decreasing and satisfies the strong Luzin condition.

Proof. Clearly F is non-decreasing. Thus it remains to be shown that F satisfies the strong Luzin condition. Let $Z \subset \Omega$ with $m(Z) = 0$ and let $\varepsilon > 0$. Pick a positive integer N such that

$$\sum_{k=N+1}^{\infty} m(F_k(\Omega)) < \frac{\varepsilon}{2}.$$

Each F_k satisfies the Luzin condition (N) and consequently there exists an open set U_k such that

$$F_k(Z) \subset U_k \quad \text{and} \quad m(U_k) < \frac{\varepsilon}{2N}.$$

By the continuity of each F_k we infer that $F_k^{-1}(U_k)$ is open and so there exists a gauge δ_k on Ω such that

$$(x - \delta_k(x), x + \delta_k(x)) \subset F_k^{-1}(U_k) \quad \text{for every } x \in Z.$$

Define $\delta := \min\{\delta_1, \ldots, \delta_N\}$ and let P be a tagged subpartition of Ω which is anchored in Z and subordinate to δ. Then

$$\begin{aligned}
\sum_{(x,I)\in P} \Delta(F, I) &= \sum_{k=1}^{N} \sum_{(x,I)\in P} \Delta(F_k, I) + \sum_{k=N+1}^{\infty} \sum_{(x,I)\in P} \Delta(F_k, I) \\
&\le \sum_{k=1}^{N} m(U_k) + \sum_{k=N+1}^{\infty} m(F_k(\Omega)) \\
&< N\frac{\varepsilon}{2N} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

3.2 Basic properties of regular primitives

Theorem 3.9. *Consider two functions $F, G\colon (a, b) \to \mathbb{R}$, both of which satisfy the strong Luzin condition, such that $DF = DG$ a.e. on (a, b). Then $F - G$ is constant.*

Proof. It will suffice to fix a compact subinterval $[c, d] \subset (a, b)$ and show that $F - G$ is constant on $[c, d]$. Define the set

$$E := \{x \in (a, b)\colon -\infty < DF(x) = DG(x) < \infty\}.$$

Ward's Theorem 2.1 guarantees that $m((a, b) \setminus E) = 0$ and since F and G both satisfy the strong Luzin condition we have $\nu(F, (a, b) \setminus E) = \nu(G, (a, b) \setminus E) = 0$. Thus there exists a gauge δ_1 on (a, b) such that

$$\sum_{(x,I)\in P_1} (|\Delta(F, I)| + |\Delta(G, I)|) < \frac{\varepsilon}{2}$$

for every tagged subpartition P_1 of (a,b) which is anchored in $(a,b)\setminus E$ and subordinate to δ_1.

Let $\xi \in [c,d]$ and $\varepsilon > 0$. Furthermore, let δ_2 be a gauge on (a,b) such that for every point-interval pair (x,I) with $x \in I\setminus E$ and $I \subset (x-\delta_2(x), x+\delta_2(x)) \cap (a,b)$, we have

$$|\Delta(F,I) - DF(x)m(I)|, |\Delta(G,I) - DG(x)m(I)| \leq \frac{\varepsilon}{4(b-a)} m(I).$$

By Cousin's Lemma 1.4 there exists a tagged subpartition P of (a,b), which is subordinate to $\min\{\delta_1,\delta_2\}$, such that

$$[c,\xi] = \bigcup_{(x,I)\in P} I.$$

With $C := F(c) - G(c)$ we have

$$\begin{aligned}
&|F(\xi) - G(\xi) - C| \\
&\quad = |F(\xi) - F(c) - (G(\xi) - G(c))| \\
&\quad = \left| \sum_{(x,I)\in P} \Delta(F,I) - \sum_{(x,I)\in P} \Delta(G,I) \right| \\
&\quad \leq \sum_{\substack{(x,I)\in P:\\ x\notin E}} (|\Delta(F,I)| + |\Delta(G,I)|) + \sum_{\substack{(x,I)\in P:\\ x\in E}} |\Delta(F,I) - \Delta(G,I)| \\
&\quad < \frac{\varepsilon}{2} + \sum_{\substack{(x,I)\in P:\\ x\in E}} (|\Delta(F,I) - DF(x)m(I)| + |\Delta(G,I) - DG(x)m(I)|) \\
&\quad \leq \frac{\varepsilon}{2} + 2 \sum_{\substack{(x,I)\in P:\\ x\in E}} \frac{\varepsilon}{4(b-a)} m(I) < \varepsilon.
\end{aligned}$$

This proves that $F - G = C$ on $[c,d]$. □

Theorem 3.10. *Consider a function $F\colon \Omega \to \mathbb{R}$. Then F is a regular primitive if and only if F is differentiable and satisfies the strong Luzin condition.*

Proof. The reverse direction follows from Theorem 3.5, we shall therefore prove the forward direction. Let $Z \subset \Omega$ with $m(Z) = 0$ and let $\varepsilon > 0$. Define the set

$$E := \{x \in \Omega\colon DF(x) = \pm\infty\}.$$

Ward's Theorem 2.1 guarantees that $m(E) = 0$ and since F satisfies the Luzin condition (N) we have $m(F(E)) = 0$. Thus by Theorem 3.7 there exists a gauge δ_1 on Ω such that

$$\sum_{(x,I)\in P_1} |\Delta(F, I)| < \frac{\varepsilon}{2}$$

for every tagged subpartition P_1 of Ω which is anchored in E and subordinate to δ_1. Define for each positive integer k the set

$$Z_k := \{x \in Z \setminus E \colon k - 1 \leq |DF(x)| < k\}.$$

Since $m(Z_k) = 0$ there exists an open set U_k such that

$$Z_k \subset U_k \subset \Omega \quad \text{and} \quad m(U_k) < \frac{\varepsilon}{k2^{k+1}}.$$

Let δ_2 be a gauge on Ω such that for every point-interval pair (x, I) with $x \in Z_k$ and $I \subset (x - \delta_2(x), x + \delta_2(x))$ we have

$$I \subset U_k \quad \text{and} \quad |\Delta(F, I)| < km(I).$$

Define $\delta := \min\{\delta_1, \delta_2\}$ and let P be a tagged subpartition of Ω which is anchored in Z and subordinate to δ. Then

$$\begin{aligned}
\sum_{(x,I)\in P} |\Delta(F, I)| &= \sum_{\substack{(x,I)\in P\colon \\ x\in E}} |\Delta(F, I)| + \sum_{\substack{(x,I)\in P\colon \\ x\notin E}} |\Delta(F, I)| \\
&< \frac{\varepsilon}{2} + \sum_{k=1}^{\infty} \sum_{\substack{(x,I)\in P\colon \\ x\in Z_k}} km(I) \\
&\leq \frac{\varepsilon}{2} + \sum_{k=1}^{\infty} km(U_k) \\
&< \frac{\varepsilon}{2} + \sum_{k=1}^{\infty} k\frac{\varepsilon}{k2^{k+1}} = \varepsilon.
\end{aligned}$$

This proves that F satisfies the strong Luzin condition. □

Corollary 3.11. *A finitely differentiable function $F \colon \Omega \to \mathbb{R}$ constitutes a regular primitive.*

Proof. We conclude that F satisfies the strong Luzin condition by replicating the proof of the forward direction of Theorem 3.10 with $E = \varnothing$. Thus we infer from Theorem 3.5 that F is continuous and satisfies the Luzin condition (N). □

Corollary 3.12. *Consider two regular primitives $F, G\colon \Omega \to \mathbb{R}$. Then $F+G$ satisfies the strong Luzin condition. Moreover, if $DF+DG$ is well-defined, then $F+G$ constitutes a regular primitive.*

Proof. This follows from Theorem 3.10 because clearly the sum of two functions which satisfy the strong Luzin condition satisfies this condition as well; and the differential operator is linear for well-defined linear combinations. □

Proof of Theorem 3.1. This follows directly from Theorems 3.9 and 3.10. □

Proof of Theorem 3.2. Let C denote the Cantor ternary set and define $C_* := C \setminus \{0,1\}$. For each positive integer i we denote by $I_{i,1}, \ldots, I_{i,2^{i-1}}$ the open intervals of length $1/3^i$ contiguous to C, indexed in ascending order. Define the function $f\colon (0,1) \to \overline{\mathbb{R}}$ by

$$f(x) := \begin{cases} i\sqrt{\dfrac{m(I_{i,j})/2}{\operatorname{dist}(x, C_*)}}, & \text{if } x \in I_{i,j};\\ \infty, & \text{if } x \in C_*. \end{cases}$$

Note that $f(x) \geq i$ whenever $x \in I_{i,j}$. From the monotone convergence theorem for the Lebesgue integral it follows that

$$\int_{(0,1)} f\,dm = \sum_{i=1}^{\infty}\sum_{j=1}^{2^{i-1}} \int_{I_{i,j}} f\,dm = \sum_{i=1}^{\infty} i \sum_{j=1}^{2^{i-1}} 2m(I_{i,j}) = \sum_{i=1}^{\infty} i\left(\frac{2}{3}\right)^i < \infty.$$

The final inequality follows from d'Alembert's criterion for convergence. Define the function $F\colon (0,1) \to \mathbb{R}$ by

$$F(x) := \int_{(0,x]} f\,dm.$$

Then F is absolutely continuous[1] and therefore $\nu(F, C_*) = 0$. Moreover, since f is continuous (in the extended sense) we have $DF = f$. Let T denote the Cantor ternary function and define the function $G := F + T|_{(0,1)}$. Then F and G possess the desired properties. □

Theorem 3.13. *Consider two functions $F, G\colon \Omega \to \mathbb{R}$ which satisfy the strong Luzin condition. Then FG satisfies the strong Luzin condition as well.*

[1]Recall that F is absolutely continuous provided that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $\sum_{I \in P} |\Delta(F, I)| < \varepsilon$ for every subpartition P of Ω which satisfies $\sum_{I \in P} m(I) < \delta$. It is easily seen that absolute continuity implies the strong Luzin condition.

Proof. Let $Z \subset \Omega$ with $m(Z) = 0$ and let $\varepsilon > 0$. Define for each positive integer k the set

$$Z_k := \{x \in Z \colon k - 1 \leq \max\{|F(x)|, |G(x)|\} < k\}.$$

Note that the sets Z_k are pairwise disjoint and $Z = \bigcup_{k=1}^{\infty} Z_k$. For each k there exists a gauge δ_k on Ω such that

$$\sum_{(x,[u,v]) \in P_k} (|F(v) - F(u)| + |G(v) - G(u)|) < \frac{\varepsilon}{k2^k}$$

for every tagged subpartition P_k of Ω which is anchored in Z_k and subordinate to δ_k. In addition we require that the following inequality is satisfied for every $x \in Z_k$ and $y \in (x - \delta_k(x), x + \delta_k(x)) \cap \Omega$:

$$\max\{|F(y)|, |G(y)|\} < k.$$

This is possible because F and G are both continuous by Theorem 3.5.

Let δ be a guage on Ω such that $\delta(x) = \delta_k(x)$ whenever $x \in Z_k$, and let P be a tagged subpartition of Ω which in anchored in Z and subordinate to δ. Then

$$\begin{aligned}
&\sum_{(x,[u,v]) \in P} |F(v)G(v) - F(u)G(u)| \\
&\qquad \leq \sum_{k=1}^{\infty} \sum_{\substack{(x,[u,v]) \in P\colon \\ x \in Z_k}} (|F(v) - F(u)||G(v)| + |F(u)||G(v) - G(u)|) \\
&\qquad \leq \sum_{k=1}^{\infty} k \sum_{\substack{(x,[u,v]) \in P\colon \\ x \in Z_k}} (|F(v) - F(u)| + |G(v) - G(u)|) \\
&\qquad < \sum_{k=1}^{\infty} k \frac{\varepsilon}{k2^k} = \varepsilon.
\end{aligned}$$

□

Corollary 3.14. *Consider two regular primitives $F, G \colon \Omega \to \mathbb{R}$. Then FG satisfies the strong Luzin condition.*

Proof. This follows directly from Theorems 3.10 and 3.13. That is, F and G both satisfy the strong Luzin condition and therefore the product FG satisfies this condition as well. □

Chapter 4

Monotonicity theorems

This chapter concerns the construction of non-decreasing regular primitives in addition to sufficient conditions for a function to be non-decreasing.

Theorem 4.1 (Zahorski–Choquet). *Let $Z \subset \Omega$ be a G_δ set with $m(Z) = 0$. Then there exists a bounded, non-decreasing, regular primitive $G \colon \Omega \to \mathbb{R}$ such that $DG(x) = \infty$ if and only if $x \in Z$.*

This theorem is readily inferred from the technical Lemma 4.11 which concerns the special case $Z \subset (0, 1)$. Constructions of this type were first advanced in the 1940s by Zahorski [17] and later Choquet [5, pp. 216–220], both of which supplied constructions based on Luzin–Menshov theorem.[1] By the aid of Bruckner [3, pp. 20–23, 86] we supply our own construction. Moreover, the conditions of Theorem 4.1 cannot be improved upon. That is, by Ward's Theorem 2.1 the points at which an arbitrary function possesses an infinite derivative form a set of measure zero; and if this function is both continuous and differentiable, then Theorem 4.12 (due to Young [16]) states that this set is of type G_δ.

Piranian [13] has supplied a very simple construction of a discontinuous non-decreasing function with an infinite derivative on a prescribed G_δ set which is countable, and with a vanishing derivative elsewhere. In particular a function of this type possesses a vanishing derivative a.e. and is non-constant whenever the G_δ set is non-empty. However, such properties cannot be possessed by a continuous primitive defined on an open interval. This is an immediate corollary of the following monotonicity theorem due to Goldowsky [10] and Tonelli [15]:

[1] The Luzin–Menshov theorem was in fact never published by neither Luzin nor Menshov. It was first published by Luzin's student Vera Bogomolova [1] who attributed this theorem to both Luzin and Menchov.

Theorem 4.2 (Goldowsky–Tonelli). *Let $F\colon (a,b) \to \mathbb{R}$ be a continuous function possessing a non-negative derivative a.e. on (a,b) and suppose there exists a countable subset $C \subset (a,b)$ such that F is differentiable on $(a,b) \setminus C$. Then F is non-decreasing.*

Our proof was inspired by the proof of Saks [14, pp. 206–207]. Note that it is readily inferred from Theorem 3.1 that a regular primitive cannot be singular on an open interval. However, since Theorem 3.2 states that Theorem 3.1 cannot be extended to all continuous primitives, we do indeed require the Goldowsky–Tonelli theorem in order to conclude that all continuous primitives possess this property as well.

Finally, by using the Goldowsky–Tonelli theorem in addition to the fact that infinite derivatives with continuous primitives constitute Darboux functions in the first class of Baire, we obtain a simple proof of the Denjoy–Clarkson theorem.

Theorem 4.3 (Denjoy–Clarkson). *Let $F\colon \Omega \to \mathbb{R}$ be a continuous primitive and let $\alpha, \beta \in \overline{\mathbb{R}}$. Then the set $E_{\alpha,\beta} := \{x \in \Omega\colon \alpha < DF(x) < \beta\}$ is either empty or has positive measure.*

The proof closely resembles the original proof of Clarkson [6] in which infinite derivatives are considered;[2] whereas the special case concerning finite derivatives was first established by Denjoy [7].

4.1 Theorems of Luzin–Menshov and Zahorski–Choquet

Our first objective is to prove the aforementioned Luzin–Menshov theorem. Bruckner [3, p. 26] mentions that his proof of this theorem is an adaptation of the proof supplied by Goffman, Neugebauer and Nishiura [9] for n-dimensional space. Excluding some minor modifications, the former proof is replicated here. For this purpose we shall begin by establishing several lemmas concerning basic properties of closed sets.

[2] It should be noted that although Clarkson makes no explicit mention of infinite derivatives; it appears that his proof remains valid for infinite derivatives (with continuous primitives). However, Clarkson does not mention the Goldowsky–Tonelli theorem by name and instead refers to a similar theorem in Lebesgue's book *Leçons sur l'intégration, 2nd ed., Paris, 1928, p. 97.* Whether or not the latter theorem concerns infinite derivatives has not been verified in the writing of this thesis. Nevertheless by using the Goldowsky–Tonelli theorem we obtain a proof which is highly similar to Clarkson's proof. Therefore we attribute to Clarkson the general version of the Denjoy–Clarkson theorem. This sentiment is shared by others as well, cf. Bruckner and Leonard [4, p. 32].

In order to proceed we require the following definitions: Let $x \in \mathbb{R}$, $P \subset \overline{\mathbb{R}}$ and $Q \subset \mathbb{R}$. The *distance* between x and P is defined by

$$\operatorname{dist}(x, P) := \begin{cases} \inf\limits_{y \in P} |x - y|, & \text{if } P \neq \varnothing; \\ 0, & \text{if } P = \varnothing. \end{cases}$$

The *diameter* of Q is defined by

$$\operatorname{diam} Q := \begin{cases} \sup\limits_{y,z \in Q} |y - z|, & \text{if } Q \neq \varnothing; \\ 0, & \text{if } Q = \varnothing. \end{cases}$$

Lemma 4.4. *Let F be a closed set and let $\{P_i\}_{i=1}^{\infty}$ be a collection of perfect sets such that one of the following conditions is met:*

(i) *There exists a positive integer n such that $F \subset \bigcup_{i=1}^{n} P_i = \bigcup_{i=1}^{\infty} P_i$, unless F is perfect, in which case we merely require that $\bigcup_{i=1}^{n} P_i = \bigcup_{i=1}^{\infty} P_i$.*

(ii) *Each P_i is non-empty and $\max_{x \in P_i} \operatorname{dist}(x, F) \to 0$ as $i \to \infty$. Moreover, if $F_0 \subset F$ denotes the subset of points which are isolated in F, then we have $\inf_i \operatorname{dist}(x, P_i) = 0$ for every $x \in F_0$.*

Then $P := F \cup \bigcup_{i=1}^{\infty} P_i$ is perfect.

Proof. The assertion is evidently valid when condition (i) is met. Moreover, the set P contains no isolated points when condition (ii) is met because each P_i is perfect and $\liminf_{i \to \infty} \operatorname{dist}(x, P_i) = 0$ for every $x \in F_0 \setminus \bigcup_{i=1}^{\infty} P_i$. It remains to be shown that P is closed in the latter case. Let $\{x_j\}_{j=1}^{\infty} \subset P$ be a convergent sequence with the limit x. Then there exists a subsequence $\{x_{j_k}\}_{k=1}^{\infty}$ for which one of the following conditions is met:

(1) we have $\{x_{j_k}\}_{k=1}^{\infty} \subset F$, in which case $x \in F \subset P$;

(2) for some i we have $\{x_{j_k}\}_{k=1}^{\infty} \subset P_i$, in which case $x \in P_i \subset P$;

(3) there exists an injective mapping $k \to i_k$ such that $x_{j_k} \in P_{i_k}$, in which case $\operatorname{dist}(x, F) \leq |x - x_{j_k}| + \operatorname{dist}(x_{j_k}, F) \leq |x - x_{j_k}| + \max_{y \in P_{i_k}} \operatorname{dist}(y, F) \to 0$ as $k \to \infty$, so that $x \in F \subset P$.

In all three cases we have $x \in P$ and so the proof is complete. $\square$

Lemma 4.5. *Let K be a closed set. Then there exists a countable set C and a perfect set P such that $C \cap P = \varnothing$ and $K = C \cup P$.*

Proof. Let R be the collection of intervals (p, q), with rational endpoints $p < q$, such that $K \cap (p, q)$ is countable. Define the sets

$$C := K \cap \bigcup_{(p,q)\in R} (p, q) \quad \text{and} \quad P := K \setminus C = K \cap \Bigg(\bigcup_{(p,q)\in R} (p, q) \Bigg)^{c}.$$

Clearly C is countable, P is perfect[3], $C \cap P = \varnothing$ and $K = C \cup P$. □

Lemma 4.6. *Suppose $E \subset \mathbb{R}$ is measurable with $m(E) < \infty$. Then for every $\varepsilon > 0$ there exists a perfect set $P \subset E$ such that $m(E \setminus P) < \varepsilon$.*

Proof. Let $\varepsilon > 0$ and note that by the regularity of the Lebesgue measure there exists a compact subset $K \subset E$ such that $m(E \backslash K) < \varepsilon$. Since K is closed we use Lemma 4.5 to obtain a countable set C and a perfect set P such that $C \cap P = \varnothing$ and $K = C \cup P$. Then $P \subset K \subset E$ and $m(E \setminus P) = m(E \setminus K) < \varepsilon$. □

Lemma 4.7. *Let $E \subset [0, 1]$ be measurable and let $x \in E$ satisfy $d(x, E) > 0$.*[4] *Then for every $\eta > 0$ there exists a perfect set P such that $x \in P \subset E$ and* $\operatorname{diam} P < \eta$.

Proof. Let $\eta > 0$. Since $d(x, E) > 0$ there exists for each positive integer k an interval $I_k \subset [0, 1]$ such that

$$x \in \partial I_k, \quad m(E \cap I_k) > 0 \quad \text{and} \quad m(I_k) < \frac{\eta}{2^k}.$$

By Lemma 4.6 and the inequality $m(E \cap I_k) > 0$ we obtain a non-empty perfect set $P_k \subset E \cap I_k$. Define the set

$$P := \{x\} \cup \bigcup_{k=1}^{\infty} P_k.$$

Then P is perfect by Lemma 4.4. Moreover, $x \in P \subset E$ and $\operatorname{diam} P < \eta$. □

Lemma 4.8. *Let $E \subset [0, 1]$ be measurable and let $C \mathrel{\dot\subset} E$ be a countable subset for which $\overline{C} \subset E$.*[5] *Then there exists a perfect set P such that $C \subset P \subset E$.*

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[3]The first expression for P ensures that P contains no isolated points; whereas the second expression ensures that P is closed.

[4]The density operator d is defined in Section 1.1.

[5]The notation $\mathrel{\dot\subset}$ is defined in Section 1.1 using the aforementioned density operator.

Proof. Write $C = \{x_k\}_k$, where the indices k are positive integers. For each k, Lemma 4.7 yields a perfect sets P_k such that $x_k \in P_k \subset E$ and $\operatorname{diam} P_k < 1/k$. Define the set

$$P := \overline{C} \cup \bigcup_k P_k.$$

Then P is perfect by Lemma 4.4. Moreover, we have $C \subset P \subset E$. □

Lemma 4.9. *Let $E \subset [0,1]$ be measurable and let $F \Subset E$ be closed. Then there exists a perfect set P such that $F \subset P \subset E$.*

Proof. Using Lemma 4.5 we obtain a countable set C and a perfect set P_1 such that $F = C \cup P_1$. Moreover, since $C \Subset E$ and $\overline{C} \subset E$ we use Lemma 4.8 to obtain a perfect set P_2 such that $C \subset P_2 \subset E$. Then $P := P_1 \cup P_2$ is perfect and satisfies $F \subset P \subset E$. □

Theorem 4.10 (Luzin–Menshov). *Let $E \subset [0,1]$ be measurable and let $F \Subset E$ be closed. Then there exists a perfect set P such that $F \Subset P \Subset E$.*

Proof. The points $x \in E$ for which $d(x, E) < 1$ form a set of measure zero, and since these points are excluded from F, we may assume that $d(x, E) = 1$ for every $x \in E$. Thus when a perfect set P with $F \Subset P \subset E$ has been obtained, then the proof will be complete since trivially $P \Subset E$. By Lemma 4.9 we shall further assume that F is perfect. Define for each positive integer k the set

$$T_k := \left\{ x \in [0,1] \colon \frac{1}{k+1} < \operatorname{dist}(x, F) \leq \frac{1}{k} \right\} \quad \text{and} \quad S_k := E \cap T_k.$$

Note that $E = F \cup \bigcup_{k=1}^{\infty} S_k$. Moreover, each T_k is a G_δ set[6] and so each S_k is measurable with $m(S_k) \leq 1$. For each positive integer k we use Lemma 4.6 to obtain a perfect set $P_k \subset S_k$ such that $m(S_k \setminus P_k) < 1/2^k$. Define the set

$$P := F \cup \bigcup_{k=1}^{\infty} P_k.$$

By definition we have $F \subset P \subset E$ and by Lemma 4.4 we infer that P is perfect. It remains to be shown that $F \Subset P$. Let $x_0 \in F$ and let $\{I_j\}_{j=1}^{\infty}$ be a collection of non-degenerate intervals such that

$$x_0 \in I_j \subset [0,1], \quad \lim_{j \to \infty} m(I_j) = 0 \quad \text{and} \quad d(x_0, P) = \lim_{j \to \infty} \frac{m(P \cap I_j)}{m(I_j)}.$$

[6] That is,

$$T_k = \bigcap_{n=1}^{\infty} \left\{ x \in [0,1] \colon \frac{1}{k+1} < \operatorname{dist}(x, F) < \frac{1}{k} + \frac{1}{n} \right\}.$$

If there exist infinitely many j such that $S_k \cap I_j = \varnothing$ for all k, then since for all such j we have $P \cap I_j = F \cap I_j = E \cap I_j$, it follows that $d(x_0, P) = d(x_0, E) = 1$. Therefore, since we incur no loss of generality by excluding a finite amount of intervals I_j, we shall assume that there exists no j such that $S_k \cap I_j = \varnothing$ for all k. Then each j corresponds to a smallest positive integer k_j such that $S_{k_j} \cap I_j \neq \varnothing$. Note that

$$\begin{aligned} m(P \cap I_j) &= m(F \cap I_j) + \sum_{k=k_j}^{\infty} m(P_k \cap I_j) \\ &\geq m(F \cap I_j) + \sum_{k=k_j}^{\infty} (m(S_k \cap I_j) - m(S_k \setminus P_k)) \\ &> m(F \cap I_j) + \sum_{k=k_j}^{\infty} \left(m(S_k \cap I_j) - \frac{1}{2^k} \right) \\ &= m(E \cap I_j) - \frac{1}{2^{k_j - 1}}. \end{aligned}$$

Since $S_{k_j} \subset T_{k_j}$, $S_{k_j} \cap I_j \neq \varnothing$ and $x_0 \in F \cap I_j$ we have

$$m(I_j) \geq \operatorname{dist}(x_0, T_{k_j}) \geq \frac{1}{k_j + 1},$$

so that

$$\frac{m(P \cap I_j)}{m(I_j)} > \frac{m(E \cap I_j) - 1/2^{k_j - 1}}{m(I_j)} \geq \frac{m(E \cap I_j)}{m(I_j)} - \frac{k_j + 1}{2^{k_j - 1}}.$$

Moreover, $k_j \to \infty$ as $j \to \infty$, because $m(I_j) \geq 1/(k_j + 1)$ and $m(I_j) \to 0$ as $j \to \infty$. Consequently

$$d(x_0, P) = \lim_{j \to \infty} \frac{m(P \cap I_j)}{m(I_j)} \geq \lim_{j \to \infty} \left(\frac{m(E \cap I_j)}{m(I_j)} - \frac{k_j + 1}{2^{k_j - 1}} \right) = d(x_0, E) = 1,$$

and since $d(x_0, P) \leq 1$, it follows that $d(x_0, P) = 1$. This proves that $F \Subset P$. $\square$

Lemma 4.11. *Let $Z \subset (0, 1)$ be a G_δ set with $m(Z) = 0$. Then there exists a positive, non-decreasing, regular primitive $G \colon \mathbb{R} \to \mathbb{R}$ whose derivative vanishes on $\mathbb{R} \setminus (0, 1)$ and for which $DG(x) = \infty$ if and only if $x \in Z$.*

Proof. Let $E := [0, 1] \setminus Z$ and note that E is an F_σ set. Thus there exists a non-decreasing collection of closed sets $\{F_k\}_{k=0}^{\infty}$ such that $E = \bigcup_{k=0}^{\infty} F_k$ and $\{0, 1\} \subset F_0$. We shall construct a collection of closed sets $\{P_\lambda\}_{\lambda \geq 0}$ such that $E =$

$\bigcup_{\lambda \geq 0} P_\lambda$ and $P_{\lambda_1} \Subset P_{\lambda_2}$ whenever $\lambda_1 < \lambda_2$. First we construct the subcollection $\{P_k\}_{k=0}^\infty$. By Lemma 4.9 and the trivial fact that $F_0 \Subset E$ we obtain a perfect set P_0 such that $F_0 \subset P_0 \subset E$.

We proceed inductively. Suppose that for a fixed k we have already determined a perfect set $P_k \subset [0,1]$ such that $F_k \subset P_k \subset E$. Let $\{I_{i,k}\}_i$ denote the open subintervals of $(0,1)$ which are contiguous to P_k. By the Luzin–Menshov theorem and the trivial fact that $F_{k+1} \cup P_k \Subset E$ we obtain a perfect set Q_{k+1} with $F_{k+1} \cup P_k \Subset Q_{k+1} \Subset E$. For each i we use Lemma 4.6 to obtain a perfect set $R_{i,k} \subset E \cap I_{i,k}$ satisfying

$$m(I_{i,k})^2 > m((E \cap I_{i,k}) \setminus R_{i,k}) = m(I_{i,k} \setminus R_{i,k}) = m(I_{i,k}) - m(R_{i,k}),$$

so that

$$\frac{m(R_{i,k})}{m(I_{i,k})} > 1 - m(I_{i,k}).$$

Define the set

$$P_{k+1} := Q_{k+1} \cup \bigcup_i R_{i,k}.$$

Then P_{k+1} is perfect by Lemma 4.4. Moreover, we have $F_{k+1} \cup P_k \Subset P_{k+1}$ since $F_{k+1} \cup P_k \Subset Q_{k+1} \subset P_{k+1}$, and in particular $P_k \Subset P_{k+1}$.

Note that $E = \bigcup_{k=0}^\infty P_k$ and for all $I_{i,k}$ we have

$$\frac{m(P_{k+1} \cap I_{i,k})}{m(I_{i,k})} \geq \frac{m(R_{i,k})}{m(I_{i,k})} > 1 - m(I_{i,k}). \tag{4.1}$$

Next we construct for each pair of non-negative integers j, k a perfect set $P_{k/2^j}$. For $j = 0$ we take the sets P_k constructed above, which satisfy

$$P_{k/2^j} \Subset P_{(k+1)/2^j}. \tag{4.2}$$

Again we proceed inductively. Suppose that for a fixed j we have already determined sets $P_{k/2^j}$ such that condition (4.2) is met. Then for each k we let $P_{(2k)/2^{j+1}} := P_{k/2^j}$; and from the Luzin–Menshov theorem we obtain a perfect set $P_{(2k+1)/2^{j+1}}$ such that

$$P_{k/2^j} \Subset P_{(2k+1)/2^{j+1}} \Subset P_{(k+1)/2^j}.$$

Now, for every $\lambda \geq 0$ we define

$$P_\lambda := \bigcap_{k/2^j \geq \lambda} P_{k/2^j}.$$

Note that for $\lambda = k/2^j$ this definition agrees with the previous definition of $P_{k/2^j}$; and since intersections of closed sets are closed, it follows that all P_λ are closed. Moreover, we have $P_{\lambda_1} \Subset P_{\lambda_2}$ whenever $\lambda_1 < \lambda_2$. To see this, note that $P_{\lambda_1} \subset P_{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$. This is clear by the definitions of P_{λ_1} and P_{λ_2}. If $\lambda_1 < \lambda_2$, then choose j, k such that

$$\lambda_1 \leq \frac{k}{2^j} < \frac{k+1}{2^j} \leq \lambda_2.$$

By (4.2) we have $P_{\lambda_1} \subset P_{k/2^j} \Subset P_{(k+1)/2^j} \subset P_{\lambda_2}$, so that $P_{\lambda_1} \Subset P_{\lambda_2}$.

Now, define the function $g \colon \mathbb{R} \to \overline{\mathbb{R}}$ by

$$g(x) := \begin{cases} \infty, & \text{if } x \in Z; \\ \inf\{\lambda \colon x \in P_\lambda\}, & \text{if } x \in E; \\ 0, & \text{if } x \in \mathbb{R} \setminus [0,1]. \end{cases}$$

Then g is lower semicontinuous[7] on $\mathbb{R}$ and continuous (in the extended sense) on $\mathbb{R} \setminus E$. The former implies that g is measurable. Moreover, we have $g(0) = g(1) = 0$ since $\{0,1\} \subset F_0 \subset P_0 \subset E$.

If k is a non-negative integer and $J \subset [0,1]$ is an interval with $\partial J \subset P_k$, then

$$\begin{aligned} m(J \setminus P_k)^2 &= m\Bigg(\bigcup_{i \colon I_{i,k} \subset J} I_{i,k} \Bigg)^2 = \Bigg(\sum_{i \colon I_{i,k} \subset J} m(I_{i,k}) \Bigg)^2 \\ &\geq \sum_{i \colon I_{i,k} \subset J} m(I_{i,k})^2 \overset{(4.1)}{\geq} \sum_{i \colon I_{i,k} \subset J} \big(m(I_{i,k}) - m(P_{k+1} \cap I_{i,k}) \big) \\ &= \sum_{i \colon I_{i,k} \subset J} m(I_{i,k} \setminus P_{k+1}) = m\Bigg(\bigcup_{i \colon I_{i,k} \subset J} I_{i,k} \setminus P_{k+1} \Bigg) \\ &= m(J \setminus P_{k+1}). \end{aligned} \tag{4.3}$$

With the notation $m_k := m([0,1] \setminus P_k)$ we have $\sum_{k=0}^{\infty} (k+1) m_k < \infty$, where the inequality follows from (4.3) and d'Alembert's criterion for convergence. Thus by the monotone convergence theorem for the Lebesgue integral it follows that g is

[7]By definition g is lower semicontinuous provided that $g^{-1}((\alpha, \infty])$ is open for every $\alpha \in \mathbb{R}$. Indeed this criterion is met because

$$g^{-1}((\alpha,\infty]) = \begin{cases} (0,1) \setminus P_\alpha, & \text{if } \alpha \geq 0; \\ \mathbb{R}, & \text{if } \alpha < 0. \end{cases}$$

This is equivalent to showing that $\liminf_{x \to x_0} g(x) \geq g(x_0)$ for every $x_0 \in \mathbb{R}$. (Upon proving that g is a derivative we may in fact conclude that $\liminf_{x \to x_0 \pm} g(x) = g(x_0)$ for every $x_0 \in \mathbb{R}$ because by Theorem 1.1 derivatives possess the intermediate value property.)

Lebesgue integrable. More precisely, $g^{-1}((k, k+1]) \subset g^{-1}((k, \infty]) = [0, 1] \setminus P_k$ and so the integral of g on $g^{-1}((k, k+1])$ does not exceed the value $(k+1)m_k$. Define the function $G\colon \mathbb{R} \to \mathbb{R}$ by

$$G(x) := \int_{(-\infty, x]} g \, dm.$$

Then G is absolutely continuous and must therefore possess all of the desired properties, except for those concerning differentiability which are addressed next. By the lower semicontinuity of g it is clear that

$$\underline{D}G(x) \geq g(x) \quad \text{for every } x \in \mathbb{R}.$$

Since g is continuous (in the extended sense) on $\mathbb{R} \setminus E$ we have $DG(x) = g(x)$ for every $x \in \mathbb{R} \setminus E$. In order to extend this equality to E it remains to be shown that $\overline{D}G(x) \leq g(x)$ for every $x \in E$. Hence we let $x_0 \in E$ and $\varepsilon > 0$. Note that if $x_0 = 0$ ($x_0 = 1$) then since g vanishes on $\mathbb{R} \setminus (0, 1)$ it will suffice to establish the inequality for the right-sided (left-sided) correspondent of $\overline{D}G(x_0)$. Define $\lambda_0 := g(x_0)$, let k_0 be the smallest integer for which $\lambda_0 + \varepsilon/3 \leq k_0$ and let $M > 0$ be such that

$$\sum_{k=k_0}^{\infty} \frac{k+1}{2^{k+M}} < \frac{\varepsilon}{3}. \tag{4.4}$$

Since $x_0 \in P_\lambda \mathrel{\subset\!\!\!\bullet} P_{\lambda_0+\varepsilon/3} \subset P_{k_0}$ for every $\lambda \in (\lambda_0, \lambda_0 + \varepsilon/3)$, there exists an interval $U \subset [0, 1]$ which contains x_0, has measure $m(U) \leq 1/2$, is open relative to $[0, 1]$ and has its endpoints in P_{k_0}, such that for every non-degenerate interval V with $x_0 \in V \subset U$ we have

$$\frac{m(V \setminus P_{\lambda_0+\varepsilon/3})}{m(V)} \leq \max\left\{\frac{\varepsilon}{3k_0}, \frac{1}{2^{k_0+M+1}}\right\}. \tag{4.5}$$

We aim to show that $\Delta(G, I)/m(I) < \lambda_0 + \varepsilon$ for every non-degenerate compact interval I with $x_0 \in I \subset U$. Hence we let I be as described and let J be the largest subinterval of U for which $P_{k_0} \cap (J \setminus I) = \varnothing$. Note that $I \subset J \subset U$ and therefore $0 < m(J) \leq 1/2$. Moreover, the interval J satisfies (4.3) for all $k \geq k_0$ since $\partial J \subset P_{k_0}$; and the inequality (4.5) is valid with both $V = I$ and $V = J$ since I and J are non-degenerate intervals satisfying $x_0 \in I \subset J \subset U$. We have

$$1 - \frac{m(I)}{m(J)} = \frac{m(J \setminus I)}{m(J)} \leq \frac{m(J \setminus P_{k_0})}{m(J)} \leq \frac{m(J \setminus P_{\lambda_0+\varepsilon/3})}{m(J)} \overset{(4.5)}{\leq} \frac{1}{2},$$

so that

$$m(J) \leq 2m(I). \tag{4.6}$$

Next we show that

$$m(J \setminus P_k) \le \frac{m(J \setminus P_{k_0})}{2^{k-k_0}} \quad \text{for each integer } k \ge k_0. \tag{4.7}$$

For $k = k_0$ the claim is evidently true. We proceed inductively and assume that the claim holds for some integer $k \ge k_0$. Then

$$m(J \setminus P_{k+1}) \overset{(4.3)}{\le} m(J \setminus P_k)^2 \overset{m(J)\le 1/2}{\le} \frac{m(J \setminus P_k)}{2} \le \frac{m(J \setminus P_{k_0})}{2^{k+1-k_0}}.$$

Thus the claim holds. Furthermore, we have

$$\begin{aligned}\frac{m(I \setminus P_k)}{m(I)} &\overset{(4.6)}{\le} \frac{m(J \setminus P_k)}{2^{-1}m(J)} \overset{(4.7)}{\le} \frac{m(J \setminus P_{k_0})}{2^{k-k_0-1}m(J)} \overset{(4.5)}{\le} \frac{1}{2^{(k-k_0-1)+(k_0+M+1)}} \\ &= \frac{1}{2^{k+M}} \quad \text{for each integer } k \ge k_0. \end{aligned} \tag{4.8}$$

Finally, $g^{-1}((0, \lambda_0 + \varepsilon/3]) \subset (0,1)$, $g^{-1}((\lambda_0 + \varepsilon/3, k_0]) \cap I \subset I \setminus P_{\lambda_0+\varepsilon/3}$ and $g^{-1}((k, k+1]) \cap I \subset I \setminus P_k$ for each integer $k \ge k_0$, so that

$$\begin{aligned}\frac{\Delta(G, I)}{m(I)} &\le \lambda_0 + \frac{\varepsilon}{3} + k_0 \frac{m(I \setminus P_{\lambda_0+\varepsilon/3})}{m(I)} + \sum_{k=k_0}^{\infty} (k+1) \frac{m(I \setminus P_k)}{m(I)} \\ &\overset{\substack{(4.8)\\(4.5)}}{\le} \lambda_0 + \frac{\varepsilon}{3} + k_0 \frac{\varepsilon}{3k_0} + \sum_{k=k_0}^{\infty} \frac{k+1}{2^{k+M}} \\ &\overset{(4.4)}{<} \lambda_0 + \varepsilon = g(x_0) + \varepsilon. \end{aligned}$$

Since $x_0 \in E$ and $\varepsilon > 0$ were arbitrary we conclude that

$$\overline{D}G(x) \le g(x) \quad \text{for every } x \in E.$$

This completes the proof. $\square$

Proof of Theorem 4.1. For each integer k we use Lemma 4.11 to obtain a positive, non-decreasing, regular primitive $G_k \colon \mathbb{R} \to \mathbb{R}$ whose derivative vanishes on $\mathbb{R} \setminus (k/2, k/2+1)$ and for which $DG(x) = \infty$ if and only if $x \in Z \cap (k/2, k/2+1)$. We may of course assume that $G_k(k/2+1) \le 1/2^{|k|}$, so that

$$\sum_{k=-\infty}^{\infty} G_k(k/2+1) \le \sum_{k=-\infty}^{\infty} \frac{1}{2^{|k|}} = 3.$$

Thus we may define a bounded function $F\colon \Omega \to \mathbb{R}$ by

$$F(x) := \sum_{k=-\infty}^{\infty} G_k(x).$$

Clearly F possesses the desired properties. □

Theorem 4.12 (Young). *Let $F\colon \Omega \to \mathbb{R}$ be a continuous primitive and define the sets $Z_- := \{x \in \Omega\colon DF(x) = -\infty\}$, $Z_+ := \{x \in \Omega\colon DF(x) = \infty\}$ and $Z := Z_- \cup Z_+$. Then Z_-, Z_+, Z are all G_δ sets with measure zero.*

Proof. From Ward's Theorem 2.1 it follows that $m(Z_-) = m(Z_+) = m(Z) = 0$. Clearly it will suffice to show that Z_+ is a G_δ set. Define for each pair of positive integers i, j the set

$$Z_{i,j} := \bigcap_{|k|>j} \left\{ x \in \Omega\colon \frac{F(x + 1/k) - F(x)}{1/k} \leq i \ \text{ and } \ \operatorname{dist}(x, \overline{\mathbb{R}} \setminus \Omega) \geq 1/j \right\}.$$

Since F is continuous it follows that each $Z_{i,j}$ is expressed as an intersection of closed sets and is therefore closed itself. Moreover, since F is differentiable we have

$$Z_+ = \bigcap_{i=1}^{\infty} \bigcap_{j=1}^{\infty} Z_{i,j}^c,$$

which is a G_δ set. □

4.2 Theorems of Goldowsky–Tonelli and Denjoy–Clarkson

Lemma 4.13. *Let $F\colon (a, b) \to \mathbb{R}$ be a continuous function possessing a non-negative derivative a.e. on (a, b). Moreover, suppose there exists a countable subset $C \subset (a, b)$, a non-degenerate subinterval $J \subset (a, b)$ and a number $M > 0$ such that $\underline{D}F > -M$ on $J \setminus C$. Then F is non-decreasing on J.*

Proof. Pick a non-degenerate compact subinterval $K \subset J$. We aim to show that $\Delta(F, K) \geq 0$. Let $\varepsilon > 0$ and define the sets

$$A := \{x \in K \setminus C\colon DF(x) \geq 0\} \quad \text{and} \quad B := K \setminus (A \cup C).$$

Since $m(B) = 0$ there exists an open set U such that

$$B \subset U \subset (a, b) \quad \text{and} \quad m(U) < \frac{\varepsilon}{4M}.$$

Write $C = \{x_n\}_n$, where the indices n are positive integers. Let δ be a gauge on Ω such that for every point-interval pair (x, I) with $I \subset (x-\delta(x), x+\delta(x)) \cap (a, b)$, the following conditions are met:

(1) $\Delta(F, I) > -\dfrac{\varepsilon}{4m(K)} m(I)$ whenever $x \in A$;

(2) $I \subset U$ and $\Delta(F, I) > -Mm(I)$ whenever $x \in B$;

(3) $\Delta(F, I) > -\dfrac{\varepsilon}{2^{n+1}}$ whenever $x = x_n \in C$.

By Cousin's Lemma 1.4 there exists a tagged subpartition P of (a, b), which is subordinate to δ, such that

$$K = \bigcup_{(x,I)\in P} I.$$

Thus we have

$$\begin{aligned}
\Delta(F, K) &= \sum_{(x,I)\in P} \Delta(F, I) \\
&= \sum_{\substack{(x,I)\in P:\\ x\in A}} \Delta(F, I) + \sum_{\substack{(x,I)\in P:\\ x\in B}} \Delta(F, I) + \sum_{\substack{(x_n,I)\in P:\\ x_n\in C}} \Delta(F, I) \\
&> -\frac{\varepsilon}{4m(K)} \sum_{\substack{(x,I)\in P:\\ x\in A}} m(I) - M \sum_{\substack{(x,I)\in P:\\ x\in B}} m(I) - \frac{\varepsilon}{2} \sum_{\substack{(x_n,I)\in P:\\ x_n\in C}} \frac{1}{2^n} \\
&> -\frac{\varepsilon}{4m(K)} m(K) - Mm(U) - \frac{\varepsilon}{2} = -\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we conclude that $\Delta(F, K) \geq 0$. $\square$

Proof of Theorem 4.2. It will suffice to show that F is non-decreasing on a fixed but arbitrary, non-degenerate, compact subinterval $K \subset (a, b)$. Without loss of generality we may assume that C is infinite so that we can write $C = \{x_n\}_{n=1}^{\infty}$. Denote by E the set of points $x \in K$ such that for every neighbourhood U of x the function F fails to be non-decreasing on $K \cap U$. It is easily seen that the set E is perfect and the function F is non-decreasing on every subinterval of $K \setminus E$.[8] Hence it remains to be shown that $E = \varnothing$.

[8] Here we used the fact that a continuous function which is locally non-decreasing on a given interval is in fact non-decreasing on the entire interval. Consequently F is non-decreasing on every subinterval of $K \setminus E$, and by continuity this claim can be extended to the closure of these subintervals. The latter implies that E possesses no isolated points. Moreover, the fact that E is closed follows directly from the definition of E.

Assume towards a contradiction that $E \neq \varnothing$. For each positive integer n we denote by P_n the set of points $x \in K$ for which the inequality $|y-x| < 1/n$, $y \in (a,b)$, implies that $F(y) - F(x) \leq -(y-x)$. Similarly, we denote by Q_n the set of points $x \in K$ for which the same inequality implies that $F(y) - F(x) \geq -2(y-x)$. Note that the sets P_n and Q_n are closed; and together with C they cover all of K. In particular $E = E \cap \bigcup_{n=1}^{\infty} P_n \cup Q_n \cup \{x_n\}$, where each $\{x_n\}$ is nowhere dense in E, and so the Baire category theorem guarantees that for some positive integer n_0, at least one of the sets P_{n_0} and Q_{n_0} coincides with E on some open interval J with $E \cap J \neq \varnothing$.

We consider first the case when $P_{n_0} \cap J = E \cap J \neq \varnothing$ and for convenience we shall assume that $m(J) < 1/n_0$. By hypothesis we have $DF \geq 0$ a.e. on (a,b) and thus P_{n_0} is nowhere dense in J. From this in addition to the fact that E is a perfect subset of K, we infer the existence of an open interval $(x,y) \subset J \cap K$ which is contiguous to P_{n_0}. The function F is non-decreasing on (x,y), and by continuity the same holds on $[x,y]$. However, this contradicts the fact that since $x, y \in P_{n_0}$ and $y - x < 1/n_0$ we have $F(y) - F(x) \leq -(y-x) < 0$.

Finally we consider the case when $Q_{n_0} \cap J = E \cap J \neq \varnothing$. Then $\underline{D}F \geq -2$ on $J \cap K^o \cap Q_{n_0}$ and $\underline{D}F \geq 0$ on $J \cap K^o \cap Q_{n_0}^c$. Again, by hypothesis we have $DF \geq 0$ a.e. on (a,b). Thus by Lemma 4.13 we infer that F is non-decreasing on $J \cap K^o$, and by continuity the same holds on $J \cap K$. However, this yields the contradiction $E \cap J = E \cap J \cap K = \varnothing$. □

Corollary 4.14. *A continuous primitive cannot be singular on an open interval.*

Proof. Assume towards a contradiction that there exists a continuous primitive $F\colon (a,b) \to \mathbb{R}$ which is singular, and therefore non-constant. By the Goldowsky–Tonelli theorem both F and $-F$ are non-decreasing, so that F is constant. Thus we have reached a contradiction. □

Proof of Theorem 4.3. It will suffice to consider the case when α and β are both finite. Assume towards a contradiction that $E_{\alpha,\beta} \neq \varnothing$ and $m(E_{\alpha,\beta}) = 0$. Pick a compact subinterval $K \subset (a,b)$ such that $E := E_{\alpha,\beta} \cap K^o \neq \varnothing$ and define the two sets

$$E_\alpha := \{x \in K^o \colon DF(x) \leq \alpha\} \quad \text{and} \quad E_\beta := \{x \in K^o \colon DF(x) \geq \beta\}.$$

We begin by showing that $E \subset \overline{E}_\alpha \cap \overline{E}_\beta$. Suppose that $x_0 \in E \setminus \overline{E}_\alpha$. Then there exists an open interval U such that $x_0 \in U \subset K^o$ and $E_\alpha \cap U = \varnothing$. Note that $DF \geq \beta$ a.e. on U. Thus by the Goldowsky–Tonelli theorem we have $DF \geq \beta$ everywhere on U. That is, the function $F - \beta I$ is non-decreasing on U, where $I\colon \Omega \to \mathbb{R}$ denotes the identity function on Ω. But then $E \cap U = \varnothing$ which contradicts the fact that $x_0 \in E \cap U$. This proves that $E \subset \overline{E}_\alpha$.

Similarly, suppose that $x_0 \in E \setminus \overline{E}_\beta$. Then there exists an open interval U such that $x_0 \in U \subset K^o$ and $E_\beta \cap U = \varnothing$. Note that $DF \leq \alpha$ a.e. on U. Thus by the Goldowsky–Tonelli theorem we have $DF \leq \alpha$ everywhere on U, i.e. $\alpha I - F$ is non-decreasing on U. But again $E \cap U = \varnothing$ which contradicts the fact that $x_0 \in E \cap U$. This proves that $E \subset \overline{E}_\beta$.

Now, since DF is a Baire-1 function[9], it follows that $DF|_{\overline{E}}$ cannot be discontinuous everywhere on $\overline{E}$. Otherwise $\overline{E}$ would be of the first category relative to itself according to Lemma 1.6; but the Baire category theorem states that $\overline{E}$ is of the second category relative to itself. Therefore, upon proving that $DF|_{\overline{E}}$ is discontinuous everywhere on $\overline{E}$, we shall obtain the desired contradiction. For every open interval $U \subset \Omega$ with $E \cap U \neq \varnothing$, the inclusion $E \subset \overline{E}_\alpha \cap \overline{E}_\beta$ implies the following inequalities:

$$\inf_{x \in U} DF(x) \leq \alpha \quad \text{and} \quad \sup_{x \in U} DF(x) \geq \beta.$$

Recall that derivatives are Darboux functions by Theorem 1.1, i.e. they possess the intermediate value property. From this we infer that

$$\inf_{x \in E \cap U} DF(x) = \alpha \quad \text{and} \quad \sup_{x \in E \cap U} DF(x) = \beta.$$

This proves that $DF|_{\overline{E}}$ is discontinuous everywhere on $\overline{E}$. □

[9] Baire-1 functions are defined in the discussion before Lemma 1.6.

Chapter 5

The strong Luzin condition

The Zahorski–Choquet theorem – which was proved in the previous chapter – is interesting by its own merits. Nevertheless it was included in this thesis in order to establish the two main results of this chapter.

Theorem 5.1. *There exists an irregular primitive $G\colon (0,1) \to \mathbb{R}$ possessing the property that for every open interval $U \subset (0,1)$ which contains a point of the Cantor ternary set, there does not exist any function $F\colon U \to \mathbb{R}$ which satisfies both the strong Luzin condition and $DF = DG$ a.e. on U.*

Thus – contrary to Theorem 3.2 – there exists an irregular primitive which cannot be locally expressed, at every point of Ω, as the sum of a singular function and a function which satisfies the strong Luzin condition. Another consequence of Theorem 5.1 concerning the Henstock–Kurzweil integral was discussed in the introduction. In summary, for every compact interval which contains a point of the Cantor ternary set in its interior, there does not exist any function which is both integrable and coincides with the derivative of G a.e. on this interval.

Theorem 5.2. *A function $F\colon \Omega \to \mathbb{R}$ satisfies the strong Luzin condition if and only if it can be expressed as the sum of two regular primitives.*

This result is obtained by first establishing Lemmas 5.3–5.10 which concern the Henstock variation ν and its connection with the more accessible Jordan variation ν_0. Recall that ν_0 describes the variation of a function on a compact interval; whereas ν describes the variation of a function on an arbitrary set. Although the reverse direction of Theorem 5.2 follows from Corollary 3.12; the forward direction depends on the aforementioned lemmas.

5.1 A pathological irregular primitive

Proof of Theorem 5.1. Let C denote the Cantor ternary set and define $C_* := C \setminus \{0,1\}$. For each positive integer i we denote by $I_{i,1},\dots,I_{i,2^{i-1}}$ the open intervals of length $1/3^i$ contiguous to C, indexed in ascending order. Define the function $H_1\colon (0,1)\to\mathbb{R}$ by

$$H_1(x) := \begin{cases} \dfrac{1}{i}\left(\dfrac{(x-\inf I_{i,j})(x-\sup I_{i,j})}{(m(I_{i,j})/2)^2}\right)^2, & \text{if } x\in I_{i,j};\\ 0, & \text{if } x\in C_*. \end{cases}$$

Since $0 < H_1(x) \le i$ whenever $x\in I_{i,j}$ it is clear that H_1 is continuous. Moreover, H_1 is finitely differentiable on $(0,1)\setminus C_*$ and has a vanishing right-sided (left-sided) derivative at the left (right) endpoint of each $I_{i,j}$. The latter can easily be shown by applying the product rule for differentiation to the polynomials in the definition of H_1.

Partition each interval $\overline{I_{i,j}}$ into non-degenerate compact subintervals $J_{i,j}^{(k)}$, $k=0,1,\dots,2r_{i,j}+1$, which are non-overlapping and indexed in ascending order, such that the following conditions are met:

(i) $L_{i,j} := m(J_{i,j}^{(0)}) = m(J_{i,j}^{(2r_{i,j}+1)}) < \frac{1}{2}m(I_{i,j})$.

(ii) $|\Delta(H_1,K)| < \frac{1}{3i}m(K)$ for every non-degenerate compact interval K which is contained in $J_{i,j}^{(0)}\cup J_{i,j}^{(2r_{i,j}+1)}$ and shares a common endpoint with $I_{i,j}$.

(iii) $|\Delta(H_1,K)| < \frac{1}{3i}L_{i,j}$ for every compact interval K which is contained in some $J_{i,j}^{(k)}, k=1,\dots,2r_{i,j}$.[1] By (i) and (ii) in addition to the fact that H_1 is monotone on $J_{i,j}^{(0)}$ and $J_{i,j}^{(2r_{i,j}+1)}$, respectively, it is easily shown that this inequality remains valid for $k=0$ and $k=2r_{i,j}+1$.

(iv) The common endpoint of $J_{i,j}^{(r_{i,j})}$ and $J_{i,j}^{(r_{i,j}+1)}$ is the midpoint of $I_{i,j}$.

For each $k=1,\dots,2r_{i,j}$ let $K_{i,j}^{(k)}$ denote a non-degenerate compact subinterval of $(J_{i,j}^{(k)})^o$. Denote by T the Cantor ternary function and define the function $H_{i,j}^{(k)}\colon (0,1)\to\mathbb{R}$ by

$$H_{i,j}^{(k)}(x) := \begin{cases} 0, & \text{if } x\in(0,\min K_{i,j}^{(k)});\\ |\Delta(H_1,J_{i,j}^{(k)})|\,T\left(\dfrac{x-\min K_{i,j}^{(k)}}{m(K_{i,j}^{(k)})}\right), & \text{if } x\in K_{i,j}^{(k)};\\ |\Delta(H_1,J_{i,j}^{(k)})|, & \text{if } x\in(\max K_{i,j}^{(k)},1). \end{cases}$$

[1] This is possible because H_1 is uniformly continuous on $\overline{I_{i,j}}$.

Note that for every $k = 1, \ldots, r_{i,j}$ ($k = r_{i,j}+1, \ldots, 2r_{i,j}$) the two functions H_1 and $H_{i,j}^{(k)}$ ($-H_1$ and $H_{i,j}^{(k)}$) are both non-decreasing on $J_{i,j}^{(k)}$ and $\Delta(H_{i,j}^{(k)}, J_{i,j}^{(k)}) = |\Delta(H_1, J_{i,j}^{(k)})| < \frac{1}{3i} L_{i,j}$. In addition $\overline{D}(H_1 - H_{i,j}^{(k)}) < \infty$ ($\underline{D}(H_1 + H_{i,j}^{(k)}) > -\infty$) on $I_{i,j}$; and $H_{i,j}^{(k)}$ is constant on each component of $(0,1) \setminus (m(K_{i,j}^{(k)})C + \min K_{i,j}^{(k)})$.

Define the function $H_2 \colon (0,1) \to \mathbb{R}$ by

$$H_2 := \sum_{i=1}^{\infty} \sum_{j=1}^{2^{i-1}} \Biggl(\sum_{k=r_{i,j}+1}^{2r_{i,j}} H_{i,j}^{(k)} - \sum_{k=1}^{r_{i,j}} H_{i,j}^{(k)} \Biggr).$$

By (i)–(iv) in conjunction with the above remarks we infer that $H_2 = H_{i,j}^{(k)}$ on each $J_{i,j}^{(k)}$, $k = 1, \ldots, 2r_{i,j}$; H_2 vanishes on C_* and also on each $J_{i,j}^{(0)} \cup J_{i,j}^{(2r_{i,j}+1)}$; and H_2 is constant on every subinterval of $(0,1)$ whose intersection with each set $m(K_{i,j}^{(k)})C + \min K_{i,j}^{(k)}$ is empty. In particular H_2 is singular. Moreover, for every non-degenerate compact interval K, which is contained in some interval $\overline{I_{i,j}}$ and shares a common endpoint with $\overline{I_{i,j}}$, we have

$$\begin{aligned} |\Delta(H_1 + H_2, K)| \le |\Delta(H_1, K \cap J_{i,j}^{(0)})| + |\Delta(H_1 + H_2, K \cap \bigcup_{k=1}^{2r_{i,j}} J_{i,j}^{(k)})| \\ + |\Delta(H_1, K \cap J_{i,j}^{(2r_{i,j}+1)})| < \frac{1}{i} m(K). \end{aligned} \tag{5.1}$$

Next we shall prove that $D(H_1 + H_2)$ exists and vanishes on C_*. Note that $H_1 + H_2$ has a vanishing right-sided (left-sided) derivative at each point of C_* which constitutes the left (right) endpoint of some interval $I_{i,j}$, because we already know that the corresponding claim concerning H_1 instead of $H_1 + H_2$ is valid, and H_2 vanishes on $J_{i,j}^{(0)} \cup J_{i,j}^{(2r_{i,j}+1)}$. It remains to be shown that for every $x_0 \in C_*$ and for every side of x_0 where C_* is accumulative, the corresponding one-sided derivative of $H_1 + H_2$ at x_0 vanishes. To see this, let $x_0 \in C_*$. We shall consider the case when C_* is accumulative to the right of x_0. The other case can be dealt with in a similar manner. Let $\varepsilon > 0$ and pick an integer $k \ge 2$ such that $1/k < \varepsilon$. The following notation will be used:

$$I_0 := \begin{cases} I_{i,j}, & \text{if } x_0 \in \partial I_{i,j} \text{ for some pair } i, j, \\ \varnothing, & \text{otherwise.} \end{cases}$$

Define $\eta := \operatorname{dist}(x_0, \bigcup_{i=1}^{k} \bigcup_{j=1}^{2^{i-1}} I_{i,j} \setminus I_0)$ and pick a point $x \in (0,1)$ such that $x_0 < x < x_0 + \eta$. If $x \in C_*$ then $H_1(x_0) = H_2(x_0) = H_1(x) = H_2(x) = 0$, in which case the difference quotient defined by the function $H_1 + H_2$ and the two points x_0 and x vanishes. Otherwise x belongs to some interval $I_{i,j}, i > k$, in which case $|\Delta(H_1 + H_2, [x_0, x])| = |H_1 + H_2|(x) = |\Delta(H_1 + H_2, [\inf I_{i,j}, x])|$, so that

$$\frac{|\Delta(H_1 + H_2, [x_0, x])|}{x - x_0} \le \frac{|\Delta(H_1 + H_2, [\inf I_{i,j}, x])|}{x - \inf I_{i,j}} \overset{(5.1)}{<} \frac{1}{i} < \frac{1}{k} < \varepsilon.$$

Thus we have proved that $D(H_1 + H_2)$ exists and vanishes on C_*.

With $C_{i,j}^{(k)} := m(K_{i,j}^{(k)})C + \min K_{i,j}^{(k)}$ and $B_{x,n} := (x - 1/n, x + 1/n)$ we define the sets

$$E_1 := \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} \bigcup_{k=1}^{r_{i,j}} C_{i,j}^{(k)} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} \bigcup_{k=1}^{r_{i,j}} \bigcup_{x \in C_{i,j}^{(k)}} B_{x,n} \cap (J_{i,j}^{(k)})^o;$$

$$E_2 := \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} \bigcup_{k=r_{i,j}+1}^{2r_{i,j}} C_{i,j}^{(k)} = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{2^{i-1}} \bigcup_{k=r_{i,j}+1}^{2r_{i,j}} \bigcup_{x \in C_{i,j}^{(k)}} B_{x,n} \cap (J_{i,j}^{(k)})^o.$$

Note that E_1 and E_2 are disjoint G_δ sets of measure zero; $D(H_1 + H_2)$ exists and is finite on $(0,1) \setminus (E_1 \cup E_2)$; $\overline{D}(H_1 + H_2) < \infty$ on E_1; and $\underline{D}(H_1 + H_2) > -\infty$ on E_2. By Theorem 4.1 we obtain two bounded, non-decreasing, regular primitives $L_1, L_2 \colon (0,1) \to \mathbb{R}$ such that $DL_1(x) = \infty$ ($DL_2(x) = \infty$) if and only if $x \in E_1$ ($x \in E_2$). Define the function

$$G := H_1 + H_2 - L_1 + L_2.$$

Then G is a continuous primitive.

Let $U \subset (0,1)$ be an open interval which contains a point of C and assume towards a contradiction that there exists a function $F \colon U \to \mathbb{R}$ which satisfies both the strong Luzin condition and $DF = DG$ a.e. on U. For convenience we shall assume that $\partial U \subset (0,1) \setminus C$, so that $C \cap U$ is perfect. We shall now argue that $F - (H_1 - L_1 + L_2)$ is constant on each interval $\overline{I_{i,j}} \cap U$. To see this, fix $I_{i,j}$. By Corollary 3.11 the function H_1 is a regular primitive on $I_{i,j} \cap U$ since it is finitely differentiable there. In addition L_1 and L_2 are two regular primitives such that $-DL_1 + DL_2$ is pointwise well-defined. Thus by Corollary 3.12 the function $H_1 - L_1 + L_2$ is a regular primitive on $I_{i,j} \cap U$. Moreover, since H_2 is singular we have $D(H_1 - L_1 + L_2) = DG = DF$ a.e. on $I_{i,j} \cap U$, so Theorem 3.9 guarantees that $F - (H_1 - L_1 + L_2)$ is constant on $I_{i,j} \cap U$. By continuity the same holds on $\overline{I_{i,j}} \cap U$. This fact will be used in a moment.

Finally, we endeavour to prove that $\nu(F, C \cap U) = \infty$, thereby obtaining the desired contradiction. Upon obtaining this contradiction we may infer that G is an irregular primitive since Theorem 3.10 states that all regular primitives satisfy the strong Luzin condition. Let δ be a gauge on Ω. For each positive integer n we define the set

$$C_n := \{x \in C \cap U \colon \delta(x) > 1/n\}.$$

Recall that $C \cap U$ is non-empty and perfect in addition to satisfying $C \cap U = \bigcup_{n=1}^{\infty} C_n$. Thus by the Baire category theorem there exists a positive integer n_0 and an open interval I such that $C \cap U \cap I \neq \varnothing$ and $C_{n_0} \cap U \cap I$ is dense in $C \cap U \cap I$. For convenience we shall assume that $I \subset U$. Let I_{i_0,j_0} be such that

$$m(I_{i_0,j_0}) < 1/n_0 \quad \text{and} \quad I_{i_0,j_0} + m(I_{i_0,j_0}) \subset I.$$

For each positive integer k we define $i_k := k + i_0$ and $j_k := 2^k j_0$, so that

$$I_{i_k,j_k} \subset I_{i_0,j_0} + m(I_{i_0,j_0}) \subset I.$$

Fix an arbitrary number $M > 0$. By the divergence of the harmonic series there exists a positive integer N such that

$$\sum_{k=1}^{N} \frac{1}{i_k} > M + \operatorname{osc}(L_1 + L_2, U) + 1.$$

Furthermore, let $\{J_k\}_{k=1}^{N}$ be a collection of non-overlapping compact intervals $J_k \subset I_{i_0,j_0} + m(I_{i_0,j_0})$ which possess the endpoints x_k, y_k, where $x_k \in C_{n_0}$ and y_k is the midpoint of I_{i_k,j_k}, in addition to satisfying $|\Delta(F, J_k \setminus I_{i_k,j_k})| < 1/2^k$. Then the collection of all pairs (x_k, J_k), which we denote by P, constitutes a tagged subpartition of U which is anchored in $C \cap U$ and subordinate to δ. Now, recall that $F - (H_1 - L_1 + L_2)$ is constant on each $\overline{I_{i,j}} \cap U$. Consequently

$$\begin{aligned}
\sum_{(x,J)\in P} |\Delta(F, J)| &> \sum_{k=1}^{N} \left(|\Delta(F, \overline{I_{i_k,j_k}} \cap J_k)| - \frac{1}{2^k} \right) \\
&> \sum_{k=1}^{N} |\Delta(H_1 - L_1 + L_2, \overline{I_{i_k,j_k}} \cap J_k)| - 1 \\
&\geq \sum_{k=1}^{N} (|\Delta(H_1, \overline{I_{i_k,j_k}} \cap J_k)| - \Delta(L_1 + L_2, \overline{I_{i_k,j_k}} \cap J_k)) - 1 \\
&\geq \sum_{k=1}^{N} \frac{1}{i_k} - \operatorname{osc}(L_1 + L_2, U) - 1 > M.
\end{aligned}$$

Since both δ and $M > 0$ were arbitrary, we conclude that $\nu(F, C \cap U) = \infty$. This completes the proof. $\square$

5.2 A characterization of the strong Luzin condition

Throughout this section we shall consider a compact subinterval $[a, b] \subset \Omega$ and a function $F\colon \Omega \to \mathbb{R}$.

Lemma 5.3. *Suppose that F is left-continuous at a and right-continuous at b. Then*

$$\nu(F, [a, b]) = \nu_0(F, [a, b]).$$

Proof. If $\nu_0(F, [a, b]) = \infty$ then the inequality $\nu(F, [a, b]) \geq \nu_0(F, [a, b])$ gives the desired result. We shall therefore assume that $\nu_0(F, [a, b]) < \infty$. Let $\varepsilon > 0$. By the continuity assumption there exist numbers $\eta_a, \eta_b > 0$ such that

$$|F(x) - F(a)| < \frac{\varepsilon}{5} \quad \text{for every } x \in (a - \eta_a, a] \cap \Omega$$

and

$$|F(x) - F(b)| < \frac{\varepsilon}{5} \quad \text{for every } x \in [b, b + \eta_b) \cap \Omega.$$

Define a constant gauge on Ω by $\delta_0(x) := \min\{\eta_a, \eta_b\}$ and let P_0 be a tagged subpartition of Ω which is anchored in $[a, b]$ and subordinate to δ_0. Then

$$\sum_{(x,I)\in P_0} |\Delta(F, I)| < \frac{2\varepsilon}{5} + \sum_{(x,I)\in P_0} |\Delta(F, I \cap [a, b])| \leq \frac{2\varepsilon}{5} + \nu_0(F, [a, b]) < \infty.$$

Thus $\nu(F, [a, b]) \leq \nu(F, [a, b], \delta_0) < \infty$, and so there exists a gauge δ_1 on Ω such that

$$\nu(F, [a, b], \delta_1) - \nu(F, [a, b]) < \frac{\varepsilon}{5}.$$

Let δ_2 be a gauge on Ω such that

$$\delta_2(x) < \begin{cases} \operatorname{dist}(x, \{a, b\}), & \text{if } x \in \Omega \setminus \{a, b\}; \\ b - a, & \text{if } x \in \{a, b\} \text{ and } a < b. \end{cases}$$

Define $\delta := \min\{\delta_1, \delta_2\}$ and let P be a tagged subpartition of Ω, which is anchored in $[a, b]$ and subordinate to δ, such that

$$\nu(F, [a, b], \delta) - \sum_{(x,I)\in P} |\Delta(F, I)| < \frac{\varepsilon}{5}.$$

We can assume that $x \in \partial I$ for each $(x, I) \in P$. Therefore, since the stipulated properties of δ_2 are inherited by δ, each $(x, I) \in P$ satisfies either $I \subset [a, b]$ or

$I \subset \Omega \setminus (a,b)$. Denote by P_{in} the subset of pairs $(x,I) \in P$ for which the former holds; and by P_{out} the subset of pairs $(x,I) \in P$ for which the latter holds. Then $P_{\text{in}} \cap P_{\text{out}} = \varnothing$ and $P = P_{\text{in}} \cup P_{\text{out}}$. Let Q be a partition of $[a,b]$ such that

$$\nu_0(F,[a,b]) - \sum_{I \in Q} |\Delta(F,I)| < \frac{\varepsilon}{5}.$$

Furthermore, let Q_{in} be the corresponding partition of P_{in} (i.e. Q_{in} consists of all intervals I which correspond to a pair $(x,I) \in P_{\text{in}}$) and let Q_{ref} be the common refinement of Q and Q_{in} (i.e. Q_{ref} consists of all intersections $I_1 \cap I_2$ for which $I_1 \in Q$ and $I_2 \in Q_{\text{in}}$). Then, for each $J \in Q_{\text{ref}}$, we use Cousin's Lemma 1.4 to obtain a tagged subpartition P_J of Ω, which is anchored in J and subordinate to δ, such that $J = \bigcup_{(x,I) \in P_J} I$. Finally, define $R := \bigcup_{J \in Q_{\text{ref}}} P_J$. Then we have

$$\begin{aligned}
|\nu(F,[a,b]) - \nu_0(F,[a,b])| &= \Bigg| \sum_{(x,I) \in P_{\text{out}}} |\Delta(F,I)| \\
&\qquad + \nu(F,[a,b]) - \nu(F,[a,b],\delta) \\
&\qquad + \nu(F,[a,b],\delta) - \sum_{(x,I) \in P_{\text{out}} \cup R} |\Delta(F,I)| \\
&\qquad + \sum_{(x,I) \in R} |\Delta(F,I)| - \nu_0(F,[a,b]) \Bigg| \\
&\le \sum_{(x,I) \in P_{\text{out}}} |\Delta(F,I)| \\
&\qquad + \nu(F,[a,b],\delta) - \nu(F,[a,b]) \\
&\qquad + \nu(F,[a,b],\delta) - \sum_{(x,I) \in P_{\text{out}} \cup R} |\Delta(F,I)| \\
&\qquad + \nu_0(F,[a,b]) - \sum_{(x,I) \in R} |\Delta(F,I)| \\
&\le \sum_{(x,I) \in P_{\text{out}}} |\Delta(F,I)| \\
&\qquad + \nu(F,[a,b],\delta_1) - \nu(F,[a,b]) \\
&\qquad + \nu(F,[a,b],\delta) - \sum_{(x,I) \in P} |\Delta(F,I)| \\
&\qquad + \nu_0(F,[a,b]) - \sum_{I \in Q} |\Delta(F,I)| \\
&< \frac{2\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} + \frac{\varepsilon}{5} = \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we obtain the desired equality. $\square$

Lemma 5.4. *Suppose that F is continuous on $[a,b]$ in addition to being constant on $(-\infty,a]\cap\Omega$ and $[b,\infty)\cap\Omega$, respectively. Let $\xi\in\Omega$. Then*[2]

$$V_F(\xi) = \nu_0(F,(-\infty,\xi]\cap[a,b]).$$

Proof. Define the sets

$$A := (-\infty,\xi]\cap\Omega \quad \text{and} \quad B := (-\infty,\xi]\cap[a,b].$$

We shall prove that $\nu(F,A)=\nu_0(F,B)$. Let δ_1 be a gauge on Ω such that

$$\delta_1(x) < \operatorname{dist}(x,\{a,\xi\}) \quad \text{for every } x\in\Omega\setminus\{a,\xi\}.$$

Consider first the case $\xi < a$. Then $\nu(F,A,\delta_1)=0$ since $x+\delta_1(x)<a$ for every $x\in A$, and so $V_F(\xi)=\nu(F,A)=0$. Moreover, $\nu_0(F,B)=\nu_0(F,\varnothing)=0$. Thus $V_F(\xi)=\nu_0(F,B)$.

Consider next the case $\xi\geq a$. By Lemma 5.3 we have $\nu(F,B)=\nu_0(F,B)$. It will therefore suffice to show that $\nu(F,A)=\nu(F,B)$. Clearly

$$\nu(F,A)\geq\nu(F,B), \quad \text{since } B\subset A.$$

If $\nu_0(F,B)=\infty$ then from the above remarks we infer that $\nu(F,A)\geq\nu(F,B)=\nu_0(F,B)=\infty$, so that $\nu(F,A)=\nu(F,B)$. Hence we assume that $\nu_0(F,B)<\infty$ and aim to establish the reverse inequality $\nu(F,A)\leq\nu(F,B)$. For every tagged subpartition P of Ω which is subordinate to δ_1 we have

$$\sum_{\substack{(x,I)\in P:\\ x\in A}} |\Delta(F,I)| = \sum_{\substack{(x,I)\in P:\\ x\in B}} |\Delta(F,B\cap I)| \leq \nu_0(F,B)<\infty.$$

Therefore $\nu(F,A,\delta)=\nu(F,B,\delta)<\infty$ for every gauge δ on Ω which is dominated by δ_1. In particular $\nu(F,A)$ and $\nu(F,B)$ are both finite. Let $\varepsilon>0$ and pick a gauge δ_2 on Ω such that

$$\nu(F,B,\delta_2)-\nu(F,B)<\varepsilon.$$

Finally, define $\delta:=\min\{\delta_1,\delta_2\}$. Then

$$\begin{aligned}
\nu(F,A)-\nu(F,B) &= \nu(F,A)-\nu(F,A,\delta)+\nu(F,B,\delta)-\nu(F,B)\\
&\leq \nu(F,B,\delta)-\nu(F,B)\\
&\leq \nu(F,B,\delta_2)-\nu(F,B)<\varepsilon.
\end{aligned}$$

[2]The total variation function V_F is defined in Section 1.2.

Since $\varepsilon > 0$ was arbitrary we have

$$\nu(F, A) \leq \nu(F, B),$$

which completes the proof. $\square$

Lemma 5.5. *Let* $[x, y] \subset [a, b]$. *Then*

$$\nu_0(F, [a, y]) = \nu_0(F, [a, x]) + \nu_0(F, [x, y]).$$

Proof. Let P be a partition of $[a, y]$ and define

$$\begin{aligned} P_1 &:= \{[u, v] \in P \colon v \leq x\} \cup \{[u, x] \colon [u, v] \in P \text{ and } u < x < v\}; \\ P_2 &:= \{[u, v] \in P \colon x \leq u\} \cup \{[x, v] \colon [u, v] \in P \text{ and } u < x < v\}. \end{aligned}$$

We have

$$\begin{aligned} \sum_{I \in P} |\Delta(F, I)| &\leq \sum_{I \in P_1} |\Delta(F, I)| + \sum_{I \in P_2} |\Delta(F, I)| \\ &\leq \nu_0(F, [a, x]) + \nu_0(F, [x, y]), \end{aligned}$$

and therefore

$$\nu_0(F, [a, y]) \leq \nu_0(F, [a, x]) + \nu_0(F, [x, y]). \tag{5.2}$$

If $\nu_0(F, [a, x]) = \infty$ or $\nu_0(F, [x, y]) = \infty$, then since $\nu_0(F, [a, y]) \geq \nu_0(F, [a, x])$ and $\nu_0(F, [a, y]) \geq \nu_0(F, [x, y])$, we must have $\nu_0(F, [a, y]) = \infty$, in which case we obtain the desired equality. Hence we assume that both $\nu_0(F, [a, x])$ and $\nu_0(F, [x, y])$ are finite. Let $\varepsilon > 0$ and let Q_1 and Q_2 be partitions of $[a, x]$ and $[x, y]$, respectively, such that

$$\nu_0(F, [a, x]) < \sum_{I \in Q_1} |\Delta(F, I)| + \frac{\varepsilon}{2}$$

and

$$\nu_0(F, [x, y]) < \sum_{I \in Q_2} |\Delta(F, I)| + \frac{\varepsilon}{2}.$$

Define $Q := Q_1 \cup Q_2$ and note that

$$\begin{aligned} \nu_0(F, [a, x]) + \nu_0(F, [x, y]) &< \sum_{I \in Q_1} |\Delta(F, I)| + \frac{\varepsilon}{2} + \sum_{I \in Q_2} |\Delta(F, I)| + \frac{\varepsilon}{2} \\ &= \sum_{I \in Q} |\Delta(F, I)| + \varepsilon \\ &\leq \nu_0(F, [a, y]) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary we conclude that

$$\nu_0(F, [a, x]) + \nu_0(F, [x, y]) \leq \nu_0(F, [a, y]). \tag{5.3}$$

From the inequalities (5.2) and (5.3) we obtain the desired equality. $\square$

Lemma 5.6. *Let $[x, y] \subset [a, b]$. Suppose that F is continuous on $[a, b]$, constant on $(-\infty, a] \cap \Omega$ and $[b, \infty) \cap \Omega$, respectively, and that $\nu_0(F, [a, b]) < \infty$. Then*

$$V_F(y) - V_F(x) = \nu_0(F, [x, y]) = \nu(F, [x, y]).$$

Proof. The first equality follows from Lemmas 5.4 and 5.5; whereas the second equality follows from Lemma 5.3. $\square$

Lemma 5.7. *Suppose that F is continuous on $[a, b]$ and that $\nu_0(F, [a, b]) < \infty$. Then for every $\varepsilon > 0$ there exists an $\eta > 0$ such that*

$$\sum_{I \in Q} |\Delta(F, I)| > \nu_0(F, [a, b]) - \varepsilon$$

for every partition Q of $[a, b]$ with $\mathrm{mesh}(Q) < \eta$.

Proof. If $a = b$ then any $\eta > 0$ will suffice, hence we assume that $a < b$. Let $\varepsilon > 0$. There exists a partition P of $[a, b]$, consisting of non-degenerate intervals and satisfying $\mathrm{mesh}(P) < b - a$, such that

$$\sum_{I \in P} |\Delta(F, I)| > \nu_0(F, [a, b]) - \frac{\varepsilon}{2}.$$

Define the set

$$W := \{w \colon w \in \{u, v\} \setminus \{a, b\} \text{ for some } [u, v] \in P\},$$

and let N denote the number of points in W. Since $\mathrm{mesh}(P) < b - a$ we have $N \geq 1$. Moreover, since F is in fact uniformly continuous on $[a, b]$ there exists an $\eta_0 > 0$ such that

$$|\Delta(F, I)| < \frac{\varepsilon}{4N}$$

for every compact subinterval $I \subset [a, b]$ with $m(I) < \eta_0$. Define

$$\eta := \min(\{m(I) \colon I \in P\} \cup \{\eta_0\}),$$

and let Q be a partition of $[a, b]$ with $\mathrm{mesh}(Q) < \eta$. Define the sets

$$\begin{aligned} Q_1 &:= \{[u, v] \in Q \colon W \cap (u, v) = \varnothing\}; \\ Q_2 &:= \{[u, w], [w, v] \colon [u, v] \in Q \text{ and } w \in W \cap (u, v)\}. \end{aligned}$$

Note that $Q_1 \cap Q_2 = \varnothing$; $Q_1 \cup Q_2$ constitutes a partition of $[a,b]$; for each interval $I \in P$ there exists a subcollection $Q_I \subset Q_1 \cup Q_2$ which constitutes a partition of I; and the number of intervals in Q_2 does not exceed $2N$. Therefore

$$\begin{aligned}
\sum_{I\in Q} |\Delta(F,I)| &\geq \sum_{I\in Q_1} |\Delta(F,I)| \\
&= \sum_{I\in Q_1\cup Q_2} |\Delta(F,I)| - \sum_{I\in Q_2} |\Delta(F,I)| \\
&\geq \sum_{I\in P} |\Delta(F,I)| - \sum_{I\in Q_2} |\Delta(F,I)| \\
&> \nu_0(F,[a,b]) - \frac{\varepsilon}{2} - 2N\frac{\varepsilon}{4N} \\
&= \nu_0(F,[a,b]) - \varepsilon.
\end{aligned}$$

□

Lemma 5.8. *Suppose that F is continuous on $[a,b]$, constant on $(-\infty,a]\cap\Omega$ and $[b,\infty)\cap\Omega$, respectively, and that $\nu_0(F,[a,b])<\infty$. Then V_F is finite and continuous.*

Proof. From Lemma 5.4 we infer that V_F is finite and constant on $(-\infty,a]\cap\Omega$ and $[b,\infty)\cap\Omega$, respectively. It remains to be shown that V_F is left-continuous on $(a,b]$ and right-continuous on $[a,b)$. Since F is in fact uniformly continuous on $[a,b]$ there exists for every $\varepsilon>0$ an $\eta_\varepsilon>0$ such that

$$|F(x)-F(y)| < \frac{\varepsilon}{2} \quad \text{for all } x,y\in[a,b] \text{ with } |x-y|<\eta_\varepsilon.$$

First we show that V_F is left-continuous on $(a,b]$. Let $x\in(a,b]$ and $\varepsilon>0$. Moreover, let P be a partition of $[a,x]$ such that

$$\nu_0(F,[a,x]) < \sum_{I\in P} |\Delta(F,I)| + \frac{\varepsilon}{2}.$$

We may assume that there exists an interval $[u,x]\in P$ with $x-\eta_\varepsilon<u<x$, so that

$$\nu_0(F,[a,x]) < \sum_{I\in P\setminus\{[u,x]\}} |\Delta(F,I)| + \varepsilon \leq \nu_0(F,[a,u]) + \varepsilon.$$

Thus by Lemma 5.5 we have

$$\nu_0(F,[u,x]) = \nu_0(F,[a,x]) - \nu_0(F,[a,u]) < \varepsilon.$$

For every $y\in[u,x]$ we use Lemma 5.6 to conclude that

$$V_F(x) - V_F(y) = \nu_0(F,[y,x]) \leq \nu_0(F,[u,x]) < \varepsilon.$$

This proves that V_F is left-continuous on $(a,b]$.

Next we show that F is right-continuous on $[a,b)$. Let $x \in [a,b)$ and $\varepsilon > 0$. Moreover, let P be a partition of $[x,b]$ such that

$$\nu_0(F,[x,b]) < \sum_{I\in P} |\Delta(F,I)| + \frac{\varepsilon}{2}.$$

We may assume that there exists an interval $[x,v] \in P$ with $x < v < x + \eta_\varepsilon$, so that

$$\nu_0(F,[x,b]) < \sum_{I\in P\setminus\{[x,v]\}} |\Delta(F,I)| + \varepsilon \le \nu_0(F,[v,b]) + \varepsilon.$$

Thus by Lemma 5.5 we have

$$\nu_0(F,[x,v]) = \nu_0(F,[x,b]) - \nu_0(F,[v,b]) < \varepsilon.$$

For every $y \in [x,v]$ we use Lemma 5.6 to conclude that

$$V_F(y) - V_F(x) = \nu_0(F,[x,y]) \le \nu_0(F,[x,v]) < \varepsilon.$$

This proves that V_F is right-continuous on $[a,b)$. □

Lemma 5.9. *Suppose that F is continuous, satisfies the Luzin condition (N), is constant on $(-\infty,a]\cap\Omega$ and $[b,\infty)\cap\Omega$, respectively, and that $\nu_0(F,[a,b]) < \infty$. Then V_F is finite, continuous and satisfies the Luzin condition (N).*

Proof. Lemma 5.8 states that V_F is finite and continuous. From Lemma 5.4 we infer that V_F is constant on $(-\infty,a]\cap\Omega$ and $[b,\infty)\cap\Omega$, respectively, and therefore satisfies the Luzin condition (N) on these sets. It remains to be shown that V_F satisfies the Luzin condition (N) on (a,b). Let $Z \subset (a,b)$ with $m(Z) = 0$ and let $\varepsilon > 0$. By Lemma 1.3 we may assume that for every $x \in Z$ the function V_F is non-constant in all neighbourhoods of x. Define the set

$$E := \{x \in Z \colon DF(x) = \pm\infty\}.$$

Moreover, for each positive integer j, let E_j be the set of points $x \in Z \setminus E$ for which j is the largest positive integer satisfying

$$|y| \ge j-1 \quad \text{for every } y \in [\underline{D}F(x), \overline{D}F(x)].$$

Note that $Z = E \cup \bigcup_{j=1}^\infty E_j$. For every $x \in Z$ and $\eta > 0$ let $I_{x,\eta}$ be a compact interval with

$$x \in I_{x,\eta} \subset (a,b) \quad \text{and} \quad 0 < m(V_F(I_{x,\eta})) < \eta.$$

In addition, if $x \in E_j$ then we require that

$$|\Delta(F, I_{x,\eta})| < jm(I_{x,\eta}). \tag{5.4}$$

Here we used the fact that since F is continuous, the derived numbers of F at x form the interval $[\underline{D}F(x), \overline{D}F(x)]$.[3] For each j there exists an open set U_j with

$$E_j \subset U_j \subset (a,b) \quad \text{and} \quad m(U_j) < \frac{\varepsilon}{j2^{j+1}}. \tag{5.5}$$

Let δ_1 be a gauge on Ω such that

$$(x - \delta_1(x), x + \delta_1(x)) \subset U_j \quad \text{for every } x \in E_j. \tag{5.6}$$

By Lemma 3.7 there exists a gauge δ_2 on Ω such that

$$\sum_{(x,I)\in P} |\Delta(F, I)| < \frac{\varepsilon}{4} \tag{5.7}$$

for every tagged subpartition P of Ω which is anchored in E and subordinate to δ_2. By Lemma 5.7 there exists an $\eta_0 > 0$ such that

$$\nu_0(F, [a,b]) - \sum_{I\in Q} |\Delta(F, I)| < \frac{\varepsilon}{4}$$

for every partition Q of $[a,b]$ with $\mathrm{mesh}(Q) < \eta_0$. Note that if Q_0 is a subpartition of $[a,b]$ with $\mathrm{mesh}(Q_0) < \eta_0$, and Q is a partition of $[a,b]$ with $Q_0 \subset Q$ and $\mathrm{mesh}(Q) < \eta_0$, then we use Lemma 5.5 to infer that

$$\begin{aligned}\sum_{I\in Q_0} (\nu_0(F, I) - |\Delta(F, I)|) &\le \sum_{I\in Q} (\nu_0(F, I) - |\Delta(F, I)|) \\ &= \nu_0(F, [a,b]) - \sum_{I\in Q} |\Delta(F, I)| < \frac{\varepsilon}{4}.\end{aligned}$$

Let δ_3 be a gauge on Ω with $\delta_3(x) < \eta_0/2$ for every $x \in [a,b]$. A consequence of the above is that if P is a tagged subpartition of Ω which is subordinate to δ_3 and has the property that $I \subset [a,b]$ for each $(x, I) \in P$, then

$$\sum_{(x,I)\in P} (\nu_0(F, I) - |\Delta(F, I)|) < \frac{\varepsilon}{4}. \tag{5.8}$$

[3]This concerns primarily the case when $j = 1$ and $\underline{D}F(x) \le -1 < 1 \le \overline{D}F(x)$ because then the extreme derivatives do not guarantee that the additional requirement (5.4) can be satisfied. In this particular case we use the fact that every point of $(-1, 1)$ constitutes a derived number of F at x. (Derived numbers are defined at the end of Section 1.3.)

Now, define $\delta := \min\{\delta_1, \delta_2, \delta_3\}$ and

$$\mathcal{V} := \{V_F(I_{x,\eta})\colon x \in Z, \eta > 0 \text{ and } I_{x,\eta} \subset (x - \delta(x), x + \delta(x))\}.$$

Note that $\mathcal{V}$ is a Vitali cover of $V_F(Z)$. Thus by the Vitali covering theorem there exists a countable subcollection $\{J_k\}_k \subset \mathcal{V}$ of strictly non-overlapping intervals, with positive integer indices k, such that

$$m\Big(V_F(Z) \setminus \bigcup_k J_k\Big) = 0. \tag{5.9}$$

For each k, let $I_k := I_{x_k,\eta_k}$ denote an interval by which J_k was defined above. Then, if J_{k_1} and J_{k_2} are distinct, we must have

$$I_{k_1} \cap I_{k_2} = \varnothing \quad \text{since} \quad V_F(I_{k_1}) \cap V_F(I_{k_2}) = J_{k_1} \cap J_{k_2} = \varnothing.$$

For each positive integer n we denote by P_n the subcollection of pairs (x_k, I_k) with $k \le n$. Then P_n constitutes a tagged subpartition of Ω which is anchored in Z, subordinate to δ, and has the property that $I \subset [a, b]$ for each $(x, I) \in P_n$. Therefore, since $m(J_k) = m(V_F(I_k)) = \Delta(V_F, I_k) = \nu_0(F, I_k)$, we have

$$\begin{aligned}
m\Big(\bigcup_{k\le n} J_k\Big) &= \sum_{(x,I)\in P_n} \nu_0(F, I) \\
&= \sum_{(x,I)\in P_n} (\nu_0(F, I) - |\Delta(F, I)|) + \sum_{(x,I)\in P_n} |\Delta(F, I)| \\
&\overset{(5.8)}{<} \frac{\varepsilon}{4} + \sum_{\substack{(x,I)\in P_n:\\ x\in E}} |\Delta(F, I)| + \sum_{j=1}^{\infty} \sum_{\substack{(x,I)\in P_n:\\ x\in E_j}} |\Delta(F, I)| \\
&\overset{(5.7)}{<} \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} \sum_{\substack{(x,I)\in P_n:\\ x\in E_j}} |\Delta(F, I)| \\
&\overset{(5.4)}{\le} \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} j \sum_{\substack{(x,I)\in P_n:\\ x\in E_j}} m(I) \\
&\overset{(5.6)}{\le} \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} j m(U_j) \\
&\overset{(5.5)}{<} \frac{\varepsilon}{2} + \sum_{j=1}^{\infty} j \frac{\varepsilon}{j2^{j+1}} = \varepsilon.
\end{aligned}$$

Consequently

$$m\Big(\bigcup_k J_k\Big) = \lim_{n\to\infty} m\Big(\bigcup_{k\le n} J_k\Big) \le \varepsilon.$$

From (5.9) and the above we infer that

$$m^*(V_F(Z)) \le m\Big(V_F(Z) \setminus \bigcup_k J_k\Big) + m\Big(\bigcup_k J_k\Big) \le \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary we conclude that $m(V_F(Z)) = 0$. $\square$

Lemma 5.10. *Suppose that F satisfies the strong Luzin condition. Let $Z \subset (a,b)$ be a set for which there exist a number $M > 0$ and a gauge δ on Ω such that $\delta(x) > b - a$ for every $x \in Z$, and*

$$\sum_{(x,[u,v])\in P} \operatorname{osc}(F,[u,v]) < M$$

for every tagged subpartition P of Ω which is anchored in Z and subordinate to δ. Then there exists a function $G\colon \Omega \to \mathbb{R}$ with the following properties:

(i) *G is constant on $(-\infty, a] \cap \Omega$ and $[b,\infty) \cap \Omega$, respectively;*

(ii) *G is continuous and satisfies the Luzin condition (N);*

(iii) *$|\Delta(F,[u,v])| \le \nu_0(G,[u,v])$ for every compact subinterval $[u,v] \subset [a,b]$ with at least one endpoint in $\{a,b\} \cup \overline{Z}$;*

(iv) *$\nu_0(G,[a,b]) < \infty$.*

Proof. Define the function $G\colon \Omega \to \mathbb{R}$ by letting $G := F$ on $\{a,b\} \cup \overline{Z}$; then $G := F(a)$ on $(-\infty,a) \cap \Omega$ and $G := F(b)$ on $(b,\infty) \cap \Omega$; and finally, on each open subinterval $(c^{(j)}, d^{(j)}) \subset (a,b)$ which is contiguous to $\{a,b\} \cup \overline{Z}$, define G as follows: Let $\{x_k^{(j)}\}_{k=-\infty}^{\infty}$ be a strictly increasing sequence of points such that

$$(c^{(j)}, d^{(j)}) = \bigcup_{k=-\infty}^{\infty} [x_k^{(j)}, x_{k+1}^{(j)}).$$

For each non-zero integer k we define

$$E_k^{(j)} := [c^{(j)}, x_{1-|k|}^{(j)}] \cup [x_{|k|-1}^{(j)}, d^{(j)}] \quad \text{and} \quad Y_k^{(j)} := m(F(E_k^{(j)})).$$

Note that $Y_{-k}^{(j)} = Y_k^{(j)}$ and $Y_k^{(j)} \to 0$ as $|k| \to \infty$. Now, for every $x \in (c^{(j)}, d^{(j)})$ we define

$$G(x) := \begin{cases} \dfrac{Y_k^{(j)} - Y_{k-1}^{(j)}}{x_k^{(j)} - x_{k-1}^{(j)}}(x - x_{k-1}^{(j)}) + F(c^{(j)}) + Y_{k-1}^{(j)}, & \text{if } x_{k-1}^{(j)} \le x < x_k^{(j)} \le x_{-1}^{(j)}; \\ \dfrac{F(d^{(j)}) - F(c^{(j)})}{x_1^{(j)} - x_{-1}^{(j)}}(x - x_{-1}^{(j)}) + F(c^{(j)}) + Y_1^{(j)}, & \text{if } x_{-1}^{(j)} \le x < x_1^{(j)}; \\ \dfrac{Y_{k+1}^{(j)} - Y_k^{(j)}}{x_{k+1}^{(j)} - x_k^{(j)}}(x - x_k^{(j)}) + F(d^{(j)}) + Y_k^{(j)}, & \text{if } x_1^{(j)} \le x_k^{(j)} \le x < x_{k+1}^{(j)}. \end{cases}$$

Next we shall establish a result which is required in order to prove (i)–(iv). Let P be a partition of some interval $[c^{(j)}, d^{(j)}]$. Note that G is non-decreasing on $[c^{(j)}, x_{-1}^{(j)}]$; monotone on $[x_{-1}^{(j)}, x_1^{(j)}]$; and non-increasing on $[x_1^{(j)}, d^{(j)}]$. Therefore

$$\begin{aligned} \sum_{[u,v]\in P} |\Delta(G,[u,v])| &\le \sum_{[u,v]\in P} \Delta(G,[u,v]\cap[c^{(j)}, x_{-1}^{(j)}]) \\ &\quad + \sum_{[u,v]\in P} |\Delta(G,[u,v]\cap[x_{-1}^{(j)}, x_1^{(j)}])| \\ &\quad + \sum_{[u,v]\in P} |\Delta(G,[u,v]\cap[x_1^{(j)}, d^{(j)}])| \\ &\le \Delta(G,[c^{(j)}, x_{-1}^{(j)}]) \\ &\quad + |\Delta(G,[x_{-1}^{(j)}, x_1^{(j)}])| \\ &\quad + |\Delta(G,[x_1^{(j)}, d^{(j)}])| \\ &\le 3\operatorname{osc}(F,[c^{(j)}, d^{(j)}]), \end{aligned}$$

so that

$$\nu_0(G,[c^{(j)}, d^{(j)}]) \le 3\operatorname{osc}(F,[c^{(j)}, d^{(j)}]) = 3Y_1. \tag{5.10}$$

We shall now verify (i)–(iv). Note that (i) holds by the definition of G. By Theorem 3.5 and the fact that $F = G$ on $\{a,b\} \cup \overline{Z}$ it follows that G satisfies the Luzin condition (N) and is continuous relative to $\{a,b\} \cup \overline{Z}$.[4] Moreover, G is continuous on each $(c^{(j)}, d^{(j)})$, right-continuous at $x = c^{(j)}$ and left-continuous at $x = d^{(j)}$. In order to conclude that condition (ii) has been met it remains to be

[4] Note that the restriction of G to each $[x_k^{(j)}, x_{k+1}^{(j)}]$ is affine. Thus if $G(x) = Ax + B$ for every $x \in [x_k^{(j)}, x_{k+1}^{(j)}]$, and if $N \subset [x_k^{(j)}, x_{k+1}^{(j)}]$ with $m(N) = 0$, then $m(G(N)) = |A|m(N) = 0$.

shown that G is continuous on $\{a,b\}\cup\overline{Z}$. Let $x_0 \in \{a,b\}\cup\overline{Z}$ and $\varepsilon>0$. By the previous remarks it will suffice to show that G is continuous on each side of x_0 where $\{a,b\}\cup\overline{Z}$ accumulative. For convenience we assume that $\{a,b\}\cup\overline{Z}$ is accumulative to the left of x_0 and aim to show that G is left-continuous at x_0. The case when G is accumulative to the right of x_0 can be dealt with in a similar manner. We have already established that G is continuous relative to $\{a,b\}\cup\overline{Z}$, so there exists some $\eta_1>0$ such that

$$|\Delta(G,[x,x_0])| < \frac{\varepsilon}{2} \quad \text{whenever } x_0-\eta_1 < x \le x_0 \text{ and } x \in \{a,b\}\cup\overline{Z}. \tag{5.11}$$

Moreover, since F is continuous by Theorem 3.5 and $\{a,b\}\cup\overline{Z}$ is accumulative to the left of x_0 by assumption, there exists some $\eta_2>0$ possessing the property that $\operatorname{osc}(F,[c^{(j)},d^{(j)}])<\varepsilon/6$ whenever $x_0-\eta_2<x\le x_0$ and $x\in(c^{(j)},d^{(j)})$ for some j. Thus by (5.10) we have

$$|\Delta(G,[x,d^{(j)}])| < \frac{\varepsilon}{2} \quad \text{whenever } x_0-\eta_2 < x \le x_0 \text{ and } x \in (c^{(j)},d^{(j)}). \tag{5.12}$$

Let $\eta:=\min\{\eta_1,\eta_2\}$ and $x\in[a,b]$ with $x_0-\eta<x\le x_0$. If $x\in\{a,b\}\cup\overline{Z}$, then (5.11) implies that $|\Delta(G,[x,x_0])|<\varepsilon/2$. Otherwise $x\in(c^{(j)},d^{(j)})$ for some j, in which case (5.11) and (5.12) imply that

$$|\Delta(G,[x,x_0])| = |\Delta(G,[x,d^{(j)}])| + |\Delta(G,[d^{(j)},x_0])| < \frac{\varepsilon}{2}+\frac{\varepsilon}{2} = \varepsilon.$$

This proves that G is left-continuous at x_0. Thus condition (ii) has been met.

To prove (iii), let $[u,v]$ be a compact subinterval of $[a,b]$ which has at least one endpoint in $\{a,b\}\cup\overline{Z}$. If $[u,v]$ has both of its endpoints in $\{a,b\}\cup\overline{Z}$, then the claim follows immediately from the fact that $F=G$ on $\{a,b\}\cup\overline{Z}$. That is,

$$|\Delta(F,[u,v])| = |\Delta(G,[u,v])| \le \nu_0(G,[u,v]).$$

We shall therefore assume that $[u,v]$ has precisely one endpoint in $\{a,b\}\cup\overline{Z}$, in which case we can write $[u,v]=[u_1,v_1]\cup[u_2,v_2]$, where $(u_1,v_1)\cap(u_2,v_2)=\varnothing$, $[u_1,v_1]\subset[c^{(j)},d^{(j)}]$ for some j such that $\{u_1,v_1\}\cap\{c^{(j)},d^{(j)}\}$ contains precisely one point, and $u_2,v_2\in\{a,b\}\cup\overline{Z}$. Note that

$$|\Delta(F,[u_1,v_1])| \le \begin{cases} \Delta(G,[c^{(j)},\min\{v_1,x_{-1}^{(j)}\}]), & \text{if } u_1=c^{(j)}; \\ |\Delta(G,[\max\{u_1,x_1^{(j)}\},d^{(j)}])|, & \text{if } v_1=d^{(j)}. \end{cases}$$

In particular $|\Delta(F,[u_1,v_1])|\le\nu_0(G,[u_1,v_1])$. Since $F=G$ on $\{a,b\}\cup\overline{Z}$ we have $|\Delta(F,[u_2,v_2])|\le\nu_0(G,[u_2,v_2])$. Thus by Lemma 5.5 we obtain the inequality

$$\begin{aligned} |\Delta(F,[u,v])| &\le |\Delta(F,[u_1,v_1])| + |\Delta(F,[u_2,v_2])| \\ &\le \nu_0(G,[u_1,v_1]) + \nu_0(G,[u_2,v_2]) = \nu_0(G,[u,v]), \end{aligned}$$

which proves (iii).

We can now prove (iv). Let Q be a partition of $[a,b]$. We aim to show that $\sum_{[u,v]\in Q} |\Delta(G,[u,v])| \leq B$ for some constant $B \geq 0$ which is independent of Q. Thus it will suffice to consider the case when $a < b$ and every interval of Q is non-degenerate. If $Z = \varnothing$ then by (5.10) it will suffice to take $B := 3Y_1$ since $[a,b]$ is the only interval of type $[c^{(j)}, d^{(j)}]$. Hence we shall assume that $Z \neq \varnothing$, so that each interval $[c^{(j)}, d^{(j)}]$ has at least one endpoint in $\overline{Z}$. Note that every interval of Q constitutes a subinterval of some $[c^{(j)}, d^{(j)}]$ or else contains a point of Z. Thus we let Q_1 consist of all intervals of Q which constitute a subinterval of some $[c^{(j)}, d^{(j)}]$, and let Q_2 consist of the remaining intervals of Q. Furthermore, we let R consist of all intervals $[c^{(j)}, d^{(j)}]$ which contain an interval of Q_1. For convenience we assume the intervals of R to be indexed in ascending order with unit increments. The intervals $[c^{(j)}, d^{(j)}] \in R$ with odd (even) indices can be expanded by a minuscule amount such that the resulting intervals $[C^{(j)}, D^{(j)}]$ contain a point of Z and remain non-overlapping.[5] Now, by hypothesis we have $\delta(x) > b - a$ for every $x \in Z$, and consequently a subpartition of $[a,b]$ consisting of intervals whose intersection with Z is non-empty can be extended to a tagged subpartition of $[a,b]$ which is anchored in Z and subordinate to δ. From this in addition to (5.10) we obtain the following:

$$
\begin{aligned}
\sum_{[u,v]\in Q} |\Delta(G,[u,v])| &= \sum_{[u,v]\in Q_1} |\Delta(G,[u,v])| + \sum_{[u,v]\in Q_2} |\Delta(G,[u,v])| \\
&< \sum_{[c^{(j)},d^{(j)}]\in R} \nu_0(G,[c^{(j)}, d^{(j)}]) + M \\
&\leq 3 \sum_{[c^{(j)},d^{(j)}]\in R} \operatorname{osc}(F,[c^{(j)}, d^{(j)}]) + M \\
&\leq 3 \sum_{\substack{[c^{(j)},d^{(j)}]\in R:\\ j \text{ odd}}} \operatorname{osc}(F,[C^{(j)}, D^{(j)}]) \\
&\quad + 3 \sum_{\substack{[c^{(j)},d^{(j)}]\in R:\\ j \text{ even}}} \operatorname{osc}(F,[C^{(j)}, D^{(j)}]) + M \\
&< 7M.
\end{aligned}
$$

That is, $\nu_0(G,[a,b]) \leq 7M < \infty$, and so (iv) holds. □

[5] For instance, suppose the expansion of an interval $[c^{(j)}, d^{(j)}] \in R$ with an odd (even) index results in the interval $[C^{(j)}, D^{(j)}]$. Then it will suffice to impose upon the quantities $c^{(j)} - C^{(j)}$ and $D^{(j)} - d^{(j)}$ the condition that they do not exceed half of the minimum distance between $[c^{(j)}, d^{(j)}]$ and the remaining intervals of R with odd (even) indices.

Proof of Theorem 5.2. The reverse direction follows from Corollary 3.12, therefore we shall prove the forward direction. Since the case $\Omega = \varnothing$ is trivial, we shall assume henceforth that $\Omega \neq \varnothing$. Define the set

$$Z := \{x \in \Omega \colon \underline{D}F(x) = -\infty\}.$$

Theorems 2.1 and 2.2 guarantee that $m(Z) = 0$ and since F satisfies the strong Luzin condition we infer that $\nu(F, Z) = 0$. Thus by Lemma 3.3 there exists a gauge δ on Ω such that

$$\sum_{(x,I)\in P} \operatorname{osc}(F, I) < 1$$

for every tagged subpartition P of Ω which is anchored in Z and subordinate to δ. For each positive integer i let $\{[a_{i,j}, b_{i,j}]\}_{j=1}^{\infty}$ be a collection of compact intervals such that

$$a_{i,j}, b_{i,j} \in \Omega \setminus Z, \quad 0 < b_{i,j} - a_{i,j} < 1/i \quad \text{and} \quad \Omega = \bigcup_{j=1}^{\infty} [a_{i,j}, b_{i,j}].$$

Define the sets

$$Z_{i,j} := \{x \in (a_{i,j}, b_{i,j}) \cap Z \colon \delta(x) > 1/i\},$$

and note that $Z = \bigcup_{i,j} Z_{i,j}$.

For each pair i, j we use Lemma 5.10 to obtain a function $F_{i,j} \colon \Omega \to \mathbb{R}$ with the following properties:

(i) $F_{i,j}$ is constant on $(-\infty, a_{i,j}] \cap \Omega$ and $[b_{i,j}, \infty) \cap \Omega$, respectively;

(ii) $F_{i,j}$ is continuous and satisfies the Luzin condition (N);

(iii) $|\Delta(F, I)| \leq \nu_0(F_{i,j}, I)$ for every compact subinterval $I \subset [a_{i,j}, b_{i,j}]$ with at least one endpoint in $\{a_{i,j}, b_{i,j}\} \cup \overline{Z_{i,j}}$;

(iv) $\nu_0(F_{i,j}, [a_{i,j}, b_{i,j}]) < \infty$.

By (i), (ii), (iv) and Lemma 5.9 it follows that $V_{F_{i,j}}$ is finite, continuous and satisfies the Luzin condition (N). Moreover, by (iii), Lemma 5.6 and the fact that $Z_{i,j} \subset (a_{i,j}, b_{i,j})$, we have

$$\liminf_{\substack{m(I)\to 0,\\ x\in\partial I}} \frac{\Delta(V_{F_{i,j}}, I) - |\Delta(F, I)|}{m(I)} \geq 0 \quad \text{for every } x \in Z_{i,j}. \tag{5.13}$$

For each pair i, j let $\{U_{i,j}^{(k)}\}_{k=1}^{\infty}$ be a collection of open intervals such that

$$U_{i,j}^{(k)} \cap V_{F_{i,j}}(\Omega) \neq \varnothing, \quad V_{F_{i,j}}(Z_{i,j}) \subset \bigcup_{k=1}^{\infty} U_{i,j}^{(k)} \quad \text{and} \quad \sum_{k=1}^{\infty} m(U_{i,j}^{(k)}) < \frac{1}{2^{i+j}}.$$

With the notation $(c_{i,j}^{(k)}, d_{i,j}^{(k)}) := (a_{i,j}, b_{i,j}) \cap V_{F_{i,j}}^{-1}(U_{i,j}^{(k)})$ we define the functions $L_{i,j}^{(k)} \colon \Omega \to \mathbb{R}$ by

$$L_{i,j}^{(k)}(x) := \begin{cases} 0, & \text{if } x \in (-\infty, c_{i,j}^{(k)}] \cap \Omega; \\ V_{F_{i,j}}(x) - V_{F_{i,j}}(c_{i,j}^{(k)}), & \text{if } x \in (c_{i,j}^{(k)}, d_{i,j}^{(k)}); \\ V_{F_{i,j}}(d_{i,j}^{(k)}) - V_{F_{i,j}}(c_{i,j}^{(k)}), & \text{if } x \in [d_{i,j}^{(k)}, \infty) \cap \Omega. \end{cases}$$

Note that each $L_{i,j}^{(k)}$ is non-negative, non-decreasing, continuous and satisfies the Luzin condition (N). Moreover, we have

$$\sum_{i,j,k=1}^{\infty} L_{i,j}(c_{i,j}^{(k)}) \leq \sum_{i,j,k=1}^{\infty} m(U_{i,j}^{(k)}) < \sum_{i,j=1}^{\infty} \frac{1}{2^{i+j}} = 1. \tag{5.14}$$

Now, define the functions $G, H, L \colon \Omega \to \mathbb{R}$ by

$$L := \sum_{i,j,k=1}^{\infty} L_{i,j}^{(k)}, \quad G := L + F \quad \text{and} \quad H := -L.$$

From (5.14) and Lemma 3.8 we infer that L is non-decreasing and satisfies the strong Luzin condition. Since G and H evidently satisfy the strong Luzin condition as well, Theorem 2.2 guarantees that both functions are differentiable a.e. on Ω. Next we show that

$$\underline{D}G(x) > -\infty \quad \text{and} \quad \overline{D}H(x) \leq 0 \quad \text{for every } x \in \Omega. \tag{5.15}$$

The latter inequality follows from the fact that H is non-increasing. In order to establish the former inequality we first note that for every $x \in Z_{i,j} \cap (c_{i,j}^{(k)}, d_{i,j}^{(k)})$ we can use (5.13) to infer that

$$\begin{aligned} \underline{D}G(x) &= \liminf_{\substack{m(I)\to 0, \\ x\in\partial I}} \frac{\Delta(G,I)}{m(I)} \\ &= \liminf_{\substack{m(I)\to 0, \\ x\in\partial I}} \frac{\Delta(L,I) + \Delta(F,I)}{m(I)} \\ &\geq \liminf_{\substack{m(I)\to 0, \\ x\in\partial I}} \frac{\Delta(L_{i,j}^{(k)},I) + \Delta(F,I)}{m(I)} \\ &\geq \liminf_{\substack{m(I)\to 0, \\ x\in\partial I}} \frac{\Delta(V_{F_{i,j}},I) - |\Delta(F,I)|}{m(I)} \geq 0. \end{aligned}$$

Moreover, for every $x \in \Omega \setminus Z$ we have

$$\underline{D}G(x) = \liminf_{\substack{m(I)\to 0,\\ x\in\partial I}} \frac{\Delta(G,I)}{m(I)} \geq \liminf_{\substack{m(I)\to 0,\\ x\in\partial I}} \frac{\Delta(F,I)}{m(I)} = \underline{D}F(x) > -\infty.$$

Denote by Z_1 the set of points $x \in \Omega$ at which F and G do not both possess derivatives (finite or infinite). Since Ω is open and $m(Z_1) = 0$ the regularity of the Lebesgue measure yields a G_δ set Z_2 such that $Z_1 \subset Z_2 \subset \Omega$ and $m(Z_2) = 0$. By Theorem 4.1 there exists a non-decreasing regular primitive $F_0 \colon \Omega \to \mathbb{R}$ such that $DF_0(x) = \infty$ for every $x \in Z_2$. Note that $G+F_0$ and $H-F_0$ both constitute regular primitives by (5.15) and Theorem 3.10. Since $F = (G+F_0)+(H-F_0)$, the proof is complete. $\square$

Bibliography

[1] V. S. Bogomolova, On a class of everywhere asymptotically continuous functions, *Sb. Math.*, 32 (1924), 152–171. (Russian)

[2] B. Bongiorno, V. A. Skvortsov and L. Di Piazza, A new full descriptive characterization of Denjoy–Perron integral, *Real Anal. Exchange*, 21:2 (1995/96), 656–663.

[3] A. M. Bruckner, *Differentiation of real functions*, CRM Monograph Series 5, Amer. Math. Soc., Providence, RI, 1994.

[4] A. M. Bruckner and J. L. Leonard, Derivatives, *Amer. Math. Monthly*, 73:4 (1966), part 2, 24–56.

[5] G. Choquet, Application des propriétés descriptives de la fonction contingent á la théorie des fonctions de variable réelle et á la géométrie différentielle des variétés cartésiennes (thesis), *J. Math. Pures Appl.* (9), 26 (1947), 115–226.

[6] J. A. Clarkson, A property of derivatives, *Bull. Amer. Math. Soc.*, 53:2 (1947), 124–125.

[7] A. Denjoy, Sur une propriété des fonctions dérivées, *Enseign. Math.*, 18 (1916), 320–328.

[8] C.-A. Faure, A short proof of Lebesgue's density theorem, *Amer. Math. Monthly*, 109:2 (2002), 194–196.

[9] C. Goffman, C. Neugebauer and T. Nishiura, Density topology and approximate continuity, *Duke Math. J.*, 28:4 (1961), 497–506.

[10] G. Goldowsky, Note sur les dérivées exactes, *Sb. Math.*, 35 (1928), 35–36.

[11] H. Hahn, Über den Fundamentalsatz der Integralrechnung, *Monatsh. Math. Phys.*, 16 (1905), 161–166.

[12] P. Y. Lee, On ACG* functions, *Real Anal. Exchange*, 15:2 (1989/90), 754–759.

[13] G. Piranian, The derivative of a monotonic discontinuous function, *Proc. Amer. Math. Soc.,* 16:2 (1965), 243–244.

[14] S. Saks, *Theory of the integral*, 2nd rev. ed., Hafner, New York, 1937.

[15] L. Tonelli, Sulle derivate esatte, *Mem. Accad. Sci. Ist. Bologna* (8), 8 (1930/31), 13–15.

[16] W. H. Young, On the infinite derivatives of a function of a single real variable, *Ark. Mat. Astr. Fys.*, 1 (1903), 201–204.

[17] Z. Zahorski, Über die Menge der Punkte in welchen die Ableitung unendlich ist, *Tôhoku Math. J.*, 48 (1941), 321–330.

[18] ——, Sur la première dérivée, *Trans. Amer. Math. Soc.*, 69 (1950), 1–54.