The History of the Dirichlet Problem for Laplace's Equation

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Abstract

This thesis aims to provide an introduction to the field of potential theory at an undergraduate level, by studying an important mathematical problem in the field, namely the Dirichlet problem. By examining the historical development of different methods for solving the problem in increasingly general contexts, and the mathematical concepts which were established to support these methods, the aim is to provide an overview of various basic techniques in the field of potential theory, as well as a summary of the fundamental results concerning the Dirichlet problem in Euclidean space.

Keywords:
Potential Theory, Dirichlet Problem, Laplace's Equation, Complex Analysis

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Sammanfattning

Målet med detta arbete är att vara en introduktion på kandidatnivå till ämnesfältet potentialteori, genom att studera ett viktigt matematiskt problem inom potentialteori, Dirichletproblemet. Genom att undersöka den historiska utvecklingen av olika lösningsmetoder för problemet i mer och mer allmänna sammanhang, i kombination med de matematiska koncepten som utvecklades för att användas i dessa lösningsmetoder, ges en översikt av olika grundläggande tekniker inom potentialteori, samt en sammanfattaning av de olika matematiska resultaten som har att göra med Dirichletproblemet i det Euklidiska rummet.

Nyckelord:
Potentialteori, Dirichletproblemet, Laplaces ekvation, komplex analys

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Nomenclature

Symbols and notation

The table below gives a summary of the notation used in this text. Some of the more complex notation and definitions are elaborated on in Chapter 2.

$\mathbb{N}$	The natural numbers, excluding zero.
$\mathbb{Z}$	The integers.
$\mathbb{Q}$	The rational numbers.
$\mathbb{R}$	The real numbers.
$[-\infty, \infty]$	The extended real numbers $\mathbb{R} \cup \{-\infty, \infty\}$.
$\overline{A}$	The closure of the set A.
∂A	The boundary of the set A.
A^c	The complement of the set A, defined as $\mathbb{R}^n \setminus A$.
A^e	The exterior of the set A, defined as $\overline{A}^c$.
$A \Subset E$	The condition that $\overline{A}$ is a compact subset of E.
B	An open ball in $\mathbb{R}^n$.
$B(x, \rho)$	The open ball with center $x \in \mathbb{R}^n$ and radius $\rho > 0$.
σ_n	The surface area of a sphere in n dimensions with radius 1.
ν_n	The volume of an n-dimensional ball with radius 1.
$\widehat{n}$	The outer unit normal to a surface.
χ_A	The characteristic (indicator) function for the set A.
$\mathcal{M}(u; x, \rho)$	The spherical mean value of u over $\partial B(x, \rho)$.
$\mathcal{A}(u; x, \rho$	The ball mean value of u in $B(x, \rho$.
$\mathcal{C}(A)$	The space of continuous functions from A to $\mathbb{R}$.
$\mathcal{C}^k(A)$	The space of k times continuously differentiable functions from A to $\mathbb{R}$.

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Chapter 1

Introduction

Two important equations in potential theory are Laplace's equation

$$\Delta u = 0$$

and Poisson's equation

$$\Delta u = f,$$

where the Laplacian is defined by

$$\Delta u(x) = \nabla \cdot \nabla u(x) = \sum_{j=1}^{n} \frac{\partial^2 u}{\partial^2 x_j}(x),$$

for a function $u\colon \mathbb{R}^n \to \mathbb{R}$. These two differential equations, the former being a special case of the latter, appear in many areas of physics. For example Newton's law of gravity can be described as the gravitational field caused by a point mass m at the origin

$$g(x) = -G\frac{m}{|x|^2}\frac{x}{|x|}.$$

This is the gradient $g = -\nabla P_g$ of a potential P_g, which obeys Laplace's equation

$$\Delta P_g(x) = 0,$$

for all $x \neq 0$. For a more general mass distribution ρ_m, it holds that

$$\nabla \cdot g(x) = -4\pi G \rho_m(x),$$

due to Gauss' law. Then the potential P_g obeys Poisson's equation

$$\Delta P_g = -4\pi G\rho_m.$$

Likewise, an electrical field E generated by a charge density ρ_q, has the electrical potential P_E such that $E = -\nabla P_E$. Maxwell's law for the potential of ρ_q can then be expressed as

$$\Delta P_E = -\nabla \cdot E = -\frac{\rho_q}{\varepsilon_0},$$

where ε_0 is the *electric constant* (permittivity of free space). As a final example of a physical quantity fulfilling Laplace's equation, we can examine the heat equation,

$$\alpha\Delta u = \frac{\partial u}{\partial t},$$

and in particular the steady-state heat equation fulfilling $\frac{\partial u}{\partial t} = 0$, so that

$$\Delta u = 0.$$

Having introduced these partial differential equations (PDEs), the question of, for example, finding the electrical potential inside a given conductor, or the gravitational field inside a given space, simply amounts to solving the corresponding PDE inside the conductor or space. In general this differential equation would be Poisson's equation, but if we assume that the conductor or space for which we wish to solve the equation contains no charge or mass density, respectively, the equation simplifies to Laplace's equation.

When solving a differential equation, a boundary condition is required to obtain unique solutions from the otherwise infinitely many possible solutions. In the case of an electrical conductor, a natural boundary condition is to provide a potential on the boundary of the conductor, just as a natural boundary condition for the heat equation is the temperature on the walls of the room. We then obtain a system of equations of the following form, letting the set of points Ω represent the conductor or room of interest and the function f the measured boundary values,

$$\begin{cases} u = f & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{in } \Omega. \end{cases}$$

This is the Dirichlet problem. Investigation of this mathematical question began around the time when the physical laws of gravity and electromagnetics were discovered. The Dirichlet problem was intensively investigated by mathematicians in the late nineteenth century and early twentieth century. During this period many of the mathematical foundations for solving the problem in increasingly general settings were developed.

Apart from these physical interpretations of the problem, potential theory also has strong connections to the study of analytic functions in complex analysis. Some discoveries within this field could, in a sense, be seen as a generalization of the results for complex analytic functions to functions in higher dimensions. This is something that will be discussed in Section 3.3.

Chapter 2

Preliminaries

Many important theorems from the fields of calculus in one and several variables, as well as measure and integration theory, will be used here without providing a proof.

2.1 Concepts from Real Analysis

When talking about limits and continuity, $\mathbb{R}^n$ should be interpreted as referring to the metric space $(\mathbb{R}^n, |\cdot|)$ with Euclidean distance

$$|x - y| = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}$$

as the metric. As such, E is used to refer to the metric subspace $(E, |\cdot|)$ where $E \subset \mathbb{R}^n$. As an illustration of this, consider the definition of the statement $\lim_{y\to x} f(y) = A$.

Definition 2.1. For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, we say that $f(y) \to A$ as $y \to x$, or equivalently $\lim_{y\to x} f(y) = A$, if for all $\varepsilon > 0$ there exists a $\delta > 0$, so that if $y \neq x$ is in the domain of f and $|y - x| < \delta$, then $|f(y) - A| < \varepsilon$.

Definition 2.2. The distance between a point x and a set A is defined as

$$\operatorname{dist}(x, A) = \inf_{y\in A} |x - y|.$$

Definition 2.3. The *open ball* $B(x, \rho)$ in $\mathbb{R}^n$ is defined as the set

$$B(x, \rho) = \{y \in \mathbb{R}^n : |x - y| < \rho\}.$$

By convention, the word ball is used to refer to the open ball.

Definition 2.4. A set G is *open* if for each point $x \in G$ there exists a neighborhood $B(x, \varepsilon) \subset G$ of some radius $\varepsilon > 0$.

A set F is *closed* if its complement F^c is open.

Note that there exist sets $E \subset A \subset X$ such that E is not open when considered as a set in the metric space (X, d), but open when considered as a set in the metric space (A, d). For example we can take $E = [0, 1)$, $A = [0, 2]$ and $X = \mathbb{R}$.

Definition 2.5. If $E \subset A$ and there exists some open G such that $E = G \cap A$, we say that E is *relatively open* in A.

Definition 2.6. The *closure* $\overline{A}$ of a set A is defined as the smallest closed set which contains A as a subset.

Definition 2.7. The *boundary* ∂A of a set A is defined as

$$\partial A = \overline{A} \cap \overline{A^c}.$$

Definition 2.8. For a given set $A \subset \mathbb{R}^n$, a collection $\{C_\lambda\}_{\lambda \in \Lambda}$ of sets is called a *cover* of A if

$$A \subset \bigcup_{\lambda \in \Lambda} C_\lambda.$$

The collection $\{C_\lambda\}_{\lambda \in \Lambda}$ may be finite or infinite. Since it may be uncountable, we use a so-called *index set* Λ which may be uncountable. We say that the cover has a *finite subcover* if there exists a collection $\{C_k\}_{k=1}^n \subset \{C_\lambda\}_{\lambda \in \Lambda}$ such that

$$A \subset \bigcup_{k=1}^n C_k.$$

Theorem 2.9 (The Heine–Borel theorem)**.** *For any set $A \subset \mathbb{R}^n$, the following two statements are equivalent:*

(a) *A is closed and bounded.*

(b) *A is compact, meaning that every cover of A consisting of open sets, has a finite subcover.*

Definition 2.10. A set A is *connected* if there do not exist two non-empty sets A_1 and A_2 such that: the sets are disjoint (i.e. $A_1 \cap A_2 = \varnothing$), each of the sets is relatively open in A, and $A = A_1 \cup A_2$.

A non-empty set which is open and connected is called a *domain*.

Definition 2.11. A subset $A_1 \subset A$ is said to be a *(connected) component* of A if A_1 is connected and there does not exist any connected $A_2 \subset A$ such that $A_1 \subset A_2$ and $A_1 \neq A_2$ (that is, A_1 is a maximal element with regard to the relation $(\subset)$).

Definition 2.12. A function $f\colon \Omega \to [-\infty, \infty]$ is said to be *upper semicontinuous* if

$$\limsup_{y \to x} f(y) \leq f(x) \quad \text{for all } x \in \Omega.$$

A function $f\colon \Omega \to [-\infty, \infty]$ is said to be *lower semicontinuous* if

$$\liminf_{y \to x} f(y) \geq f(x) \quad \text{for all } x \in \Omega.$$

It is apparent that a function is continuous if and only if a function is both upper and lower semicontinuous as well as real-valued, because

$$f(x) \leq \liminf_{y \to x} f(y) \leq \limsup_{y \to x} f(y) \leq f(x) \quad \iff \quad \lim_{y \to x} f(y) = f(x).$$

Note that functions $f\colon \Omega \to [-\infty, \infty]$ are not considered to be continuous if they attain the values $\pm\infty$ at any point, even if $\lim_{y \to x} f(y) = f(x)$ at these points.

Proposition 2.13. *If* $\{f_k\}$ *is a collection of continuous functions, then* $\inf\{f_k\}$ *is upper semicontinuous and* $\sup\{f_k\}$ *is lower semicontinuous.*

Weierstrass' theorem, also called the extreme value theorem, says that a continuous function must attain a maximum and a minimum on any compact set.

Theorem 2.14 (Weierstrass' theorem)**.** *Consider* $f\colon A \to [-\infty, \infty]$ *where* A *is a compact set. If* f *is upper semicontinuous, then*

$$f(x) = \sup_{z \in A} f(z) \quad \textit{for some } x \in A. \tag{2.1}$$

If f *is lower semicontinuous, then*

$$f(y) = \inf_{z \in A} f(z) \quad \textit{for some } y \in A. \tag{2.2}$$

If f *is continuous on* A*, then both* (2.1) *and* (2.2) *hold.*

The following simple lemma gives a sufficient condition for when a function has a continuous extension to the closure of its domain.

Lemma 2.15. *Let $u\colon \Omega \to \mathbb{R}$ be continuous in the open set Ω and assume that*

$$f(x) = \lim_{\Omega \ni y \to x} u(x) \quad \textit{for each } x \in \partial\Omega. \tag{2.3}$$

for some $f\colon \partial\Omega \to \mathbb{R}$. Then the function

$$g(x) = \begin{cases} u(x) & \textit{for } x \in \Omega, \\ f(x) & \textit{for } x \in \partial\Omega, \end{cases}$$

is continuous on $\overline{\Omega}$ and f is continuous on $\partial\Omega$.

Proof. By hypothesis, g is continuous at any $x \in \Omega$. Take any $x \in \partial\Omega$, and any $\varepsilon > 0$. By (2.3) there exists a $\delta > 0$ such that

$$|g(y) - g(x)| = |u(y) - f(x)| < \frac{\varepsilon}{2} \quad \text{for all } y \in B(x, \delta) \cap \Omega.$$

If $z \in B(x, \delta) \cap \partial\Omega$, then by (2.3) again, there exists a $\delta_z > 0$ such that

$$|g(y) - g(z)| = |u(y) - f(z)| < \frac{\varepsilon}{2} \quad \text{for all } y \in B(z, \delta_z) \cap \Omega.$$

Thus, for any $z \in B(x, \delta) \cap \partial\Omega$, there exists a $y \in B(x, \delta) \cap B(z, \delta_z) \cap \Omega$, such that

$$\begin{aligned} |g(z) - g(x)| &= |g(z) - g(y) + g(y) - g(x)| \\ &\le |g(z) - g(y)| + |g(x) - g(y)| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Therefore, if $z \in B(x, \delta)$ then $|f(z) - f(x)| < \varepsilon$, i.e. g is continuous at $x \in \partial\Omega$. Finally, because $f = g|_{\partial\Omega}$, it is also continuous. □

Here we also define *uniform continuity*, which is a stronger version of continuity, and state a sufficient condition for a function to be uniformly continuous. Note that the difference from ordinary continuity is that δ does not depend on x or y.

Definition 2.16. A function $f\colon \Omega \to \mathbb{R}$ is said to be *uniformly continuous* in Ω if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(y) - f(x)| < \varepsilon \quad \text{whenever } |y - x| < \delta.$$

Theorem 2.17 (Heine-Cantor theorem)**.** *If $f\colon E \to \mathbb{R}$ is continuous in the compact set E, then f is uniformly continuous in E.*

2.2 Concepts from Multivariable Calculus

This section contains many of the results which are useful for computations involving derivatives in more than one dimension, such as the Laplacian. These results should be familiar from a course in multivariable calculus, and many of the formulas can be found in reference books such as *Physics Handbook* [26], for instance. For integrals, the following notations will be used.

Definition 2.18. For a surface $S \subset \mathbb{R}^3$ which can be parametrised by functions in $\mathcal{C}^1(\mathbb{R}^2)$, sometimes called $\mathcal{C}^1$-*surfaces*, we write $\int_S f\, d\sigma$ for the surface integral of the function $f\colon S \to \mathbb{R}$.

For a set $A \subset \mathbb{R}^3$, we write $\int_A f\, d\lambda$ for the volume integral of the function $f\colon A \to \mathbb{R}$.

The concept of integrals will be extended to sets which cannot necessarily be parametrised this way in Section 2.3. When working with spheres and balls in a general number of dimensions, it is useful to have a simple representation of their surface area and volume.

Proposition 2.19. *The volume of the n-dimensional ball $B(0,1)$ is given by*

$$\nu_n = \int_{B(0,1)} 1\, d\lambda = \begin{cases} \dfrac{\pi^{n/2}}{(n/2)!}, & n \text{ even},\\ \dfrac{2^{(n+1)/2}\pi^{(n-1)/2}}{1\cdot 3\cdots(n-2)\cdot n}, & n \text{ odd and } n \geq 3. \end{cases}$$

The volume of the ball $B(0,\rho)$ is given by $\nu_n \rho^n$.

Proposition 2.20. *The surface area of the $(n-1)$-dimensional sphere $\partial B(0,1)$ in $\mathbb{R}^n$ is given by $\sigma_n = n\nu_n$. The surface area of the sphere $\partial B(0,\rho)$ is given by $\sigma_n \rho^{n-1}$.*

Proof. Consider the ball with radius ρ as the union of all spheres with radius $0 < r \leq \rho$. The volume of the ball can then be written

$$\nu_n \rho^n = \int_0^\rho \sigma_n r^{n-1}\, dr = \frac{\sigma_n}{n}\rho^n.$$

Differentiating this equation gives $n\nu_n\rho^{n-1} = \sigma_n \rho^{n-1}$ and thus $n\nu_n = \sigma_n$. □

Here we introduce the three most common notions of derivatives in a general number of dimensions: the gradient, divergence and Laplacian.

Definition 2.21. The *gradient* of a function $f \in \mathcal{C}^1(\Omega)$ is defined as

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

The *directional derivative* of f along the unit vector w is written $\frac{\partial f}{\partial w}$ and defined as $\frac{\partial f}{\partial w} = \nabla f \cdot w$.

Definition 2.22. The *divergence* of a vector-valued function $F\colon \Omega \to \mathbb{R}^n$ such that the components F_k satisfy $F_k \in \mathcal{C}^1(\Omega)$ for all $1 \le k \le n$, is defined as

$$\nabla \cdot F = \sum_{k=1}^{n} \frac{\partial F_k}{\partial x_k}.$$

Definition 2.23. The *Laplacian* of a function $f \in \mathcal{C}^2(\Omega)$, is defined as

$$\Delta f = \sum_{k=1}^{n} \frac{\partial^2 f}{\partial x_k^2}.$$

Note that $\Delta f = \nabla \cdot \nabla f$. Therefore the Laplacian is sometimes written ∇^2, though we will always use Δ.

Proposition 2.24. *The Laplacian commutes with differentiation, translation, rotation and integration.*

In particular, for any function $f\colon \Omega_x \times \Omega_y \to \mathbb{R}$ such that $f(\cdot, y) \in \mathcal{C}^2(\Omega_x)$ and for any signed measure (see Definition 2.33) $\mu\colon \mathcal{B}(\Omega_y) \to \mathbb{R}$ such that f is integrable relative to μ over Ω_y,

$$\Delta_x \int_{\Omega_y} f(x,y)\, d\mu(y) = \int_{\Omega} \Delta_x f(x,y)\, d\mu(y).$$

Proof. The first two properties follow directly from the same properties for componentwise differentiation.

The rotational invariance can be proved as follows. Let $u \in \mathcal{C}^2(\mathbb{R}^n)$ be a function and $M\colon \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal matrix, i.e. $MM^T = I$. Let the coordinates $\{y_i\}$ be given by $y = Mx$, i.e.

$$y_i = \sum_{j=1}^{n} m_{ij} x_j.$$

Then

$$\frac{\partial (u \circ M)}{\partial x_i} = \sum_{j=1}^{n} m_{ji} \frac{\partial u}{\partial y_j},$$

and

$$\frac{\partial^2 (u \circ M)}{\partial x_i^2} = \sum_{k=1}^{n} m_{ki} \frac{\partial}{\partial y_k} \sum_{j=1}^{n} m_{ji} \frac{\partial u}{\partial y_j} = \sum_{k=1}^{n} \sum_{j=1}^{n} m_{ki} m_{ji} \frac{\partial^2 u}{\partial y_k \partial y_j}.$$

Because

$$\sum_{i=1}^{n} m_{ki} m_{ji} = \begin{cases} 1 & \text{if } k = j, \\ 0 & \text{if } k \neq j, \end{cases}$$

we can conclude that

$$\Delta_x (u \circ M)(x) = \sum_{i=1}^{n} \frac{\partial^2 (u \circ M)}{\partial x_i^2} = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial y_i^2} = \Delta_y u(y).$$

To prove that we may exchange the order of integration and the Laplacian operator, we can apply the linearity of the integral and exchange the order of each partial differential operator $\frac{\partial^2}{\partial x_k^2}$ with the integration. This is allowed by the so-called Leibniz integral rule, which has as a consequence the fact that if f and $\frac{\partial^2 f}{\partial x_k^2}$ are both integrable with respect to y on Ω_y, then

$$\frac{\partial^2}{\partial x_k^2} \int_{\Omega_y} f(x, y)\, d\mu(y) = \int_{\Omega} \frac{\partial^2 f}{\partial x_k^2}(x, y)\, d\mu(y).$$

See for example Corollary 5.9 in [4, p. 46]. □

To prove other useful identities for the Laplacian, we make use of Gauss' divergence theorem, which gives a relation between a volume integral of the divergence and a surface integral of the normal derivative for a function.

Theorem 2.25 (Gauss' divergence theorem)**.** *If the divergence for the vector-valued function F exists on the bounded open set Ω, with a piecewise smooth boundary, then*

$$\int_{\Omega} \nabla \cdot F\, d\lambda = \int_{\partial\Omega} F \cdot \widehat{n}\, d\sigma,$$

where $\widehat{n}$ denotes the outer unit normal to the surface $\partial\Omega$.

The following basic identities for vector derivatives are useful for computing the Laplacian of various functions. The rules (a) and (b) can be seen as extensions of the chain rule and product rule for the first derivative, respectively. The same is true for (c) and (d), but in this case for the second derivative.

Proposition 2.26. *Given a domain $\Omega \subset \mathbb{R}^n$ and functions $g \in \mathcal{C}^1(\mathbb{R})$, $f \in \mathcal{C}^1(\Omega)$ and $F\colon \Omega \to \mathbb{R}^n$ such that $F_k \in \mathcal{C}^1(\Omega)$, the following identities hold:*

(a) $\nabla(g \circ f) = (g' \circ f)\nabla f$,

(b) $\nabla \cdot (fF) = (\nabla f) \cdot F + f(\nabla \cdot F)$.

For any functions $g \in \mathcal{C}^2(\mathbb{R}), f \in \mathcal{C}^2(\Omega)$ and $g \in \mathcal{C}^2(\Omega)$, the following identities also hold:

(c) $\Delta(g \circ f) = (g' \circ f)\Delta f + (g'' \circ f)|\nabla f|^2$,

(d) $\Delta(fg) = f\Delta g + g\Delta f + 2(\nabla f \cdot \nabla g)$.

Proof. The first identity follows from a simple calculation with the chain rule from calculus in one dimension.

$$\begin{aligned}
\nabla(g \circ f) &= \left(\frac{\partial}{\partial x_1}(g \circ f), \ldots, \frac{\partial}{\partial x_n}(g \circ f)\right) \\
&= \left((g' \circ f)\frac{\partial f}{\partial x_1}, \ldots, (g' \circ f)\frac{\partial f}{\partial x_n}\right) \\
&= (g' \circ f)\nabla f.
\end{aligned}$$

Similarly, the second identity follows from the product rule from calculus in one dimension.

$$\begin{aligned}
\nabla \cdot (fF) &= \frac{\partial}{\partial x_1}(fF_1) + \ldots + \frac{\partial}{\partial x_n}(fF_n) \\
&= \frac{\partial f}{\partial x_1}F_1 + f\frac{\partial F_1}{\partial x_1} + \ldots + \frac{\partial f}{\partial x_n}F_n + f\frac{\partial F_n}{\partial x_n} \\
&= \left(\frac{\partial f}{\partial x_1}F_1 + \ldots + \frac{\partial f}{\partial x_n}F_n\right) + \left(f\frac{\partial F_1}{\partial x_1} + \ldots + f\frac{\partial F_n}{\partial x_n}\right) \\
&= (\nabla f) \cdot F + f(\nabla \cdot F).
\end{aligned}$$

The third identity can be derived from the first two by rewriting the Laplacian as the divergence of the gradient, which gives

$$\begin{aligned}
\Delta(g \circ f) &= \nabla \cdot \nabla(g \circ f) \\
&= \nabla \cdot ((g' \circ f)\nabla f) \\
&= \nabla(g' \circ f) \cdot \nabla f + (g' \circ f)(\nabla \cdot \nabla f) \\
&= (g'' \circ f)\nabla f \cdot \nabla f + (g' \circ f)(\nabla \cdot \nabla f) \\
&= (g' \circ f)\Delta f + (g'' \circ f)|\nabla f|^2,
\end{aligned}$$

while the fourth identity also requires the formula $\nabla(fg) = f\nabla g + g\nabla f$, which can be proven by a straightforward componentwise application of the product rule:

$$\begin{aligned}\Delta(fg) &= \nabla \cdot \nabla(fg)\\ &= \nabla \cdot (f\nabla g + g\nabla f)\\ &= \nabla f \cdot \nabla g + f(\nabla \cdot \nabla g) + \nabla g \cdot \nabla f + g(\nabla \cdot \nabla f)\\ &= f\Delta g + g\Delta f + 2(\nabla f \cdot \nabla g).\end{aligned}$$

The requirements on the differentiability of the involved functions are sufficient for all the calculations to be well defined. □

Like the well-known Green's theorem in two dimensions, there exists a similar equality in an arbitrary dimension, called Green's identity.

Theorem 2.27 (Green's identity)**.** *For a bounded set* Ω *and functions* $u, v\colon \overline{\Omega} \to \mathbb{R}$, $u, v \in \mathcal{C}^2(\overline{\Omega})$,

(a) $\displaystyle\int_\Omega (\nabla u \cdot \nabla v + u\Delta v)\, d\lambda = \int_{\partial\Omega} u(\nabla v \cdot \widehat{n})\, d\sigma,$

(b) $\displaystyle\int_\Omega (u\Delta v - v\Delta u)\, d\lambda = \int_{\partial\Omega} (u(\nabla v \cdot \widehat{n}) - v(\nabla u \cdot \widehat{n}))\, d\sigma\,.$

Proof. We begin by observing that Proposition 2.26 (b) gives

$$\nabla \cdot (u\nabla v) = \nabla u \cdot \nabla v + u(\nabla \cdot \nabla v) = \nabla u \cdot \nabla v + u\Delta v.$$

Applying Gauss' theorem (2.25) to the integral of this expression yields

$$\int_\Omega (\nabla u \cdot \nabla v + u\Delta v)\, d\lambda = \int_\Omega \nabla \cdot (u\nabla v)\, d\lambda = \int_{\partial\Omega} (u\nabla v)\cdot \widehat{n}\, d\sigma = \int_{\partial\Omega} u(\nabla v \cdot \widehat{n})\, d\sigma,$$

so the proof of (a) is done. Subtracting $v\Delta u$ and rearranging the terms gives

$$\int_\Omega (u\Delta v - v\Delta u)\, d\lambda = \int_{\partial\Omega} (u(\nabla v \cdot \widehat{n}) - v(\nabla u \cdot \widehat{n}))\, d\sigma + \int_\Omega (\nabla u \cdot \nabla v - \nabla v \cdot \nabla u)\, d\lambda,$$

where the integrand of the last integral is zero, which proves (b). □

Finally, we discuss some different types of mean values of a function. The spherical mean value and ball mean value can be used to prove many properties of functions whose Laplacian is zero, as we will see in Chapter 3.

Definition 2.28. The *spherical mean value* of a function $f\colon \Omega \to \mathbb{R}$ on the sphere $\partial B(x, \rho)$ is denoted

$$\mathcal{M}(f; x, \rho) = \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B(x,\rho)} f\, d\sigma\,.$$

If f is continuous, then $\mathcal{M}(f;x,\rho)$ is continuous with respect to ρ, and we can define the mean value for a sphere of radius zero as

$$\mathcal{M}(f;x,0) = \lim_{\rho\to 0} \mathcal{M}(f,x,\rho) = f(x).$$

Definition 2.29. The *ball mean value* of a function $f\colon \Omega \to \mathbb{R}$ in the ball $B(x,\rho)$ is denoted

$$\mathcal{A}(f;x,\rho) = \frac{1}{\nu_n \rho^n} \int_{B(x,\rho)} f \, d\lambda \,.$$

If f is continuous, then $\mathcal{A}(f;x,\rho)$ is continuous with respect to ρ, and we can define the mean value for a ball of radius zero as

$$\mathcal{A}(f;x,0) = \lim_{\rho\to 0} \mathcal{A}(f,x,\rho) = f(x).$$

Note also that the mean values of continuous functions are continuous also with respect to x. We now prove that the mean values of k times continuously differentiable functions are k times continuously differentiable functions of the radius ρ.

Proposition 2.30. *If $f \in \mathcal{C}^k(\Omega)$ and $B(x,r) \Subset \Omega$ then the functions $\rho \mapsto \mathcal{M}(f;x,\rho)$ and $\rho \mapsto \mathcal{A}(f;x,\rho)$ are in $\mathcal{C}^k([0,r])$.*

Proof. If $f \in \mathcal{C}^1(\Omega)$ then

$$\begin{aligned}
\frac{\partial}{\partial\rho}\mathcal{M}(f;x,\rho) &= \frac{\partial}{\partial\rho}\frac{1}{\sigma_n\rho^{n-1}}\int_{\partial B(x,\rho)} f(y)\,d\sigma(y) \\
&= \frac{1-n}{\sigma_n\rho^n}\int_{\partial B(x,\rho)} f(y)\,d\sigma(y) + \frac{1}{\sigma_n\rho^{n-1}}\frac{\partial}{\partial\rho}\int_{\partial B(x,\rho)} f(y)\,d\sigma(y) \\
&= \frac{1-n}{\rho}\mathcal{M}(f;x,\rho) + \frac{1}{\sigma_n\rho^{n-1}}\frac{\partial}{\partial\rho}\int_{\partial B(0,1)} \rho f(x+\rho z)\,d\sigma(z)
\end{aligned}$$

By differentiating under the integral sign, we get

$$\frac{\partial}{\partial\rho}\int_{\partial B(0,1)} \rho f(x+\rho z)\,d\sigma(z) = \int_{\partial B(0,1)} \frac{\partial}{\partial\rho}\rho f(x+\rho z)\,d\sigma(z),$$

and as the integral of a continuous function is continuous, $\frac{d}{d\rho}\mathcal{M}(M;x,\rho)$ is the sum of two continuous functions, and is therefore continuous itself.

For the ball mean we have analogously,

$$\begin{aligned}
\frac{\partial}{\partial\rho}\mathcal{A}(f;x,\rho) &= \frac{\partial}{\partial\rho}\frac{1}{\nu_n\rho^n}\int_{B(x,\rho)} f(y)\,d\lambda(y) \\
&= \frac{-n}{\nu_n\rho^{n+1}}\int_{B(x,\rho)} f(y)\,d\lambda(y) + \frac{1}{\nu_n\rho^n}\frac{\partial}{\partial\rho}\int_{B(x,\rho)} f(y)\,d\lambda(y) \\
&= \frac{-n}{\rho}\mathcal{A}(f;x,\rho) + \frac{1}{\nu_n\rho^n}\frac{\partial}{\partial\rho}\int_{B(0,1)} \rho f(x+\rho z)\,d\lambda(z) \\
&= \frac{-n}{\rho}\mathcal{A}(f;x,\rho) + \frac{1}{\nu_n\rho^n}\int_{B(0,1)} \frac{\partial}{\partial\rho}\rho f(x+\rho z)\,d\lambda(z).
\end{aligned}$$

This argument can be iterated to prove that if the kth derivative of f exists and is continuous (in all coordinate components x_1 through x_n), then the kth derivative of the sphere mean value and the ball mean value, exists and is continuous.

Note that if $f \in \mathcal{C}^\infty(\Omega)$, induction may be used to prove that the functions $\rho \mapsto \mathcal{M}(f;x,\rho)$ and $\rho \mapsto \mathcal{A}(f;x,\rho)$ are in $\mathcal{C}^\infty([0,r])$, with a similar method as in this proof. □

The preceding two definitions are special cases of the more general mean value which can be taken over any bounded set. This general mean value will not be used in this text, however it is included for completeness.

Definition 2.31. The *mean value* of a function $f\colon \Omega \to \mathbb{R}$ defined in a bounded set $A \subset \Omega$ with positive volume, is denoted

$$⨍_A f\,d\lambda = \frac{1}{\lambda(A)}\int_A f\,d\lambda,$$

where $\lambda(A)$ is the volume of the set (see Definition 2.36).

2.3 Concepts from Measure and Integration Theory

As noted in Definition 2.18, the Riemann integral cannot be used on a set which cannot be parametrised with $\mathcal{C}^1(\mathbb{R})$ functions. To alleviate this shortcoming of the integral, many different generalizations have been invented, such as the Riemann–Stieltjes integral, the Daniell integral and the Lebesgue integral. The latter two have been frequently used in potential theory in the early twentieth century, for example in [36] and [28], respectively. For the purposes of this text, establishing the basic definitions of measure theory and the Lebesgue integral is sufficient.

We begin with the definition of a measure, which is a a function that gives the "size" of a set. In this definition we use the symbol $\mathcal{P}(A)$ to denote the set of all subsets of A, called the *power set*.

Definition 2.32. For a set of subsets of the space, $\Sigma \subset \mathcal{P}(\mathbb{R}^n)$, a *measure* is a function $\mu\colon \Sigma \to [0, \infty]$ that is zero for the empty set, $\mu(\varnothing) = 0$, and additive for countable collections of pairwise disjoint sets $(A_k)_{k=1}^{\infty}$ in Σ,

$$\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k) \quad \text{where } A_k \cap A_j = \varnothing \text{ for } k \neq j.$$

The definition of a measure can easily be extended to functions into the extended reals, so that the measure can be both positive and negative:

Definition 2.33. If a function $\mu\colon \Sigma \to [-\infty, \infty]$ can be expressed as $\mu = \mu_+ - \mu_-$, where $\mu_+ = \max\{\mu, 0\}$ and $\mu_- = \max\{-\mu, 0\}$ are measures, then μ is called a *signed measure*. Note that at most one of μ_+ and μ_- may be infinite. For signed measures we define the "absolute value" as $|\mu| = \mu_+ + \mu_-$.

We would wish to define a measure $V\colon \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ which is invariant under rotation and translation, and corresponds to the natural notion of volume in n dimensions. Using the Axiom of Choice, one can prove that there must exist subsets of $\mathbb{R}^n$ for which this measure cannot give a meaningful value. One such example using such non-measurable sets in three dimensions is the Banach–Tarski paradox. For this reason the domain of a measure corresponding to volume, can at best be a strict subset Σ of the power set $\mathcal{P}(\mathbb{R}^n)$. To formalise which sets are measurable and which are not measurable, the definition of a σ-algebra is introduced.

Definition 2.34. A set Σ is called a *σ-algebra* on X if it contains X and is closed under complementation and countable unions and intersections, i.e. it satisfies the following:

(a) If $A \in \Sigma$ then $A^c \in \Sigma$.

(b) If $A_1, A_2, \ldots \in \Sigma$ then $\bigcup_{k=1}^{\infty} A_k \in \Sigma$.

(c) If $A_1, A_2, \ldots \in \Sigma$ then $\bigcap_{k=1}^{\infty} A_k \in \Sigma$.

Note that

$$\bigcap_{k=1}^{\infty} A_k = \left(\bigcup_{k=1}^{\infty} A_k^c\right)^c,$$

so the third condition is redundant, as it follows from the first two.

Definition 2.35. For a given set X, the *Borel σ-algebra* $\mathcal{B}(X)$ is defined as the intersection of all σ-algebras which contain all open subsets of X. The Borel σ-algebra is therefore the smallest σ-algebra which contains all open subsets of X. The Borel algebra also contains all closed subsets of X, which is easily seen as it is closed under complementation.

As non-Borel subsets of $\mathbb{R}^n$ are relatively complicated to construct, the Borel algebra can intuitively be thought of as containing all "reasonable" subsets of $\mathbb{R}^n$. In the remainder of this text, all sets which are discussed will be assumed to be members of the Borel algebra, unless otherwise noted.

On the Borel algebra we can define the Lebesgue measure, which corresponds to the usual notions of length in $\mathbb{R}$, area in $\mathbb{R}^2$ or volume in $\mathbb{R}^3$.

Definition 2.36. The *Lebesgue measure* $\lambda\colon \mathcal{B}(\mathbb{R}^n) \to [0, \infty)$ is defined as

$$\lambda(A) = \inf_{\mathcal{R}} \sum_{E \in \mathcal{R}} (\text{volume of the box } E),$$

where the infimum is taken over all countable collections $\mathcal{R}$ of n-dimensional (axis-parallel) boxes

$$E = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times \cdots \times [\alpha_n, \beta_n]$$

such that $A \subset \bigcup_{E \in R} E$.

Definition 2.37. A set $A \subset \mathbb{R}^n$ is said to be *Lebesgue measurable* if

$$\lambda(A) = \lambda(A \cap E) + \lambda(A \cap E^c) \quad \text{for all } E \subset \mathbb{R}^n.$$

It can be shown that all sets in the Borel algebra are Lebesgue measurable, and even that there exist sets which are Lebesgue measurable but do not belong to the Borel algebra, e.g. any non-Borel subset of a set with measure zero.

A tuple (X, Σ) of a set X and a σ-algebra Σ is called a *measurable space*. A tuple (X, Σ, μ) of a set X, a σ-algebra Σ and a measure μ is called a *measure space*. Where appropriate, $\mathbb{R}^n$ will refer to either the measurable space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ or the measure space $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n), \lambda)$.

Definition 2.38. A function $f\colon \Omega \to \mathbb{R}$ is said to be *measurable* if for all $t \in \mathbb{R}$ the set $\{x : f(x) > t\}$ is measurable.

Having obtained a mathematically consistent notion of length, area and volume for all sets in the Borel algebra, we may now define the Lebesgue integral. The

Lebesgue integral can be defined using *Cavalieri's principle*, which is the formula (2.4).

Definition 2.39. Given the Lebesgue measure λ and a measurable function $f\colon \Omega \to [0, \infty)$ (on a measurable set Ω), the *Lebesgue integral* is given in terms of the Riemann integral as

$$\int_\Omega f \, d\lambda = \int_0^\infty \lambda(\{x : f(x) > t\}) \, dt, \tag{2.4}$$

i.e. the Riemann integral with respect to t of the measure of the preimage $f^{-1}((t, \infty))$.

Informally, the Lebesgue integral corresponds to adding up the measures (multiplied by dt) of the sets which map to $(t, t+dt)$, for all t where dt is an infinitesimal step.

Definition 2.40. For a measurable function $f\colon \Omega \to \mathbb{R}$, the Lebesgue integral on a (measurable) set $E \subset \Omega$ is defined by

$$\int_E f \, d\lambda = \int_E f_+ \, d\lambda - \int_E f_- \, d\lambda, \tag{2.5}$$

where $f = f_+ - f_-$, for the non-negative functions $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$, whose integrals can be evaluated by (2.4). For this to be well-defined, at most one of the two integrals in the right-hand side can be infinite, meaning that

$$\int_E f_+ \, d\lambda < \infty \quad \text{or} \quad \int_E f_- \, d\lambda < \infty.$$

Definition 2.41. A function $f\colon \Omega \to \mathbb{R}$ is said to be *Lebesgue integrable* (on Ω) if it is measurable and the integral of the function's absolute value is finite, i.e.

$$\int_\Omega |f| \, d\lambda < \infty.$$

The space of functions which are Lebesgue integrable on Ω is often written $L^1(\Omega)$.

Note that if a function f is Lebesgue integrable on Ω, then

$$\int_E |f| \, d\lambda = \int_E f_+ \, d\lambda + \int_E f_- \, d\lambda < \infty,$$

which implies

$$\int_E f_+ \, d\lambda < \infty \quad \text{and} \quad \int_E f_- \, d\lambda < \infty$$

for all sets $E \subset \Omega$, so the integral of f as defined by (2.5) exists.

The integral with respect to an arbitrary measure μ can be constructed in the same way as above (Definitions 2.39–2.41), simply replacing the Lebesgue measure λ with μ. We say that a measurable function f is integrable on Ω with respect to μ, if $\int |f| \, d\mu < \infty$.

It is evident that the value of a function on a set of measure zero does not affect the value of the integral of the function, meaning that if $\mu(E) = 0$ then

$$\int_A f \, d\mu = \int_{A \cup E} f \, d\mu$$

for all integrable functions $f \colon A \cup E \to \mathbb{R}$.

Definition 2.42. If a property holds on a set A except on a subset $E \subset A$ with measure zero, $\mu(E) = 0$, we say that the property holds *almost everywhere* on A with respect to the measure μ, abbreviated $a.e.(\mu)$. When the set A is the whole space, it is often omitted. Likewise, when the measure is the Lebesgue measure λ, it is omitted.

Sets with (Lebesgue) measure zero are for example all finite sets in $\mathbb{R}^n$ and the set of rationals in $\mathbb{R}$. The Cantor set, constructed by iteratively removing the middle third of each component interval in $[0, 1]$, has measure zero. However, the Smith–Voltera–Cantor set, is constructed by removing an interval of width $(1/4)^n$ from each component, where n is the number of iterations, in $[0, 1]$. The iterations are

$$\begin{aligned} I_0 &= [0, 1], \\ I_1 &= [0, 3/8] \cup [5/8, 1], \\ I_2 &= [0, 5/32] \cup [7/32, 3/8] \cup [5/8, 25/32] \cup [27/32, 1], \\ &\vdots, \end{aligned}$$

and at each step 2^n intervals of length $(1/4)^n$ are removed, giving the Lebesgue measure of the limit

$$\lambda\left(\lim_{n\to\infty} I_n\right) = \lim_{n\to\infty} \lambda(I_n) = 1 - \sum_{n=1}^{\infty} 2^n (1/4)^n = 1/2.$$

Taking the complement of the Smith–Voltera–Cantor set with respect to $(0, 1)$ gives an open set whose boundary is the Smith–Voltera–Cantor set. It is therefore an example of a domain whose boundary has positive Lebesgue measure, something which one might not at first expect to exist. More “ordinary” sets such as finite unions of balls, cubes or similar, all have boundaries with measure zero.

Another important measure is the Dirac measure, which only depends on if a single point is a member of the set to be measured.

Definition 2.43. The *Dirac measure* δ_a at a point $a \in \mathbb{R}^n$ is defined as

$$\delta_a(A) = \begin{cases} 1, & \text{if } a \in A, \\ 0, & \text{if } a \notin A. \end{cases}$$

All sets are Dirac measurable, because they must either contain the point a or not.

This measure is closely related to the Dirac impulse, which is used in distribution theory. Compare the integral of a function $T\colon \mathbb{R} \to \mathbb{R}$ with respect to the Dirac measure,

$$\int_{\mathbb{R}} T(x)\, d\delta_a(x) = T(a),$$

with the usual definition of the Dirac impulse $\delta_a\colon \mathcal{C}^\infty(\mathbb{R}) \to \mathbb{R}$,

$$\delta_a(T) = \text{``} \int_{-\infty}^{\infty} T(x)\delta_a(x)\, dx \text{''} = T(a).$$

The integral within quotes above is not a well defined Riemann or Lebesgue integral, but a common intuitive interpretation of the evaluation a distribution for the test function $T \in \mathcal{C}^\infty(\mathbb{R})$ is as an integral of their product over the reals.

Chapter 3

Harmonic Functions

The name *harmonic* comes from the set of functions called the *solid harmonics (in spherical coordinates)*, which are a collection of solutions to Laplace's equation on the unit ball in three dimensions,

$$\begin{aligned}&u\colon B(0,1)\to\mathbb{R},\\&\Delta u=0.\end{aligned}$$

This definition was first introduced by William Thomson and Peter Guthrie Tait in their 1867 textbook *Treatise on Natural Philosophy*, wherein the function u was additionally required to be homogeneous of some degree l, i.e.

$$u(\alpha r,\theta,\varphi)=\alpha^l u(r,\theta,\varphi).$$

A family of solutions to this equation is given by

$$u(r,\theta,\varphi)=r^l P_l^m(\cos\theta)e^{im\varphi},$$

where $P_l^m(x)$ is the *associated Legendre function* of degree $l\in\{0,1,\dots\}$ and order $m\in\{-l,\dots,l\}$. These functions are named after Adrien-Marie Legendre, who together with Pierre Simon de Laplace developed solutions to the Laplace equation in three dimensions during the 1780s [23, pp. 72–74]. See Section 4.1 for a summary of the derivation of these solutions.

The solid harmonics have the useful property that they form an orthogonal basis for the space of functions on the sphere. This means that in a similar manner as the Taylor series of a function gives a representation of the function as a linear combination of the monomials x^k, or the Fourier series of a function gives a representation as a linear combination of the periodic functions $e^{i2\pi\omega k}$, any function u which is harmonic in the sphere has a (unique) representation as a linear combination of solid harmonics, namely

$$u(r,\theta,\varphi)=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}C_{l,m}r^lP_l^m(\cos\theta)e^{im\varphi},$$

where $C_{l,m}$ are real constants. These functions were first studied in relation to Newton's gravitational potential, but the solid harmonics are also used in modern physics. For example they appear as the solutions to the Schrödinger equation for the hydrogen atom [26, p. 296].

3.1 The Definition of Harmonicity

The requirement for a function to be harmonic was at first said to be that the function has zero Laplacian for a given set of points. In modern potential theory however, the property of harmonicity is sometimes, e.g. by Brelot in 1965 [7], defined using the mean value over a sphere or ball, instead of the Laplacian of the function. That is, a function is harmonic if at each point, the value of the function is equal to the mean value over each sphere surrounding the point. The two definitions result in exactly the same family of functions being regarded as harmonic in Euclidean space, as we will see in Theorem 3.4.

Definition 3.1. A twice continuously differentiable function $u\colon \Omega\to\mathbb{R}, u\in\mathcal{C}^2(\Omega)$, is said to be *harmonic* in Ω if it has the *spherical mean-value property*:

$$u(x)=\mathcal{M}(u;x,\rho)\quad\text{for all } x\in\Omega, \rho>0 \text{ such that } B(x,\rho)\Subset\Omega.$$

Definition 3.2. The space of functions that are harmonic in Ω is denoted $\mathcal{H}(\Omega)$.

The property of harmonicity can, as mentioned above, also be defined by using the ball mean value, and these two definitions are completely equivalent.

Proposition 3.3. *A function $u\in\mathcal{C}^2(\Omega)$ is harmonic if and only if it has the* ball mean-value property*:*

$$u(x)=\mathcal{A}(u;x,\rho)\quad\textit{for all } x\in\Omega, \rho>0 \textit{ such that } B(x,\rho)\Subset\Omega.$$

Proof. Assume that u is harmonic. Then, using spherical coordinates,

$$\mathcal{A}(u;x,\rho)=\frac{n}{\rho^n}\int_0^\rho r^{n-1}\mathcal{M}(u;x,r)\,dr\,. \tag{3.1}$$

Substituting $\mathcal{M}(u;x,r) = u(x)$ for all $0 \le r \le \rho$ gives

$$\mathcal{A}(u;x,\rho) = \frac{n}{\rho^n}\int_0^\rho r^{n-1}u(x)\,dr = u(x)\frac{n}{\rho^n}\int_0^\rho r^{n-1}\,dr = u(x).$$

Conversely, we have seen in Proposition 2.30 that the ball mean and sphere mean of u are twice continuously differentiable with respect to the radius ρ because $u \in \mathcal{C}^2(\Omega)$. Thus, differentiating (3.1) and substituting $\mathcal{A}(u;x,\rho) = u(x)$ gives

$$\mathcal{M}(u,x,\rho) = \frac{1}{\rho^{n-1}}\frac{\partial}{\partial\rho}\frac{\rho^n}{n}\mathcal{A}(u;x,\rho) = \frac{1}{\rho^{n-1}}\rho^{n-1}u(x) = u(x). \qquad \square$$

As mentioned, in Euclidean space the characterization of harmonic functions as solutions to the Laplace equation and as functions with the mean-value property are equivalent.

Theorem 3.4. *A function $u \in \mathcal{C}^2(\Omega)$ is harmonic if and only if its Laplacian is identically zero in Ω.*

This proof is given by Armitage and Gardiner [2, pp. 3–4].

Proof. First we note that for each ball $B(x,\rho) \Subset \Omega$, Theorem 2.27 (a) gives

$$\int_{B(x,\rho)} (\nabla v \cdot \nabla u + v\Delta u)\,d\lambda = \int_{\partial B(x,\rho)} v(\nabla u \cdot \widehat{n})\,d\sigma$$

for any $v \in \mathcal{C}^2(\Omega)$, in particular $v \equiv 1$, which simplifies the expression to

$$\int_{B(x,\rho)} \Delta u\,d\lambda = \int_{\partial B(x,\rho)} \frac{\partial u}{\partial \widehat{n}}\,d\sigma\,.$$

Differentiating under the integral sign gives

$$\begin{aligned}\int_{B(x,\rho)} \Delta u\,d\lambda &= \rho^{n-1}\int_{\partial B(x,1)} \frac{\partial}{\partial\rho}u(x+\rho y)\,d\sigma(y)\\ &= \rho^{n-1}\frac{d}{d\rho}\int_{\partial B(x,1)} u(x+\rho y)\,d\sigma(y).\end{aligned}$$

Rewriting this using Definitions 2.28 and 2.29 gives

$$\nu_n\rho^n\mathcal{A}(\Delta u;x,\rho) = \sigma_n\rho^{n-1}\frac{d}{d\rho}\mathcal{M}(u;x,\rho), \tag{3.2}$$

and substituting $\Delta u = 0$ gives

$$0 = \frac{d}{d\rho}\mathcal{M}(u;x,\rho),$$

so $\mathcal{M}(u;x,\rho)$ is constant with respect to ρ. The constant value is obtained by $\mathcal{M}(u;x,0) = u(x)$.

We now prove the implication

$$\Delta u \equiv 0 \text{ in } \Omega \quad \Longrightarrow \quad u \in \mathcal{H}(\Omega).$$

To prove the converse,

$$u \in \mathcal{H}(\Omega) \quad \Longrightarrow \quad \Delta u \equiv 0 \text{ in } \Omega,$$

assume that $u(x) = \mathcal{M}(u;x,r)$ whenever $B(x,r) \Subset \Omega$. If $\rho < r$ then (3.2) holds. Dividing by $\nu_n \rho^n$ and substituting $u(x)$ for $\mathcal{M}(u;x,\rho)$ gives

$$\mathcal{A}(\Delta u;x,\rho) = \frac{n}{\rho}\frac{d}{d\rho}\mathcal{M}(u;x,\rho) = \frac{n}{\rho}\frac{d}{d\rho}u(x) = 0.$$

Because $u \in \mathcal{C}^2(\Omega)$, both sides of the equality are continuous (see Definition 2.29) and we may take the limit as $\rho \to 0$, giving

$$\Delta u(x) = \mathcal{A}(\Delta u;x,0) = 0.$$

As Ω is open, there exists for any point $x \in \Omega$ a ball $B(x,r) \Subset \Omega$, and therefore we can conclude that $\Delta u \equiv 0$ in Ω. □

A basic property of the space of harmonic functions is that it is closed under linear combinations and limits.

Proposition 3.5. *The space $\mathcal{H}(\Omega)$ is a (linear) vector space, meaning that if u, v are harmonic in Ω and $c \in \mathbb{R}$ then the functions cu and $u+v$ are harmonic in Ω.*

Proof. Because the Laplacian is linear the theorem is proved by computing the Laplacian for each of the functions.

$$\Delta(cu) = c\Delta u = 0,$$
$$\Delta(u+v) = \Delta u + \Delta v = 0.$$

It can also be proved by using the mean value property:

$$cu(x) = c\mathcal{M}(u;x,\rho) = \mathcal{M}(cu;x,\rho),$$
$$u(x) + v(x) = \mathcal{M}(u;x,\rho) + \mathcal{M}(v;x,\rho) = \mathcal{M}(u+v;x,\rho).$$

□

The following result about the convergence of sequences of harmonic functions can be compared to uniform convergence for integrable functions.

Proposition 3.6. *If $u_1, u_2, \dots$ is a sequence of functions which are harmonic in Ω, that converges uniformly in Ω to a function u, then the function u is harmonic. Further, for any differential operator $\frac{\partial}{\partial x_j}$ the sequence $(\frac{\partial}{\partial x_j} u_k)_{k=1}^{\infty}$ converges to $\frac{\partial}{\partial x_j} u$.*

Proof. Because the terms are harmonic and therefore in $\mathcal{C}^2(\Omega)$, uniform convergence gives that $u \in \mathcal{C}^2(\Omega)$. Uniform convergence also gives that $u(x) = \mathcal{M}(u; x, \rho)$ whenever $B(x, \rho) \Subset \Omega$. This is because uniform convergence of a sequence of integrable functions is sufficient for the convergence of the sequence of integrals

$$\mathcal{M}(u_k; x, \rho) = \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B} u_k \, d\sigma \to \frac{1}{\sigma_n \rho^{n-1}} \int_{\partial B} u \, d\sigma = \mathcal{M}(u; x, \rho).$$

So u satisfies the mean-value property and therefore $u \in \mathcal{H}(\Omega)$.

The second part of the theorem follows from a similar argument, but relies on the fact that harmonic functions are infinitely differentiable, a fact which will be established in Theorem 3.18. □

3.2 The Fundamental Solution for Laplace's Equation

Recalling Maxwell's equations from electrostatics, we see that u is harmonic if it is the potential of an electric field $E = \nabla u$ which fulfills Gauss' law in vacuum: $\nabla \cdot E = \Delta u = 0$. Therefore, the potential for the point charge $q \in \mathbb{R}$ at $y \in \mathbb{R}^3$, given by Coulomb's law

$$\frac{q}{4\pi\varepsilon_0 |x - y|}$$

is a harmonic function on its domain, $\mathbb{R}^3 \setminus \{y\}$. For simplicity we will discard the constant $4\pi\varepsilon_0$, where ε_0 is the electric constant.

This can be generalized to $y \in \mathbb{R}^n, n \geq 3$, by the Newtonian potential

$$\Gamma_y(x) = |x - y|^{2-n},$$

for the moment disregarding the constant q. To compute the Laplacian of Γ_y, we first define the function

$$\begin{aligned} &r \colon \mathbb{R}^n \to \mathbb{R}, \\ &r(x) = |x - y|. \end{aligned}$$

Observing that for all $k \in \mathbb{R}$,

$$\nabla r^k = k r^{k-2}(x-y) \quad \text{and} \quad \Delta r^k = k(k+(n-2))r^{k-2} \tag{3.3}$$

we get

$$\begin{aligned} \Delta \Gamma_y = \Delta r^{2-n} &= (2-n)((2-n)+(n-2))r^{-n} \\ &= 0 \quad \text{when } r \neq 0, \text{ i.e. } x \neq y. \end{aligned} \tag{3.4}$$

In general, any function $u\colon \mathbb{R}^n \to \mathbb{R}$, which depends only on $|x-y|$ can be written as $u = g \circ r$, for some function $g\colon \mathbb{R} \to \mathbb{R}$. The function u is harmonic if and only if

$$\begin{aligned} \Delta u = \Delta(g \circ r) &= (g'' \circ r)|\nabla r|^2 + (g' \circ r)\Delta r \\ &= (g'' \circ r) + (g' \circ r)(n-1)r^{-1} = 0, \end{aligned}$$

or stated as an ordinary differential equation in the real variable r,

$$g''(r) + \frac{(n-1)}{r} g'(r) = 0.$$

The solutions to this equation are obtained by transforming the expression to

$$\frac{d \ln g'(r)}{dr} = \frac{g''(r)}{g'(r)} = \frac{1-n}{r},$$

integrating to obtain

$$\ln g'(r) = (1-n)\ln r + C$$

and exponentiating both sides to get

$$g'(r) = e^C r^{1-n},$$

for some constant $C \in \mathbb{R}$. Ultimately, the solutions are given by

$$g(r) = \begin{cases} a r^{2-n} + b, & \text{if } n \geq 3, \\ a \ln r + b, & \text{if } n = 2, \end{cases}$$

for some real constants a and b, which implies that

$$u(x) = \begin{cases} a r(x)^{2-n} + b, & \text{if } n \geq 3, \\ a \ln r(x) + b, & \text{if } n = 2. \end{cases}$$

Note that these solutions approach infinity as $x \to 0$.

Definition 3.7. The function

$$\Gamma_y(x) = \begin{cases} |x-y|^{2-n}, & \text{for } n \geq 3, \\ \ln|x-y|, & \text{for } n = 2, \end{cases}$$

is known as the *fundamental solution to Laplace's equation* on $\mathbb{R}^n \setminus \{y\}$.

Proposition 3.8. *For either $\Omega = \mathbb{R}^n \setminus \{y\}$ or $\Omega = B(y,\rho) \setminus \{y\}$, any function $u\colon \Omega \to \mathbb{R}$ which is harmonic and depends only on $|x-y|$ can be expressed as*

$$u(x) = a\Gamma_y(x) + b \quad \textit{for } x \in \Omega \setminus \{y\}$$

for some real constants a and b.

Proof. For the case $\Omega = \mathbb{R}^n \setminus \{y\}$, see the discussion above. For the case $\Omega = B(y,\rho) \setminus \{y\}$, the function $u\colon B(y,\rho) \setminus \{y\} \to \mathbb{R}$ can be written as $u = g \circ r$ where $g\colon \mathbb{R} \to \mathbb{R}$ and $r\colon B(y,\rho) \to \mathbb{R}$. The same methods as in the discussion above can then be applied, with the additional requirement that $r < \rho$. □

As the Laplacian is linear, we can observe that for any finite sequence of points $(y_i)_{i=1}^m$ and "point charges", i.e. real coefficients $(q_i)_{i=1}^m$, the sum

$$u(x) = \sum_{i=1}^{m} \Gamma_{y_i}(x) q_i$$

is a harmonic function in $\mathbb{R} \setminus \{y_i\}_{i=1}^m$. If we replace the finite set of points with an infinite closed set of points E, and replace the point charges with a signed measure of charge, $\mu\colon \mathcal{B}(E) \to \mathbb{R}$, we get the integral

$$u(x) = \int_E \Gamma_y(x)\, d\mu(y).$$

This suggests that the function given by the above formula should be harmonic in $\mathbb{R}^n \setminus E$, a fact which will be proven in Proposition 3.10.

Definition 3.9. Let $\mu\colon \mathcal{B}(E) \to \mathbb{R}$ be an signed measure on the closed set $E \subset \mathbb{R}^n$. When $n \geq 3$, the *Newtonian potential* of μ is defined in $\mathbb{R}^n \setminus E$ as

$$u(x) = \int_E \Gamma_y(x)\, d\mu(y).$$

When $n = 2$, the function given above is called the *logarithmic potential.*

The part of Γ_y that is used when $n \geq 3$ is sometimes called the *Newton kernel*, likewise the part that is used when $n = 2$ is called the *logarithmic kernel.*

Proposition 3.10. *The Newtonian potential (or logarithmic potential if $n = 2$) of μ in $E \subset \mathbb{R}^n$ is harmonic in $\mathbb{R}^n \setminus E$.*

Proof. Let u be the Newtonian potential of μ in A. Assume that $x \in \mathbb{R}^n \setminus E$. According to Proposition 2.24 we can write

$$\Delta u(x) = \Delta \int_E \Gamma_y(x)\, d\mu(y) = \int_E \Delta\Gamma_y(x)\, d\mu(y).$$

From (3.4) we get the fact that $\Delta\Gamma_y(x) = 0$ when $x \neq y$, so the integrand is identically zero. □

3.3 Connections to Complex Analysis

The space of harmonic functions in two dimensions is closely related to the space of analytic functions in the complex plane.

Definition 3.11. A function $f \colon \Omega_{\mathbb{C}} \to \mathbb{C}$, where $\Omega_{\mathbb{C}} \subset \mathbb{C}$, is said to be *complex analytic* or simply *analytic* if for every $y \in \Omega_{\mathbb{C}}$, the function can be written as a convergent power series of the form

$$f(x) = \sum_{k=0}^{\infty} c_k (x-y)^k, \tag{3.5}$$

for some sequence of coefficients $(c_k)_{k=0}^{\infty}$ in $\mathbb{C}$.

A function $f \colon \Omega \to \mathbb{R}$ is said to be *real analytic* if for every $y \in \Omega$, it can be written as a convergent power series of the form (3.5) for some sequence of coefficients $(c_k)_{k=0}^{\infty}$ in $\mathbb{R}$.

Any function which is real or complex analytic is infinitely differentiable, which can be shown by termwise differentiation of the power series. For complex functions the converse is also true, i.e. the space of complex analytic functions is identical to $\mathcal{C}^\infty(\mathbb{C})$ [1, p. 48], but for real analytic functions this is not the case. For example, the function (3.8), which we show is infinitely differentiable in Lemma 3.19, is not analytic at the origin.

Recalling the Cauchy–Riemann equations from complex analysis, it can be shown that the real and imaginary components of every analytic function in the complex plane, belong to $\mathcal{H}(\mathbb{R}^2)$. In the following theorem and the remainder of this chapter, we will use the notation $\Omega_{\mathbb{C}}$ for the subset of $\mathbb{C}$ given by

$$\Omega_{\mathbb{C}} = \{x + iy : (x, y) \in \Omega\}$$

for some $\Omega \subset \mathbb{R}^2$.

Theorem 3.12 (The Cauchy–Riemann equations)**.** *A complex function of one variable $f(x+iy) = u(x,y) + iv(x,y)$ is analytic in $\Omega_{\mathbb{C}} \subset \mathbb{C}$ if and only if its components are differentiable, and both equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied in $\Omega \subset \mathbb{R}^2$.

Proposition 3.13. *If a complex function $f = u + iv$ is analytic in $\Omega_{\mathbb{C}}$, then its components u and v are harmonic in $\Omega \subset \mathbb{R}^2$.*

Proof. Employing the Cauchy–Riemann equations as well as the mixed partial derivative property from multivariable calculus, we get

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right) = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0$$

and

$$\Delta v = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\frac{\partial u}{\partial x} = -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y} = 0.$$

Note that the second derivatives of u and v are well defined because an analytic function f is in $\mathcal{C}^\infty(\mathbb{C})$, and therefore its components are also infinitely differentiable. ☐

The converse of the theorem is not true in general, i.e. for two harmonic functions $u, v \in \mathcal{H}(\Omega)$ the function $f = u + iv$ need not be analytic in $\Omega_{\mathbb{C}}$. For example, for the harmonic functions $u(x,y) = v(x,y) = \ln\sqrt{x^2+y^2} \in \mathcal{H}(\mathbb{R}^2)$, the Cauchy–Riemann equations are not satisfied when $(x,y) \neq (0,0)$, since then at least one of the following inequalities hold:

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{x}{x^2+y^2} \neq \frac{y}{x^2+y^2} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= \frac{y}{x^2+y^2} \neq \frac{-x}{x^2+y^2} = -\frac{\partial v}{\partial x}. \end{aligned}$$

The following theorems for harmonic functions, the Maximum principle and Liouville's theorem, have direct analogues for analytic functions in complex analysis. Firstly, a harmonic or analytic function cannot attain an extremum in an open connected set without being constant.

Theorem 3.14 (The maximum principle)**.** *Let $u\colon \Omega \to \mathbb{R}$ be a harmonic function.*

(a) *If u attains a local extremum at x, then u is constant in some neighbourhood of x.*

(b) *If u attains a global extremum in Ω, and Ω is connected, then u is constant in Ω.*

This proof is given by Armitage and Gardiner [2, p. 5].

Proof. The proof is identical in structure regardless of whether the extremum is a minimum or maximum. Assume that u attains a local maximum at x. Choose ρ so that $B(x,\rho) \subset \Omega$ and $u(y) \leq u(x)$ for all $y \in B(x,\rho)$. Because $u(x) = \mathcal{A}(u;x,\rho)$, u must be identically $u(x)$ in $B(x,\rho)$, so (a) is satisfied.

To prove (b), let $M = \{z : u(z) = \sup_{y\in\Omega} u(y)\}$, which is non-empty by the assumption that there is a global maximum in Ω. Since u is continuous, M is relatively closed in Ω, but (a) implies that for each $y \in M$, there exists a $B(y,\rho_y) \subset M$ so M is open. Connectedness of Ω gives $M = \Omega$, so (b) is satisfied. □

A proof of the same property for analytic functions can be made with a similar structure as the proof given for harmonic functions above. Here only a proof that the absolute value of the function is constant is provided, but this implies that the function itself is constant, see for example [1, p. 13]. To prove this we first need the sub-mean-value property of analytic functions.

Lemma 3.15 (Mean-value and sub-mean-value property of analytic functions)**.** *If $f\colon \Omega_{\mathbb{C}} \to \mathbb{C}$ is an analytic function, then*

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(z+\rho e^{i\theta})\,d\theta, \tag{3.6}$$

and

$$|f(z)| \leq \frac{1}{2\pi}\int_0^{2\pi} |f(z+\rho e^{i\theta})|\,d\theta, \tag{3.7}$$

for all z and $\rho > 0$ such that $B(z,\rho) \subset \Omega_{\mathbb{C}}$.

Proof. Let $f(x+iy) = u(x,y) + iv(x,y)$ where $u, v\colon \Omega \to \mathbb{R}$ are harmonic according to Proposition 3.13. The mean-value property for harmonic functions together with the linearity of integration gives

$$\begin{aligned} f(x+iy) &= u(x,y) + iv(x,y) \\ &= \mathcal{M}(u;(x,y),\rho) + i\mathcal{M}(v;(x,y),\rho) \\ &= \mathcal{M}(u+iv;(x,y),\rho) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(x+iy+\rho e^{i\theta})\, d\theta\,. \end{aligned}$$

We have thus shown (3.6). Then the triangle inequality for integrals immediately gives (3.7). □

See also [1, p. 56] for a proof using only concepts from complex analysis.

Theorem 3.16. *Let $f\colon \Omega_{\mathbb{C}} \to \mathbb{C}$ be an analytic function.*

(a) *If $|f|$ attains a local extremum at z, then $|f|$ is constant in some neighbourhood of z.*

(b) *If $|f|$ attains a global extremum in $\Omega_{\mathbb{C}}$, and $\Omega_{\mathbb{C}}$ is connected, then $|f|$ is constant in $\Omega_{\mathbb{C}}$.*

Proof. Assume that $|f|$ attains a local maximum at $z \in \Omega_{\mathbb{C}}$. Take ρ so that the circle $C_\rho = \{z + \rho e^{i\theta} : 0 \le \theta < 2\pi\}$ is in $\Omega_{\mathbb{C}}$, and

$$|f(z+\rho e^{i\theta})| \le |f(z)| \quad \text{for all } 0 \le \theta < 2\pi.$$

Combining this with the sub-mean-value property (3.7), gives that $|f|$ is constant in the circle C_ρ. Repeating this argument for all circles of radius $r \in (0,\rho)$ shows that $|f|$ is constant in the neighbourhood $\{re^{i\theta} : 0 \le r < \rho, 0 \le \theta < 2\pi\}$. □

Secondly, harmonic or analytic functions cannot be bounded on the entire space without being constant. This property is called Liouville's theorem, after Joseph Liouville, who stated it for analytic functions. Note that if a complex analytic function is bounded, then its real and imaginary parts are bounded harmonic functions, so Liouville's theorem for harmonic functions implies Liouville's theorem for analytic functions. The proof which is used here was given by Edward Nelson in 1961 [25].

Theorem 3.17 (Liouville's theorem)**.** *If a function is harmonic and bounded on $\mathbb{R}^n$, then the function is constant.*

Proof. Assume that $u \in \mathcal{H}(\mathbb{R}^n)$ and $|u| \le M$ for some constant $M \in \mathbb{R}$. Then $u(x) = \mathcal{A}(u;x,\rho)$ according to Proposition 3.3. For any two points x and y, the balls $B(x,\rho)$ and $B(y,\rho)$ can be made to differ by an arbitrarily small proportion of their volume, i.e. for any $\varepsilon > 0$ there exists a $\rho > 0$ such that $\lambda(K)/\nu_n\rho^n < \varepsilon$ where K is the symmetric difference of the balls,

$$K = (B(x,\rho) \setminus B(y,\rho)) \cup (B(y,\rho) \setminus B(x,\rho)).$$

Therefore

$$\begin{aligned} |u(x) - u(y)| &= |\mathcal{A}(u;x,\rho) - \mathcal{A}(u;y,\rho)| \\ &= \left| \frac{1}{\nu_n\rho^n} \int_K u\,d\lambda \right| \leq \frac{1}{\nu_n\rho^n} \int_K |u|\,d\lambda \\ &\leq \frac{1}{\nu_n\rho^n} \int_K M\,d\lambda = \frac{1}{\nu_n\rho^n} \lambda(K)M < \varepsilon M, \end{aligned}$$

and since ε can be made arbitrarily small, $u(x) = u(y)$, so u is constant. □

Finally, harmonic functions are guaranteed to be infinitely differentiable, and therefore real analytic. For this result we use a more general theorem which states that all continuous and mean-valued functions are infinitely differentiable (note that $\mathcal{H}(\Omega) \subset \mathcal{C}^2(\Omega)$).

If a function $f\colon \Omega_{\mathbb{C}} \to \mathbb{C}$ is twice differentiable and fulfils the Cauchy–Riemann equations, then by Proposition 3.13 its components are harmonic and thus infinitely differentiable, so the function itself is infinitely differentiable. The following theorem can therefore be used to establish Theorem 3.12, under the additional requirement that f be twice differentiable instead of once differentiable.

Theorem 3.18. *If $u \in \mathcal{C}(\Omega)$ and $u(x) = \mathcal{M}(u;x,\rho)$ for all $B(x,\rho) \Subset \Omega$, then $u \in \mathcal{C}^\infty(\Omega)$.*

To prove this, we first investigate the regularity of an auxiliary function η.

Lemma 3.19. *The function $\eta\colon \mathbb{R} \to \mathbb{R}$, given by*

$$\eta(t) = \begin{cases} Ce^{-1/t}, & \textit{for } t > 0, \\ 0, & \textit{for } t \leq 0, \end{cases} \tag{3.8}$$

for some constant $C \in \mathbb{R}$, is infinitely differentiable.

Proof. Let $g_1(t) = 1/t^2$. We see that, for $t > 0$,

$$\frac{d}{dt} Ce^{-1/t} = \frac{1}{t^2} Ce^{-1/t} = g_1(t) Ce^{-1/t}$$

and

$$\frac{d^2}{dt^2}Ce^{-1/t} = \left(\frac{d}{dt}g_1(t)\right)Ce^{-1/t} + g_1(t)\left(\frac{d}{dt}Ce^{-1/t}\right) = g_2(t)Ce^{-1/t}$$

where $g_2(t) = \frac{d}{dt}g_1(t) + g_1(t)g_1(t)$. Iterating this application of the chain rule, we get

$$\frac{d^k}{dt^k}Ce^{-1/t} = g_k(t)Ce^{-1/t},$$

where the function g_k is given by the recursive formula

$$g_k(t) = \frac{d}{dt}g_{k-1}(t) + g_{k-1}(t)g_1(t).$$

We see that the right derivative of η at zero is given by

$$\lim_{t\to 0+}\frac{\eta(t)-\eta(0)}{t} = \lim_{s\to\infty} s(Ce^{-s}-0) = 0,$$

and therefore equals the left derivative at zero. If we assume that $\eta^{(k)}(0) = 0$, then the right $(k+1)$-derivative at zero is given by

$$\lim_{t\to 0+}\frac{\eta^{(k)}(t)-\eta^{(k)}(0)}{t} = \lim_{s\to\infty}\frac{Csg_k(1/s)}{e^s} = 0,$$

because $sg_k(1/s)$ is smaller than some polynomial for large s. Therefore we can conclude by induction that $\eta \in \mathcal{C}^\infty(\mathbb{R})$. □

Having established the regularity of η, the following proof is given by Armitage and Gardiner [2, pp. 3–5].

Proof of Theorem 3.18. Assume that $u \in \mathcal{H}(\Omega)$. Let $\eta \in \mathcal{C}^\infty(\mathbb{R})$ be the function given by (3.8) where $C \in \mathbb{R}$ is chosen so that

$$\sigma_n \int_0^1 t^{n-1}\eta(1-t^2)\,dt = 1.$$

For each $k \in \mathbb{N}$, define $\eta_k \colon \mathbb{R}^n \to \mathbb{R}$ by

$$\eta_k(x) = k^n\eta(1-k^2|x|^2).$$

Then $\eta_k \in \mathcal{C}^\infty(\mathbb{R}^n)$ because it is the composition of functions in $\mathcal{C}^\infty$. If $\Omega \neq \mathbb{R}^n$, let $\Omega_k = \{x \in \Omega : \operatorname{dist}(x,\partial\Omega) > 1/k\}$. Otherwise, let $\Omega_k = \mathbb{R}^n$. Define $U_k \colon \Omega_k \to \mathbb{R}$ by

$$U_k(x) = \int_\Omega \eta_k(x-y)u(y)\,d\lambda(y)$$

for $x \in \Omega_k$. Then $U_k \in \mathcal{C}^\infty(\Omega_k)$, because η_k and all its derivatives are bounded on $\mathbb{R}^n$ and vanish outside $B(0, 1/k)$. Further, because $u(x) = \mathcal{M}(u; x, t)$,

$$\begin{aligned}
U_k(x) &= \int_0^{1/k} k^n \eta(1 - k^2 t^2) \int_{\partial B(x,t)} u \, d\sigma \, dt \\
&= u(x)\sigma_n \int_0^{1/k} k^n t^{n-1} \eta(1 - k^2 t^2) \, dt \\
&= u(x)\sigma_n \int_0^1 s^{n-1} \eta(1 - s^2) \, ds \\
&= u(x)
\end{aligned}$$

in Ω_k. Therefore

$$u \in \mathcal{C}^\infty \left(\bigcup_{k=1}^{\infty} \Omega_k \right) = \mathcal{C}^\infty(\Omega). \qquad \square$$

3.4 Super- and Subharmonic Functions

The concepts of super- and subharmonic functions have been in use in potential theory since the beginning of the study of the Dirichlet problem. In 1890, Henri Poincaré made use of something similar to these in his method of sweeping out [29]. This method will be discussed in-depth in Section 4.3.

To justify the definitions of super- and subharmonic functions, we first study the one-dimensional case. Harmonic functions in one dimension are simply linear functions $f(x) = ax + b$ for some constants a and b, as can be easily verified by integrating

$$\Delta f = \frac{\partial^2 f}{\partial x^2} = 0$$

twice. Superharmonic functions in one dimension are simply the concave functions, and subharmonic functions in one dimension are the convex functions. An equation like (3.4) for the super- and subharmonic functions does exist, however due to the fact that super- and subharmonic functions are not necessarily twice differentiable, or even continuous, we must make use of the *weak Laplacian* for this. See Definition 3.25 and the subsequent discussion.

If a function is both concave and convex, it is a linear function. Likewise, if a function is both super- and subharmonic, it is harmonic.

Definition 3.20. A function $\psi \colon \Omega \to (-\infty, \infty]$ is said to be *superharmonic* if:

(a) ψ is *super-mean-valued*, i.e. $\psi(x) \geq \mathcal{M}(\psi; x, \rho)$ whenever $B(x, \rho) \Subset \Omega$,

(b) $\psi \not\equiv \infty$ in any component of Ω,

(c) ψ is lower semicontinuous (Definition 2.12) in Ω.

Definition 3.21. A function $\varphi\colon \Omega \to [-\infty, \infty)$ is said to be *subharmonic* if:

(a) φ is *sub-mean-valued*, i.e. $\varphi(x) \leq \mathcal{M}(\varphi; x, \rho)$ whenever $B(x, \rho) \Subset \Omega$,

(b) $\varphi \not\equiv -\infty$ in any component of Ω,

(c) φ is upper semicontinuous in Ω.

Note that φ is subharmonic if and only if $-\varphi$ is superharmonic. Thus, all of the following results about superharmonic functions have direct analogues for subharmonic functions.

The definition of subharmonicity using the sub-mean-value property was first given by Frigyes (sometimes spelled Frédéric) Riesz in 1926 [31, p. 331] as: A function $u\colon \Omega \to \mathbb{R}$, where $\Omega \subset \mathbb{R}^2$, is subharmonic if for every point $(x_0, y_0) \in \Omega$, there exists some $r > 0$ such that

$$u(x_0, y_0) \leq \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r\cos\varphi, y_0 + r\sin\varphi)\, d\varphi = \mathcal{M}(u; (x_0, y_0), r).$$

Earlier definitions of subharmonicity, such as in [28, p. 43], relied on the function u being less than or equal to its Poisson modification, $M_B f$, (see Definition 4.12 and cf. (4.7)) for each disc $B \subset \mathbb{R}^2$. This however results in the same set of functions being considered subharmonic (to see this, consider Theorem 3.31 together with Proposition 4.9).

As stated above, one of the reasons super- and subharmonic functions are useful is that they can be used as a characterization for harmonicity.

Proposition 3.22. *A function is harmonic in Ω if and only if it is both super- and subharmonic in Ω.*

Proof. Proving that harmonic functions are super- and subharmonic is simply a matter of noticing that all the conditions in the definition are satisfied.

Conversely, assume that u is super- and subharmonic in Ω. Then, u is upper semicontinuous and lower semicontinuous, so it is continuous in Ω (as it is never allowed to attain the values $\pm\infty$ by the definition of super- and subharmonicity). Likewise, u has the spherical mean-value property in Ω because it is both super-mean-valued and sub-mean-valued. By Theorem 3.18, $u \in \mathcal{C}^2(\Omega) \subset \mathcal{C}^\infty(\Omega)$, so all conditions in Definition 3.1 are fulfilled. □

As one would hope, the definition of superharmonicity can be straightforwardly related to the Laplacian.

Proposition 3.23. *If $\psi \in \mathcal{C}^2(\Omega)$ and $\Delta\psi \leq 0$ in Ω, then*

$$\psi(x) \geq \mathcal{M}(u; x, \rho)$$

whenever $B(x,\rho) \Subset \Omega$, which implies that ψ is superharmonic.

Proof. We require that the function u is twice continuously differentiable for the Laplacian to be well defined. This immediately gives (b) and (c) in Definition 3.20.

Following the same steps as in the proof of Theorem 3.4, we apply Green's identity and differentiate under the integral sign to get (3.2),

$$\int_{B(x,\rho)} \Delta\psi \, d\lambda = \sigma_n \rho^{n-1} \frac{d}{d\rho} \mathcal{M}(\psi; x, \rho),$$

for all balls $B(x,\rho) \Subset \Omega$. Substituting $\Delta\psi \leq 0$ gives

$$0 \geq \frac{d}{d\rho} \mathcal{M}(\psi; x, \rho),$$

so $\mathcal{M}(\psi; x, \rho)$ is a non-increasing function, and as $\mathcal{M}(\psi; x, 0) = \psi(x)$ we get

$$\psi(x) \geq \mathcal{M}(\psi; x, \rho). \qquad \square$$

Note that unlike harmonic functions, there exist superharmonic functions which are not $\mathcal{C}^2(\mathbb{R}^n)$, or even $\mathcal{C}(\mathbb{R})$, so the Laplacian cannot be used to prove properties of superharmonic functions in the same extent as for harmonic functions, without using the weak formulation of the Laplacian, which we introduce briefly below.

Definition 3.24. For a function $f\colon \mathbb{R}^n \to \mathbb{R}$, its *support* is the set

$$\operatorname{supp} f = \overline{\{x \in \mathbb{R}^n : f(x) \neq 0\}}.$$

For the set of smooth functions with compact support, we use the symbol

$$\mathcal{C}_c^\infty(\mathbb{R}^n) = \{\varphi \in \mathcal{C}^\infty(\mathbb{R}^n) : \operatorname{supp} \varphi \text{ is compact}\}.$$

We use the notation

$$\langle u, \varphi \rangle = \int_{\mathbb{R}^n} u\varphi \, d\lambda,$$

where $u\colon \mathbb{R}^n \to \mathbb{R}$ is locally integrable and $\varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n)$. The function φ (which should not be confused with a subharmonic function, for which we also use the

Greek letter phi) is called a *test function*. Let $A \subset \mathbb{R}^n$ be an open set which contains the support of φ. Then by use of Theorem 2.27 (b), we get for each $u \in \mathcal{C}^2(\mathbb{R})$, that

$$\begin{aligned}\langle \Delta u, \varphi \rangle &= \int_{\mathbb{R}^n} \varphi \Delta u \, d\lambda = \int_A \varphi \Delta u \, d\lambda \\ &= \int_A u \Delta \varphi \, d\lambda + \int_{\partial A} u(\nabla \varphi \cdot \widehat{n}) - \varphi(\nabla u \cdot \widehat{n}) \, d\sigma \\ &= \int_A u \Delta \varphi \, d\lambda = \langle u, \Delta \varphi \rangle,\end{aligned}$$

as $\nabla \varphi$ and φ both vanish on ∂A. This justifies defining the Laplacian of a distribution $u \in (\mathcal{C}_c^\infty(\mathbb{R}^n))'$ as follows:

Definition 3.25. For any distribution $u \in (\mathcal{C}_c^\infty(\mathbb{R}^n))'$, its Laplacian Δu is given by the formula

$$\langle \Delta u, \varphi \rangle = \langle u, \Delta \varphi \rangle \quad \text{for all } \varphi \in \mathcal{C}_c^\infty(\mathbb{R}^n).$$

We can now note that each superharmonic function $\psi \colon \mathbb{R} \to (-\infty, \infty]$ has non-positive Laplacian in a weak sense, i.e. $\Delta \psi_D \leq 0$, where $\psi_D \in (\mathcal{C}_c^\infty)'$ is the distribution corresponding to the function ψ. Likewise, each subharmonic function $\varphi \colon \mathbb{R} \to [-\infty, \infty)$ has non-negative Laplacian in a weak sense, i.e. $\Delta \varphi_D \geq 0$. However this, along with the weak solutions to the Dirichlet problem, will not be discussed further in this text.

Superharmonic functions obey the minimum principle, while subharmonic functions obey the maximum principle. This can be compared with with the maximum principle for harmonic functions, Theorem 3.14.

Theorem 3.26 (The minimum principle)**.** *Let $\psi \colon \Omega \to \mathbb{R}$ be a superharmonic function.*

(a) *If ψ attains a local minimum at x, then ψ is constant in some neighbourhood of x.*

(b) *If ψ attains a global minimum in Ω, and Ω is connected, then ψ is constant in Ω.*

Proof. Assume that ψ attains a local minimum at x. Choose ρ so that $B(x, \rho) \Subset \Omega$ and $\psi \geq \psi(x)$ in $B(x, \rho)$. Because $\psi(x) \leq \mathcal{M}(u; x, \rho)$, the function ψ must be identically $\psi(x)$ on $B(x, \rho)$, so (a) is satisfied.

Moving on to the second part, we note that since ψ is lower semicontinuous, the set $M = \{y : \psi(y) = \psi(x)\}$ is closed, but from (a), M is open. By connectedness, $M = \Omega$, so (b) is satisfied. $\square$

Proposition 3.27. *The space of superharmonic functions is a so-called* convex cone*, meaning that if ψ_1 and ψ_2 are superharmonic functions and $c \in \mathbb{R}, c \geq 0$, is a non-negative constant, then $c\psi_1$ and $\psi_1 + \psi_2$ are superharmonic functions.*

Note that the use of the term convex in the theorem above has no relation to the appearance of the functions themselves, as it simply states that the space is closed under convex combinations, i.e. linear combinations of the form $t\psi_1 + (1-t)\psi_2$ for $0 \leq t \leq 1$. The term cone is used to mean that the space is closed under scalings of the form $t\psi$ for $t \geq 0$. Note that e.g. $\psi_1 - \psi_2$ need not be superharmonic.

Proof. The functions $c\psi_1$ and $\psi_1 + \psi_2$ are clearly lower semicontinuous and greater than $-\infty$ in Ω, given that ψ_1 and ψ_2 are. Multiplying the inequality in Definition 3.20 (a) with c, we get

$$c\psi_1(x) \geq c\mathcal{M}(\psi_1; x, \rho) = \mathcal{M}(c\psi_1; x, \rho)$$

whenever $B(x, \rho) \Subset \Omega$. Likewise,

$$\psi_1(x) + \psi_2(x) \geq \mathcal{M}(\psi_1; x, \rho) + \mathcal{M}(\psi_2; x, \rho) = \mathcal{M}(\psi_1 + \psi_2; x, \rho)$$

whenever $B(x, \rho) \Subset \Omega$. $\square$

The relation between superharmonic functions and concave functions which was discussed earlier partially holds in higher dimensions, as concave functions are always superharmonic. However, the converse does not necessarily hold. To prove this, we first define the notion of convex and concave functions in higher dimensions.

Definition 3.28. A function $f \colon \Omega \to \mathbb{R}$ defined in a convex set Ω, is said to be *convex* if for any $x, y \in \Omega$ and $t \in [0, 1]$,

$$f((1-t)x + ty) \leq (1-t)f(x) + tf(y).$$

A function f is said to be *concave* if $-f$ is convex. This is equivalent to the condition that

$$f((1-t)x + ty) \geq (1-t)f(x) + tf(y)$$

for any $x, y \in \Omega$ and $t \in [0, 1]$.

Proposition 3.29. *If a function $f \in \mathcal{C}^2(\Omega)$ is concave, then it is superharmonic.*

Proof. This follows from the fact that a twice differentiable function is concave if and only if the *Hessian matrix*,

$$Hf = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & & \vdots \\ \vdots & & \ddots & \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix},$$

is negative definite or negative semidefinite, i.e. all of its eigenvalues are non-positive [14, Section 7.0]. The Laplacian is equal to the *trace* of the Hessian,

$$\Delta f = \operatorname{tr}(Hf) = \frac{\partial^2 f}{\partial x_1^2} + \cdots + \frac{\partial^2 f}{\partial x_n^2},$$

and since the trace of a matrix is equal to the sum of the eigenvalues of the matrix, the Laplacian is non-positive. Applying Proposition 3.23 then gives the desired result. □

The following function is an example of a function which is harmonic, and therefore both super- and subharmonic, but neither concave nor convex.

$$\begin{aligned} &u \colon \mathbb{R}^2 \to \mathbb{R}, \\ &u(x_1, x_2) = x_1^2 - x_2^2. \end{aligned} \tag{3.9}$$

The function is harmonic, as $\Delta u = 0$, but as seen in Figure 3.1 we can easily find chords which lie both below and above the plot of the function. It is therefore neither convex nor concave.

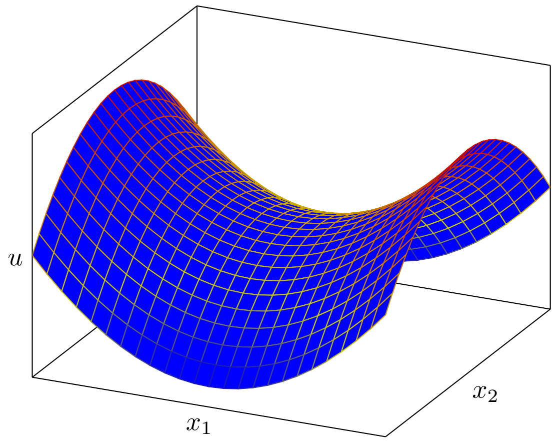

Figure 3.1: The function given in (3.9).

We also note the somewhat obvious property that a function which is harmonic on a set Ω, is also harmonic on all open subsets E of Ω. This property also holds for super- and subharmonic functions, with a slight caveat to avoid the case of functions which are identically ∞.

Proposition 3.30. *If ψ is superharmonic in Ω and $E \subset \Omega$ is a domain, then ψ is either superharmonic or identically ∞ in E.*

Proof. If $B \Subset E$ then $B \Subset \Omega$, so ψ is super-mean-valued in E. The function is also trivially lower semicontinuous in E. Therefore, if the function is not identically ∞ in E, all conditions in Definition 3.20 are fulfilled, so the function is superharmonic. □

The case $\psi \equiv \infty$ is not actually necessary, because (as we shall see) a superharmonic function can only have the value ∞ on a so-called *polar* set, which always has measure zero, and thus cannot be a domain. This will be discussed at the end of Section 5.5, in particular in Proposition 5.19.

There is an alternative definition of superharmonicity: a function is superharmonic if it lies above any harmonic function with equal or lower boundary values. Here we state it as a theorem. Likewise, a function is subharmonic if and only if it lies below any harmonic function with equal or greater boundary values. See e.g., [13, p. 62–63] for a proof.

Theorem 3.31. *A function $\psi \colon \Omega \to (-\infty, \infty]$ is superharmonic if and only if* *it holds that*

$$\psi \geq u \text{ on } \partial K \quad \text{implies} \quad \psi \geq u \text{ in } K,$$

for any harmonic $u \in \mathcal{C}(K)$ *where* $K \Subset \Omega$.

Finally, if a function is locally super-mean-valued, then it satisfies the minimum principle and in fact it is superharmonic.

Definition 3.32. We say that a function $u\colon \Omega \to [-\infty, \infty]$ is *locally super-mean-valued* in Ω if u is not identically ∞ in any component of Ω and for any $x \in \Omega$, there exists a $\rho > 0$ such that $B(x, \rho) \Subset \Omega$ and

$$u(x) \geq \mathcal{M}(u; x, r) \quad \text{for all } 0 < r < \rho.$$

Theorem 3.33. *If* u *is locally super-mean-valued in* Ω*, then* u *satisfies the minimum principle in* Ω.

The above is shown in e.g., [13, pp. 61–62]. Now we can prove the following theorem, which is essentially the converse of Proposition 3.30.

Theorem 3.34. *If* u *is locally super-mean-valued in* Ω*, then* u *is superharmonic in* Ω.

Proof. If suffices to show that u is superharmonic in each open and connected $K \Subset \Omega$. Let v be a continuous function in $\overline{K}$, harmonic in K and such that $u \geq h$ on ∂K. The function $u - h$ is locally super-mean-valued, so by Theorem 3.33, satisfies the minimum principle in K. Because $u - h \geq 0$ on ∂K, $u - h \geq 0$ in K by the minimum principle. Thus u is greater than any harmonic function with the same boundary values, so by Theorem 3.31, is superharmonic in K. □

The above proof is based on [13, p. 65].

By using Theorem 3.34 and its corresponding result for locally sub-mean-valued functions, we can show the following.

Corollary 3.35. *If* u *is harmonic in every ball* $B \Subset \Omega$*, then* u *is harmonic in* Ω.

Proof. As u is harmonic and therefore mean-valued in each $B \Subset \Omega$, clearly it is both locally super-mean-valued and locally sub-mean-valued. Thus u is both superharmonic and subharmonic, and by Proposition 3.22, harmonic in Ω. □

3.5 The Dirichlet Problem

The Dirichlet problem asks the question of whether, given a measurement of the electrical potential on the surface of an electrical conductor, or mathematically speaking, the boundary of a domain, we are able to reconstruct the potential in the interior or exterior of this conductor.

The Dirichlet problem in its original form was first studied in the early nineteenth century by Peter Gustav Lejeune Dirichlet, after whom it is named, as well by as William Thomson (also known as Lord Kelvin) and Carl Friedrich Gauss.

Definition 3.36. Given a bounded open set $\Omega \subset \mathbb{R}^n$, and a continuous function $f\colon \partial\Omega \to \mathbb{R}$ on the boundary $\partial\Omega$, solving the *(interior) Dirichlet problem* consists of finding a function $u\colon \Omega \to \mathbb{R}$ which is harmonic in Ω and agrees with f on the boundary, i.e. $\lim_{y\to x} u(y) = f(x)$ for each $x \in \partial\Omega$.

A variant of the problem is called the exterior Dirichlet problem, which is solved by essentially the same theory.

Definition 3.37. Given a bounded open set $\Omega \subset \mathbb{R}^n$, a continuous function $f\colon \partial\Omega \to \mathbb{R}$, solving the *exterior Dirichlet problem* consists of finding a function $u\colon \Omega^e \to \mathbb{R}$ which is harmonic in the exterior of Ω, agrees with f on the boundary of Ω^e, and vanishes at infinity, i.e. $\lim_{|y|\to\infty} u(y) = 0$.

For the question to make sense as stated, we must require that the so-called *boundary condition* f is continuous (if it is real-valued).

Theorem 3.38. *If $u\colon \Omega \to \mathbb{R}$ is harmonic, f is real-valued, and*

$$\lim_{\Omega \ni y \to x} u(y) = f(x) \quad \textit{for all } x \in \partial\Omega \tag{3.10}$$

then f must be continuous.

Proof. If u is harmonic and therefore continuous in Ω, and equation (3.10) holds, the function given by

$$g(x) = \begin{cases} u(x), & \text{for } x \in \Omega, \\ f(x), & \text{for } x \in \partial\Omega, \end{cases}$$

is continuous on $\overline{\Omega}$ by Lemma 2.15. As $f = g|_{\partial\Omega}$ is the restriction of a continuous function, f must be continuous. □

It is relatively straightforward to prove the uniqueness of solutions to the Dirichlet problem using the maximum principle. The following approach is taken from Helms [13, p. 24].

Theorem 3.39. *If both u and v are solutions to the Dirichlet problem for the continuous boundary condition f and the bounded open set Ω, then $u = v$.*

Proof. If Ω is not connected, then the following argument can be made for each connected component of Ω separately. Therefore we assume without loss of generality that Ω is connected. Let u and v be two solutions and let

$$w(x) = \begin{cases} u(x) - v(x) & \text{for } x \in \Omega, \\ 0 & \text{for } x \in \partial\Omega. \end{cases}$$

Then w is harmonic according to Proposition 3.5 and continuous on the closure $\overline{\Omega}$ since u and v are continuous in Ω and for any $x \in \partial\Omega$, we have

$$\lim_{\Omega \ni y \to x} w(y) = \lim_{\Omega \ni y \to x} (u(y) - v(y)) = f(x) - f(x) = 0 = w(x).$$

Assume without loss of generality that the solutions differ at $z \in \Omega$ so that $u(z) > v(z)$. Since w is continuous on the compact set $\overline{\Omega}$ it must attain a maximum on the set, by Theorem 2.14. By definition, w is zero on the boundary. Because the maximum must be at least as large as $w(z) = u(z) - v(z) > 0$, it is attained in the interior Ω. However, the maximum principle (Theorem 3.14) leads to a contradiction since w is not constant, and so u and v cannot differ at any point. $\square$

Theorem 3.40. *If both u and v are solutions to the exterior Dirichlet problem for the continuous boundary condition f and the open set Ω, then $u = v$.*

Proof. Assume without loss of generality that Ω^e is connected. Take two solutions u, v and define

$$w(x) = \begin{cases} u(x) - v(x) & \text{for } x \in \Omega^e, \\ 0 & \text{for } x \in \partial\Omega^e. \end{cases}$$

As in the previous proof, $w \in \mathcal{H}(\Omega^e)$ and $w \in \mathcal{C}(\overline{\Omega^e})$. Assume that the solutions differ at $z \in \Omega^e$ so that $u(z) > v(z)$. This gives

$$\sup_{x \in \overline{\Omega^e}} w(x) \geq u(z) - v(z) > 0.$$

Because $w(x) \to 0$ as $|x| \to \infty$, there is some $R > 0$ such that

$$w(x) < u(z) - v(z) \quad \text{for all } |x| > R.$$

This implies that the supremum of w over $\overline{\Omega^e}$ is equal to the supremum of w over $\overline{\Omega^e} \cap \overline{B(0, R)}$. Because the latter set is compact, this supremum must be attained at some point in the set, and because $w \equiv 0$ on $\delta\Omega^e \cap \overline{B(0, R)}$, this point must lie in the interior $\Omega^e \cap \overline{B(0, R)}$. However, this contradicts the maximum principle as w is not constant. Thus the assumption that u and v differ at z must be false and therefore $u \equiv v$ in Ω^e. $\square$

A partial counterexample to the above theorem is obtained by taking $\Omega = B(0,1) \subset \mathbb{R}^2$ and $f = 0$. Then both $u(x) = 0$ and $v(x) = \ln|x|$ are harmonic in Ω^e and have f as the limit on the boundary. However, $\lim_{|x|\to\infty} v(x) = \infty$, so v is not a solution to the exterior Dirichlet problem in the sense of Definition 3.37. This example justifies the requirement of a certain limit at infinity. Note further that it is not enough to require that the limit be finite, as for example the functions $u(x) = |x|^{-1}$ and $v(x) = 1$ are both harmonic outside $\Omega = B(0,1) \subset \mathbb{R}^3$ and have boundary values $f = 1$. Here v fails to be a solution as in Definition 3.37 because $\lim_{|x|\to\infty} v(x) = 1 \neq 0$.

Chapter 4

Solving the Dirichlet Problem

In this chapter, we will see that several methods exist which are able to assign a harmonic function to any continuous boundary condition on any set fulfilling certain conditions.

4.1 Solid Harmonics

Here we consider the Dirichlet problem in two and three dimensions, in order to provide some concrete justification for the Poisson integration formula which will be investigated in higher dimensions in the following section. A set of solutions that fulfil

$$\Delta u = 0 \quad \text{in } \Omega, \tag{4.1}$$

for the three-dimensional unit ball $\Omega = B(0,1) \subset \mathbb{R}^3$, can be computed using separation of variables in a spherical coordinate system. We write the function as a product of a radial part and an angular part,

$$u(r,\theta,\varphi) = R(r)Y(\theta,\varphi),$$

for $r \in [0,1)$, $\theta \in [0,2\pi)$, $\varphi \in [0,\pi)$. Not all solutions to (4.1) can be written this way, but as noted in the beginning of Chapter 3, the subset of solutions which can, form an orthogonal basis for the set of solutions. Computing the Laplacian in spherical coordinates by the formula [26, p. 425]

$$\Delta u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2 \sin^2\varphi}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r^2 \sin\varphi}\frac{\partial}{\partial \varphi}\left(\sin\varphi\frac{\partial u}{\partial \varphi}\right)$$

gives the equation

$$
\begin{aligned}
0 = \Delta u = & \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)Y(\theta,\varphi)\right) \\
& + \frac{1}{r^2\sin^2\varphi}\frac{\partial^2}{\partial^2\theta}R(r)Y(\theta,\varphi) \\
& + \frac{1}{r^2\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}R(r)Y(\theta,\varphi)\right).
\end{aligned} \tag{4.2}
$$

Dividing by $R(r)Y(\theta,\varphi)$, multiplying by r^2 and using the fact that $Y(\theta,\varphi)$ is constant with respect to r, while $R(r)$ is constant with respect to θ and φ, we get

$$
\begin{aligned}
0 = & \frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)\right) \\
& + \frac{1}{Y(\theta,\varphi)}\left(\frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial^2\theta}Y(\theta,\varphi) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}Y(\theta,\varphi)\right)\right).
\end{aligned}
$$

By assuming that the above equation holds for some point, $\Delta u(r_1,\theta,\varphi) = 0$ and another point, $\Delta u(r_2,\theta,\varphi) = 0$, we can reason that since the part depending only on θ and φ has not changed, the part depending on r cannot have changed either, when r_1 is replaced by r_2. Because the equation should hold for all r, we conclude that

$$
\frac{1}{R(r)}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)\right) = K, \tag{4.3}
$$

or after multiplying by $R(r)$,

$$
\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)\right) = KR(r), \tag{4.4}
$$

for some constant K, which also gives

$$
\frac{1}{\sin^2\varphi}\frac{\partial^2}{\partial^2\theta}Y(\theta,\varphi) + \frac{1}{\sin\varphi}\frac{\partial}{\partial\varphi}\left(\sin\varphi\frac{\partial}{\partial\varphi}Y(\theta,\varphi)\right) = -KY(\theta,\varphi) \tag{4.5}
$$

for the same K.

The there is no problem with a potential division by zero, because if $R(r) = 0$ for some r, then (4.2) is equivalent to

$$
0 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}R(r)\right)Y(\theta,\varphi)
$$

which is true if either $Y(\theta, \varphi) = 0$, which implies (4.2), or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r} R(r)\right) = 0,$$

which is equivalent to (4.4), when $R(r) = 0$.

How exactly these two resulting equations are solved is out of scope for this text, but briefly, (4.4) can be written as the second-order Cauchy–Euler equation (a special type of ordinary differential equation),

$$r^2 \frac{\partial^2}{\partial r^2} R(r) + 2r \frac{\partial}{\partial r} R(r) - KR(r) = 0,$$

resulting in

$$R(r) = Cr^l,$$

while (4.5) can be solved by applying separation of variables again,

$$Y(\theta, \varphi) = \Theta(\theta)\Phi(\varphi),$$

and making use of the constraint that it be continuous over the sphere. From this a family of solutions is obtained,

$$u_{l,m}(r, \theta, \varphi) = Cr^l e^{im\theta} P_{l,|m|}(\cos\varphi)$$

where $l \in \mathbb{N} \cup \{0\}$ and $m \in \mathbb{Z}$, $|m| \le l$, are indices which give a convenient name to each spherical harmonic. These are called the *degree* and *order*, respectively. $C \in \mathbb{R}$ is an arbitrary constant. The functions $P_{l,k}$ are called the *associated Legendre polynomials*. A table of the first few associated Legendre polynomials can be found in e.g. [26, p. 297].

In the two-dimensional unit disc $\Omega = B(0, 1) \subset \mathbb{R}^2$, the situation further simplifies. Inserting $\varphi = \pi/2$, the angular part (4.5) simply becomes

$$\frac{\partial^2}{\partial \theta^2}\Theta(\theta) + K\Theta(\theta) = 0$$

and we obtain the family of solutions [32, pp. 165–168]

$$u_m = Cr^m e^{\pm im\theta} \tag{4.6}$$

for $m = 0, 1, 2, \ldots$, where C is an arbitrary constant.

When solving the Dirichlet problem, we are interested in the solutions that satisfy

$$u = f \quad \text{on } \partial\Omega,$$

for some given $f\colon \partial\Omega \to \mathbb{R}$, which can be written in polar coordinates as

$$u(1,\theta) = f(\theta) \quad \text{for } \theta \in [0, 2\pi).$$

Assuming for the moment that the boundary condition f has an absolutely convergent Fourier series expansion (which, by elementary Fourier analysis, is true almost everywhere for any continuous f),

$$f(\theta) = \sum_{m\in\mathbb{Z}} C_m e^{im\theta},$$

where the coefficients are given by

$$C_m = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)e^{-im\xi}\,d\xi,$$

we can let

$$u(r,\theta) = \sum_{m\in\mathbb{Z}} C_m r^{|m|} e^{im\theta},$$

so that terms of the above sum are of the form (4.6). We interchange the order of integration and summation (which can be justified because the series in the resulting expression is absolutely convergent), to get

$$\begin{aligned} u(r,\theta) &= \sum_{m\in\mathbb{Z}} \frac{1}{2\pi}\int_0^{2\pi} f(\xi)e^{-im\xi}\,d\xi\, r^{|m|}e^{im\theta} \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left(\sum_{m\in\mathbb{Z}} r^{|m|}e^{im(\theta-\xi)}\right)d\xi\,. \end{aligned}$$

The value of the series can be found as a pair of simple geometric series,

$$\begin{aligned} K(r,\theta,\xi) &:= \sum_{m\in\mathbb{Z}} r^{|m|}e^{im(\theta-\xi)} \\ &= 1 + \sum_{m=1}^{\infty} r^m e^{im(\theta-\xi)} + \sum_{m=1}^{\infty} r^m e^{im(\xi-\theta)} \\ &= 1 + \frac{re^{i(\theta-\xi)}}{1-re^{i(\theta-\xi)}} + \frac{re^{i(\xi-\theta)}}{1-re^{i(\xi-\theta)}} \\ &= \frac{1-r^2}{1-2r\cos(\theta-\xi)+r^2}. \end{aligned}$$

This function is called the *Poisson kernel* for the unit disc. In conclusion, given that f has a convergent Fourier series, the solution to the Dirichlet problem for the unit disc with boundary condition f is given by the *Poisson integral*,

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)K(r,\theta,\xi)\,d\xi\,.$$

As we shall see in the following section, it turns out that this gives a solution for any continuous boundary condition, and that this can be proved directly, without the use of Fourier analysis.

4.2 Poisson Integration on a Sphere

If the set Ω in which we wish to solve the Dirichlet problem is simply a ball, the solution can be found by the so-called Poisson integral, named after Siméon Denis Poisson. This method was known to Dirichlet as early as 1850 [10]. The use of Poisson integration to obtain solutions is the basis for two of the more general methods which we will examine: Poincaré's method of sweeping out and Perron's method of super- and subharmonic functions.

Definition 4.1. The *Poisson kernel* in $B(y,\rho)$ is defined as

$$K_{B(y,\rho)}(x,z) = \frac{1}{\sigma_n\rho}\frac{\rho^2-|y-x|^2}{|z-x|^n},$$

where $x \in B(y,\rho)$ and $z \in \partial B(y,\rho)$.

Theorem 4.2. *For any ball B and continuous boundary condition $f\colon \partial B \to \mathbb{R}$, the function given by the Poisson integral of f,*

$$u(x) = \int_{\partial B} K_B(x,z)f(z)\,d\sigma(z)$$

is the solution to the Dirichlet problem for B and f.

To prove this theorem we first introduce and prove a number of lemmas relating to the Poisson integral.

Lemma 4.3. *If μ is a signed measure for sets in $\mathcal{B}(\partial B(y,\rho))$ such that $|\mu|(\partial B(y,\rho)) < \infty$, then*

$$u(x) = \int_{\partial B(y,\rho)} K_{B(y,\rho)}(x,z)\,d\mu(z) \quad \textit{for } x \in B(y,\rho)$$

is harmonic in $B(y,\rho)$.

Proof. By Proposition 2.24, we can write

$$\begin{aligned}\Delta u(x) &= \Delta_x \int_{\partial B(y,\rho)} K_{B(y,\rho)}(x,z)\,d\mu(z)\\ &= \int_{\partial B(y,\rho)} \Delta_x K_{B(y,\rho)}(x,z)\,d\mu(z)\,.\end{aligned}$$

For convenience, let the functions v and w be defined as

$$\begin{aligned}v(x) &= |z-x|^{-n},\\ w(x) &= \rho^2 - |y-x|^2.\end{aligned}$$

Then from Proposition 2.26 (d), we have

$$\Delta_x K_{B(y,\rho)}(x,z) = \frac{1}{\sigma_n \rho}\Delta(vu) = \frac{1}{\sigma_n\rho}\big(v\Delta w + w\Delta v + 2\nabla v \nabla w\big),$$

where computations similar to those in Section 3.2 and in particular the identity (3.3) give

$$\begin{aligned}\nabla v &= n|z-x|^{-n-2}(z-x),\\ \Delta v &= 2n|z-x|^{-n-2},\\ \nabla w &= 2(y-x),\\ \Delta w &= -2n,\end{aligned}$$

which in turn gives

$$\begin{aligned}\Delta_x K_{B(y,\rho)}(x,z) &= \frac{1}{\sigma_n\rho}\frac{2n}{|z-x|^{n+2}}\big(\rho^2 - |z-x|^2 - |y-x|^2 + 2(y-x)\cdot(z-x)\big)\\ &= \frac{1}{\sigma_n\rho}\frac{2n}{|z-x|^{n+2}}\big(\rho^2 - |z-y|^2\big),\end{aligned}$$

and since $z \in \partial B(y,\rho)$ the distance is simply $|z-y|^2 = \rho^2$, and $z \neq x$ as $x \in B(y,\rho)$ which does not contain its boundary, so we get

$$\Delta_x K_{B(y,\rho)}(x,z) = 0. \qquad \square$$

Corollary 4.4. *If $f\colon \Omega \to [-\infty,\infty]$ is integrable on the surface $\partial B(y,\rho) \Subset \Omega$ then*

$$u(x) = \int_{\partial B(y,\rho)} K_{B(y,\rho)}(x,z) f(z)\,d\sigma(z)$$

is harmonic in $B(y,\rho)$.

Proof. Define the signed measure

$$\begin{aligned}\mu(S)\colon \mathcal{B}(\partial B(y,\rho)) &\to \mathbb{R}\\ \mu(S) &= \int_S f\, d\sigma\end{aligned}$$

and use Lemma 4.3. $\square$

Because continuous functions are integrable on any compact set, we can now introduce an operator $P_B\colon \mathcal{C}(\partial B) \to \mathcal{H}(B)$ which produces a harmonic function in the ball B from a continuous boundary condition on the sphere ∂B.

Definition 4.5. If f is a continuous function on the sphere ∂B, then $P_B f$ is the harmonic function defined in B by

$$P_B f(x) = \int_{\partial B} K_B(x,z) f(z)\, d\sigma(z)\,.$$

Lemma 4.6. *If $c \in \mathbb{R}$ is a constant, then $P_B c = c$.*

Proof. Let for the ball $B = B(y,\rho)$,

$$u = P_B c = \int_{\partial B} \frac{1}{\sigma_n \rho} \frac{\rho^2 - |y-x|^2}{|z-x|^n}\, d\sigma(z),$$

which is harmonic in $B \supset B \setminus \{y\}$. By Proposition 3.8, we know that since u only depends on $|y-x|$, there exist constants $a, b \in \mathbb{R}$ so that

$$u = a\Gamma_y + b,$$

but because u is finite in the whole of B, and $\Gamma_y(x) \to \infty$ as $x \to y$, it must be the case that $a = 0$ and $b = c$. $\square$

Lemma 4.7. *If $f \in \mathcal{C}(\partial B)$ is integrable relative to surface area and $u = P_B f$, then for all points $x_0 \in \partial B$,*

$$\lim_{B \ni x \to x_0} u(x) = f(x_0).$$

The proof below is similar to the one given in [13, pp. 25–26].

Proof. Let $B = B(y,\rho)$. Take some $m > f(x_0)$, and choose $\delta > 0$ such that

$$f(z) \le m \quad \text{for all } z \in B(x_0,\delta) \cap \partial B.$$

Because of the linearity of integration, we may split the Poisson integral, using the characteristic function $\chi_{B(x_0,\delta)}$, in the following manner:

$$u = P_B f = P_B(\chi_{B(x_0,\delta)} f) + P_B((1 - \chi_{B(x_0,\delta)}) f).$$

Since $f(z) \le m$ for $z \in B(y,\delta) \cap \partial B$,

$$P_B(\chi_{B(y,\delta)} f) \le P_B m = m,$$

by Lemma 4.6.

If $x \in B(x_0, \delta/2)$, $z \in \partial B$, and $|z - x_0| > \delta$, then $|z - x| > \delta/2$, because

$$\delta < |z - x_0| \le |z - x| + |x - x_0| < |z - x| + \delta/2.$$

Therefore

$$\begin{aligned} |P_B((1 - \chi_{B(x_0,\delta)}) f)(x)| &\le \int_{\partial B \setminus B(x_0,\delta)} \frac{1}{\sigma_n \rho} \frac{\rho^2 - |y - x|^2}{(\delta/2)^n} |f(z)| \, d\sigma(z) \\ &\le \frac{1}{\sigma_n \rho} \frac{\rho^2 - |y - x|^2}{(\delta/2)^n} \int_{\partial B} |f(z)| \, d\sigma(z), \end{aligned}$$

and since the numerator $\rho^2 - |y - x|^2$ goes to zero as $x \to x_0 \in \partial B(y, \rho)$, we get

$$P_B((1 - \chi_{B(x_0,\delta)}) f)(x) \to 0 \qquad \text{as } x \to x_0 \in \partial B(y,\rho).$$

We have thus established that

$$\limsup_{x \to x_0} u(x) \le \limsup_{x \to x_0} P_B(\chi_{B(y,\delta)} f) + \limsup_{x \to x_0} P_B((1 - \chi_{B(x_0,\delta)}) f) \le m,$$

and letting $m \to f(x_0)$, we get

$$\limsup_{x \to x_0} u(x) \le f(x_0).$$

Applying this to $-u$ we get

$$\limsup_{x \to x_0} (-u(x)) = \limsup_{x \to x_0} P_B(-f) \le -f(x_0),$$

so

$$\liminf_{x \to x_0} u(x) \ge f(x_0),$$

and thus

$$f(x_0) \le \liminf_{x \to x_0} u(x) \le \limsup_{x \to x_0} u(x) \le f(x_0),$$

so $\lim_{x \to x_0} u(x) = f(x_0)$. □

We thus see that the Dirichlet problem on a ball is solved by Poisson integration.

Proof of Theorem 4.2. This now follows directly from Lemma 4.7. □

Proposition 4.8. *P_B is a linear operator from $\mathcal{C}(\partial B)$ to $\mathcal{H}(B)$, meaning that if f and g are continuous functions on ∂B and $c \in \mathbb{R}$ is a constant, then $P_B(f+g) = P_B f + P_B g$ and $P_B(cf) = cP_B f$.*

Proof. By the linearity of integration we get

$$\begin{aligned} P_B(f+g) &= \int_{\partial B} K_B(x,z)(f(z)+g(z))\,d\sigma(z) \\ &= \int_{\partial B} K_B(x,z)f(z)\,d\sigma(z) + \int_{\partial B} K_B(x,z)g(z)\,d\sigma(z) \\ &= P_B f + P_B g \end{aligned}$$

and

$$\begin{aligned} P_B(cf) &= \int_{\partial B} K_B(x,z)(cf(z))\,d\sigma(z) \\ &= c\int_{\partial B} K_B(x,z)f(z)\,d\sigma(z) = cP_B f. \end{aligned}$$ □

Proposition 4.9. *If $u \in \mathcal{H}(\Omega)$, then for each ball $B \Subset \Omega$*

$$u = P_B u \quad \textit{in } B.$$

Proof. This follows from the uniqueness of solutions to the Dirichlet problem, Theorem 3.39, as the functions $u|_B$ and $P_B u$ are both harmonic in B, and approach the same boundary values, namely $u|_{\partial B}$. □

This gives us an alternative justification for the equivalence of Definition 3.1 and Theorem 3.4: We see that if $\Delta u = 0$ in Ω then

$$u(x) = P_B u(x) = \int_{\partial B} K_B(x,z)f(z)\,d\sigma(z),$$

for any $B \Subset \Omega$, and in particular if $B = B(0,\rho)$,

$$\begin{aligned} u(0) = P_B u(0) &= \int_{\partial B} \frac{1}{\sigma_n \rho}\frac{\rho^2}{|z|^n} f(z)\,d\sigma(z), \\ &= \frac{1}{\sigma\rho^{n-1}}\int_{\partial B} f\,d\sigma = \mathcal{M}(f;0,\rho). \end{aligned} \tag{4.7}$$

We can get the spherical mean-value property for points $x \neq 0$ by translating the above equations.

Having established a “canonical” representation of any harmonic function in a fixed ball $B(y, \rho)$, we can prove Harnack's inequality, which is a statement about how much a harmonic function can maximally vary from the value of $u(y)$ in the ball $B(y, \rho)$.

Theorem 4.10 (Harnack's inequality)**.** *For any non-negative harmonic function $u\colon B(y, \rho) \to [0, \infty)$, the following inequalities hold for all $x \in B(y, \rho)$,*

$$\frac{(\rho - |y-x|)\rho^{n-2}}{(\rho + |y-x|)^{n-1}} u(y) \leq u(x) \leq \frac{(\rho + |y-x|)\rho^{n-2}}{(\rho - |y-x|)^{n-1}} u(y)$$

Proof. We verify by calculation that the Poisson kernel satisfies the inequality

$$\frac{\rho - |y-x|}{\sigma_n \rho(\rho + |y-x|)^{n-1}} \leq K_{B(y,\rho)}(x, z) = \frac{\rho^2 - |y-x|^2}{\sigma_n \rho |z-x|^n} \leq \frac{\rho + |y-x|}{\sigma_n \rho(\rho - |y-x|)^{n-1}}, \tag{4.8}$$

when $x \in B(y, \rho)$ and $z \in \partial B(y, \rho)$. Because $\rho + |y-x| \geq |z-x| \geq \rho - |y-x|$ we get

$$\frac{\rho - |y-x|}{(\rho + |y-x|)^{n-1}} \leq \frac{(\rho - |y-x|)(\rho + |y-x|)}{|z-x|^n} \leq \frac{\rho + |y-x|}{(\rho - |y-x|)^{n-1}},$$

which is precisely (4.8) after multiplying by $\sigma_n \rho$. So, writing

$$u(x) = \int_{\partial B(y,\rho)} K_{B(y,\rho)}(x, z) u(z)\, d\sigma(z)$$

and integrating gives the desired inequality. □

An application of this, called Harnack's principle, states that a sequence of harmonic functions converges to a harmonic function under certain conditions [27, p. 615].

Theorem 4.11 (Harnack's principle)**.** *Consider a monotone non-decreasing sequence $u_1 \leq u_2 \leq \dots$ of functions which are harmonic in the domain Ω. If there exists a point $x \in \Omega$ at which the sequence is bounded above,*

$$u_k(x) \leq M \text{ for all } k \in \mathbb{N},$$

for some constant $M \in \mathbb{R}$, then the sequence (u_k) converges locally uniformly to a function u, which is harmonic in Ω.

The same holds for a monotone non-increasing sequence $u_1 \geq u_2 \geq \dots$ of harmonic functions, which is bounded below.

For a proof of this theorem, see for example [2, p. 15] or [13, p. 56].

4.3 Sweeping Out Method

The sweeping out method, also called the *balayage* method from the French word for sweeping, was introduced in 1890 by the French mathematician and physicist Henri Poincaré (1854–1912) [29, pp. 211–227]. The method involves approximating the domain Ω, for which the problem is to be solved, by a collection of balls. By iterating a transformation of the function, which is called *sweeping* the function, a function which is harmonic on the domain may be obtained by starting with a non-harmonic function which fulfils the boundary condition.

The sweeping of a function is given by an operator $M_B\colon \mathcal{C}(\Omega) \to \mathcal{C}(\Omega)$, which is defined by use of the Poisson integral in the following fashion:

Definition 4.12. If $f \in \mathcal{C}(\Omega)$, let for any ball $B \Subset \Omega$ the function $M_B f \in \mathcal{C}(\Omega)$ be defined by

$$M_B f(x) = \begin{cases} f(x), & \text{if } x \in \Omega \setminus B, \\ P_B f(x), & \text{if } x \in B. \end{cases}$$

The function $M_B f$ is called the *Poisson modification* of f.

The continuity of $M_B f$ in $\Omega \setminus B$ is obvious, and because harmonic functions are continuous, we have continuity in B. By Lemma 4.7 we have continuity on the boundary of B, so $M_B f$ is indeed continuous in Ω. By Proposition 4.8 we know that M_B is a linear operator from $\mathcal{C}(\Omega)$ to $\mathcal{C}(\Omega)$. In a sense, this operator makes a function "more harmonic", because the resulting function is guaranteed to be harmonic in the ball B.

If we can iterate this process over a set of balls whose union covers the entirety of Ω, the function resulting from this iteration can reasonably be expected to be harmonic. Aiming to prove this, we start by considering the effect of M_B on superharmonic functions. The proof of the following lemma is adapted from the proof of [3, Theorem 11.5].

Lemma 4.13. *If ψ is a superharmonic continuous function in Ω, then for any ball $B \Subset \Omega$, the function $M_B\psi$ is superharmonic, continuous, and $M_B\psi \le \psi$ in Ω.*

Proof. We first prove that $M_B\psi \le \psi$ in Ω. Because the functions ψ and $M_B\psi$ are identical in $\Omega \setminus B$, we only need to consider B. The function $-M_B\psi$ is harmonic in B, so it is also necessarily superharmonic in B. Therefore the sum $\psi + (-M_B\psi)$ is superharmonic, by Proposition 3.27. It is also continuous, as it is the sum of two continuous functions. Because $\psi - M_B\psi = 0$ on ∂B, the minimum principle (Theorem 3.26) implies that $0 \le \psi - M_B\psi$, or equivalently

$$M_B\psi(x) \le \psi(x) \quad \text{for all } x \in \Omega. \tag{4.9}$$

We proceed by proving that $M_B\psi$ is superharmonic in Ω. It is clear that $M_B\psi$ is lower semicontinuous and $M_B\psi \not\equiv \infty$ in (all components of) Ω, i.e. the latter two conditions of Definition 3.20 are fulfilled. Therefore we only need to show that

$$M_B\psi(x) \ge \mathcal{M}(M_B\psi; x, \rho) \tag{4.10}$$

whenever $x \in \Omega$ and ρ is such that $B(x,\rho) \Subset \Omega$. See Figure 4.1 for a sketch of this, where $D = B(x,\rho)$.

We start by examining the case $x \in \Omega \setminus B$. By the inequality (4.9), together with the fact that ψ is superharmonic, we have

$$M_B\psi(x) = \psi(x) \ge \mathcal{M}(\psi; x, \rho) \ge \mathcal{M}(M_B\psi; x, \rho).$$

For the case $x \in B$, M_B is harmonic in B, so

$$M_B\psi(x) = \mathcal{M}(M_B\psi; x, r) \quad \text{for all } r \text{ such that } B(x,r) \Subset B.$$

Thus, u is locally super-mean-valued in the entire set Ω, and by Theorem 3.34, is superharmonic in Ω. □

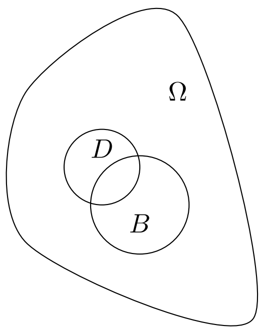

Figure 4.1: Sketch of the sets Ω, $D = B(x,\rho)$ and B which are used in the proof of Lemma 4.13.

The effect on subharmonic functions is that $M_B\varphi$ continues to be subharmonic and becomes greater than φ. This is proved in an analogous manner. Using an infinite sequence of such sweepings, a solution to the Dirichlet problem can be obtained, as seen in the following theorem.

Poincaré assumes that $\Omega \subset \mathbb{R}^3$ and that $\int_{\partial\Omega} f = 1$, and lets the initial function u_0 be the potential caused by a distribution of charge of $1/(4\pi R)$ over the surface of a large sphere $\partial B(0,R)$ such that $\Omega \Subset B(0,R)$. This function will be given by

$$u_0(x) = \begin{cases} 1, & \text{for } x \in B(0,R), \\ \frac{R}{|x-y|}, & \text{for } x \in \mathbb{R}^n \setminus B(0,R). \end{cases}$$

In particular we have $0 < u_0 \leq 1$, and $u_0(x) \to 0$ as $|x| \to \infty$.

Theorem 4.14. *Let Ω be a domain for which there exists a sequence of balls $(B_k)_{k=1}^{\infty}$ such that each ball is contained in the set, $B_k \subset \Omega$, the balls cover the interior of the set, $\bigcup_{k=1}^{\infty} B_k = \Omega$, and each ball B_k occurs infinitely many times in the sequence, e.g.*

$$(B_k)_{k=1}^{\infty} = (D_1, D_2, D_1, D_2, D_3, D_1, D_2, D_3, D_4, D_1, \dots),$$

where all of D_k are balls.

Let $u_0 \colon \Omega \to \mathbb{R}$, $u_0 \in \mathcal{C}^2(\Omega)$, be a bounded function which fulfils the following:

(a) $\lim_{\Omega \ni y \to x} u_0(y) = f(x)$ *for all* $x \in \partial\Omega$,

(b) $\Delta u_0 < 0$ *in* Ω.

Then the limit of the sequence of functions given by the recursive formula

$$u_k = M_{B_k} u_{k-1} \tag{4.11}$$

is harmonic on each component of Ω, and solves the interior Dirichlet problem for the set Ω and f.

Proof (sketch). By Lemma 4.13, we know that if u_{k-1} is superharmonic, then u_k is superharmonic and $u_k \leq u_{k-1}$. Because the function u_0 is superharmonic, we can use induction to show that $(u_k)_{k=0}^{\infty}$ is a decreasing sequence of superharmonic functions.

Consider for some fixed $k_0 \in \mathbb{N}$ the subsequence $(B_{k_j})_{j=1}^{\infty}$ which is such that $B_{k_0} = B_{k_j}$ for all $j \in \mathbb{N}$. Because each u_{k_j} is harmonic in B_{k_0}, Harnack's principle (Theorem 4.11) implies that the sequence $(u_{k_j})_{j=1}^{\infty}$ converges uniformly to the harmonic function u, if it is bounded below at any point $x \in B_{k_0}$. Hence the limit must either be $u \equiv -\infty$ or harmonic on B_{k_0}, and this argument can be repeated for each $k_0 \in \mathbb{N}$.

Given that u_0 is bounded, i.e. that there exists an M such that $|u_0| \leq M$, we can show that $|u_k| \leq M$ for all k. This can be done by induction on the facts that in $\Omega \setminus B_k$, $u_k = u_{k-1}$ implies $|u_k| = |u_{k-1}|$, and in B_k, u_{k-1} is harmonic, which implies $|u_k| \leq \sup_\Omega |u_{k-1}| \leq M$ by the maximum principle. Thus the limit u is harmonic on each component of Ω.

One can show that the boundary values are unchanged by this process, that is

$$\lim_{\Omega \ni y \to x} u(y) = \lim_{\Omega \ni y \to x} u_0(y) = f(x) \quad \text{for all } x \in \partial\Omega$$

meaning that $u = \lim_{k\to\infty} u_k$ is the solution to the Dirichlet problem for the set Ω and boundary condition f. □

4.4 Perron's Method

The Perron method, also called the Perron–Wiener–Brelot method, was independently developed in 1923 by the German mathematicians Oskar Perron (1880–1975) and Robert Remak (1888–1942), see [30, footnote 1, on p. 126]. The method involves over- and underestimating the solution using the so-called upper and lower classes, consisting of super- and subharmonic functions, respectively [28]. Perron and Remak provided proofs in $\mathbb{R}^2$, while Wiener worked in $\mathbb{R}^3$ [39, p. 22] and Brelot in $\mathbb{R}^n, n \geq 2$ [6, p. 134]

In the definition of upper and lower functions Perron makes use of the sweeping operator, M_B as defined in Definition 4.12.

Definition 4.15. For a given domain Ω and a bounded function $f \colon \partial\Omega \to \mathbb{R}$ an *upper function* $\psi \colon \Omega \to \mathbb{R}$ is a continuous function such that

(a) $\liminf_{y\to x} \psi(y) \geq f(x)$ for every $x \in \partial\Omega$,

(b) $\psi \geq M_B\psi$ in Ω, for each ball $B \Subset \Omega$.

The set of such functions is called the *upper class* and is written Ψ^f_Ω.

Definition 4.16. A *lower function* is a continuous function $\varphi \colon \Omega \to \mathbb{R}$ such that

(a) $\limsup_{y\to x} \varphi(y) \leq f(x)$ for every $x \in \partial\Omega$,

(b) $\varphi \leq M_B\varphi$ in Ω, for each ball $B \Subset \Omega$.

The set of such functions is called the *lower class* and is written Φ^f_Ω.

Note that all constant functions $g \equiv c$ where $c \geq \sup_{\partial\Omega} f$ are members of the upper class Ψ^f_Ω. Likewise, $g \in \Phi^f_\Omega$ for all $g \equiv c$ such that $c \leq \inf_{\partial\Omega} f$.

To justify the usefulness of these definitions, we can note that if u is harmonic in Ω and takes the boundary values given by f, i.e. $\lim_{\Omega \ni y \to x} u(y) = f(x)$ for all $x \in \partial\Omega$, then
$$\varphi \leq u \leq \psi \quad \text{in } \Omega,$$
for all $\varphi \in \Phi_\Omega^f$ and all $\psi \in \Psi_\Omega^f$. In this case, we also have $u \in \Phi_\Omega^f \cap \Psi_\Omega^f$.

We also see that the upper class consists of superharmonic functions, and that the lower class consists of subharmonic functions.

Proposition 4.17. *All upper functions $\psi \in \Psi_\Omega^f$ are superharmonic.*

Proof. It is clear that the function ψ is upper semicontinuous and not identically ∞. To prove that ψ is super-mean-valued, take any ball $B = B(x, \rho) \Subset \Omega$. For the centre of the ball we have
$$\psi(y) \geq M_B\psi(y) = P_B\psi(y) = \int_B \frac{1}{\sigma_n \rho} \frac{\rho^2}{|z-y|^n} \psi(z)\, d\sigma(z) = \mathcal{M}(\psi; y, \rho). \qquad \square$$

Definition 4.18. The *upper Perron solution* and the *lower Perron solution* are defined as $\overline{H}_\Omega f = \inf \Psi_\Omega^f$ and $\underline{H}_\Omega f = \sup \Phi_\Omega^f$, respectively.

Theorem 4.19. *For any bounded Ω and continuous boundary values $f\colon \partial\Omega \to \mathbb{R}$, the lower Perron solution is smaller than the upper Perron solution,*
$$\underline{H}_\Omega f \leq \overline{H}_\Omega f.$$

Proof. Suppose that $\psi \in \Psi_\Omega^f$ and $\varphi \in \Phi_\Omega^f$. Then for any $x \in \partial\Omega$, we have
$$\limsup_{y \to x} (\varphi(y) - \psi(y)) \leq \limsup_{y \to x} \varphi(y) - \limsup_{y \to x} \psi(y) \leq f(x) - f(x) = 0,$$
where we require that $y \in \Omega$ in each limit. So $\varphi - \psi$ is a subharmonic function which is less than or equal to zero near the boundary, and the maximum principle then gives
$$\varphi \leq \psi \quad \text{in } \Omega. \qquad \square$$

The definitions of the upper and lower classes have changed a number of times. For example, Perron and Remak originally only considered continuous, and hence bounded, boundary functions. Wiener explored discontinuous boundary functions; for example, in 1923 he proposed a method of solution for these [36]. Later Marcel Brelot examined the case of unbounded boundary functions. In his 1939 paper Brelot gave a new definition of upper functions (and a corresponding definition of lower functions), adding the condition that the functions in the upper class be bounded from below [6, p. 146], while relaxing the condition that

f be bounded. The latter condition implies the former: if f is bounded, then there exists an $M > 0$ such that $|f| \leq M$ on $\partial\Omega$, and then by Definition 4.15 (a), we know that

$$\liminf_{y \to x} \psi(y) \geq f(x) \geq -M \quad \text{for } x \in \partial\Omega,$$

that is, any upper function ψ is bounded from below, by the minimum principle for superharmonic functions. This property of upper functions is desirable because it is required for Theorem 4.19 to hold, that is, $\underline{H}_\Omega f \leq \overline{H}_\Omega f$.

We could for example consider $\Omega = B(0,1) \setminus 0$ and

$$f(x) = \begin{cases} \infty, & x = 0, \\ 0, & x \in \partial B(0,1). \end{cases}$$

For this boundary condition, $u_a = -a \ln |x|$ is a member of both the lower and upper class for any $a > 0$, as it is a harmonic function such that

$$\lim_{\Omega \ni y \to x} u_a(x) = f(x) \quad \text{for } x \in \partial\Omega.$$

This gives the lower Perron solution

$$\underline{H}_\Omega f = \sup \Phi_\Omega^f \geq \sup_{a>0} u_a \equiv \infty,$$

and upper Perron solution

$$\overline{H}_\Omega f = \inf \Psi_\Omega^f \leq \inf_{a>0} u_a \equiv 0,$$

so for this example it would not hold that the upper solution is greater than or equal to the lower solution, without Brelot's condition.

Another example is that Perron [28, p. 43] and Wiener [39, p. 23] required that the upper and lower functions be continuous, while Brelot [6, p. 145] considered Φ_Ω^f to consist of all "general subharmonic" functions φ on Ω such that $\limsup_{\Omega \ni y \to x} \varphi(y) \leq f(x)$, hence including also discontinuous superharmonic functions in the lower functions.

We will however restrict ourselves to bounded boundary conditions f and can therefore use Definitions 4.15 and 4.16.

We now wish to prove that the upper and lower Perron solutions are harmonic. To do this we first note that the upper class Ψ_Ω^f is closed under taking the pointwise minimum, and closed under the Poisson modification operator M_B for balls $B \Subset \Omega$. The lower class Φ_Ω^f is likewise closed under taking the pointwise maximum, and the operator M_B.

Lemma 4.20. *If $\psi_1, \dots, \psi_m$ are upper functions for Ω and f, then $\psi = \min\{\psi_1, \dots, \psi_m\}$ is an upper function for Ω and f.*

Proof. The pointwise minimum of a finite number of continuous functions is continuous. Since for any x, there is a k such that $\psi(x) = \psi_k(x) \geq \limsup_{y\to x} f(y)$, we know that $\psi(x) \geq \limsup_{y\to x} f(y)$.

For any $k = 1, \dots, m$ we have $\psi \leq \psi_k$, so

$$M_B\psi \leq M_B\psi_k \leq \psi_k \quad \text{for any ball } B \Subset \Omega.$$

Thus ψ is an upper function. □

Theorem 4.21. *For any bounded set Ω and continuous bounded boundary condition $f\colon \partial\Omega \to \mathbb{R}$, $\overline{H}f$ and $\underline{H}f$ are harmonic in Ω.*

Perron [28] made use of the Lebesgue integral to prove Theorem 4.21, while Remak [30] provided a proof using only the Riemann integral.

Proof. If $\Psi_\Omega^f = \varnothing$ then $\overline{H}f = \infty$, so we assume that $\Psi_\Omega^f \neq \varnothing$. Take a ball B such that $B \Subset \Omega$. As f is bounded, there exists an $M > 0$ such that $|f| \leq M$ on $\partial\Omega$. Let $\psi \in \Psi_\Omega^f$ be arbitrary and note that

$$v = \min\{\psi, M\}$$

is a minimum of two functions in Ψ_Ω^f and therefore a member of Ψ_Ω^f.

Let $\psi' = M_B v$, so that $\psi' = P_B v$ in B and $\psi' \leq v \leq \psi$ in Ω, by Lemma 4.13. The function ψ' is also a member of Ψ_Ω^f as it is superharmonic and has the same boundary values as v. Thus

$$\overline{H}f = \inf_{\psi\in\Psi_\Omega^f} \psi = \inf_{\psi\in\Psi_\Omega^f} \psi',$$

and because all ψ' are harmonic and therefore continuous in B, $\overline{H}f$ is upper semicontinuous, by Proposition 2.13.

Now choose some dense countable subset $E = \{x_j\}_{j=1}^\infty$ of Ω, for example $E = \Omega \cap \mathbb{Q}^n$. Because $\overline{H}f$ is the pointwise infimum of $(\psi')_{\psi\in\Psi_\Omega^f}$, we can find a sequence $(\psi_{j,k})_{k=1}^\infty \subset \Psi_\Omega^f$ of the type of functions of the form ψ' above, such that $\psi_{j,k}(x_j)$ decreases to $\overline{H}f(x_j)$ as $k \to \infty$. Consider the sequence

$$\psi_i = \min_{j\leq i, k\leq i} \psi_{j,k}.$$

for which it holds that $\psi_i \in \Psi_\Omega^f$ and

$$\psi_i(x_j) \leq \psi_{j,i}(x_j) \to \overline{H}f(x_j) \text{ as } i \to \infty \quad \text{for each } j.$$

Thus $\psi_i(x_k)$ decreases to $\overline{H}f(x_k)$ as $i \to \infty$, for each k.

Let $u(x) = \lim_{i\to\infty} \psi_i(x)$ for $x \in \Omega$, where the limit exists as the sequence is pointwise decreasing. As $|f| \le M$ on $\partial\Omega$, each $\psi \in \Psi_\Omega^f$ satisfies $\psi \ge -M$ in Ω. In B, u is the pointwise limit of a decreasing sequence of harmonic functions, all of which are greater than $-M$, so by Theorem 4.11, u is harmonic in B.

Clearly, $u = \overline{H}f$ on E. As each ψ_i is an upper function for f in Ω, we also see that $\overline{H}f \le u$ in Ω. As u is continuous in B and $\overline{H}f$ is upper semicontinuous in B, they are equal in B. This can be seen by taking some $x \in B \setminus E$, for which there exists a sequence $(x_i)_{i=1}^\infty$ which converges to x. Then

$$\overline{H}f(x) \ge \lim_{i\to\infty} \overline{H}f(x_i) = \lim_{i\to\infty} u(x_i) = u(x) \ge \overline{H}f(x)$$

by upper semicontinuity and continuity.

Therefore $\overline{H}f$ is harmonic in any arbitrary ball $B \Subset \Omega$, and as harmonicity is a local property (see Corollary 3.35), $\overline{H}f$ is harmonic in Ω. Likewise, we can show that $\underline{H}f$ is harmonic in Ω. □

Definition 4.22. If for some function f and bounded open set Ω, the upper Perron solution is equal to the lower Perron solution, and real everywhere, we say that f is a *resolutive* boundary function. In this case the *Dirichlet solution* is defined as

$$H_\Omega f = \overline{H}_\Omega f = \underline{H}_\Omega f.$$

Note that one of the properties of the Perron solutions is that if they are not identically $\pm\infty$ in a connected component of Ω, then they are real in that component. Thus the requirement that $\overline{H}_\Omega f$ is real everywhere can be relaxed to the requirement that it is not identically $+\infty$ in any component of Ω, and likewise for $\underline{H}_\Omega f$.

The operator H_Ω defined above is a linear operator from the vector space of resolutive boundary functions in Ω to $\mathcal{H}(\Omega)$. This makes the Perron method very useful because solutions to complicated boundary functions can be obtained by decomposing the problem into smaller problems and using the linearity to combine the partial solutions. Linear operators also have other useful properties in functional analysis, which are not discussed here.

Theorem 4.23. *If f and g are resolutive boundary functions on $\partial\Omega$ and $c \in \mathbb{R}$ is a constant, then:*

(a) *cf is resolutive, with $H_\Omega(cf) = cH_\Omega f$.*

(b) *$f + g$ is resolutive, with $H_\Omega(f + g) = H_\Omega f + H_\Omega g$.*

The preceding theorem follows from a stronger variant given below.

Theorem 4.24. *If f and g are any real-valued boundary functions on $\partial\Omega$, then:*

(a) *If $c \geq 0$ then $\overline{H}_\Omega(cf) = c\overline{H}_\Omega f$ and $\underline{H}_\Omega(cf) = c\underline{H}_\Omega f$.*

(b) $\overline{H}_\Omega(-f) = -\underline{H}_\Omega f$.

(c) $\overline{H}_\Omega(f+g) \leq \overline{H}_\Omega f + \overline{H}_\Omega g$ *and* $\underline{H}_\Omega(f+g) \geq \underline{H}_\Omega f + \underline{H}_\Omega g$

For a proof of this, see [13, p. 79].

4.5 Perron Solutions for the Exterior Dirichlet Problem

Like the interior Dirichlet problem, the exterior Dirichlet problem is not always solvable for a given continuous boundary function. For example, with $\Omega = B(0,1)$ and

$$f(x) = 1 \quad \text{for } x \in \partial B(0,1),$$

the Perron method can be used to obtain a solution. This is because the Perron method works in the unbounded set Ω^e as well as in the bounded Ω, which is proved in e.g. [2, Section 6.2]. However this Perron solution, which is $H_\Omega f \equiv 1$, does not vanish at infinity. The point at infinity is thus an *irregular point* for this Ω^e, see Definition 5.2. Zaremba's example of an irregular point for the interior Dirichlet problem, Example 5.1, has some similarity to this example.

To see that $H_\Omega f \equiv 1$, we can note that $1 \in \Psi_\Omega^f$, as 1 is a superharmonic function with boundary values greater than or equal to f on $\partial\Omega$, and with a limit at infinity which is greater than zero. The requirement that upper and lower functions lie above or below (respectively) the boundary value at infinity is important to guarantee that $\underline{H}f \leq \overline{H}f$ (cf. Theorem 4.19). Further, Φ_Ω^f contains all functions u_a given by

$$u_a(x) = 1 - a\Gamma_0(x) = 1 - a\ln|x| \quad \text{for any } a > 0,$$

because each of these functions is harmonic, equal to f on $\partial\Omega$, and $\lim_{|x|\to\infty} u_a(x) = -\infty \leq 0$. Thus, for all $x \in \Omega^e$,

$$1 = \sup_{a>0} u_a(x) = \underline{H}f(x) \leq \overline{H}f(x) \leq 1.$$

One may show that ∞ is an irregular point for Ω^e given any domain Ω in two dimensional space, but there exist unbounded sets in $\mathbb{R}^2$ which have ∞ as a regular point on the boundary, e.g. the half-plane $\{(x, y) \in \mathbb{R}^2 : x > 0\}$ [2, p. 189, Example 6.7.4(ii)]. In $\mathbb{R}^n$ for $n \geq 3$, it can be shown that the point at infinity is a regular point when considered as a point on the boundary of any unbounded open set [2, p. 188, Theorem 6.7.1]. In particular it is a regular point on the boundary of the exterior Ω^e of any bounded open $\Omega \subset \mathbb{R}^n$.

4.6 Harmonic Functions as Energy Minimizers

Harmonic functions can be seen as minimizers of the Dirichlet energy, a representation of the total variation of a function. This can be compared with the electrostatic energy of a field E with potential v,

$$\frac{1}{2}\int \varepsilon_0 E \cdot E \, d\lambda = \frac{\varepsilon_0}{2}\int \nabla v \cdot \nabla v \, d\lambda \,.$$

For simplicity we discard the constant $\frac{\varepsilon_0}{2}$, where ε_0 is the electric constant.

Definition 4.25. For a function $v \in \mathcal{C}^2(\Omega)$ the *Dirichlet energy* is defined as

$$E(v) = \int_\Omega |\nabla v|^2 \, d\lambda \,.$$

Note that because $|\nabla v|^2 \geq 0$ the energy is always non-negative.

Theorem 4.26. *If u is the solution to the Dirichlet problem with boundary condition f in Ω, then*

$$E(u) = \inf_v E(v), \tag{4.12}$$

where the infimum is taken over all functions $v \in \mathcal{C}^2(\Omega)$ that have boundary values $v|_{\partial\Omega} = f$.

This can be used to find solutions in certain cases, by use of the following converse, which was named Dirichlet's principle by Bernhard Riemann in 1857 [24, p. 33]. In 1856 and 1857 Dirichlet gave lectures where he showed the result [24, p. 30]. William Thomson and Carl Friedrich Gauss also investigated the problem around the same time.

Theorem 4.27 (Dirichlet's principle)**.** *Assume that u is an energy minimizer for a given boundary condition f and domain Ω, i.e., $u|_{\partial\Omega} = f$ and (4.12) holds. Then u is harmonic, so it is the solution to the Dirichlet problem for f and Ω.*

Because the energy $E(v)$ is bounded from below (by 0), the greatest lower bound $\inf_v E(v)$ is guaranteed to exist. Riemann assumed that there always exists a v such that the infimum is attained, but Karl Weierstrass showed in 1870 [24,

p. 36] that there exists a functional which does not attain its minimum, casting doubt on the validity of this assumption, but not disproving Dirichlet's principle directly, as the functional being minimized was different, denoted J below.

The example is as follows: Let $\Omega = (-1, 1)$, $f(-1) = a$ and $f(1) = b$ for some real constants such that $a < b$. Define the functional

$$J(v) = \int_{-1}^{1} \left(t \frac{dv}{dt} \right)^2 dt,$$

for any $v \in \mathcal{C}(\Omega) \cap \mathcal{C}^1(\Omega \setminus A)$ where A has measure zero (and may depend on v). Consider for example the function

$$v(t) = \begin{cases} a, & \text{when } -1 < t < -c, \\ \frac{a+b}{2} + t\frac{b-a}{2c}, & \text{when } -c \leq t \leq c, \\ b, & \text{when } c < t < 1, \end{cases}$$

for some $c \in (0, 1)$, which has derivative

$$v'(t) = \begin{cases} 0, & \text{when } -1 < t < -c, \\ \frac{b-a}{2c}, & \text{when } -c < t < c, \\ 0, & \text{when } c < t < 1. \end{cases}$$

Note that the derivative is undefined at the points $t = \pm c$, however this is a set of measure zero, so it does not affect the value of the integral. Applying the functional J we obtain the value

$$J(v) = \int_{-c}^{c} \left(t \frac{b-a}{2c} \right)^2 dt = c\frac{(b-a)^2}{6},$$

i.e. $\inf_v J(v) = 0$ by letting $c \to 0$. However, one can also show that there is no $v \in \mathcal{C}^1(\Omega)$ such that $J(v) = 0$, as this would imply that $\frac{dv}{dt} \equiv 0$ on Ω.

As we will see in the following chapter, there exist sets for which the Dirichlet problem has no solution, e.g. Zaremba's example. Therefore it is not possible for the Dirichlet principle to hold in these cases. However, in 1901 it was shown by Hilbert [15], that in some cases the Dirichlet principle remains valid, using a property which is in modern functional analysis known as the *sequential compactness* of the class of allowed functions, which is also expanded to include functions other than only $\mathcal{C}^2(\Omega)$. This will not be discussed in greater detail here, but can be found in e.g. [24, pp. 55–60].

Chapter 5

Characterizations of Regular Points

Since the methods discussed in the previous chapter are guaranteed to give harmonic functions on the set in question, the remaining difficulty in solving the Dirichlet problem lies mainly in finding out whether a candidate for a solution agrees with the boundary condition, or can possibly be made to agree with the boundary function.

Here we shall only examine the question when the boundary condition is continuous, however a number of characterizations for the solvability of the Dirichlet problem with discontinuous boundary conditions have been formulated, see e.g. [36].

A simple example of a situation in which it is impossible to obtain a solution which agrees with the boundary condition was given in 1911 by the Polish mathematician and engineer Stanisław Zaremba (1863–1942) [40, pp. 308–310].

Example 5.1. Consider the domain

$$\Omega = B(0,1) \setminus \{0\} \subset \mathbb{R}^2 \tag{5.1}$$

and the boundary function

$$f(x) = \begin{cases} 0, & \text{if } x \in \partial B(0,1), \\ 1, & \text{if } x = 0. \end{cases} \tag{5.2}$$

We examine the family of functions $u_a = \min\{-a \ln|x|, 1\}$ for $a > 0$. The functions $-a\ln|x| = -a\Gamma_0(x)$ and 1 are clearly harmonic, and because the minimum of two superharmonic functions is superharmonic ([13, p. 68, Theorem 2.4.5], cf. Lemma 4.20), the functions u_a are superharmonic functions with the correct boundary values, i.e. they lie in the set Ψ_Ω^f. Thus the upper Perron solution must be

$$\overline{H}_\Omega f \le \inf_{a>0} u_a \equiv 0$$

on Ω, but then as f must be continuous, it cannot "reach" the boundary value $f(0) = 1$, so the Dirichlet problem for Ω and f cannot have a solution.

We can also note that the lower Perron solution is

$$\underline{H}_\Omega f \ge 0,$$

as the zero function is a member of Φ_Ω^f, so therefore by Theorem 4.19,

$$H_\Omega f = \underline{H}_\Omega f = \overline{H}_\Omega f \equiv 0.$$

In Zaremba's original example, the boundary function is given as an arbitrary nonzero constant on the circle, and zero at the origin, but the principle holds for any constants. The proof of the nonexistence of a solution is also different to that presented here, as Perron's method had not been discovered at the time. See also [38, pp. 24–45] for a concise proof similar to Zaremba's.

Points such as $x = 0$ in this example are called *irregular points*.

Definition 5.2. For a given bounded set Ω, a *regular point* is a point $x \in \partial\Omega$ such that for every continuous boundary condition $f : \partial\Omega \to \mathbb{R}$, the Perron solution $H_\Omega f$ of the Dirichlet problem attains its boundary value at x, i.e.

$$\lim_{\Omega \ni y \to x} H_\Omega f(y) = f(x).$$

An *irregular point* is a point $x \in \partial\Omega$ which is not regular.

5.1 The Lebesgue Spine

In 1912, Henri Lebesgue gave another example of a domain for which the Dirichlet problem is not solvable for continuous boundary conditions [20]. This differs from Zaremba's example in that the boundary of Lebesgue's domain contains no isolated points. Lebesgue's example can in fact be modified slightly to give a domain which is 2-connected, a concept which will be discussed later.

The domain was described by Lebesgue as follows:

Example 5.3. First consider a line segment

$$L = \{(x_1, 0, 0) \in \mathbb{R}^3 : 0 \le x_1 \le 1\}$$

with electrical density of charge $q(x_1, 0, 0) = x_1$ at each point of the segment. Let $v(x)$ denote the potential at x which is generated by the conductor L. The potential can be computed by considering the formula for the electrostatic potential around a point charge at $(l, 0, 0)$,

$$v_l(x_1, x_2, x_3) = \frac{q(l, 0, 0)}{|(x_1, x_2, x_3) - (l, 0, 0)|} = \frac{l}{\sqrt{(l - x_1)^2 + x_2^2 + x_3^2}}$$

and applying the superposition principle to get the total potential from the integral

$$\begin{aligned} v(x_1, x_2, x_3) &= \int_0^1 v_l(x_1, x_2, x_3)\, dl \\ &= \int_{-x_1}^{1-x_1} \frac{t + x_1}{\sqrt{t^2 + x_2^2 + x_3^2}}\, dt \\ &= \ln r_1 - \ln r_0 + x_1 \ln(r_1 + 1 - x_1) - x_1 \ln(r_0 - x_1), \end{aligned}$$

where $r_1 = \sqrt{(1 - x_1)^2 + x_2^2 + x_3^2}$ and $r_0 = \sqrt{x_1^2 + x_2^2 + x_3^2} = |x|$, for $(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus L$.

The equipotential surfaces $v = \alpha$ are then surfaces of revolution around the segment L, and for $\alpha \ge 1$ these surfaces come arbitrarily close to the origin, i.e., $\operatorname{dist}(0, \{x \in \mathbb{R}^3 : v(x) = \alpha\}) = 0$. The potential at the origin, $v(0)$, is not defined and the limit of $v(x)$ as $x \to 0$ only exists if the origin is approached through certain paths (namely paths contained in one of these equipotential surfaces).

Let the domain Ω be the set delimited by the two equipotential surfaces $v = 2$ and $v = 1/2$. We try to solve the Dirichlet problem in Ω with the boundary condition

$$f(x) = \begin{cases} v(x), & x \in \partial\Omega \setminus \{0\}, \\ 2, & x = 0, \end{cases}$$

which is continuous on $\partial\Omega$ because we take $f(0) = 2$. Because v is the Newtonian potential, see Section 3.2, it is harmonic in Ω, so the solution to the Dirichlet problem would be given by v.

Figure 5.1 shows the cross-sections of the equipotential surfaces $v = 1/2$, $v = 2$ and $v = 3$. Surfaces with higher potential are closer to L, as expected. In the figure we can graphically see that the irregularity arises due to the narrow thorn-like shape with its endpoint at 0. If we approach the endpoint by a curve C_α in the surface $v = \alpha$ we get the limit

$$\lim_{\substack{y \to 0 \\ y \in C_\alpha}} v(y) = \alpha.$$

Because the limit depends on which curve is taken, $\lim_{\Omega \ni y \to 0} v(y)$ does not exist. Thus the Dirichlet problem is not solvable for these Ω and f.

The set which Lebesgue defined is simply connected, meaning that every closed curve can be shrunk to a point by a continuous mapping, without leaving the set. This forbids the set from having any holes, in the topological sense of the word. This concept can be generalized to n-connectedness. Informally, a set is n-connected if every surface of dimension $1 \leq k \leq n$ in the set can be shrunk to a point by a continuous mapping, without leaving the set. This is equivalent to the condition that every surface of dimension k can be continuously transformed to any other surface of dimension k in the set. Note that 1-connectedness is the same as simple connectedness and 0-connectedness is the same as path-connectedness, as the existence of a path between two points is equivalent to the existence of a continuous transformation between the points. Lebesgue notes that the example which he gave can be modified slightly to obtain at set which is 2-connected, although he uses the term simply connected for this property. This can be done by removing points around the positive x_1 axis, so that the inner and outer boundary join up to form a single boundary, as in (5.3).

The family of shapes with this behaviour are called *Lebesgue spines*. In modern usage, a domain in $\mathbb{R}^n, n \geq 3$ which contains a Lebesgue spine is often given as

$$\mathbb{R}^3 \setminus \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : x_3 > 0, x_1^2 + x_2^2 \leq e^{-1/x_3} \right\}. \tag{5.3}$$

For example, by Armitage and Gardiner [2, p. 187]. A plot of the cross-section of this curve in $\mathbb{R}^3$ can be found in Figure 5.2. For a proof of the irregularity of the point 0 for this set, see [2, p. 186, Theorem 6.6.16].

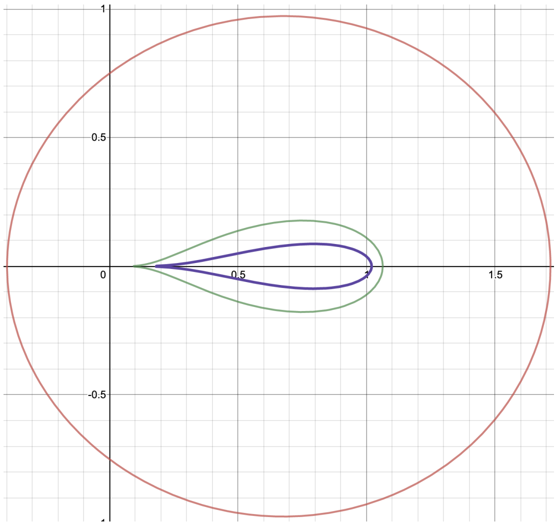

Figure 5.1: Lebesgue's original example. On red curve $v = 1/2$, on the green $v = 2$, and on the blue $v = 3$.

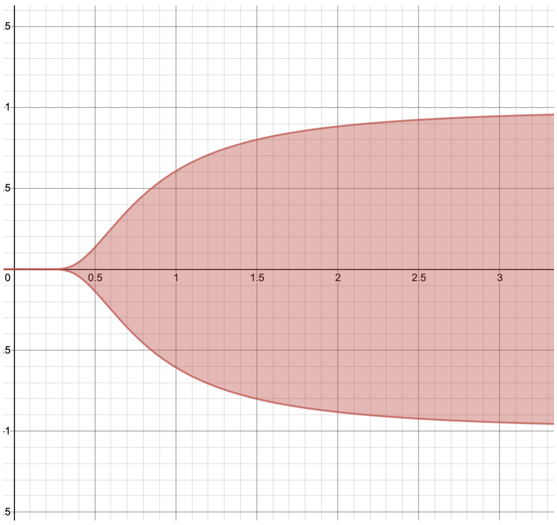

Figure 5.2: A modern example of a Lebesgue spine.

5.2 Poincaré's Criterion

Many different criteria for the regularity of boundary points have been developed during the research of the Dirichlet problem.

One of the oldest criteria for the regularity of a point was given by Henri Poincaré in 1890 [29, pp. 226–227]. Poincaré showed that a harmonic function with the correct boundary value at a point $x \in \partial\Omega$ could be produced by his sweeping-out method, as long as the point x could be "touched" by a sphere lying outside of Ω. This was shown in $\mathbb{R}^3$.

Theorem 5.4 (Poincaré's criterion)**.** *If for a point x on the boundary of Ω there exists a ball $B(y,\rho) \subset \Omega^c$ such that $x \in \partial B(y,\rho)$, then the point x is regular.*

We will not prove this theorem here.

5.3 Zaremba's Criterion

In 1909 Stanisław Zaremba gave a slightly improved version of Poincaré's condition by observing that the ball could be replaced with a cone [40, p. 311]. Zaremba proved this in $\mathbb{R}^2$.

Definition 5.5. A *cone* with vertex 0 is a set

$$C = \{(x_1, \dots, x_n) : |(x_1, \dots, x_{n-1})| \le x_n\},$$

or some linear transformation of this set. A cone with vertex y is a set

$$D = C + y = \{(x_1 + y_1, \dots, x_1 + y_n) : x \in C\},$$

where C is a cone with vertex 0.

The condition, which is called the *cone condition*, is stated as follows:

Theorem 5.6 (Zaremba's cone condition)**.** *If for a given point $x \in \partial\Omega$, where Ω is open and bounded, there exists a closed conical surface $C \subset \Omega^c$, such that the vertex of C is x, then the point x is regular.*

As regularity is a local property (see Proposition 5.10), the theorem may be stated in a slightly more general way. In essence, the cone can be cut off at an arbitrary radius ρ to prevent it from intersecting other parts of the set Ω. See Figure 5.3 for an illustration of this.

Theorem 5.7. *If for a given point x there exists a closed conical surface C with vertex in x, and a radius $\rho > 0$, such that $C \cap B(x,\rho) \subset \Omega^c$, then the point x is regular.*

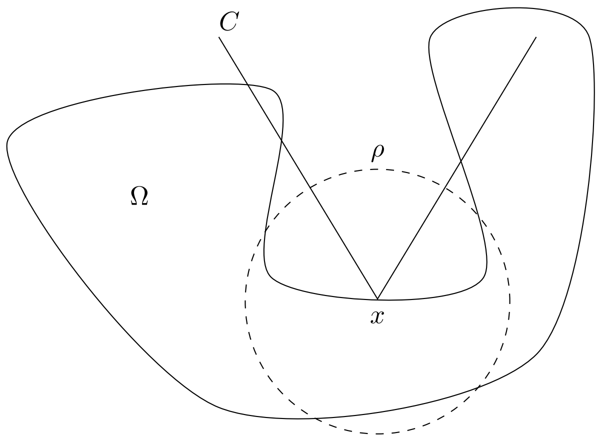

Figure 5.3: A simple sketch of Zaremba's cone condition, Theorem 5.7

5.4 Barrier Condition

A regular point can be characterized by the existence of a special function, called a barrier. Barriers, in the form of *barrier sets* were studied by Lebesgue [19, p. 335] and Kellogg [16, pp. 528–529], among others. The following definition of a *barrier function* was first given by Bouligand [8, p. 88–90].

Definition 5.8. Consider a point $x \in \partial\Omega$. A superharmonic function

$$v\colon B(x,\rho) \cap \Omega \to \mathbb{R},$$

defined in some neighbourhood $B(x,\rho)$, such that

$$v(y) > 0 \quad \text{for all } y \in B(x,\rho) \cap \Omega$$

and

$$v(y) \to 0 \qquad \text{as } y \to x, y \in B(x,\rho) \cap \Omega,$$

is called a *barrier* for x.

Theorem 5.9. *A point $y \in \partial\Omega$ is regular if and only if there exists a barrier for y.*

A characterization of regular points similar to this, was also given by Lebesgue in [21, p. 353]. Because the barrier function is defined only on a neighbourhood of the point in question, we see directly that regularity is a local property. This is mentioned in e.g. [8, p. 90]

Proposition 5.10. *A point $x \in \partial\Omega$ is regular with respect to Ω if and only it is regular with respect to $\Omega \cap B(x, \rho)$ for some $\rho > 0$.*

Proof. Assume x is regular with respect to Ω, so there exists a barrier

$$v\colon B(x, \rho_0) \cap \Omega \to \mathbb{R}.$$

Then, given any $\rho > 0$,

$$v\colon B(x, \min\{\rho_0, \rho\}) \cap \Omega \to \mathbb{R},$$

is a barrier, so x is regular with respect to $\Omega \cap B(x, \rho)$. A similar argument gives the converse implication. $\square$

5.5 Capacity

The capacity of a set is a concept which is based on the physical property of capacitance. Just as the capacitance of a conductor can be used to predict how charges and voltages interact with the conductor, the capacity of a set can be used to predict how harmonic functions interact with the set.

The mathematical definition of capacity was first given by Wiener in [37, p. 26], stated as follows. Let $E \subset \mathbb{R}^3$ be a bounded set and u be the solution to the exterior Dirichlet problem in E^e, with boundary value $f = 1$ on ∂E. Then the capacity of the set is given by

$$C(E) = -\frac{1}{4\pi} \int_{\partial\Omega} \frac{\partial u}{\partial \widehat{n}}\, d\sigma,$$

for any Ω with smooth boundary such that $E \subset \Omega$.

To explain this definition and extend it to higher dimensions, we introduce the concept of charge. We saw how to find the potential of a certain charge in Section 3.2. To find the charge of a certain potential, we can use Gauss' flux theorem from electromagnetics, stated in [9, p. 110] as: *The total outward electric flux over any surface is equal to the total free charge enclosed in the surface.* As capacity in $\mathbb{R}^2$ is slightly different from higher dimensions, we leave this case for later.

Definition 5.11. Given a function u which is harmonic in $\Omega \backslash E$, where $\Omega \subset \mathbb{R}^n$, for $n \geq 3$, is a bounded domain containing E, the value

$$-\frac{1}{(n-2)\sigma_n} \int_{\partial\Omega} \frac{\partial u}{\partial \widehat{n}}\, d\sigma,$$

is called the *charge* of u on E. The charge is independent of Ω and depends only on u and E [37, p. 40]

For example, let $u\colon \mathbb{R}^n \to \mathbb{R}$, $n \geq 3$, be a linear combination of two fundamental solutions,

$$u = q_1\Gamma_{y_1} + q_2\Gamma_{y_2}$$

which is harmonic on $\mathbb{R}^n \setminus \{y_1, y_2\}$. Take $\Omega = B(y_1, \rho)$ with radius ρ such that $y_2 \notin \Omega$. Then the charge of u on $E = \{y_1\}$ is, by the linearity of divergence and integration, as well as Gauss' divergence theorem

$$\begin{aligned}
&-\frac{1}{(n-2)\sigma_n}\int_{\partial\Omega}\frac{\partial u}{\partial\widehat{n}}\,d\sigma \\
&\quad= -\frac{1}{(n-2)\sigma_n}\int_{\partial\Omega}(\nabla(q_1\Gamma_{y_1})\cdot\widehat{n} + \nabla(q_2\Gamma_{y_2})\cdot\widehat{n})\,d\sigma \\
&\quad= -\frac{1}{(n-2)\sigma_n}\left(\int_{\partial\Omega}\nabla(q_1\Gamma_{y_1})\cdot\widehat{n}\,d\sigma + \int_{\Omega}\nabla\cdot\nabla(q_2\Gamma_{y_2})\,d\lambda\right),
\end{aligned}$$

and as Γ_{y_2} is harmonic in $\mathbb{R}^n \setminus \{y_2\}$,

$$\nabla\cdot\nabla(q_2\Gamma_{y_2}) = \Delta q_2\Gamma_{y_2} = 0$$

in Ω, so the charge of u in E is

$$\begin{aligned}
&-\frac{1}{(n-2)\sigma_n}\int_{\partial\Omega}\nabla(q_1\Gamma_{y_1})\cdot\widehat{n}\,d\sigma \\
&\quad= -\frac{1}{(n-2)\sigma_n}\int_{\partial B(y_1,\rho)}\nabla(q_1|x-y_1|^{2-n})\cdot\widehat{n}\,d\sigma \\
&\quad= -\frac{1}{(n-2)\sigma_n}\int_{\partial B(y_1,\rho)} q_1(2-n)|x-y_1|^{-n}(x-y_1)\cdot\frac{(x-y_1)}{|x-y_1|}\,d\sigma \\
&\quad= \frac{q}{\sigma_n\rho^{n-1}}\int_{\partial B(y_1,\rho)}d\sigma \\
&\quad= q_1,
\end{aligned}$$

as one would expect of a sensible definition of charge.

Definition 5.12 (Newtonian capacity)**.** If $K \subset \mathbb{R}^n$, for $n \geq 3$, is a compact set such that the exterior Dirichlet problem is solvable in $\mathbb{R}^n \setminus K$ for the boundary condition $f = 1$ on ∂K, the *capacity* $C(K)$ of the set is given by the charge, i.e., the uniquely determined bounded harmonic function $u\colon K^e \to \mathbb{R}$ such that

$$\lim_{y\to x} u(y) = 1 \quad \text{for all regular } x \in \partial K$$

and

$$\lim_{|x|\to\infty} u(x) = 0,$$

i.e. the solution to the exterior Dirichlet problem for K and f.

The capacity can be computed by

$$C(K) = -\frac{1}{(n-2)\sigma_n}\int_S \frac{\partial u}{\partial \widehat{n}}\, d\sigma,$$

where S is a smooth surface enclosing K. The function u is called the capacity potential for K.

This definition can be compared to a capacitor from electrostatics, whose capacitance is computed by $\frac{q}{v}$ where q is the charge on the surface of the capacitor that is required to obtain the potential v around the capacitor. If we set the potential to $v = 1$ on the capacitor, relative to 0 at infinity, the capacitance is simply q, which is essentially what is done in the definition of capacity.

We now examine the case of $\mathbb{R}^2$. We begin by noting that we cannot use Definition 5.12. Consider for example the case $K = B(0,1) \subset \mathbb{R}^2$. Since the exterior Dirichlet problem for this set and boundary conditions as above is rotationally invariant, we see (as in Section 3.2) that any solution must be of the form

$$u(x) = a \ln|x| + b,$$

for some constants a and b. However, there are no constants such that $u(x) \to 0$ as $|x| \to \infty$, and therefore the problem lacks solution, meaning that we cannot define the capacity of K.

However, the definition of capacity can be extended by the use of *Green functions* (see e.g. Helms [13, pp. 107–111] or Armitage and Gardiner [2, p. 89]). We will not explain the theory necessary for this definition of capacity in greater detail, as it additionally requires introducing the concept of a *regularized reduction* (see e.g. [13, p. 159] or [2, p. 129]). We simply state it here to give an idea of how the notion of capacity can be extended.

Definition 5.13 (Green capacity)**.** If $K \subset \Omega$ where $\Omega \subset \mathbb{R}^n$, let μ_K be the measure (cf. Theorem 6.3) such that

$$\widehat{R}_1^K(x) = \int_\Omega G_\Omega(x,y)\, d\mu(y),$$

where $\widehat{R}_1^K$ is the *regularized reduction* of the constant function 1 relative to K in Ω, and $G_\Omega(\cdot,\cdot)$ is the *Green function* for the set Ω. The *capacity* of K is then given by

$$C_G(K) = C_G(K,\Omega) = \mu_K(\Omega).$$

It can be shown that the Green capacity for $n \geq 3$ agrees with the Newtonian capacity, for all sets for which they are both defined [13, pp. 164–165], in the sense that

$$C(K) = \lim_{r\to\infty} C_G(K, B(0,r)).$$

Note that $C_G(K, B(0,r))$ is only defined when $K \subset B(0,r)$, however this holds for all sufficiently large r, as K is compact. We note also that, for compact K, it holds that $C(K) = 0$ if $C(K, B) = 0$ for any ball $B \Supset K$.

For the case of $\mathbb{R}^2$, we can give the following expression for the Green function [13, pp. 14–16].

Definition 5.14 (Green function for a disc in $\mathbb{R}^2$)**.** If $B = B(0,\rho)$ then the *Green function* $G_B \colon B \times B \to (0,\infty]$ for B is given by

$$G_B(x,z) = \begin{cases} \ln \frac{|x|}{\rho}\frac{|z-x^*|}{|z-x|}, & z \in B, z \neq x \\ \infty, & z = x, \end{cases}$$

where

$$x^* = \frac{\rho^2}{|x|^2}x$$

i.e., the *inverse* of x with respect to ∂B. The case of $B = B(y,\rho)$ follows by translation.

Just as the value of f on a set of measure zero does not affect the value of the integral $\int f\, d\mu$, we will see in Theorem 6.2 that the value of f on a set of capacity zero does not affect the solution to the Dirichlet problem with boundary condition f. Such sets are called polar sets. See e.g. [13, pp. 149 ff.]

Definition 5.15. A set $K \subset \Omega$ is *polar* if there exists a non-constant superharmonic function $u \colon \Omega \to [-\infty,\infty]$ such that

$$K \subset \{x : u(x) = \infty\}.$$

Example 5.16. For example, all countable sets $\{y_1, \dots\} \subset \mathbb{R}^n$ are polar. To see this, let

$$\mu = \sum_{k=1}^{\infty} 2^{-k}\delta_{y_k},$$

where δ_y is the Dirac measure at y (see Definition 2.43) and note that $\mu(\mathbb{R}^n) = \sum_{k=1}^{\infty} 2^{-k} < \infty$. Then

$$\sum_{k=1}^{\infty} 2^{-k}\Gamma_{y_k} = \int_{\mathbb{R}^n} \Gamma_y(x)\, d\mu(y),$$

is a non-constant harmonic function that has value ∞ at each y_k, where $\Gamma_y(x)$ is the fundamental solution to Laplace's equation (see Definition 3.7), so $\{y_1, \dots\}$ is a polar set.

It can be proved that all polar sets have Lebesgue measure 0. It can also be proved that polar sets in $\mathbb{R}^2$ are totally disconnected, meaning that every connected component of a polar set consists of a single point [2, p. 155, Corollary 5.8.9].

Definition 5.17. If a property holds on a set A except on a subset $E \subset A$ of capacity zero, $C(E) = 0$, we say that the property holds *quasi-everywhere*, abbreviated *q.e.*

Proposition 5.18. *For any superharmonic function $\psi\colon \Omega \to [-\infty, \infty]$, it holds that $\psi \neq \infty$ q.e., and in particular, $\psi \not\equiv \infty$ on any domain $E \subset \Omega$.*

As a corollary of this, we get an improved version of Proposition 3.30, namely:

Proposition 5.19. *If φ is superharmonic in Ω and $E \subset \Omega$ is a domain, then φ is superharmonic in E.*

Proof. We know from Proposition 3.30 that either φ is superharmonic in E, or $\varphi \equiv \infty$ in E, but by Proposition 5.18, the former must be the case, as E is a domain. $\square$

5.6 Wiener's Criterion

In his 1924 paper [38, p. 130], Wiener gave a criterion which is both necessary and sufficient for the regularity of a point, called the Wiener criterion.

Theorem 5.20 (Wiener's criterion)**.** *Let x be a point on the boundary of Ω. Take any $0 < \lambda < 1$. Let γ_k be the capacity of the set $\{y \in \Omega^c : \lambda^k \leq |y - x| \leq \lambda^{k-1}\}$. Then x is a regular point if the series*

$$\sum_{k=1}^{\infty} \frac{\gamma_k}{\lambda^k} = \frac{\gamma_1}{\lambda} + \frac{\gamma_2}{\lambda^2} + \dots$$

diverges. If the series converges, then x is an irregular point of $\partial\Omega$.

5.7 Kellogg Property

In a 1929 paper [18] by the American mathematician Oliver Dimon Kellogg (1878–1932) in collaboration with the Romanian mathematician Florin Vasilescu

(1897–1958), the following result, called the Kellogg property, was conjectured to hold in any number of dimensions.

Theorem 5.21 (The Kellogg property)**.** *If $A \subset \partial\Omega$ is a closed and bounded set with positive capacity, then A contains at least one regular point. Equivalently, any set consisting of only irregular points has capacity zero.*

The theorem was proven in any number of dimensions in 1933 by the American mathematician Griffith Conrad Evans (1887–1973) [12]. Using the Kellogg property, Kellogg and Vasilescu proved that the solutions to the Dirichlet problem are unique even in the presence of irregular points, see Theorem 6.2. Kellogg had proved that the theorem holds in two dimensions the preceding year [17].

Definition 5.22. A set Ω is *reduced* if it contains no points x, such that

$$C(\partial\Omega \cap B(x,\rho)) = 0 \quad \text{for some } \rho > 0.$$

Since the set of such points for any given Ω is a set of capacity zero, we may in many cases assume without loss of generality that a set Ω is reduced. Since any isolated point is such a point, a reduced set has no isolated points.

Chapter 6

The Extended Dirichlet Problem

As we have seen, even when the Dirichlet problem is not solvable in the sense of the original formulation, a harmonic function may be found which fulfils the boundary condition at every regular point. This extended formulation of the Dirichlet problem makes sense even for discontinuous boundary functions, as we will shortly see.

Definition 6.1. Given a bounded domain Ω and a bounded function $f\colon \partial\Omega \to \mathbb{R}$, solving the *extended Dirichlet problem* consists of finding a bounded function $u\colon \Omega \to \mathbb{R}$, which is harmonic in Ω and agrees with the boundary condition quasi-everywhere,

$$\lim_{y\to x} u(y) = f(x) \quad \text{for q.e. } x \in \partial\Omega.$$

The generalized solution to the Dirichlet problem can in many cases be found with the Perron method. Consider for instance Zaremba's counterexample, given in (5.1) and (5.2). We may find an upper function ψ arbitrarily close to 0, in the sense that there exists a sequence $(\psi_k)_{k=1}^{\infty}$ in the upper class Ψ_Ω^f such that $\lim_{n\to\infty} \psi_n = 0$. One such sequence is

$$\varphi(x_1, x_2) = \begin{cases} 1 - nx_1^2 - nx_2^2, & \text{if } nx_1^2 + nx_2^2 < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Because the Laplacian is less than zero almost everywhere for every function in this sequence, and $\limsup_{y\to x} \psi(y) \geq f(x)$ for all points $x \in \partial\Omega$, each element of the sequence is in Ψ_Ω^f. It is also easy to see that the function $\varphi \equiv 0$ is a lower function. Therefore, the function $0 = H_\Omega f = \overline{H}_\Omega f = \underline{H}_\Omega f$ is a solution to the extended Dirichlet problem for Zaremba's counterexample.

A theorem stating that the solution to the Dirichlet problem is uniquely determined even in the presence of irregular points, was first introduced by Kellogg in 1929 [18, p. 516]. We present it here without proof.

Theorem 6.2. *Given a domain Ω and a boundary condition $f\colon \partial\Omega \to \mathbb{R}$, the function which is bounded and harmonic in Ω and agrees with f at every regular point is unique, if it exists.*

6.1 Brelot's Resolutivity Theorem

In his 1925 paper [39, p. 29], Wiener gives an apparent example of an instance of the generalized Dirichlet problem where the Perron method cannot give a generalized solution. The example is as follows: let $\Omega = B(0,1) \subset \mathbb{R}^2$ and let f be 1 at the points which have a rational angle component and 0 otherwise, i.e.

$$f(x_1, x_2) = \begin{cases} 1, & \text{if } \arctan \dfrac{x_2}{x_1} \in \mathbb{Q}, \\ 0, & \text{otherwise.} \end{cases}$$

Wiener states that any upper function must be at least 1 and any lower function must be at most 0, and therefore that the Perron method cannot yield a unique generalized solution to the generalized Dirichlet problem. This was commonly accepted among the mathematicians of the time, as Wiener was regarded as an authority in the mathematical world. However, in 1937 when Brelot was preparing a lecture which included this example, he realized that the upper solution is in fact 0, meaning that the example has a unique generalized Perron solution [22, pp. 558–559].

Brelot also provided a characterization of resolutive functions in terms of the *harmonic measure*. As we have seen in Section 4.3, for each continuous function $f\colon \partial\Omega \to \mathbb{R}$ on the boundary of a given open bounded set Ω, there exists a generalized solution to the Dirichlet problem, given by $H_\Omega f$. For a fixed $x \in \partial\Omega$, we may consider $H_\Omega f(x)\colon \mathcal{C}(\partial\Omega) \to \mathbb{R}$ as a linear functional from the space of continuous functions on the boundary. Using this functional, we can define a measure as follows.

Theorem 6.3. *For a given point x in the bounded open set Ω, there exists a* *measure* μ_x*, called the* harmonic measure*, for which*

$$H_\Omega f(x) = \int_{\partial\Omega} f \, d\mu_x$$

for each continuous boundary condition $f \in \mathcal{C}(\partial\Omega)$*.*

This result will not be discussed in detail. For a full proof, see for example [13, pp. 216–217] or [2, pp. 172–177].

Theorem 6.4 (Brelot's theorem)**.** *Assume that the function* $f \colon \partial\Omega \to \mathbb{R}$ *is integrable with respect to* μ_x *for some* $x \in \Omega$*, where* μ_x *is the harmonic measure for the connected set* Ω *at* x*. Then* f *is resolutive and the solution to the Dirichlet problem for* f *and* Ω *is given by*

$$H_\Omega f = \int_{\partial\Omega} f \, d\mu_x,$$

Conversely, if f *is resolutive then* f *is integrable with respect to* μ_x *for all* $x \in \Omega$*.*

We can note that if Ω is a connected set, then f is integrable with respect to μ_x for some $x \in \Omega$ if and only if f is integrable with respect to μ_y for some $y \in \Omega$. In other words if f is integrable with respect to the harmonic measure at one point it is integrable with respect to the harmonic measure at any other point [2, p. 175, Corollary 6.4.7].

Chapter 7

Conclusions and Discussion

We have considered the history of the Dirichlet problem for the Laplacian, from the beginnings of its formulation to its solution by various methods, including the Dirichlet principle, Poisson integration and most generally by the method of Perron solutions. We have also investigated some early examples of irregular points, those being Zaremba's example and the Lebesgue spine, as well as some criteria and characterizations of regular points, including Poincaré's criterion, Zaremba's criterion, barrier characterizations and the Wiener criterion. We have striven to present these ideas and developments in a consistent notation and as part of a coherent historical narrative.

7.1 Further Developments

There are many directions in which this work can be expanded. For instance, it would be interesting to investigate the theory deeper and provide full proofs of more of the stated theorems. Another interesting angle is to move forward to more modern potential theory, perhaps using the historical framework to explain some of the results regarding generalizations of the Dirichlet problem which have been made more recently.

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