Capacities, Poincaré inequalities and gluing metric spaces.
Capacities, Poincaré inequalities and gluing metric spaces

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Abstract

This thesis consists of an introduction, and one research paper with results related to potential theory both in the classical Euclidean setting, as well as in quite general metric spaces.

The introduction contains a theoretical and historical background of some basic concepts, and their more modern generalisations to metric spaces developed in the last 30 years. By using upper gradients it is possible to define such notions as first order Sobolev spaces, $p$-harmonic functions and capacity on metric spaces. When generalising classical results to metric spaces, one often needs to impose some structure on the space by making additional assumptions, such as a doubling condition and a Poincaré inequality.

In the included research paper, we study a certain type of metric spaces called bow-ties, which consist of two metric spaces glued together at a single designated point. For a doubling measure $\mu$, we characterise when $\mu$ supports a Poincaré inequality on the bow-tie, in terms of Poincaré inequalities on the separate parts together with a variational $p$-capacity condition and a quasiconvexity-type condition. The variational $p$-capacity condition is then characterised by a sharp measure decay condition at the designated point.

We also study the special case when the bow-tie consists of the positive and negative hyperquadrants in $\mathbb{R}^n$, equipped with a radial doubling measure. In this setting, we characterise the validity of the $p$-Poincaré inequality in various ways, and then provide a formula for the variational $p$-capacity of annuli centred at the origin.
Sammanfattning


Vi kan derivera en sådan funktion i en given punkt i koordinatssystemet, med avseende på var och en av koordinaterna för sig, givet att funktionen är deriverbar (man brukar säga att funktionen är tillräckligt slät, med detta menas att funktionen inte har några plötsliga hopp eller vassa kanter). Detta kan upprepas för att erhålla högre ordningars derivator. Givet vad vi vet om situationen, till exempel utifrån naturlagar eller andra för situationen styrende principer, kan vi sedan ställa upp samband mellan dessa partiella derivator, och vi får då en partiell differentialekvation.


I denna avhandling presenteras några grundläggande begrepp från ickeelinjär potentialteori på metriska rum. Vi studerar särskilt en viss typ av metriska rum, så kallade bow-ties (flugor). Dessa är rum som bildas genom att “limma ihop” två rum, så att de möts i en enda punkt (namnet kommer från att en tidig variant konstruerades genom att limma ihop två triangelformade rum, så att resultatet blev något som liknar en fluga). Det huvudsakliga resultatet består i att vi reder ut när Poincaré-olikheter gäller på flugan, i förhållande till när de gäller på vart och ett av de två ihoplimmade rummen.
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1 – Background

This thesis deals with some topics related to analysis on metric spaces in general and studies a special type of metric spaces called bow-ties.

We will focus on characterising the validity of Poincaré inequalities in different settings, and give some results related to variational capacities. As we shall see, capacity conditions play a role also in the characterisations of Poincaré inequalities. The results are a selection from the paper Björn–Björn–Christensen [Paper A], which contains more results than those presented here.

Poincaré inequalities and a doubling condition are common assumptions when dealing with analysis in more general settings than unweighted $\mathbb{R}^n$. For example, an extensive nonlinear potential theory was developed in Heinonen–Kilpeläinen–Martio [20], for weighted $\mathbb{R}^n$ equipped with a weight $w$ such that the associated measure $d\mu = w \, dx$ meets such conditions, yielding a so-called $p$-admissible weight. Since the introduction of upper gradients by Heinonen–Koskela [21], [22], in the 90s, this theory has also developed in the metric space setting, again with these standing assumptions. There are a few books that treat the general field, see for example Björn–Björn [4], Heinonen [19] and Heinonen–Koskela–Shanmugalingam–Tyson [23].

Bow-ties are constructed by gluing two metric spaces together at one point. They are the simplest examples of a type of construction that appeared in Heinonen–Koskela [22, Section 6.14], where spaces supporting Poincaré inequalities were glued along different sets. Perhaps the simplest example of a bow-tie is the union of the first and third closed quadrants in $\mathbb{R}^2$. When intersected with the set $\{(x_1, x_2) : |x_1| + |x_2| \leq 1\}$, one obtains a space which resembles the shape of a bow-tie.

Bow-ties often provide examples of metric spaces with interesting properties. The following selection of references contains such examples: [1, p. 985], [2, Examples 5.1 and 5.4], [3, Section 8], [4], [5, Example 6.2], [7, p. 51], [8, p. 1189], [9, Example 6.1], [10, Example 4.5], [11, Example 5.2], [24, p. 814], [26, Remark 5.2], [27, p. 102], [28, Example 6.2], [29, Example 4.3.1] and [33, Example 1].

In [26] and [28], these types of spaces were also called Gehring bow-ties, stemming from Fred Gehring (an analyst in Ann Arbor), who was always wearing a bow-tie.
Outline of the introduction

In Chapter 2, we introduce the basic concepts related to analysis and potential theory on weighted $\mathbb{R}^n$, such as the variational capacity $\text{cap}_p$. We also present some capacity estimates, as well as some results regarding $A_p$-weights (which provide examples of $p$-admissible weights).

In Chapter 3 we explain how the theory is generalised to metric spaces. Along with that, we present some results about bow-ties in weighted $\mathbb{R}^n$.

In Chapter 4 we present more general results in the metric space setting, and discuss some geometrical properties that are assumed of the space for these results to hold.

Delimitations and possible outlooks

The results presented here resolve some unanswered questions. Among other things, we give a full characterisation of the validity of the Poincaré inequality on quite general bow-ties, related to its validity on the separate parts.

A somewhat related, possible expansion could be to explore if it is possible to achieve similar results on more general gluing spaces, for example spaces glued together along a line, or some other simple set.

Another way to generalise the results could be to consider more general capacities than $\text{cap}_p$, such as Orlicz capacities $\text{cap}_\Phi$, where $\Phi$ belongs to some suitable family of functions.

Included research paper

2 – Sobolev spaces and weighted \( \mathbb{R}^n \)

In this chapter we will discuss first order Sobolev spaces on \( \mathbb{R}^n \) (denoted \( W^{1,p} \)), and on weighted \( \mathbb{R}^n \) (denoted \( H^{1,p} \)). Apart from being essential for even defining weak solutions to PDE’s, another nice property of Sobolev functions is that they can be approximated by smooth functions, which makes it possible to prove existence theorems for weak solutions to certain PDE’s. Exactly which Sobolev space should be considered depends on the problem. See for example Chapter 6 in Evans [15] for a treatment of second order elliptic equations with prescribed boundary values, where the solutions are in \( W^{1,2} \).

2.1 Sobolev spaces on \( \mathbb{R}^n \)

When constructing first order Sobolev spaces, one makes no a priori assumptions on the smoothness of the functions, and so weak derivatives are used in the definition. Let \( \Omega \subset \mathbb{R}^n \) be open. The (closed) support of a function \( \phi : \Omega \to \mathbb{R} \), is the closure (within \( \Omega \)) of the set of points in \( \Omega \) where \( \phi \) is nonzero. If this set is compact, then \( \phi \) has compact support in \( \Omega \). We denote the space of functions that are in \( C^\infty(\Omega) \) with compact support in \( \Omega \) by \( C^\infty_0(\Omega) \). We say that the function \( g \in L^1_{\text{loc}}(\Omega) \) is a weak derivative of \( f \in L^1_{\text{loc}}(\Omega) \) with respect to \( x_i \) in \( \Omega \), written \( g = D_if \), if

\[
\int_{\Omega} \phi(x)g(x) \, dx = -\int_{\Omega} f(x)\frac{\partial \phi(x)}{\partial x_i} \, dx
\]

for all \( \phi \in C^\infty_0(\Omega) \). The weak gradient of \( f \) is then given by

\[
\nabla f = (D_1f, D_2f, \ldots, D_nf),
\]

and the first order Sobolev space \( W^{1,p}(\Omega) \) by

\[
W^{1,p}(\Omega) = \left\{ f : \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)} < \infty \right\}.
\]

Here, \( \|f\|_{L^p(\Omega)} \) denotes the \( L^p \)-norm of \( f \) on the set \( \Omega \), i.e.

\[
\|f\|_{L^p(\Omega)} = \left( \int_{\Omega} |f|^p \, dx \right)^{1/p}.
\]

We assume throughout the thesis that \( 1 \leq p < \infty \).

Note that higher order Sobolev spaces can be defined (replacing the 1 in \( W^{1,p} \) with a larger integer). This definition requires a more convoluted definition of (higher order) weak derivatives however, and will be of no interest to us, as we will only generalise first order Sobolev spaces to metric spaces later on. We are not
Aware of any reasonable generalisation of higher order Sobolev spaces to metric spaces.

Sobolev functions have many nice properties. For instance, the weak derivative defined above obeys the Leibniz rule, i.e.,

\[ D_i(uv) = uD_i v + vD_i u \]

in \( \Omega \) provided that both \( u \) and \( v \) are Sobolev functions, and that one of them is in \( C_0^\infty(\Omega) \).

## 2.2 Measures

We assume that \( \mu \) is a Borel regular measure such that all balls have positive and finite measure. We say that \( \mu \) is doubling on \( \mathbb{R}^n \) if there exists a doubling constant \( C > 0 \) such that for all balls \( B = B(x, r) := \{ y : |x - y| < r \} \) in \( \mathbb{R}^n \),

\[ 0 < \mu(2B) \leq C\mu(B) < \infty, \]

where \( cB = B(x, cr) \). In this chapter, all points are assumed to be in \( \mathbb{R}^n \), but later more general settings will be considered. Next we wish to introduce Poincaré inequalities on \( \mathbb{R}^n \) equipped with some measure, but first we need to define Lipschitz functions.

**Definition 2.1.** Let \( E \subset \mathbb{R}^n \). A function \( f : E \to \mathbb{R}^n \) is \( L \)-Lipschitz, \( L \geq 0 \), if for every \( a, b \in E \),

\[ |f(a) - f(b)| \leq L|a - b|. \]

If the constant \( L \) is of no importance, then the function is simply referred to as a Lipschitz function. If \( f \) is Lipschitz on every compact subset of \( E \), then we say that \( f \) is locally Lipschitz on \( E \).

Locally Lipschitz functions are differentiable a.e. on open subsets of \( \mathbb{R}^n \) by Rademacher’s theorem, so the gradient in (2.1) below is the regular gradient, see for example Lemma 1.11 in Heinonen–Kilpeläinen–Martio [20].

Let \( 1 \leq p < \infty \). We say that \( \mu \) supports a \( p \)-Poincaré inequality if there exist constants \( C > 0 \) and \( \lambda \geq 1 \) such that for all balls \( B = B(x, r) \subset \mathbb{R}^n \), and all locally Lipschitz functions \( f \) on \( \lambda B \),

\[ \int_B |f - f_B| d\mu \leq C\mu \left( \int_{\lambda B} |
abla f|^p d\mu \right)^{1/p}, \quad (2.1) \]

where

\[ f_B := \frac{1}{\mu(B)} \int_B f d\mu := \frac{1}{\mu(B)} \int_B f d\mu. \]

### 2.2.1 Weighted measures

A *weight* \( w \) on \( \mathbb{R}^n \) is a nonnegative locally integrable function such that the measure \( \mu \), canonically defined by \( d\mu = w dx \), is a Borel regular measure. If \( \mu \) is doubling and supports a \( p \)-Poincaré inequality on \( \mathbb{R}^n \), then \( w \) is called a \( p \)-admissible
weight. There are other definitions in the literature that are equivalent, see Corollary 20.9 in [20] (which is only in the second edition) and Proposition A.17 in Björn–Björn [4].

A weight \( w \) on \( \mathbb{R}^n \) is a (Muckenhoupt) \( A_p \)-weight if there exists \( C > 0 \) such that

\[
\int_B w \, dx < C \left( \int_B w^{1/(1-p)} \, dx \right)^{1-p}, \quad \text{if } 1 < p < \infty,
\]

\[
\essinf_B w, \quad \text{if } p = 1,
\]

for all balls \( B \subset \mathbb{R}^n \). \( A_p \)-weights are interesting to us, since they are \( p \)-admissible, see Heinonen–Kilpeläinen–Martio [20, Theorem 15.21] (for \( p > 1 \)) and Björn [12, Theorem 4] (for \( p = 1 \)). The \( A_p \)-condition also provides a convenient way to check whether certain concrete weights, such as the logarithmic power weights in Section 2.2.2 are \( p \)-admissible.

In [20], Sobolev spaces are defined on weighted \( \mathbb{R}^n \) as the completion of the set of functions in \( C^\infty(\Omega) \) with finite Sobolev norm

\[
\|f\|_{1,p} := \|f\|_{L^p(\Omega; \mu)} + \|\nabla f\|_{L^p(\Omega; \mu)},
\]

with respect to the norm \( \|\cdot\|_{1,p} \). We shall denote these spaces by \( H^{1,p}(\Omega; \mu) \), as in [20].

### 2.2.2 Logarithmic power weights

In this section, we will take a look at certain weights, to give concrete examples of measures that support a 1-Poincaré inequality and the doubling condition.

Using elementary estimates, it is straightforward to show the following result from [Paper A, Proposition 7.2].

**Proposition 2.2.** Let

\[
w(|x|) = |x|^\alpha(\max\{1, - \log |x|\})^\beta
\]

be a weight on \( \mathbb{R}^n \), where \( \alpha > -n \) and \( \beta \in \mathbb{R} \). Then \( w \) is an \( A_1 \)-weight if and only if \( \alpha < 0 \) or \( \alpha = 0 \leq \beta \).

Using this and Björn–Björn–Lehrbäck [8, Remark 10.6] we can give conditions on \( \alpha \) and \( \beta \) which yield 1-admissible weights [Paper A, Corollary 7.3]. (This was earlier known when \( \beta = 0 \), see Heinonen–Kilpeläinen–Martio [20, p. 10].)

**Corollary 2.3.** Let

\[
w(|x|) = |x|^\alpha(\max\{1, - \log |x|\})^\beta
\]

be a weight on \( \mathbb{R}^n \), \( n \geq 2 \), where \( \alpha > -n \) and \( \beta \in \mathbb{R} \). Then \( w \) is 1-admissible.

For \( n = 1 \) this is false, since any 1-admissible weight on \( \mathbb{R} \) is an \( A_1 \)-weight by Theorem 2 in Björn–Buckley–Keith [13], and that would then contradict Proposition 2.2.
2.3 Capacity

The notion of capacity first introduced by Wiener [31], was formulated differently from the one we will use. It is inspired by physics, as the name suggests, more specifically it stems from electromagnetics. If applied correctly, it can be used to calculate the energy stored in a capacitor.

The Wiener criterion for regular points (i.e. points on the boundary of a domain, at which the Perron solution attains the given boundary data for the Dirichlet problem for that domain), first published in Wiener [32], was formulated in terms of this capacity.

We will use the definition of capacity given in Heinonen–Kilpeläinen–Martio [20]. A Wiener-type criterion can be formulated for more complicated, nonlinear problems using this capacity as well, see for example Chapters 6, 12 and 21 in [20].

2.3.1 Capacities in weighted $\mathbb{R}^n$

Following Heinonen–Kilpeläinen–Martio [20], suppose $\mu$ is a weighted measure. Let $\Omega \subset \mathbb{R}^n$ be open and $K \subset \Omega$ compact. The variational $p$-capacity of $K$ with respect to $\Omega$ is

$$\text{cap}_{\mathbb{R}^n}^p(K, \Omega) = \inf_u \int_{\Omega} |\nabla u| d\mu,$$

where the infimum is taken over all $u \in C_0^\infty(\Omega)$, such that $u \geq 1$ on $K$.

Next we can define the capacity for an open set $U$ by

$$\text{cap}_{\mathbb{R}^n}^p(U, \Omega) = \sup_K \text{cap}_{\mathbb{R}^n}^p(K, \Omega),$$

where the supremum is taken over all compact $K \subset U$. At last, for an arbitrary set $E \subset \Omega$, we define

$$\text{cap}_{\mathbb{R}^n}^p(E, \Omega) = \inf_U \text{cap}_{\mathbb{R}^n}^p(U, \Omega),$$

where the infimum is taken over all open $U$ such that $E \subset U \subset \Omega$. It is worth noting that we may as well assume that $u = 1$ in $K$, and that $0 \leq u \leq 1$ in $\Omega$.

In general, calculating capacities of sets is difficult, and often one has to settle with estimates. See for example Björn–Björn–Lehrbäck [8] for capacity estimates of annuli. In certain cases, it is however possible to provide an explicit formula for it.

In [20, Example 2.12], the calculation of the following is shown.

**Example 2.4.** Let $\mu$ denote Lebesgue measure in $\mathbb{R}^n$. If $0 < r < R < \infty$, then

$$\text{cap}_{\mathbb{R}^n}^p(\overline{B(x_0, r)}, B(x_0, R)) = \begin{cases} \omega_{n-1} \left( \frac{|n - p|}{n - 1} \right)^{p-1} |R^{\frac{n-p}{p}} - r^{\frac{n-p}{p}}|^{1-p}, & \text{if } p \neq n, \\ \omega_{n-1} \left( \log \frac{R}{r} \right)^{1-n}, & \text{if } p = n, \end{cases}$$
where $\omega_{n-1}$ is the surface measure of the unit sphere in $\mathbb{R}^n$.

Note that from the definition of capacity above, we can replace $B(x_0, r)$ by $\partial B(x_0, r)$ or $B(x_0, r)$ in the formula above. In particular, this example also shows that
\[
cap_{\rho, \mu}^R(\partial B(x_0, r), B(x_0, R)) > 0
\]
for such $r$ and $R$, while the Lebesgue measure of $\partial B(x_0, r)$ in $\mathbb{R}^n$ is of course 0. In general, the variational capacity is finer than the underlying measure in the sense that if $\cap_{\rho, \mu}^R(E, \Omega) = 0$ for some bounded neighbourhood $\Omega$ of $E$, then $\mu(E) = 0$.

Next, we present a result from [Paper A, Theorem 1.4], which is an improvement of [8, Proposition 10.8]. Here, we look at annuli centred at the origin, and take the opportunity to introduce the following notation for balls centred at the origin:
\[
B_r = B(0, r).
\]
Note that we can replace $B_r$ by its closure or boundary in the following result, as discussed in the preceding example.

**Proposition 2.5.** Assume that $d\mu = w \, dx$, where $w(x) = w(|x|)$ is a radial weight on $\mathbb{R}^n$. If $0 < r < R$, then
\[
cap_{\rho, \mu}^R(B_r, B_R) = \begin{cases} 
\left( \int_r^R \bar{w}(\rho)^{1/(1-p)} \, d\rho \right)^{1-p}, & \text{if } p > 1, \\
\essinf_{r<\rho<R} \bar{w}(\rho), & \text{if } p = 1,
\end{cases}
\]
where $\bar{w}(\rho) := \omega_{n-1} w(\rho)^{n-1}$ and $\omega_{n-1}$ is the surface area of the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$ (with $\omega_0 = 2$), and
\[
cap_{\rho, \mu}^R(\{0\}, B_r) = \begin{cases} 
\left( \int_0^r \bar{w}(\rho)^{1/(1-p)} \, d\rho \right)^{1-p}, & \text{if } p > 1, \\
\essinf_{0<\rho<r} \bar{w}(\rho), & \text{if } p = 1.
\end{cases}
\]

### 2.3.2 Capacity estimates for logarithmic power weights

We finish this section by presenting a result characterising when
\[
cap_{\rho, \mu}^R(\{0\}, B_r) > 0 \quad \text{or} \quad \cap_{\rho, \mu}^R(\{0\}, B_r) \simeq r^{-p} \mu(B_r)
\]
holds for the logarithmic power weights from Section 2.2.2. This characterisation plays an important role later in Theorems 3.10 and 4.7.

**Proposition 2.6.** Let
\[
w(|x|) = |x|^{\alpha} \left( \max\{1, - \log |x|\} \right)^\beta
\]
be a weight on $\mathbb{R}^n$, $n \geq 1$, with $\alpha > -n$ and $\beta \in \mathbb{R}$. Then
\[
cap_{\rho, \mu}^R(\{0\}, B_r) > 0
\]
for some, or equivalently all, $r > 0$ if and only if one of the following conditions holds:
(a) $\alpha < p - n$,
(b) $\alpha = p - n$ and $\beta > p - 1$,
(c) $p = 1$, $\alpha = 1 - n$ and $\beta \geq 0$.

Moreover, with a fixed $0 < R_0 \leq \infty$, we have
\[
\text{cap}_{p,\mu}^R(0, B_r) \gtrsim \frac{\mu(B_r)}{r^p} \quad \text{for all } 0 < r < R_0
\]
if and only if (a) or (c) holds.

**Remark 2.7.** A clarification of the notation may be in order. We write $a \lesssim b$ if there is $C > 0$ such that $a \leq Cb$, where $C$ is depending only on the fixed parameters. Naturally, $a \gtrsim b$ if $b \lesssim a$ and $a \asymp b$ if $a \lesssim b$ and $a \gtrsim b$. 
In this chapter, we will define and study bow-ties in $\mathbb{R}^n$. Since these are not open subsets of $\mathbb{R}^n$, we need to set the stage by defining Sobolev spaces on general metric spaces. It should be mentioned that there are other possible ways to define Sobolev type spaces on metric spaces, see for example Cheeger [14] and Hajlasz [16] for two different versions (or Appendix B in Björn–Björn [4] for a short presentation of them both). In what follows, we will however only consider the most prevailing one introduced by Shanmugalingam [30].

3.1 Metric spaces

We start by reminding the reader of the definition of a metric space.

**Definition 3.1.** A metric space is a pair $(X, d)$ where $X$ is a set and $d : X \times X \to [0, \infty]$ is called a metric, and satisfying for all $x, y, z \in X$ the following:

- $d(x, y) = 0$ if and only if $x = y$,
- $d(x, y) = d(y, x)$,
- $d(x, y) \leq d(x, z) + d(z, y)$.

We let $B(x, r)$ denote the open ball centred at the point $x \in X$ with radius $r$, i.e.

$$B(x, r) = \{ y \in X : d(x, y) < r \}.$$

We assume that the metric space $X$ is equipped with a measure $\mu$, and just as in the previous chapter we assume that $\mu$ is Borel regular. The doubling condition is the same as in $\mathbb{R}^n$, i.e. there is a constant $C > 0$ such that

$$0 < \mu(2B) \leq C \mu(B) < \infty$$

for all balls $B \subset X$.

3.2 Upper gradients

We now wish to define Sobolev spaces on metric spaces, but first we need a replacement for the absolute value of the gradient in the definition. This replacement will be defined using curves. A curve in $X$ is a continuous mapping from an interval to $X$, and a rectifiable curve is a curve of finite length. We will only consider curves which are nonconstant, compact and rectifiable, and as such each curve can be parameterised by its arc length $ds$. 

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Now, consider an open set $\Omega \subset \mathbb{R}^n$ and let $\gamma$ be a curve in $\Omega$ which is rectifiable, with length $l_\gamma$, and parameterised by $ds$. If $f \in C^1(\Omega)$, then
\[
|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma |\nabla f| \, ds.
\]
The idea now is to define a quantity replacing $|\nabla f|$ in the above inequality. Following Heinonen–Koskela [22], we introduce upper gradients as follows. (They are called very weak gradients in [22].)

**Definition 3.2.** A Borel function $g : X \to [0, \infty]$ is an upper gradient of a function $f : X \to [-\infty, \infty]$ if for all curves $\gamma : [0, l_\gamma] \to X$,
\[
|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds,
\]
where the left-hand side is considered to be $\infty$ whenever at least one of the terms therein is infinite.

Notice that it follows more or less immediately from the definition that if $g_1$ and $g_2$ are upper gradients of $u_1$ and $u_2$ respectively and $a \in \mathbb{R}$, then $g_1 + g_2$ is an upper gradient of $u_1 + u_2$ and $|a|g_i$ is an upper gradient of $au_i$.

### 3.3 Newtonian spaces

Following Shanmugalingam [30], we define Newtonian spaces on $X$ as follows.

**Definition 3.3.** For a measurable function $f : X \to [-\infty, \infty]$, let
\[
\|f\|_{N^{1,p}(X)} = \left( \int_X |f|^p \, d\mu + \inf_g \int_X g^p \, d\mu \right)^{1/p},
\]
where the infimum is taken over all upper gradients $g$ of $f$. The **Newtonian space** on $X$ is
\[
N^{1,p}(X) = \{ f : \|f\|_{N^{1,p}(X)} < \infty \}.
\]

The quotient space $N^{1,p}(X)/\sim$, where
\[
f \sim h \quad \text{if and only if} \quad \|f - h\|_{N^{1,p}(X)} = 0,
\]
is a Banach space and a lattice, see [30]. We need to assume that functions in $N^{1,p}(X)$ are defined everywhere, not just up to an equivalence class in the corresponding function space, so that the definition of upper gradients makes sense.

When considering a metric space $X$, any measurable set $A \subset X$ can be viewed as a metric space itself. This enables us to make sense of the following definition.

**Definition 3.4.** We say that $f \in N^{1,p}_\text{loc}(X)$, if for every $x \in X$ there is $r > 0$ such that $f \in N^{1,p}(B(x, r))$.

When $\Omega$ is $\mathbb{R}^n$ (or an open subset thereof) equipped with a $p$-admissible weight, $N^{1,p}(\Omega)$ and $H^{1,p}(\Omega; \mu)$ are the same up to equivalence classes. More precisely,
\[
H^{1,p}(\Omega; \mu) = \{ u : \text{there is } v \in N^{1,p}(\Omega) \text{ such that } u = v \text{ a.e.} \},
\]
see [4, Proposition A.12] for a proof.
3.4 Poincaré inequalities

When moving from the Euclidean setting directly to general metric spaces, we lose too much of the structure to be able to prove interesting results. It turns out that assuming a $p$-Poincaré inequality, together with assuming that the measure $\mu$ is doubling, provides just enough good qualities to be able to prove many generalisations of classical results. In the setting of metric spaces, the Poincaré inequalities we will consider are slightly different from the ones in the previous chapter, and we consider a wider range of inequalities as follows.

**Definition 3.5.** Let $1 \leq q < \infty$. We say that $X$ or $\mu$ supports a $(q, p)$-Poincaré inequality if there exist constants $C > 0$ and $\lambda \geq 1$ such that for all balls $B = B(x, r)$, all integrable functions $f$ on $X$, and all upper gradients $g$ of $f$,

$$
\left( \frac{1}{|B|} \right)^{1/q} \int_B |f - f_B|^q \, d\mu \leq Cr^{1/p} \int_{\lambda B} g^p \, d\mu,
$$

(3.1)

where $f_B := \frac{1}{|B|} \int_B f \, d\mu := \int_B f \, d\mu / \mu(B)$.

The case $q = 1$, will often be of special interest, and in that case we just say that the space supports a $p$-Poincaré inequality.

**Remark 3.6.** It is worth mentioning a few facts about the parameters $p$ and $q$ in Definition 3.5. First of all, the weakest inequality in terms of $q$ is $q = 1$, in the sense that if $X$ supports a $(q, p)$-Poincaré inequality, then it immediately follows from Hölder’s inequality that $X$ supports a $(\tilde{q}, p)$-Poincaré inequality for every $1 \leq \tilde{q} \leq q$. However, Hajlasz–Koskela showed in [17] and [18] that we can recover a $(p, p)$-Poincaré inequality from a $(1, p)$-Poincaré inequality together with the doubling condition for $\mu$, albeit with a worse dilation constant (in fact, even a $(q, p)$-Poincaré inequality for some $q > p$). See also Björn–Björn [4, Corollary 4.24] for a proof. Under additional assumptions, we can get arbitrarily large $q$, see [4, Corollary 4.26].

Similarly for the parameter $p$, a larger $p$ yields a weaker inequality. Thus, we typically want to assume only a $(1, p)$-Poincaré inequality (i.e. a $p$-Poincaré inequality), with $p$ as large as possible. We cannot improve the parameter $p$ in the same way as $q$, even if we allow worse dilation constants. However, due to a result by Keith–Zhong [25], it can always be improved somewhat, if the space $X$ is complete and equipped with a doubling measure. In that case, if $X$ supports a $p$-Poincaré inequality, $p > 1$, then there exists $\varepsilon > 0$ such that $X$ supports a $\tilde{p}$-Poincaré inequality for every $\tilde{p} \in (p - \varepsilon, \infty)$.

3.5 Sobolev capacity

**Definition 3.7.** The Sobolev $p$-capacity of an arbitrary set $E \subset X$ is

$$
C_p(E) = \inf_u \|u\|_{N^{1/p}(X)}^p,
$$

where the infimum is taken over all \( u \in N^{1,p}(X) \) such that \( u \geq 1 \) on \( E \).

The capacity is finer than the measure \( \mu \), in the sense that if \( C_p(E) = 0 \) then \( \mu(E) = 0 \), while the converse need not be true. In fact, it is easy to see that \( \mu(E) \leq C_p(E) \) for arbitrary \( E \subset X \). Other immediate properties are that \( C_p(\emptyset) = 0 \) and that \( C_p(E_1) \leq C_p(E_2) \) whenever \( E_1 \subset E_2 \). \( C_p \) is also subadditive, and an outer measure, see [4, Theorem 1.27(iv)].

### 3.6 Variational capacity

**Definition 3.8.** Let \( \Omega \subset X \) be open. The variational \( p \)-capacity of \( E \subset \Omega \) with respect to \( \Omega \) is

\[
\text{cap}^{X}_{p,\mu}(E, \Omega) = \inf_{u, g} \int_{\Omega} g^p \, d\mu,
\]

where the infimum is taken over all \( u \in N^{1,p}(X) \), such that \( u = 1 \) in \( E \) and \( u = 0 \) on \( X \setminus \Omega \), and all upper gradients \( g \) of \( u \). We call such a function \( u \) admissible for testing \( \text{cap}^{X}_{p,\mu}(E, \Omega) \).

The variational capacity \( \text{cap}^{X}_{p,\mu} \) has many nice properties similarly to \( C_p \), see [4, Theorems 6.17 and 6.19] for a list. If \( d\mu = w \, dx \) on \( \mathbb{R}^n \) we also write

\[
\text{cap}^{\mathbb{R}^n}_{p,w}(E, \Omega) = \text{cap}^{\mathbb{R}^n}_{p,\mu}(E, \Omega).
\]

If \( \mathbb{R}^n \) is equipped with a \( p \)-admissible weight \( w \), then \( \text{cap}^{\mathbb{R}^n}_{p,w} \) is the usual variational capacity as in Section 2.3.1, see Björn–Björn [5, Theorem 5.1].

### 3.7 Bow-ties in weighted \( \mathbb{R}^n \)

We now consider bow-ties \( X_{\mathbb{R}^n} \) in weighted \( \mathbb{R}^n \), consisting of the positive and negative hyperquadrants in \( \mathbb{R}^n \), i.e.

\[
X_{\mathbb{R}^n} = X_{\mathbb{R}^n}^+ \cup X_{\mathbb{R}^n}^-,
\]

where \( X_{\mathbb{R}^n}^+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \pm x_j \geq 0, \ j = 1, \ldots, n\} \).

We equip \( \mathbb{R}^n \) with a Borel regular measure \( d\mu = w \, dx \), where \( w \) is a weight on \( \mathbb{R}^n \).

We say that a weight \( w \) is radial if there is a function \( w : [0, \infty) \rightarrow [0, \infty] \) such that \( w(x) = w(|x|) \). (We also say that \( \mu \) is radial if \( d\mu = w \, dx \), where \( w \) is radial weight.) Note that this includes the unweighted case, in which the \( p \)-Poincaré inequality does not hold for this bow-tie if \( 1 \leq p \leq n \), by Björn–Björn [4, Example 5.6]. It does however hold on each of the parts, as the following result from [Paper A, Theorem 6.2] shows.

**Theorem 3.9.** Assume that \( \mu \) is a doubling radial measure on \( \mathbb{R}^n \). Then the following are equivalent:

(a) \( \mu \) supports a \( (q,p) \)-Poincaré inequality on \( X_{\mathbb{R}^n}^+ \);

(b) \( \mu \) supports a \( (q,p) \)-Poincaré inequality on \( \mathbb{R}^n \).

We finish the chapter with the following result from [Paper A, Theorem 1.2].
Theorem 3.10. Let \( d\mu = w \, dx \) be a doubling measure on \( \mathbb{R}^n \), where \( w(x) = w(|x|) \) is a radial weight. Then the following are equivalent:

(a) \( \mu \) supports a \( p \)-Poincaré inequality on \( X_{\mathbb{R}^n} \);
(b) \( \mu \) supports a \( p \)-Poincaré inequality on \( \mathbb{R}^n \) and
\[
\text{cap}^{\mathbb{R}^n}_{p, \mu}(\{0\}, B_r) \preceq r^{-p} \mu(B_r) \quad \text{for all } r > 0; \tag{3.2}
\]
(c) \( w \) is an \( A_p \)-weight on \( \mathbb{R}^n \) and (3.2) holds;
(d) \( \tilde{w}(\rho) := |\rho|^{n-1}w(|\rho|) \) is an \( A_p \)-weight on \( \mathbb{R} \).

The capacity condition (3.2) in (b) is needed, since otherwise it is not equivalent with (a) as we saw above. It is also needed for the equivalence with (c), which can be seen for example by letting \( p = 1 \) and \( w(x) = |x|^{\alpha} \). Then \( \mu \) supports a 1-Poincaré inequality on \( \mathbb{R}^n \) for all \( \alpha > -n \) while it is only an \( A_1 \)-weight for \( -n < \alpha \leq 0 \) by Proposition 2.2.
4 – Some results on general metric spaces

We will now turn our attention to metric spaces and, in particular, to bowties. The results presented here are quite general. We will assume throughout this chapter that $X = X_+ \cup X_-$, where $X_+ \cap X_- = \{x_0\}$ is a fixed designated point and $X_\pm \neq \{x_0\}$ are closed subsets of $X$, equipped with the same metric and measure. Since we will often construct balls centred at the point $x_0$, we introduce special notation for these in this chapter; we let $B_r = B(x_0, r)$ and $B^\pm_r = B_r \cap X_\pm$.

4.1 Geometrical properties

We will now discuss some geometrical properties of metric spaces, and how they are related to the standard assumptions discussed in earlier chapters. We begin by noting that any metric space $X$, supporting a $p$-Poincaré inequality, is necessarily connected. A space is not connected if and only if it can be partitioned into disjoint open sets $X_1$ and $X_2$. Let $f$ be the characteristic function of $X_1$. Since any curve $\gamma$ in $X$ lives either entirely on $X_1$ or on $X_2$ (due to continuity), it is always the case that $f(\gamma(0)) = f(\gamma(l_\gamma))$, so that $g = 0$ is an upper gradient of $f$. Now if we take a ball with centre in $X_1$ large enough to intersect $X_2$ (note that the openness implies that this intersection has nonzero measure), then $0 < f_B < 1$ so that $|f - f_B| > 0$ everywhere on $B$. Hence, the left-hand side in (3.1) is positive while the right-hand side is zero in this case, violating the Poincaré inequality.

It is natural to wonder then, if and when it is possible for a space to support a Poincaré inequality if it consists of two separate parts that intersect only at a single point. This is studied in great detail in Paper A, and we will return to this topic in Section 4.2 where we present the results. However, we need to introduce a few other concepts before we can do that. It turns out that a quasiconvexity-type condition (see Section 4.1.1 below) and a capacity condition (as in Proposition 2.6) play a key role in characterising the validity of the Poincaré inequality on such spaces in terms of its validity on its parts $X_+$ and $X_-$. 

4.1.1 Quasiconvexity

Here we will look at a geometrical property of a metric space which is stronger than it being connected. In short, a space being quasiconvex requires not only that any two points can be joined by a curve (path), which in itself implies connectedness (see Lemma 4.37 in [4]), but also that there is a bound on the ratio of the shortest path and the distance between the points given by the metric. The following definition of quasiconvexity is taken from [4].
**Definition 4.1.** A metric space $X$ is $L$-quasiconvex, $L \geq 1$, if for all $x, y \in X$ there is a (possibly constant) arc-length parameterised curve $\gamma : [0, l_\gamma] \to X$ such that $\gamma(0) = x$, $\gamma(l_\gamma) = y$ and $l_\gamma \leq Ld(x, y)$. A metric space is quasiconvex if it is $L$-quasiconvex for some $L$, and geodesic if it is 1-quasiconvex.

If $X$ is complete, and $\mu$ is a doubling measure supporting a $p$-Poincaré inequality, then $X$ is quasiconvex. Furthermore, the constant $L$ in the definition can be estimated in terms of the constants $C_\mu$ and $C_{PI}$ from the doubling condition and Poincaré inequality, respectively. For proofs of these results, and many others related to quasiconvexity and Poincaré inequalities, see Sections 4.4–4.8 in [4]. See also the paper by Korte [27].

The following examples illustrate some different situations. Example 4.2 is trivial, and it is easy to see that the distance (given by the metric) between points on either side of the slit in Example 4.5 can be made arbitrarily small, while keeping the length of the shortest path between the points larger than some fixed value. Finally, the constant $\sqrt{2}$ in Example 4.4 follows from solving an elementary maximising problem.

**Example 4.2.** $\mathbb{R}^n$, $n \geq 1$, is 1-quasiconvex.

**Example 4.3.** $\mathbb{R}^2 \setminus \{(x, y) : x = 0, y \neq 0\}$ is not quasiconvex.

**Example 4.4.** $\{(x, y) : x \geq 0, y \geq 0\} \cup \{(x, y) : x \leq 0, y \leq 0\}$ is $\sqrt{2}$-quasiconvex.

![Example 4.2.](image1.png)
![Example 4.3.](image2.png)
![Example 4.4.](image3.png)

Figure 4.1. In the figure, the dashed lines indicate the shortest path connecting the two points (dots). The solid lines indicate the distance between the points given by the metric.

### 4.2 Bow-ties on metric spaces

Although a bow-tie is not necessarily connected, it will be so as long as both $X_+$ and $X_-$ are (and only then). Whenever a Poincaré inequality is assumed either on $X$ itself or on both of $X_\pm$, the connectivity will follow from that as we saw earlier in Section 4.1. This will play a role in the proofs of our results.

The following quasiconvexity-type condition, which is weaker than quasiconvexity, will play a key role here. There is $\Lambda$ so that

$$d(x_+, x_0) + d(x_0, x_-) \leq \Lambda d(x_+, x_-) \quad \text{for all } x_\pm \in X_\pm. \quad (4.1)$$
4.2.1 The doubling condition

We wish to establish a relationship between the doubling condition on $X$ and $X_{\pm}$. We need to assume connectedness here, but since we need this result for the characterisation of the Poincaré inequality later, this is not a problem. In general doubling on $X$ and $X_{\pm}$ are not equivalent, as the following example shows.

**Example 4.5.** Let $X = X_{-} \cup X_{+}$ be a weighted bow-tie in $\mathbb{R}^{2}$, where

$$X_{-} = \{(x_1, x_2) \in \mathbb{R}^{2} : x_1 \leq -\frac{1}{2} \text{ or } |x_2| \leq -x_1\}$$

is equipped with the weight $w_{-}(x) = |x|$, and

$$X_{+} = \{(x_1, x_2) \in \mathbb{R}^{2} : x_1 \geq 0\}$$

is equipped with the weight $w_{+}(x) = x_1$. Then, it is easily seen that both of $X_{\pm}$ are equipped with doubling measures. However, $X$ is not, which can be seen by letting $z_i = (1, i)$ and considering the balls $B^{i} = B(z_i, 1)$, $i = 1, 2, 3, \ldots$. Denoting the measure on $X$ by $\mu$ (i.e. $d\mu = w(x) \, dx$ where $w = w_{-}$ on $X_{-}$ and $w = w_{+}$ on $X_{+}$), we see that $\mu(B^{i})$ is constant with respect to $i$, whereas $\mu(2B^{i})$ grows without bound as $i$ increases. The last claim follows from the fact that the Lebesgue measure of $2B^{i} \cap X_{-}$ is positive and constant with respect to $i$, and $w(x) > i - 2$ on $2B^{i} \cap X_{-}$.

![Figure 4.2. Illustration of Example 4.5.](image-url)
We have the following result from [Paper A, Proposition 3.1], describing the relationship between doubling on the bow-tie \( X \) and its parts \( X_\pm \).

**Proposition 4.6.** Assume that \( X \) is connected and that (4.1) holds. Then the measure \( \mu \) is doubling on \( X \) if and only if the following conditions hold:

(a) \( \mu \) is doubling on \( X_+ \) and on \( X_- \);

(b) \[ \mu(B_+^r) \approx \mu(B_-^r) \quad \text{for } 0 < r < \min\{\text{diam } X_+, \text{diam } X_-\} \]

The bow-tie in Example 4.5 meets conditions (a) and (b) in the proposition above, but not condition (4.1).

The proof of Proposition 4.6 is rather straightforward, but a bit lengthy as it requires dealing with different cases. One difficulty arising is that \( X_+ \) and \( X_- \) may have very different diameters. Thus, the requirement on \( r \) in condition (b) is necessary. Furthermore, the assumption (4.1) assures that the separation between \( X_- \) and \( X_+ \) increases sufficiently as we get further from the designated point \( x_0 \), which allows for dilations of balls far from \( x_0 \) in one of the spaces, without hitting the other one. Finally, the connectedness assures us that for any \( r \) as in condition (b), there are points in both of \( X_\pm \) at distance \( \frac{1}{2}r \) from \( x_0 \).

### 4.2.2 Poincaré inequalities

We end the chapter by presenting the main result of the included research paper [Paper A, Theorem 1.3], which is a characterisation of when Poincaré inequalities hold on \( X \) in terms of their validity on \( X_\pm \).

**Theorem 4.7.** Assume that \( \mu \) is doubling on \( X \). Then \( \mu \) supports a \((q,p)\)-Poincaré inequality on \( X \) if and only if the following conditions hold:

(i) \( \mu \) supports a \((q,p)\)-Poincaré inequality on \( X_+ \) and on \( X_- \);

(ii) (4.1) holds;

(iii) for all \( 0 < r < \frac{1}{4} \min\{\text{diam } X_+, \text{diam } X_-\} \),

\[ \text{cap}_{p,\mu}^{X_+}(x_0, B_+^r) \approx \frac{\mu(B_+^r)}{r^p} \quad \text{and} \quad \text{cap}_{p,\mu}^{X_-}(x_0, B_-^r) \approx \frac{\mu(B_-^r)}{r^p} \]

If, in addition, \( p > 1 \) and there is a locally compact open set \( G \ni x_0 \), then condition (iii) can equivalently be replaced by

(iii') \( p > \max\{Q^{X_+}, Q^{X_-}\} \), where \( R_0 = \frac{1}{4} \min\{\text{diam } X_+, \text{diam } X_-\} \) and

\[ Q^{X_+} := \inf\left\{ Q > 0 : \frac{\mu(B_+^r)}{\mu(B_+^{2r})} \gtrsim \left( \frac{2}{r} \right)^Q \quad \text{for all } 0 < \rho < r < R_0 \right\} \]
Bibliography


Papers

The papers associated with this thesis have been removed for copyright reasons. For more details about these see:

https://doi.org/10.3384/9789180751858
Capacities, Poincaré inequalities and gluing metric spaces.

Andreas Christensen