Generating Functions: Powerful Tools for Recurrence Relations. Hermite Polynomials Generating Function

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LiTH-MAT-EX–2023/04–SE

Credits: **16 hp**

Level: **G2**

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Linköping: **June 2023**

Abstract

In this report we will plunge down in the fascinating world of the generating functions. Generating functions showcase the "power of power series", giving more depth to the word "power" in power series. We start off small to get a good understanding of the generating function and what it does. Also, off course, explaining why it works and why we can do some of the things we do with them. We will see alot of examples throughout the text that helps the reader to grasp the mathematical object that is the generating function.

We will look at several kinds of generating functions, the main focus when we establish our understanding of these will be the "ordinary power series" generating function ("ops") that we discuss before moving on to the "exponential generating function" ("egf"). During our discussion on ops we will see a "first time in literature" derivation of the generating function for a recurrence relation regarding "branched coverings". After finishing the discussion regarding egf we move on the Hermite polynomials and show how we derive their generating function. Which is a generating function that generates functions. Lastly we will have a quick look at the "moment generating function".

Keywords:

Ordinary power series generating function, Exponential generating function, Moment generating function, Hermite polynomials.

URL for electronic version:

`Theurltothethesis`

Acknowledgements

I would like to thank my supervisor: Milagros Izquierdo for helping me throughout this work. Her knowledge and skill alone make her a mathematical "role model" and she is someone that I look up to.

Big thanks to Ludwig, for supplying me this template, without it I would not have gotten far.

Last, but surely not least, a thanks to Kalle for helping me out with my figures and for being a constant "ballpark" with whom I always can ball idéas with.

Nomenclature

$G(x)$	Ordinary power series, or exponential, generating function
$M_X(s)$	Moment generating function
$[x^n]G(x)$	The coefficient in front of x^n in the power series expansion of G(x)
$[x^n/n!]G(x)$	The coefficient in front of $\frac{x^n}{n!}$ in the power series expansion of G(x)
D	Differentiation operator
xD	Differentiate first, then multiply by x operator
$(1+D)$	Differentiate then add the identity operator
$\{a_n\}_{n=0}\infty$	A sequence of real numbers
F_n	The n:th Fibonacci number
$S(n,k)$	Stirling numbers of the second kind
C_n	The n:th Catalan number
Q_n	Number of branched coverings with dihedral monodromy of degree $2n$
$H_n(x)$	Hermite polynomial of degree n
H_{2n}	Hermite numbers
X	Random variable/probabilistic distribution
$E(X)$	Expectation or mean of random variable X
$V(X)$	Variance of random variable X
$X \sim Ber(p)$	Random variable X is Bernoulli distributed
$r.h.s.$	Right hand side
$l.h.s.$	Left hand side

Contents

Chapter 1

Introduction

Generating functions have been around for quite some time. In 1718 Abraham DeMoivre solved the Fibonacci recurrence relation using the generating function. Leonhard Euler extended the technique in his study of partitions of integers in 1748. Later the generating functions developed further when they were used together with probability theory and the moment generating function arose and were presented by Pierre-Simon de Laplace in 1812. These historical examples are taken from [6].

Generating functions are sort of an infinitely long measuring tape that, instead of keeping track of units of length, it keeps track of numbers in a sequence. They bookkeep this information by coding the terms of a sequence as coefficients in a formal power series, this is a very efficient way to represent a sequence. They are also useful when solving many types of counting problems[16], for instance, how many integer solutions are there to the equation

$$x_1 + x_2 + \cdots + x_n = C, \quad \textit{for some} \quad C \in \mathbb{N},$$

with various constrains on x_i [6]. This is often what we first learn when reading literature that introduces generating functions [6] [16]. Apart from solving counting problems generating functions can be used to solve recurrence relations, which is something that we focus a lot on in this work.

Some generating functions generates functions instead of a sequence. These generating functions are often more interesting in a physical point of view since physicists are often interested in solving differential equations, and such solutions comes in the form of functions.

A different kind of generating function is the "Moment generating function", they are used in probability and statistics where they offer a good way to represent probabilistic distributions. As the name implies the moment generating function generates moments, and to possess the moments of a random variable is to possess all the magnitudes that characterize how it is distributed. This have applications in data analysis and Artificial Intelligence.

Chapter 2

Generating functions

A generating function keeps track of numbers in a sequence. Sort of an infinitely long measuring tape that, in oppose to telling us: $0, 1, 2, \ldots$ units of length, tells us: $a_0, a_1, a_2, \cdots = \{a_n\}_{n=0}^{\infty}$ a sequence of numbers. The generating function comes in the form of a formal power series (here we follow [16]) where the $n : th$ power of x acts as a "placeholder" for the number a_n.

They have a wide variety of uses and applications, for instance, they can be used to solve many different types of counting problems. Such as the number of non-negative integer solutions to

$$x_1 + x_2 + \cdots + x_n = C, \quad \textit{for some} \quad C \in \mathbb{N}. \tag{2.1}$$

In this text however, we will focus more on another use of generating functions, namely their ability to solve recurrence relations in an effective way.

Throughout the report we will see many different examples and applications to different areas when we use generating functions to solve different types of recurrence relations. However, we will start of with some definitions that can be found in [6] and [16].

2.1 Generating functions and power series

Definition 1 *Let $a_0, a_1, a_2, \ldots$ be a sequence of real numbers. Then the function*

$$G(x) = a_0 + a_1 x + a_2 x^2 + \cdots = \sum_{n=0}^{\infty} a_n x^n \tag{2.2}$$

is called the ordinary power series generating function ("ops") to the given sequence $\{a_n\}_{n=0}^{\infty}$.

$$G(x) = a_0 + a_1 x + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!} \tag{2.3}$$

is called the exponential generating function ("egf") to the given sequence $\{a_n\}_{n=0}^{\infty}$.

Some clarification to some of the notation used in this report.

$G \overset{ops}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ means that the power series G is the ordinary power series ("ops") generating function for the given sequence $\{a_n\}_{n=0}^{\infty}$, i.e. $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

$G \overset{egf}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ means that the power series G is the exponential generating function for the given sequence $\{a_n\}_{n=0}^{\infty}$, i.e. $G(x) = \sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$.

This notation is also used in [19].

Example 1 *Since* $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$ *a geometric series for* $|x| < 1$ *and we know the formula for such a series to be*

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{2.4}$$

Therefore the generating function for the sequence $\{1\}_{n=0}^{\infty}$ *is* $G(x) = \frac{1}{1-x}$, *and we denote it by*

$$\frac{1}{1-x} \overset{ops}{\longleftrightarrow} \{1\}_{n=0}^{\infty}. \tag{2.5}$$

In addition since

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{2.6}$$

for every real number x *we have that* e^x *is the exponential generating function for the same sequence* $\{1\}_{n=0}^{\infty}$, *instead denoted by*

$$e^x \overset{egf}{\longleftrightarrow} \{1\}_{n=0}^{\infty}. \tag{2.7}$$

Now that we have some basic understanding of what a generating function is and how it may appear lets continue and talk about when we can use this and in what ways we can use this.

A very important property regarding generating functions is that they are considered to be *formal power series* and because of that we can view them as algebraic objects, meaning we do not have to worry about their radius of convergence [16]. An introduction to the theory of formal power series as algebraic objects can be found in [13].

We started this chapter with a metaphor, describing a generating function as a measuring tape, in the sense that for a fixed n the x^n in the power series $\sum_{n=0}^{\infty} a_n x^n$ just acts as a way of keeping track of a_n. Clearly the value of x does not effect the sequence $\{a_n\}_{n=0}^{\infty}$ meaning we can consider any x such that the power series converge and when a series converge we know that the series in fact represent an analytic function [2], and analytic functions are easy to work with. So all we need is that the series at least converge somewhere and then just assume that x is within that radius.

For example, for $z \in \mathbb{C}$ such that $|z| < 1$

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z} \tag{2.8}$$

is an analytic function. As well as

$$\sum_{n=0}^{\infty} (az)^n = \frac{1}{1-az} \tag{2.9}$$

is analytic for $|z| < 1/a$. But since we are considering formal power series that converge somewhere we are free to "forget" about $|z|$ and we can just focus on the result instead of when these results are legit. In this manner we can, for instance, take the derivative of a power series. Consider 2.8 and take the derivative on both sides, we get

$$\sum_{n=1}^{\infty} nz^{n-1} = \sum_{n=0}^{\infty} (n+1)z^n = \frac{1}{(1-z)^2}. \tag{2.10}$$

This "freedom" gives us many useful identities for our generating functions. Some of them are listed here [19] [6].

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \tag{2.11}$$

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots \tag{2.12}$$

$$\frac{1}{(1-x)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{n} x^n = 1 + \binom{1+k}{1} x + \binom{2+k}{2} x^2 + \binom{3+k}{3} x^3 + \cdots \tag{2.13}$$

$$\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n = 1 + ax + a^2x^2 + a^3x^3 + \cdots \tag{2.14}$$

$$\frac{1}{(1-ax)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} a^k x^k = 1 + \binom{n}{1}x + \binom{n+1}{2}x^2 + \binom{n+2}{3}x^3 + \cdots \tag{2.15}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{2.16}$$

Adding, and also subtracting, power series as well as multiply with a constant $c \in \mathbb{C}$ is quite trivial [2]:

$$\sum_{n=0}^{\infty} a_n z^n \pm \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n \quad \text{and} \quad c\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} (ca_n) z^n.$$

Multiplying two power series also behaves as expected, it follows the Cauchy product rule [2]:

$$\sum_{n=0}^{\infty} a_n z^n \sum_{n=0}^{\infty} b_n z^n = \sum_{n=0}^{\infty} c_n z^n \quad \text{where} \quad c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

So far, so good, we can breath out knowing that as long as we keep a formal view on our power series we can manipulate them without thinking twice if the series converge. However that does not mean we never have to take into account if a series converge. If we need to draw some analytical conclusions we will have to have convergence. When we have convergence our formal power series is just as any convergent power series and we can use known results regarding such series to draw our conclusions.

We will run into a situation like that later when we need to consider the limit of a generating function. When doing that we must respect the radius of convergence and not let $z \longrightarrow a$ if a is not contained within that radius.

2.2 Recurrence relations

It is a common property of many sequences used in algorithms that the $n : th$ number depends in some way of previous numbers in the sequence. When a sequence have this property that the number a_n depends on a_{n-1}, or several a_{n-i} for $i < n$, we call this relation a "recurrence relation" and the sequence may be expressed recursively using the previous numbers [16]. For example

$$a_n = a_{n-1} \tag{2.17}$$

defines the constant sequence $\{a_0\}_{n=0}^{\infty}$. But here a_0 could be any number, meaning we also need some initial condition for the recurrence to be meaningful. Initial conditions are values of the startup terms, satisfying the recurrence relation, before the relation takes effect [16]. So if we want to describe $\{1\}_{n=0}^{\infty}$ with a recurrence relation we would write

$$a_n = a_{n-1}, \quad n \geq 1, \;\; a_0 = 1. \tag{2.18}$$

Recurrence relations are central in the field of combinatorics where they arise naturally when counting "stuff". For instance, consider the following problem:

You have 3 vertical rods on a table and n discs with different diameters and with holes in the center so that they can "slide" down on the rods. The n discs are placed on rod nr.1 in order of their diameter, the largest one being at the bottom and the smallest one being on top. Now you have to move all these discs from rod nr.1 to one of the other rods. You may only move one disc at the time and you may never place a larger disc on top of a smaller one.

The question that seeks answer is: What is the fewest amount of moves needed to achive this?

This problem is called *"The towers of Hanoi"* and it was made famous by E. Lucas, a french mathematician of the nineteenth century [3]. Here a recurrence relation is obtained by the following way of reasoning.

Let a_n denote the smallest number of moves needed to move n discs from one rod to another. It is easy to see that $a_0 = 0$ and $a_1 = 1$, also $a_2 = 3$ is not very hard to see. But what about a_n for any $n \geq 3$?

In order to move disc nr. n (the largest disc) clearly it must be that we have all the other $n-1$ discs stacked on one of the rods, say rod nr.2, leaving rod nr.3 empty.

Now the smallest possible moves required to move those $n-1$ discs to rod nr.2 is, by definition, a_{n-1}. Since rod nr.3 is empty we can move disc nr.n there, yielding one extra move. After moving that disc we will never touch it again, and now all that is left is to move the $n-1$ discs from rod nr.2 to rod nr.3, this we know can be done in a_{n-1} moves.
Adding all the moves together we get

$$a_n = 2a_{n-1} + 1, \quad n \geq 1, \quad a_0 = 0. \tag{2.19}$$

Which is the sought after recurrence relation.

Even though 2.19 is a nice formula in all its simplicity it do have the downside of requiring one to calculate every number a_i, $i < n$ in the sequence to get the value of a_n. It is at this stage that the generating function will prove useful, especially when it comes to more "complicated" recurrence relations [19]. But before we look at how we use generating functions to solve recurrence relations we do a small recap on how we solve them using the method of homogeneous and particular solution.

The material regarding the method of homogeneous and particular solutions can be found in [16].

In short, the method of homogeneous and particular solutions builds upon the fact that a linear, and homogeneous, recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} \tag{2.20}$$

has the solution $a_n = r^n$. Note that the coefficients c_i are just constant's.

This is quite easy to see since if we plug in r^n in 2.20 we get

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_k r^{n-k}. \tag{2.21}$$

And, with the condition $r \neq 0$, we can divide 2.21 by r^{n-k} which yields

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \cdots - c_k = 0. \tag{2.22}$$

Of course 2.22 is true if and only if $r \neq 0$ and r is a root in 2.22. Also 2.22 is called the *"characteristic equation"* of 2.20 and r is called a *"characteristic root"*.

In addition, a linear combination of two, or more, solutions to a linear and homogeneous recurrence relation is also a solution [16]. Lets keep this in mind while looking at a more general recurrence relation

$$a_n + c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} = f(n). \tag{2.23}$$

Here $f(n)$ is some function depending only on n, possibly $f(n) \equiv 0$, in which case 2.23 is homogeneous. Assuming that $f(n) \not\equiv 0$ and let $a_n^{(p)}$ be a particular solution to 2.23 and $a_n^{(h)}$ be the solution to the associated homogeneous recurrence relation, which is obtained by putting $f(n) \equiv 0$ in 2.23. Then every solution to 2.23 is given by

$$a_n = a_n^{(p)} + a_n^{(h)} \tag{2.24}$$

Since adding $a_n^{(h)}$ in 2.24 is basically just adding zero.

Now, there is considerably more "depth" to this theory than that I have described here and I suggest that the reader that wants a more rigorous explanation of the theory behind all this consults [16].

We will now finish this section by solving 2.19 using this method.

Example 2 *We can rewrite 2.19 as*

$$a_n - 2a_{n-1} = 1, \quad n \geq 1, \quad a_0 = 0. \tag{2.25}$$

This form agrees very well with 2.23 with $k = 1$, $c_1 = -2$ and $f(n) = 1$.

First we get $a_n^{(h)}$ by solving the characteristic equation, in this case it is trivial to see that $r = 2$ and the homogeneous solution falls out as

$$a_n^{(h)} = A(2)^n, \quad A \in \mathbb{R}. \tag{2.26}$$

Next we search for a particular solution $a_n^{(p)}$. Since the r.h.s. in 2.25 is a constant, then it is reasonable to believe that a particular solution here also will be a constant, say C. If we plug in C as the solution in 2.25 we get

$$C - 2C = 1. \tag{2.27}$$

We see that $C = -1$ is in fact a particular solution and

$$a_n = a_n^{(p)} + a_n^{(h)} = A(2)^n - 1. \tag{2.28}$$

Using our initial condition $a_0 = 0$ we find that $A = 1$ and the solution to 2.19 is

$$a_n = 2^n - 1. \tag{2.29}$$

So, we have glanced at the method of homogeneous and particular solutions and as we saw in the example above it does the job to find an explicit formula very well. We now leave this topic and will instead look at an alternative way to solve 2.19. One that focus on the essence of this paper, the generating functions.

2.2.1 The method of generating functions

In this section we will first talk about the idea behind the method of generating functions and explain how we execute it in general followed by an example where we solve, again, 2.19. This time using the generating function.

To solve a recurrence relation using the method of generating functions one first finds the generating function as a power series for the recurrence relation. This involves multiplying the recurrence relation with x^n for each n and then adding them together [19]. Doing this will yield something that looks very much like a generating function. With some manipulations we get the generating function expressed as an analytic function that we can expand in a power series to find an explicit solution to the recurrence relation, namely the coefficient of x^n. As we discussed in the first section of this chapter we will not bother ourselves with questions about convergence when doing all this, we will just assume x is sufficiently small. Expanding our generating functions in a power series can be done in several ways, partial fraction expansion or Maclaurin expansion for instance[19].

To illustrate this consider a recurrence relation

$$a_n = ca_{n-1}, \quad n \geq 1, \quad a_0 = a, \quad a, c \in \mathbb{R}. \tag{2.30}$$

Now this is a simple equation, actually it is the equation for exponential growth and has the solution $a_n = (c)^n a$ but nevertheless, it works fine to demonstrate the method on.

For each $n \geq 1$ we just have an equation, if we multiply each of those equations with x^n we get

$$\begin{aligned}
a_1 x &= ca_0 x & (n = 1) \\
a_2 x^2 &= ca_1 x^2 & (n = 2) \\
a_3 x^3 &= ca_2 x^3 & (n = 3) \\
&\vdots & \vdots
\end{aligned}$$

Now add all these equations and we get

$$\sum_{n=1}^{\infty} a_n x^n = c \sum_{n=1}^{\infty} a_{n-1} x^n. \tag{2.31}$$

Here we clearly see something that resembles 2.2 though it is not the same, but it is easy to rewrite in terms of a generating function since, let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then 2.31 can be expressed as

$$G(x) - a = cxG(x) \tag{2.32}$$

From 2.32 we get an expression for $G(x)$, namely

$$G(x) = \frac{a}{1 - cx}. \tag{2.33}$$

Here we have our generating function expressed as an analytic function, and if we want to get an explicit formula for a_n we would expand the right hand side (from now on we will write *r.h.s.* for right hand side and *l.h.s.* for left hand side) of 2.33 in a power series. Since the formula for a_n is the same as the formula for the coefficient in front of x^n which we from now on will denote as $[x^n]G(x)$. This is shown in the example that follows where we solve 2.19 using this method.

Example 3 *First of all let* $G(x) = \sum_{n=0}^{\infty} a_n x^n$ *and multiply both sides of 2.19 by* x^n *and sum over the* $n : s$ *for which the recurrence relation is defined. In this case* $n \geq 1$. *This yields*

$$\sum_{n=1}^{\infty} a_n x^n = 2\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n. \tag{2.34}$$

Looking at the l.h.s. of 2.34 we see

$$\sum_{n=1}^{\infty} a_n x^n = a_1 x + a_2 x^2 + a_3 x^3 + \cdots = G(x) - a_0 = G(x) \tag{2.35}$$

since $a_o = 0$.

Now look at the r.h.s. of 2.34.

$$2\sum_{n=1}^{\infty} a_{n-1} x^n + \sum_{n=1}^{\infty} x^n = 2xG(x) + \frac{x}{1-x}, \tag{2.36}$$

since

$$2\sum_{n=1}^{\infty} a_{n-1} x^n = 2(a_0 x + a_1 x^2 + a_2 x^3 + \cdots) = 2x(a_0 + a_1 x + a_2 x^2 + \cdots) = 2xG(x) \tag{2.37}$$

and

$$\sum_{n=1}^{\infty} x^n = x + x^2 + x^3 + \cdots = x(1 + x + x^2 + \cdots) = \frac{x}{1-x} \tag{2.38}$$

Taking what we just learned we can rewrite 2.34 in terms of $G(x)$

$$\begin{aligned} G(x) &= 2xG(x) + \frac{x}{1-x} \\ \Leftrightarrow G(x) - 2xG(x) &= \frac{x}{1-x} \\ \Leftrightarrow G(x) &= \frac{x}{(1-x)(1-2x)} \end{aligned} \tag{2.39}$$

This is our analytic form of $G(x)$ *and the solution we seek is* $[x^n]G(x)$. *To find this expression we expand* $G(x)$ *in a power series using partial fraction* *expansion.*

$$\begin{aligned}
\frac{x}{(1-x)(1-2x)} &= x(\frac{2}{1-2x} - \frac{1}{1-x}) \\
&= x(2 + 2^2x + 2^3x^2 + \cdots) - x(1 + x + x^2 + \cdots) \\
&= (2-1)x + (2^2-1)x^2 + (2^3-1)x^3 + \cdots \\
&= \sum_{n=0}^{\infty}(2^n - 1)x^n.
\end{aligned} \tag{2.40}$$

From 2.40 it is easy to see that $[x^n]G(x) = 2^n - 1$ *which is the explicit formula for* a_n *that we were looking for.*

With this example we finish this section. We will, throughout the text, consider many important sequences of different kinds. Both homogeneous, linear, non-linear and of more then one variable. All of these will be solved using this method and by doing so we illustrate the power of the generating functions when it comes to solving recurrence relations.

We will treat the different generating functions separately, in Chapter 3 we focus on the ordinary power series generating function and in Chapter 4 we will instead look at the exponential generating function.

Chapter 3

Ordinary Power Series Generating Function

In this chapter we will look at some recursively defined sequences of great importance and solve their recurrence relations with the method described in *Chapter 2*, also, along the way we will establish some "rules of computation" that will come in handy as we progress through the examples.

One of the most famous sequences we have is the Fibonacci numbers. This sequence has been around for 800 years or so and it was first discovered by Fibonacci when he presented a counting-problem involving the population of rabbits [16]. However, to say that the Fibonacci numbers model the population of rabbits may be to exaggerate a bit, since in Fibonacci´s problem no rabbits ever die but just keep on breeding. The sequence do however model things that can be interesting in a combinatorial point of view. For example, let F_n denote the n:th Fibonacci number and let $S = \{1, 2, 3, \ldots, n\}$ then the number of subsets of S containing no consecutive integers is given by F_{n+2}. Or in how many ways can one tile a $2 \times n$ units path using tiles of dimension 2×1 and 1×2 units? The answer is F_{n+1} ways [6].

The Fibonacci sequence is recursively defined as

$$F_{n+1} = F_n + F_{n-1}, \qquad n \geq 1, \quad F_0 = 0, \quad F_1 = 1, \tag{3.1}$$

a second order, linear and homogeneous recurrence relation. In the following example we will first find the generating function for the Fibonacci sequence, that is $F(x) = \sum_{n=0}^{\infty} F(n)x^n$ then we will solve for an explicit formula for F_n.

Example 4 (Fibonacci) *First we multiply 3.1 by x^n and sum over $n \geq 1$. Giving us the l.h.s.*

$$\sum_{n=1}^{\infty} F_{n+1}x^n = \frac{F(x) - F_0 - F_1 x}{x} = \frac{F(x) - x}{x} \tag{3.2}$$

since $F_0 = 0$ and $F_1 = 1$. On the r.h.s. we obtain

$$\sum_{n=1}^{\infty}(F_n + F_{n-1})x^n = \sum_{n=1}^{\infty} F_n x^n + \sum_{n=1}^{\infty} F_{n-1}x^n = F(x) + xF(x) \tag{3.3}$$

Again since $F_0 = 0$ we can write $\sum_{n=1}^{\infty} F_n x^n = \sum_{n=0}^{\infty} F_n x^n = F(x)$. By 3.2 and 3.3 we get

$$\begin{aligned} \frac{F(x) - x}{x} &= F(x) + xF(x) \\ \Leftrightarrow F(x) &= F(x)(x + x^2) + x \\ \Leftrightarrow F(x)(1 - x - x^2) &= x. \end{aligned} \tag{3.4}$$

From here we find the closed form of the generating function

$$F(x) = \frac{x}{1 - x - x^2} \tag{3.5}$$

and our first step is done. Now to find an explicit formula we use partial fraction expansion, just as we did with the Hanoi Towers, only this one is a little trickier in terms of "nice numbers". But with no fancier tools then completing the square and the use of some symbols instead of root-expressions we get

$$\frac{x}{1 - x - x^2} = \frac{x}{(1 - \alpha x)(1 - \beta x)} = \frac{1}{\alpha - \beta}\left(\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right). \tag{3.6}$$

Where $\alpha = \frac{1+\sqrt{5}}{2}$ and $\beta = \frac{1-\sqrt{5}}{2}$ meaning we can write 3.6 as

$$\frac{1}{\sqrt{5}}\left(\frac{1}{1 - \alpha x} - \frac{1}{1 - \beta x}\right). \tag{3.7}$$

Expanding 3.7 in a power series using the same technique as we did in 2.40 we soon realise that

$$[x^n]F(x) = F_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \quad \alpha = \frac{1 + \sqrt{5}}{2}, \beta = \frac{1 - \sqrt{5}}{2}. \tag{3.8}$$

We have now illustrated how the method of generating functions can be used to solve the Fibonacci recurrence relation. Was it any better then the method of homogeneous and particular solutions? In this case perhaps not, but at the same time it did not make the computations harder, just a little bit different and it shows that generating functions have what it takes to solve such equations. Later we will show how generating functions possess the power to solve other kinds of recurrence relations, quite easy, that would be hard to solve otherwise [19].

But first let us establish a few rules that will make computations faster. The rules that we formulate here can also be found in [19] without proof. Here we also provide proofs for them.

Recall that $G \overset{ops}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ means that $G(x)$ is the ordinary power series generating function for the sequence $\{a_n\}_{n=0}^{\infty}$ i.e. $G(x) = \sum_{n=0}^{\infty} a_n x^n$.

Theorem 1 (Rules of computation (ops)) *Let* $G \overset{ops}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ *and* $T \overset{ops}{\longleftrightarrow} \{b_n\}_{n=0}^{\infty}$ *Then*

1. $\frac{G(x)-a_0-a_1x-\cdots-a_{k-1}x^{k-1}}{x^k} \overset{ops}{\longleftrightarrow} \{a_{n+k}\}_{n=0}^{\infty}$

2. $(xD)^k G(x) \overset{ops}{\longleftrightarrow} \{n^k a_n\}_{n=0}^{\infty}$

   (where xD means first differentiate then multiply by x)

3. $GT \overset{ops}{\longleftrightarrow} \{\sum_{r=0}^{n} a_r b_{n-r}\}_{n=0}^{\infty}$

Proof.

1. Let $G \overset{ops}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ and let $H \overset{ops}{\longleftrightarrow} \{a_{n+k}\}_{n=0}^{\infty}$. This implies

$$G(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x - a_2 x^2 + \cdots \tag{3.9}$$

and

$$H(x) = \sum_{n=0}^{\infty} a_{n+k} x^n = a_k + a_{k+1} x + a_{k+2} x^2 + \cdots \tag{3.10}$$

That is

$$H(x) = \frac{G(x) - a_0 - a_1 x - \cdots - a_{k-1} x^{k-1}}{x^k} \tag{3.11}$$

2. Let $G \overset{ops}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ and let $H \overset{ops}{\longleftrightarrow} \{n^k a_n\}_{n=0}^{\infty}$ then for $H(x)$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} n^k a_n x^n &= a_1 x + 2^k a_2 x^2 + 3^k a_3 x^3 + \cdots \\ &= xD(a_0 + a_1 x + 2^{k-1} a_2 x^2 + 3^{k-1} a_3 x^3 + \cdots) \\ &= (xD)(xD)(a_0 + a_1 x + 2^{k-2} a_2 x^2 + 3^{k-2} a_3 x^3 + \cdots) \\ &= (xD)^2 (a_0 + a_1 x + 2^{k-2} a_2 x^2 + 3^{k-2} a_3 x^3 + \cdots) \\ &\vdots \\ &= (xD)^k (a_0 + a_1 x + 2^{k-k} a_2 x^2 + 3^{k-k} a_3 x^3 + \cdots) \\ &= (xD)^k (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots) \\ &= (xD)^k G(x).\end{aligned} \tag{3.12}$$

3. Let $G \overset{ops}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ and $T \overset{ops}{\longleftrightarrow} \{b_n\}_{n=0}^{\infty}$. Assume that both of them converge in some domain $O \in \mathbb{C}$. Then by theorem 4.28 [2]

$$GT = \sum_{n=0}^{\infty} c_n x^n, \quad \text{where } c_n = \sum_{r=0}^{n} a_r b_{n-r} \tag{3.13}$$

■

Soon we are ready to look at a bit more complicated sequence and we will do so in the example to follow, this will involve a sequence depending on two independent variables. But before we do that we should point at some properties of this notation (see [19]). Recall that $[x^n]G(x)$ denotes the coefficient in front of x^n in the power series $G(x)$. Its follows that

$$[x^n](x^a G(x)) = [x^{n-a}]G(x). \tag{3.14}$$

It also follows that

$$[\alpha x^n]G(x) = \frac{1}{\alpha}[x^n]G(x). \tag{3.15}$$

3.1 Stirling numbers of the second kind

Let A be a finite set such that $|A| = n$. In how many ways can one partition A into a fix number k parts?

The answer to that question is the *Stirling numbers of the second kind* (see [3]), we denote this number $S(n,k)$. It is quite clear that for all $n \geq 1$ it holds that

$$S(n,1) = S(n,n) = 1. \tag{3.16}$$

Since there is only one way to partition A in one part, namely A itself. Likewise, since no part of the partition can be empty there is only one way to partition A into n parts.

Also it is not possible to partition A into zero parts, nor is it possible to partition A into k parts if $n < k$ since no parts can be empty. From this we get

$$S(n,0) = 0, \quad \forall n \geq 1 \tag{3.17}$$

$$S(n,k) = 0, \text{ if } n < k. \tag{3.18}$$

Note as well the special case $S(0,0)$. This number is a bit strange when thinking of partitioning an empty set into zero parts, but similar to the convention to put $0! = 1$ it is a useful convention to define $S(0,0) = 1$ [3].

The Stirling numbers of the second kind is then defined, recursively, through

$$S(n,k) = S(n-1,k-1) + kS(n-1,k), \quad 1 \leq k < n \tag{3.19}$$

For a derivation of the formula one can read [3] or [19]. Now let us solve this with the help of generating functions.

Example 5 *Consider, for every $k \geq 0$ the sum*

$$A_k(x) = \sum_{n=0}^{\infty} S(n,k)x^n. \tag{3.20}$$

Let this be our unknown generating function. Note that 3.20 is the l.h.s. of 3.19 multiplied by x^n and summed over $n \geq 0$. For $k = 0$ we get

$$A_0 = S(0,0) + S(1,0)x + S(2,0)x^2 + \cdots = S(0,0) + 0 + 0 + \cdots = 1. \tag{3.21}$$

If we now express 3.19 in terms of $A_k(x)$ we get, for $k \geq 1$,

$$A_k(x) = \sum_{n=1}^{\infty} S(n-1,k-1)x^n + k\sum_{n=1}^{\infty} S(n-1,k)x^n, \quad A_0 = 1. \tag{3.22}$$

Let us look at the terms in the r.h.s. one by one.

$$\begin{aligned}\sum_{n=1}^{\infty} S(n-1,k-1)x^n &= S(0,k-1)x + S(1,k-1)x^2 + S(2,k-1)x^3 + \cdots \\ &= x\sum_{n=0}^{\infty} S(n,k-1)x^n = xA_{k-1}(x)\end{aligned} \tag{3.23}$$

$$
\begin{aligned}
k\sum_{n=1}^{\infty} S(n-1,k)x^n &= k(S(0,k)x + S(1,k)x^2 + S(2,k)x^3 + \cdots) \\
&= kx\sum_{n=0}^{\infty} S(n,k)x^n = kxA_k(x).
\end{aligned}
\tag{3.24}
$$

That is 3.22 can be written as

$$A_k(x) = xA_{k-1}(x) + kxA_k(x), \quad k \geq 1,\ A_0 = 1. \tag{3.25}$$

Which yields

$$A_k(x) = A_{k-1}(x)\frac{x}{1-kx}. \tag{3.26}$$

Iterating 3.26 from $k \geq 1$ *our previously unknown generating function falls out*

$$A_k(x) = \frac{x^k}{(1-x)(1-2x)\cdots(1-kx)} \quad k \geq 0. \tag{3.27}$$

Note that 3.27 works for $k = 0$ *as well if we in that case take the denominator to be* $1 - 0x$*. Again we use partial fraction decomposition to find an explicit formula for* $S(n,k)$*.*

$$\frac{1}{(1-x)(1-2x)\cdots(1-kx)} = \sum_{i=1}^{k}\frac{\alpha_i}{1-ix}, \quad \alpha_i \in \mathbb{R}. \tag{3.28}$$

Here we can find each α_i *simply by multiplying both sides of 3.28 with* $1 - rx$ *and simultaneously letting* $x = \frac{1}{r}$ *for every* $1 \leq r \leq k$*. Computing this we get*

$$\alpha_r = (-1)^{k-r}\frac{r^{k-1}}{(r-1)!(k-r)!}, \quad 1 \leq r \leq k. \tag{3.29}$$

Now we will take advantage of the notation we use, remember we are looking for

$$S(n,k) = [x^n]\left(\frac{x^k}{(1-x)\cdots(1-kx)}\right) = [x^n]x^k\left(\frac{1}{(1-x)\cdots(1-kx)}\right). \tag{3.30}$$

By 3.14 and 3.28

$$[x^n]x^k\left(\frac{1}{(1-x)\cdots(1-kx)}\right) = [x^{n-k}]\sum_{r=1}^{k}\alpha_r\frac{1}{1-rx}. \tag{3.31}$$

From before we know that $\frac{1}{1-rx}$ *is easy to expand in a power series and* $[x^{n-k}]\frac{1}{1-rx} = r^{n-k}$ *so we get*

$$
\begin{aligned}
[x^{n-k}]\sum_{r=1}^{k}\alpha_r\frac{1}{1-rx} &= \sum_{r=1}^{k}\alpha_r r^{n-k} \\
&= \sum_{r=1}^{k}(-1)^{k-r}\frac{r^{k-1}}{(r-1)!(k-r)!}r^{n-k} \qquad (3.32)\\
&= \sum_{r=1}^{k}(-1)^{k-r}\frac{r^{n}}{r!(k-r)!}
\end{aligned}
$$

To convince yourself that 3.32 truly is an explicit formula one could take a look at $S(n,2)$ for every $n \geq 2$. According to [3] $S(n,2) = 2^{n-1} - 1$ for every $n \geq 2$. Lets evaluate $S(n,2)$ using our formula to see that it gives us the correct answer.

$$S(n,2) = (-1)\frac{1^n}{1!\,1!} + (1)\frac{2^n}{2!\,0!} = 2^{n-1} - 1. \qquad (3.33)$$

Now, in the example above we considered the generating function $A_k(x) = \sum_{n=0}^{\infty} S(n,k)x^n$, where we fixate k and multiply with x^n and sum over n. What if we had chosen to fixate n, multiply by x^k and then sum over k instead so we got $B_n(x) = \sum_{k=0}^{\infty} S(n,k)x^k$. It is absolutely possible to do this, but it will not yield an explicit formula for $S(n,k)$ [19]. Let us look at what we get when considering $B_n(x)$ instead.

$$
\begin{aligned}
B_n(x) &= \sum_{k=0}^{\infty} S(n-1,k-1)x^k + \sum_{k=0}^{\infty} kS(n-1,k)x^k \\
&= xB_{n-1}(x) + (x\frac{d}{dx})B_{n-1}(x) \qquad (3.34)\\
&= [x(1+D)]B_{n-1}(x), \quad n > 0,\ B_0(x) = 1
\end{aligned}
$$

Here we used Theorem1. From 3.34 we see that we get $B_n(x)$ by doing the operation $x(1+D)$ on $B_{n-1}(x)$. This yields

$$
\begin{array}{rcc}
B_1(x) = & x(1+D)1 = & x \\
B_2(x) = & x(1+D)x = & x^2 + x \\
B_3(x) = & x(1+D)(x^2+x) = & x^3 + 3x^2 + x \\
\vdots & \vdots & \vdots
\end{array}
$$

We see that this generating function generates a polynomial of degree n and $[x^k]B_n(x) = S(n,k)$ but we cannot derive an explicit formula for $S(n,k)$ from this. But we can however use this function to prove that the sequence $\{S(n,k)\}_{k=0}^{n}$ is "unimodal"[19], meaning that if n is fixed and k varies from 0 to n, the numbers $S(n,k)$ first increase, up to a maximum, then they decrease. This is an example of another valuable application of the generating functions.

3.2 Catalan numbers

Lets say we have a circle, on the circumference of this circle there are $2n$ dots enumerated $1, 2, 3, \ldots, 2n$. Now, using n lines, connect all the dots in pairs such that no lines intersects another line. In how many ways can one do that?

To start let C_n denote the number of ways to pair up $2n$ dots. Note that we always have an even number of dots so we will never have a dot left on its own after pairing. Also if $n = 0$ we say that there is one way to pair up zero dots, that is we don´t do anything at all, so $C_0 = 1$. For $n = 1$ there is one way to connect the two dots on the circle, if $n = 2$ there is two ways to pair them, see Figure 3.1.

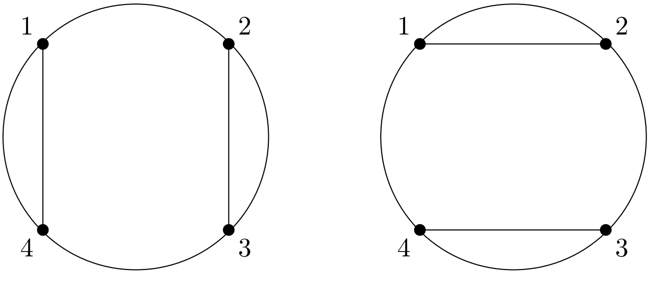

Figure 3.1:

Now consider the circle with $2n$ dots, see Figure 3.2, for simplicity start in dot nr. 1 and pick a dot to pair with. It is clear we cannot pick an odd numbered dot since that will result in an odd number of dots remaining on both sides of the line, meaning we will be forced to intersect that line to pair up all dots.

Instead pick an even numbered dot, say dot nr. $2r$ for some $1 \leq r \leq n$. This results in a line, splitting the circle in two, on one side of the line we have $2r - 2 = 2(r-1)$ dots and on the other side we have $2n - 2r = 2(n-r)$ dots. Meaning we can view these two halves as two separate circles with $2(r-1)$ and $2(n-r)$ dots respectively (see Figure 3.3).

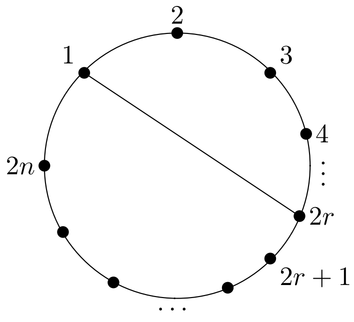

Figure 3.2:

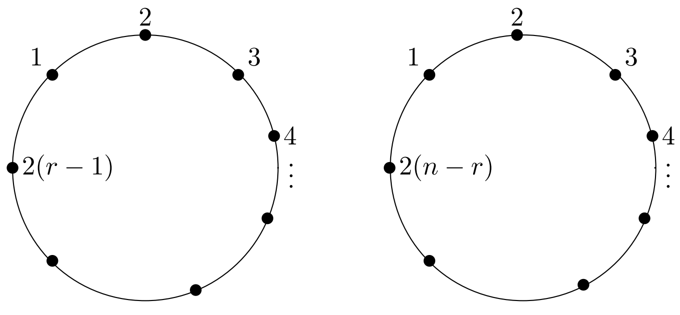

Figure 3.3:

So for this choice of r there is, by the multiplication principle, $C_{r-1}C_{n-r}$ different ways to pair up the dots and we can pick any r such that $1 \leq r \leq n$. Hence the addition principle yields

$$C_n = \sum_{r=1}^{n} C_{r-1}C_{n-r} = \sum_{r=0}^{n-1} C_r C_{n-r-1}, \quad n \geq 1, C_0 = 1. \tag{3.35}$$

This, nonlinear, recurrence relation describes the *Catalan numbers*[3]. Named after E.C. Catalan, a Belgian mathematician from the 19:th century [3]. Though the sequence was also studied prior to Catalan by Euler for instance [13].

The Catalan numbers model a great deal of different structures, here we list some of the most useful ones [7].

1. C_n is the number of binary sequences of length $2n$ containing exactly n zeros and n ones, such that at each stage of the sequence the number of

ones does not exceed the number of zeros.

2. C_n is the number of ways to parenthesize the product of $n+1$ variables in a computer operational system, to specify the order of operation.

3. (Euler´s Triangulations of Polygons) C_n is the number of ways to divide a convex $(n+2)$-gon into triangles by drawing $n-1$ non intersecting diagonals.

4. C_n is the number of "up-right paths" on a square grid from the origin $O(0,0)$ to the point $A(n,n)$ which never crosses the the diagonal.

5. C_n is the number of rooted binary trees with n nodes.

It is hard to argue that the Catalan numbers is the solution to many counting problems. But their recurrence formula is nonlinear, meaning it seems hard to solve for an explicit formula for C_n. No need to worry, generating functions will take on this task with a breeze.

Example 6 *Let $G(x)=\sum_{n=0}^{\infty} C_n x^n$. Without loss of generality we can rewrite 3.35 as*

$$C_{n+1}=\sum_{r=0}^{n} C_r C_{n-r}, \quad n \geq 0, C_0=1. \tag{3.36}$$

Multiply 3.36 by x^n and sum over $n \geq 0$ and use Theorem 1 we get, without hardly any effort at all,

$$\frac{G(x)-1}{x}=G^2(x). \tag{3.37}$$

We see that $G(x)$ is a solution to the quadratic equation

$$xG^2(x)-G(x)+1=0. \tag{3.38}$$

Which has the following solutions

$$\begin{aligned} G_1(x) = &\quad \tfrac{1-\sqrt{1-4x}}{2x} \\ G_2(x) = &\quad \tfrac{1+\sqrt{1-4x}}{2x}, \end{aligned}$$

only one of the solutions work and that is $G_1(x)$. To see this consider the limits of $G_1(x)$ and $G_2(x)$ respectively as $x \to 0$. For $G_1(x)$ we have, with the help of "L´Hospital´s Rule" [1]

$$\lim_{x\to 0} \frac{1-\sqrt{1-4x}}{2x} = 1 = C_0. \tag{3.39}$$

While for $G_2(x)$ we clearly have

$$\lim_{x\to 0} \frac{1+\sqrt{1-4x}}{2x} = \infty. \tag{3.40}$$

So we pick $G(x) = G_1(x)$ as our generating function. To find an explicit formula for C_n, first consider the Maclaurin expansion of $\sqrt{1-4x}$.

$$\begin{aligned}(1-4x)^{1/2} &= \sum_{k=0}^{\infty} \binom{1/2}{k} (-4)^k x^k \\ &= 1 - \frac{1}{2}\cdot 4x - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4^2x^2}{2} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{4^3x^3}{3!} - \cdots.\end{aligned} \tag{3.41}$$

Hence

$$\begin{aligned}G(x) &= \frac{1}{2x}\left(1 - \left(1 - \frac{1}{2}\cdot 4x - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4^2x^2}{2} - \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{4^3x^3}{3!} - \cdots\right)\right) \\ &= \frac{1}{2x}\left(\frac{1}{2}\cdot 4x + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4^2x^2}{2} + \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{4^3x^3}{3!} + \cdots\right) \\ &= 1 + \frac{1}{2}\cdot\frac{4x}{2} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{4^2x^2}{3!} + \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{4^3x^3}{4!} + \cdots\end{aligned} \tag{3.42}$$

Here we see $[x^n]G(x) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}4^n$. Now, to rewrite this more compact

$$\begin{aligned}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n(n+1)!}4^n &= \frac{2^n}{(n+1)!}1\cdot 3\cdot 5\cdots(2n-1) \\ &= \frac{2^n}{(n+1)!}\frac{1\cdot 3\cdot 5\cdots(2n-1)\cdot 2\cdot 4\cdot 6\cdots 2n}{2\cdot 4\cdot 6\cdots 2n} \\ &= \frac{2^n}{(n+1)!}\frac{(2n)!}{2^n n!} \\ &= \frac{1}{n+1}\frac{(2n)!}{n!n!} = \frac{1}{n+1}\binom{2n}{n}.\end{aligned} \tag{3.43}$$

Which is a neat formula for C_n.

Before closing this chapter I would like to just give one more example, this is a rather special one. It is a recurrence relation used by M. Izquierdo in her article *On Klein Surfaces and Dihedral Groups* [8]. The following goes far beyond the scope of this work and include things like complex geometry.

How many non-biconformally equivalent Klein-surfaces are coverings of the hyperbolic quotient space, whose fundamental group admits the dihedral group D_n as a group of automorphisms?

The interested reader finds very good account on Klein-surfaces, branched coverings and complex geometry in Jones and Singerman [10], Stillwell [18], Lages Lima [12] and Fulton [5].

What is special about it is that no one has ever yielded the generating function for it in literature, until now, thus, in literature, no one has solved it using the method of generating functions. So let us do just that.

The recurrence relation is as follows [8]:

$$Q_n = (p-2)Q_{n-1} + (p-1)Q_{n-2}, \quad n \geq 5, \tag{3.44}$$

with initial conditions

$$Q_3 = p(p-1)(p-2) \quad \text{and} \quad Q_4 = p(p-1)[(p-1) + (p-2)^2], \tag{3.45}$$

where p is an odd prime.

An interesting observation one can do is that 3.44 looks very much like the Fibonacci equation, it is just some scaling involved. This resemblance will persist also in the generating function, as we will soon see.

Example 7 ***Theorem 2*** *The generating function of*

$$Q_n = (p-2)Q_{n-1} + (p-1)Q_{n-2}, \quad n \geq 5$$

is

$$G(x) = \frac{p - xp(p-2)}{1 - (p-2)x - (p-1)x^2}$$

Proof. *Equation 3.44 can, equivalently, be rewritten as*

$$(p-1)Q_{n-2} = (p-2)Q_{n-1} - Q_n \tag{3.46}$$

From this we can derive values for Q_i, $0 \leq i \leq 2$ so that we can start the sequence from Q_0 instead of Q_3. Compute these and we find $Q_0 = p$, $Q_1 = 0$ and $Q_2 = p(p-1)$. Also we rewrite the indices in 3.44 so that we can use Theorem 1. This yields

$$Q_{n+2} = (p-2)Q_{n+1} + (p-1)Q_n, \quad n \geq 0, \; Q_0 = p, \; Q_1 = 0. \tag{3.47}$$

Now let $G(x) = \sum_{n=0}^{\infty} Q_n x^n$ *and execute the method. Then by Theorem 1 we get*

$$\frac{G(x) - p}{x^2} = \frac{(p-2)(G(x) - p)}{x} + (p-1)G(x), \tag{3.48}$$

from which we get

$$G(x) = \frac{p - xp(p-2)}{1 - (p-2)x - (p-1)x^2}. \tag{3.49}$$

■

Note that we can see some resemblance to the generating function of the Fibonacci sequence, at least in the sense that it is a first degree polynomial divided by a second degree polynomial. If we rewrite a little bit we can actually take a shortcut and use a result from the Fibonacci example when we are determining the explicit formula for Q_n*. First consider this second degree polynomial*

$$1 - ax - bx^2 = (1-\alpha x)(1-\beta x) \quad \textit{where} \quad \alpha = \frac{a + \sqrt{4b + a^2}}{2}, \ \beta = \frac{a - \sqrt{4b + a^2}}{2}. \tag{3.50}$$

If we plug in $a = (p-2)$ *and* $b = (p-1)$ *we get the denominator of 3.49, hence we can rewrite 3.49 as*

$$G(x) = \frac{p - xp(p-2)}{(1-\alpha x)(1-\beta x)} = \frac{p}{(1-\alpha x)(1-\beta x)} - p(p-2)\left(\frac{x}{(1-\alpha x)(1-\beta x)}\right) \tag{3.51}$$

From Example 4 we know that

$$\frac{x}{(1-\alpha x)(1-\beta x)} = \frac{1}{\alpha - \beta}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right). \tag{3.52}$$

Partial fraction decomposition yields

$$\frac{p}{(1-\alpha x)(1-\beta x)} = p\left(\frac{1}{(1-\alpha x)(1-\beta x)}\right) = \frac{p}{\alpha - \beta}\left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right). \tag{3.53}$$

Hence

$$\begin{aligned} G(x) &= \frac{p}{\alpha-\beta}\left(\frac{\alpha}{1-\alpha x} - \frac{\beta}{1-\beta x}\right) - \frac{p(p-2)}{\alpha-\beta}\left(\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right) \\ &= \frac{p\alpha}{\alpha-\beta}\left(\frac{1}{1-\alpha x}\right) - \frac{p(p-2)}{\alpha-\beta}\left(\frac{1}{1-\alpha x}\right) + \frac{p(p-2)}{\alpha-\beta}\left(\frac{1}{1-\beta x}\right) - \frac{p\beta}{\alpha-\beta}\left(\frac{1}{1-\beta x}\right) \\ &= \left(\frac{p\alpha - p(p-2)}{\alpha-\beta}\right)\left(\frac{1}{1-\alpha x}\right) + \left(\frac{p(p-2) - p\beta}{\alpha-\beta}\right)\left(\frac{1}{1-\beta x}\right). \end{aligned} \tag{3.54}$$

With the use of 2.14 we get the following formula for Q_n:

$$Q_n = \left(\frac{p\alpha - p(p-2)}{\alpha - \beta}\right)\alpha^n + \left(\frac{p(p-2) - p\beta}{\alpha - \beta}\right)\beta^n, \tag{3.55}$$

where

$$\alpha = \frac{(p-2) + \sqrt{4(p-1) + (p-2)^2}}{2} \quad \textit{and} \quad \beta = \frac{(p-2) - \sqrt{4(p-1) + (p-2)^2}}{2} \tag{3.56}$$

Chapter 4

Exponential Generating Functions

In this chapter we will talk about the exponential generating function. It is basically the ordinary power series generating function multiplied with $1/n!$. We will study a few examples and discuss if there are times where it is a better idea to use this generating function instead of the ordinary one, or if it is just a matter of taste. As far as the method goes, we do not do anything different when we solve recurrence relations, except that in the end we are interested in $[x^n/n!]G(x)$ as opposed to $[x^n]G(x)$. In this chapter we follow [19]

Recall that $G \overset{egf}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ means that $G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$ and $[x^n/n!]G(x)$ is the coefficient in front of $\frac{x^n}{n!}$ in that power series, in this case a_n.

Example 8 *Consider the recurrence relation*

$$a_{n+2} = 2a_{n+1} - a_n, \quad n \geq 0, \quad a_0 = 0, \quad a_1 = 1. \tag{4.1}$$

Define our unknown generating function $G(x) = \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$, multiply 4.1 with $x^n/n!$ and sum over $n \geq 0$ yields

$$\sum_{n=0}^{\infty} a_{n+2} \frac{x^n}{n!} = 2 \sum_{n=0}^{\infty} a_{n+1} \frac{x^n}{n!} - \sum_{n=0}^{\infty} a_n \frac{x^n}{n!}. \tag{4.2}$$

Note that

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n+1}\frac{x^n}{n!} &= a_1 + a_2x + a_3\frac{x^2}{2} + a_4\frac{x^3}{3!} + \cdots \\ &= \frac{d}{dx}(a_0 + a_1x + a_2\frac{x^2}{2} + a_3\frac{x^3}{3!} + \cdots) = \frac{d}{dx}G(x).\end{aligned} \tag{4.3}$$

In addition

$$\begin{aligned}\sum_{n=0}^{\infty} a_{n+2}\frac{x^n}{n!} &= a_2 + a_3x + a_4\frac{x^2}{2} + a_5\frac{x^3}{3!} + \cdots \\ &= \frac{d}{dx}(a_1 + a_2x + a_3\frac{x^2}{2} + a_4\frac{x^3}{3!} + \cdots) = \frac{d^2}{dx^2}G(x).\end{aligned} \tag{4.4}$$

So 4.1 expressed in terms of its generating function $G(x)$ *turns out to be*

$$G''(x) = 2G'(x) - G(x). \tag{4.5}$$

A homogeneous, second order, differential equation. This we can easily solve. It has the solution

$$G(x) = (c_1x + c_2)e^x, \quad \text{for some constants} \quad c_1,\ c_2 \in \mathbb{C}. \tag{4.6}$$

We get the initial conditions we need from the initial conditions of 4.1. Since $G(0) = a_0 = 0$ *and* $G'(0) = a_1 = 1$*, it follows that*

$$G(x) = xe^x \tag{4.7}$$

Using the Maclaurin expansion for xe^x *we find an explicit formula for* a_n*:*

$$xe^x = x + x^2 + \frac{x^3}{2} + \frac{x^4}{3!} + \cdots = \sum_{n=1}^{\infty}\frac{x^n}{(n-1)!} = \sum_{n=0}^{\infty} n\frac{x^n}{n!}. \tag{4.8}$$

Clearly $[x^n/n!]G(x) = n$ *and we are finished.*

This example demonstrates the method using a exponential generating function. We can see that it is quite similar to what we did in the previous chapter. One difference we can note was that when we had terms of the form a_{n+1} it was the same as taking the derivative of the generating function as opposed to subtracting a_0 and dividing by x as we did with the ops. Just as with the ops we can make this into a rule of computation. Actually we can translate Theorem 1 so that it is true for the exponential generating function. These rules can be found in [19].

Theorem 3 (Rules of computation (egf)) *Let* $G \overset{egf}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ *and* $T \overset{egf}{\longleftrightarrow} \{b_n\}_{n=0}^{\infty}$. *Then*

1. $D^k G(x) \overset{egf}{\longleftrightarrow} \{a_{n+k}\}_{n=0}^{\infty}$ *(where D is the differentiation operator)*

2. $(xD)^k G(x) \overset{egf}{\longleftrightarrow} \{n^k a_n\}_{n=0}^{\infty}$

   (where xD means first differentiate then multiply by x)

3. $GT \overset{egf}{\longleftrightarrow} \{\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\}_{n=0}^{\infty}$

Proof.

1. Let $G \overset{egf}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ and for $1 \le r \le k$ calculate $D^r G(x)$

$$\begin{aligned}
DG(x) &= D(a_0 + a_1 x + a_2 \frac{x^2}{2} + a_3 \frac{x^3}{3!} + \cdots) = a_1 + a_2 x + a_3 \frac{x^2}{2} + a_4 \frac{x^3}{3!} + \cdots \\
D^2 G(x) &= a_2 + a_3 x + a_4 \frac{x^2}{2} + a_5 \frac{x^3}{3!} + \cdots \\
&\vdots \\
D^k G(x) &= a_k + a_{k+1} x + a_{k+2} \frac{x^2}{2} + a_{k+3} \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} a_{n+k} \frac{x^n}{n!}
\end{aligned} \tag{4.9}$$

2. Same proof as for its counterpart from Theorem 1.

3. Let $G \overset{egf}{\longleftrightarrow} \{a_n\}_{n=0}^{\infty}$ and $T \overset{egf}{\longleftrightarrow} \{b_n\}_{n=0}^{\infty}$, then their product

$$GT = \sum_{n=0}^{\infty} c_n x^n \quad \text{where} \quad c_n = \sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!}. \tag{4.10}$$

   So

$$GT = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!} \right) x^n \tag{4.11}$$

and

$$\begin{aligned}[x^n/n!]GT &= n!\left(\sum_{k=0}^{n} \frac{a_k b_{n-k}}{k!(n-k)!}\right) \\ &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a_k b_{n-k} = \sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}.\end{aligned} \tag{4.12}$$

$\because\ GT \overset{egf}{\longleftrightarrow} \{\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\}_{n=0}^{\infty}$.

■

When we discussed the ordinary power series generating function one of the first recurrence relations we solved was the one of Fibonacci. Lets solve it again, this time using the exponential generating function and then look at how the two examples differ from each other.

Example 9 *We previously stated that the Fibonacci sequence was defined through 3.1. But it is also possible to change the indices to make it a better fit for us to use Theorem 3. Equivalent to 3.1 we have the following equation:*

$$F_{n+2} = F_{n+1} + F_n, \quad n \geq 0, \quad F_0 = 0, \quad F_1 = 1. \tag{4.13}$$

From 4.13 and with the use of Theorem 3 we immediately get the homogeneous differential equation

$$G'' - G' - G = 0, \quad G(0) = 0, \quad G'(0) = 1. \tag{4.14}$$

It has solutions

$$G(x) = c_1 e^{\alpha x} + c_2 e^{\beta x} \ \ \textit{where} \ \ \alpha = \frac{1+\sqrt{5}}{2}, \ \beta = \frac{1-\sqrt{5}}{2} \ \textit{and} \ c_1, \, c_2 \in \mathbb{C}. \tag{4.15}$$

The initial conditions $G(0) = 0$ and $G'(0) = 1$ gives us the unique solution

$$G(x) = \frac{1}{\sqrt{5}} e^{\alpha x} - \frac{1}{\sqrt{5}} e^{\beta x} = \frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x}). \tag{4.16}$$

From which it is easy to get the formula for F_n:

$$[x^n/n!]\left(\frac{1}{\sqrt{5}}(e^{\alpha x} - e^{\beta x})\right) = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n), \tag{4.17}$$

where , $\alpha = \frac{1+\sqrt{5}}{2}$, $\beta = \frac{1-\sqrt{5}}{2}$.

So after solving the same problem twice, using two different kinds of generating functions, what conclusions can we make? Well first of all, they both work and yields the same solution. That is what is most important here. Another thing we can notice is that in Example 4 we only needed to solve a ordinary, algebraic equation, while in Example 9 we were forced to solve a differential equation. So it seems that the ops version is somewhat easier, in this case, to obtain the generating function. At the same time, in Example 4, we had to do some partial fraction decomposition to obtain the explicit formula while in Example 9 we just applied $[x^n/n!]G(x)$ and we were done. So the egf version seems, in this case, to have an easier way of obtaining the explicit formula once the generating function is known [19].

Chapter 5

Generating Function for Hermite Polynomials

The following is taken from [17].

The quantum harmonic oscillator equation, which in itself is a one-dimensional version of the Schrödinger's equation, is a partial differential equation of great importance in physics. The solutions to this equation comes in the form a wave function that consists of a very special family of polynomials that are called the *Hermite Polynomials*. These polynomials possess alot of nice proterties, some of them we will discuss in this chapter. The main focus, and what we strive for, though will be to derive their generating function. In order to do that we first must introduce them.

Lets say we want to know $\frac{d^n}{dx^n}e^{-x^2}$. The first few are quite easy to just calculate. The first derivative is

$$\frac{d}{dx}e^{-x^2} = -2xe^{-x^2}, \tag{5.1}$$

the second is

$$\frac{d^2}{dx^2}e^{-x^2} = \frac{d}{dx}\left(-2xe^{-x^2}\right) = (4x^2-2)e^{-x^2} \tag{5.2}$$

and the third is

$$\frac{d^3}{dx^3}e^{-x^2} = \frac{d}{dx}\left((4x^2-2)e^{-x^2}\right) = -(8x^3-12x)e^{-x^2}. \tag{5.3}$$

But this soon gets pretty tedious to calculate. Instead note that when taking the n:th derivative of e^{-x^2} we get a polynomial of degree n times e^{-x^2}. Since we see this clear pattern we can say

$$D^n e^{-x^2} = p_n(x)e^{-x^2} \tag{5.4}$$

where $p_n(x)$ is a polynomial in x of degree n. This is where Hermite polynomials come in. They are almost these $p_n(x)$ except they have their highest order degree term being positive, thus

$$D^n e^{-x^2} = (-1)^n H_n(x)e^{-x^2} \tag{5.5}$$

where $H_n(x)$ is the n:th Hermite polynomial.

From 5.5 we get the following definition of the Hermite polynomials:

$$H_n(x) = (-1)^n e^{x^2} D^n e^{-x^2} \tag{5.6}$$

Actually there are other ways of writing these polynomials, one may multiply the exponent in 5.6 with some constant in order to make them more suitable for what ever purpose the user has. The ones that we consider here are sometimes called the *physicists Hermite polynomials* and they have their leading coefficient equal to 2^n, another version of the polynomials uses $e^{-\frac{x^2}{2}}$ instead and therefor have their leading coefficient equal to one, these are usually referred to as the *probabilists Hermite polynomials* [17].

Before we take on the challenge to derive the generating function for $H_n(x)$ we will look at some of the properties of these polynomials. For instance, what happens if we take the derivative of $H_n(x)$? With the product rule we get

$$H_n'(x) = (-1)^n \left[2xe^{x^2}\frac{d^n}{dx^n}e^{-x^2} + e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2}\right] \tag{5.7}$$

Note how the *r.h.s.* of 5.7 almost looks like the sum of two Hermite polynomials. In fact

$$2x(-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2} = 2x(-1)^n e^{x^2} D^n e^{-x^2} = 2xH_n(x) \tag{5.8}$$

and

$$(-1)^n e^{x^2}\frac{d^{n+1}}{dx^{n+1}}e^{-x^2} = (-1)(-1)^{n+1}e^{x^2}D^{n+1}e^{-x^2} = -H_{n+1}(x) \tag{5.9}$$

Thus 5.7 can be rewritten as

$$H_n'(x) = 2xH_n(x) - H_{n+1}(x). \tag{5.10}$$

So by taking the derivative of $H_n(x)$ we obtain a quite nice recurrence relation. But we can make it even better with the help of the following generalization of the product rule. See [14].

Theorem 4 (The general Leibniz rule) *Let f and g be n-times differentiable functions, then the product fg is also n-times differentiable and its $n:th$ derivative is given by*

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} g^{(k)}, \tag{5.11}$$

where $f^{(j)}$ denotes the $j:th$ derivative of f. In particular $f^{(0)} = f$.

This can be shown using induction.

Now consider the *l.h.s.* of 5.9:

$$(-1)^n e^{x^2} \frac{d^{n+1}}{dx^{n+1}} e^{-x^2} = (-1)^n e^{x^2} \frac{d^n}{dx^n} \left(-2xe^{-x^2}\right). \tag{5.12}$$

The only thing we did was to take one derivative. Now with the use of Theorem 4 we get:

$$\begin{aligned}
&(-1)^n e^{x^2} \tfrac{d^n}{dx^n} \left(-2xe^{-x^2}\right) = (-1)^n e^{x^2} (-2) \textstyle\sum_{k=0}^{n} \binom{n}{k} x^{(k)} (e^{x^2})^{(n-k)} \\
&= (-1)^n e^{x^2} (-2) \left(\binom{n}{0} x (e^{x^2})^{(n)} + \binom{n}{1} (1) (e^{x^2})^{(n-1)} + \binom{n}{2} (0) (e^{x^2})^{(n-2)} + 0 + 0 + \cdots\right) \\
&= (-1)^n e^{x^2} (-2) \left(x \tfrac{d^n}{dx^n} e^{-x^2} + n \tfrac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right) \\
&= (-2x)(-1)^n e^{x^2} \tfrac{d^n}{dx^n} e^{-x^2} - 2n(-1)^n \tfrac{d^{n-1}}{dx^{n-1}} e^{-x^2}
\end{aligned} \tag{5.13}$$

Plug this expression back in 5.7 and we get

$$\begin{aligned}
H_n'(x) &= (-1)^n \left[2xe^{x^2} \frac{d^n}{dx^n} e^{-x^2} - 2xe^{x^2} \frac{d^n}{dx^n} e^{-x^2} - 2ne^{x^2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right] \\
&= (-1)^n \left[-2ne^{x^2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\right] \\
&= 2n(-1)^{n-1} e^{x^2} \frac{d^{n-1}}{dx^{n-1}} e^{-x^2} \\
&= 2n(-1)^{n-1} e^{x^2} D^{n-1} e^{-x^2} = 2nH_{n-1}(x).
\end{aligned} \tag{5.14}$$

Equations 5.10 and 5.14 gives us the very nice recurrence relation for $n \geq 1$

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x), \tag{5.15}$$

with initial conditions

$$H_0(x) = 1, \quad H_1(x) = 2x \tag{5.16}$$

Another property of the Hermite polynomials worth mentioning since it is a very important property, even though we do not use this property here, is that they are orthogonal with respect to the weight function $w(x) = e^{-x^2}$ [11]. Meaning that

$$\int_{-\infty}^{\infty} H_n(x)H_m(x)e^{-x^2}\, dx = 0, \quad \text{whenever} \quad n \neq m, \tag{5.17}$$

this means Hermite polynomials can be used as an orthogonal basis.

The Hermite polynomials also have the property of being odd functions whenever n is an odd number and being even functions whenever n is an even number [11]. Recall that a function f is odd if it has the property:

$$f(-x) = -f(x), \quad \text{for every } x \text{ for which } f \text{ is defined} \tag{5.18}$$

in addition, f is an even function if

$$f(-x) = f(x), \quad \text{for every } x \text{ for which } f \text{ is defined} \tag{5.19}$$

This property is something we will have use of later so let us convince ourselves that this is truly the case.

$$\begin{aligned} H_n(-x) &= (-1)^n e^{(-x)^2} \frac{d^n}{d(-x)^n} e^{-(-x)^2} \\ &= (-1)^n e^{x^2} (-1)^n \frac{d^n}{dx^n} e^{-x^2} = (-1)^n H_n(x) \end{aligned} \tag{5.20}$$

Here we used the fact that taking a derivative with respect to $-x$ yields

$$\frac{d}{d(-x)} e^{-(-x)^2} = e^{-(-x)^2}(-2(-x)) = -(-2xe^{-x^2}) = -\frac{d}{dx} e^{-x^2} = -De^{-x^2}. \tag{5.21}$$

We clearly see in 5.20 that $H_n(x)$ is an odd function whenever n is an odd number and respectively $H_n(x)$ is an even function whenever n is an even number.

Also it might be interesting, at least it will be for us, to look at what happens with $H_n(x)$ at the origin. $H_n(0)$ is called the *Hermite Numbers* and are denoted H_n [11]. Since $H_n(x)$ is a polynomial of degree n we know that $H_n(0)$ is well defined. Combine this together with 5.20 we have that $H_n(0) = 0$ whenever n is an odd number. We write this

$$H_{2n+1}(0) = 0, \quad n = 0, 1, 2, \ldots \tag{5.22}$$

But what about when n is even? Let us write that case as $H_{2n}(x)$ ($n = 0, 1, 2, \ldots$). This one is a bit trickier and we will approach this in a similar way as we did when we derived the recurrence relation for $H_n(x)$.

$$H_{2n}(0) = (-1)^{2n} e^{x^2} D^{2n} e^{-x^2}\Big|_{x=0} = D^{2n} e^{-x^2}\Big|_{x=0}, \tag{5.23}$$

take one derivative and use Theorem 4:

$$
\begin{aligned}
D^{2n}e^{-x^2}\Big|_{x=0} &= \frac{d^{2n}}{dx^{2n}}e^{-x^2}\Big|_{x=0} \\
&= \frac{d^{2n-1}}{dx^{2n-1}}(-2xe^{-x^2})\Big|_{x=0} \\
&= (-2)\sum_{k=0}^{2n-1}\binom{2n-1}{k}x^{(k)}(e^{-x^2})^{(2n-1-k)}\Big|_{x=0} \\
&= (-2)\left(x\frac{d^{2n-1}}{dx^{2n-1}}e^{-x^2} + (2n-1)\frac{d^{2(n-1)}}{dx^{2(n-1)}}e^{-x^2}\right)\Big|_{x=0} \\
&= (-2)(2n-1)\frac{d^{2(n-1)}}{dx^{2(n-1)}}e^{-x^2}\Big|_{x=0} \\
&= (2-4n)\frac{d^{2(n-1)}}{dx^{2(n-1)}}e^{-x^2}\Big|_{x=0} = (2-4n)H_{2(n-1)}(0).
\end{aligned}
\tag{5.24}
$$

So to summarize, we obtain the following recurrence relation:

$$
H_{2n} = (2-4n)H_{2(n-1)}, \qquad n \geq 1, \quad H_0 = 1. \tag{5.25}
$$

Here using the notation for the Hermite Numbers. From this we get

$$
\begin{aligned}
H_2 = &\quad (2-4)H_0 = (2-4) \\
H_4 = &\quad (2-8)H_2 = (2-8)(2-4) \\
H_6 = &\quad (2-12)H_4 = (2-12)(2-8)(2-4) \\
\vdots & \\
H_{2n} = &\quad (2-4n)(2-4(n-1))(2-4(n-2))\cdots\ (2-8)(2-4) \\
= &\quad 2^n(1-2n)(1-2(n-1))(1-2(n-2))\cdots\ (1-4)(1-2) \\
= &\quad 2^n(1-2n)(3-2n)(5-2n)\cdots\ (-3)(-1) \\
= &\quad (-2)^n(2n-1)(2n-3)(2n-5)\cdots\ 3\cdot 1.
\end{aligned}
\tag{5.26}
$$

This expression can be rewritten as follows:

$$\begin{aligned}(-2)^n(2n-1)(2n-3)(2n-5)\cdots\ 3\cdot 1 &= \frac{(-2)^n(2n)!}{2n(2n-2)(2n-4)\cdots\ 4\cdot 2}\\ &= \frac{(-2)^n(2n)!}{2n\cdot 2(n-1)\cdot 2(n-2)\cdots\ 2(2)\cdot 2(1)}\\ &= \frac{(-2)^n(2n)!}{2^n(n!)}\\ &= (-1)^n\frac{(2n)!}{n!}\end{aligned} \tag{5.27}$$

This shows that

$$H_{2n}(0) = (-1)^n\frac{(2n)!}{n!}, \quad n = 0, 1, 2, \ldots \tag{5.28}$$

With these results we are now ready to approach the end goal of this work. That is to derive the generating function for the Hermite polynomials. We will approach this one a bit differently then we have done with previous recurrence relations.

First of all, let us define the generating function, since $H_n(x)$ is a polynomial in x we need to pick another variable as our "bookkeeping" variable and we define the generating function as

$$G(x,t) = \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \tag{5.29}$$

which is a multi-variable generating function. Take the derivative with respect to x and we get

$$\frac{\partial}{\partial x}G(x,t) \;=\; \frac{d}{dx}\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} \;=\; \sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} \tag{5.30}$$

Now, since $H_0'(x) = 0$ we can instead choose to let n start from 1, this together with 5.14 yields

$$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} \;=\; \sum_{n=1}^{\infty} 2nH_{n-1}(x)\frac{t^n}{n!}. \tag{5.31}$$

With the following change of variables: $m = n - 1$ we get

$$\sum_{m=0}^{\infty} 2(m+1)H_m(x)\frac{t^{m+1}}{(m+1)!} = \sum_{m=0}^{\infty} 2H_m(x)\frac{t^{m+1}}{m!} = 2tG(x,t) \tag{5.32}$$

So what did we learn? We learned that

$$\frac{\partial}{\partial x}G(x,t) \;=\; 2tG(x,t) \tag{5.33}$$

Which is a differential equation, which is a very strong result since differential equations model the world. Looking at 5.33 we see that we have a function that when we take its derivative with respect to x we get the same function back along with the factor $2t$. This implies

$$G(x,t) = e^{2tx}f(t), \tag{5.34}$$

where $f(t)$ is a function depending only on t. To determine $f(t)$, consider

$$G(0,t) = e^{2t0}f(t) = f(t). \tag{5.35}$$

So if we know $G(0,t)$ we will know $f(t)$.

$$G(0,t) = \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!} \tag{5.36}$$

And since we know from before what $H_n(0)$ is, we can just plug the results from 5.22 and 5.28 into 5.36 and we get

$$\begin{aligned} f(t) &= \sum_{n=0}^{\infty} H_{2n}(0)\frac{t^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}(-1)^n\frac{(2n)!}{n!}\frac{t^{2n}}{(2n)!} \\ &= \sum_{n=0}^{\infty}\frac{(-t^2)^n}{n!} = e^{-t^2}. \end{aligned} \tag{5.37}$$

Giving us the generating function

$$G(x,t) = e^{2tx}e^{-t^2} = e^{2tx-t^2} \tag{5.38}$$

Now that is a neat and compact function in comparison to the polynomials it is generating. Many of the good properties you can read from the generating function. Good properties of the generating function will translate to the polynomials.

We have now derived the generating function for the Hermite polynomials defined as 5.6, which is what I wanted us to do. It is not uncommon that you instead define the Hermite polynomials by the generating function 5.38 and instead derive identities from the generating function instead [11]. For instance, when we were interested in $H_n(0)$, if we had known the generating function we could have written

$$e^{2t0-t^2} = \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!}. \tag{5.39}$$

Expand the $l.h.s.$ in a power series and get

$$\sum_{k=0}^{\infty} \frac{(-t^2)^k}{k!} = \sum_{n=0}^{\infty} H_n(0)\frac{t^n}{n!}. \tag{5.40}$$

At once realising that $H_{2n+1}(0) = 0$ since the $l.h.s.$ of 5.40 do only contain even powers of t. To find $H_{2n}(0)$ we make the change of variables: $n = 2k$ and get

$$\sum_{k=0}^{\infty} \frac{(-1)^k t^{2k}}{k!} = \sum_{k=0}^{\infty} H_{2k}(0)\frac{t^{2k}}{(2k)!}, \tag{5.41}$$

from which it is pretty clear that $H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}$. Again illustrating the strength of generating functions.

Chapter 6

Moment Generating Functions

When it comes to random variables and probabilistic distributions the magnitudes called *moments* are of great importance. The first moment is actually the mean of a random variable. The second moment is a measure of spread, how a random variable varies and is connected to the *variance* of a random variable. The third moment is a measure of asymmetry and is connected to the *skewness* of the distributions graph. The fourth moment is a measure of "taildness" and is connected to *kurtosis* it gives information on how the graph looks at its tails.

The following is taken from *Moment Generating Functions*, MIT, Lecture 13 Fall 2018, [15].

As you may have already guessed, the moment generating function is used in probability theory and statistics and they provide a good way of representing probability distributions through a function of a single variable. Moment generating functions do not have a connection to recursion as the other generating functions we have studied do, instead they are connected to derivatives as they generate moments through derivation.

Moment generating functions are useful in several ways, to name a few:

1. It is easy to calculate the moments of a distribution using the moment generating function.

2. Characterizing the distribution of the a sum of independent random variables is easy using moment generating functions.

3. When dealing with the distribution of the sum of a random number of independent random variables they provide some powerful tools.

Also, according to [20], the moment generating function seems to be effective when it comes to probabilistic inference algorithms with latent count variables. That is where one do not have complete control of all variables.

Formal definition of a moment generating function, taken from [15], elementary probabilistic expressions, such as: random variable and expectation, can be found in [4].

Definition 2 *The moment generating function associated with a random variable X is a function $M_X(s) : \mathbb{R} \longrightarrow [0, \infty]$ defined by*

$$M_X(s) = E(e^{sX}), \tag{6.1}$$

where $E(X)$ is the expectation, or mean, of X. Meaning that

$$M_X(s) = \sum_x e^{sx} p_X(x) \tag{6.2}$$

if X is a discrete random variable with *Probability Mass Function* (PMF) $p_X(x)$ and

$$M_X(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\, dx \tag{6.3}$$

if X is a continuous random variable with *Probability Density Function* (PDF) $f_X(x)$.

Example 10 *Let X be a random variable taking the values 1 and 0 with the probability p and $1-p$ respectively, this is the same as saying X is Bernoulli distributed and we denote it $X \sim Ber(p)$.*

In this case we get the moment generating function

$$M_X(s) = e^{s0} p_X(0) + e^s p_X(1) = pe^s - p + 1. \tag{6.4}$$

Now an important property of the moment generating function is its ability to generate moments [15]. Recall that the *first moment* of a random variable X is $E(X)$, the *second moment* is $E(X^2)$, naturally the *k:th moment* is $E(X^k)$.

Let $M_X(s)$ be a moment generating function for a discrete random variable X. Then by Definition 2 and the Maclaurin expansion of e^x we get

$$\begin{aligned} M_X(s) &= E(1 + sX + \frac{s^2}{2}X^2 + \frac{s^3}{3!}X^3 + \cdots) \\ &= 1 + sE(X) + \frac{s^2}{2}E(X^2) + \frac{s^3}{3!}E(X^3) + \cdots . \end{aligned} \tag{6.5}$$

Look at what happens when we take the derivative, with respect to s and evaluate it at $s = 0$:

$$\frac{d}{ds}(1 + sE(X) + \frac{s^2}{2}E(X^2) + \frac{s^3}{3!}E(X^3) + \cdots)\Big|_{s=0} = E(X). \tag{6.6}$$

So we obtain the mean (first moment) of X by just taking the first derivative of $M_X(s)$ and evaluate at $s = 0$. If we wanted to know the second moment $E(X^2)$ we would just take the second derivative of $M_X(s)$ and evaluate at $s = 0$. It is not to hard to see that the moment generating function "generates" the moment $E(X^k)$ if we apply the operator D^k to it and evaluate at $s = 0$ [15]:

$$\begin{aligned}
D^k M_X(s)\big|_{s=0} &= D^k(1 + sE(X) + \frac{s^2}{2}E(X^2) + \frac{s^3}{3!}E(X^3) + \cdots)\Big|_{s=0} \\
&= D^{k-1}(0 + E(X) + sE(X^2) + \frac{s^2}{2}E(X^3) + \cdots)\Big|_{s=0} \\
&= D^{k-2}(0 + 0 + E(X^2) + sE(X^3) + \frac{s^2}{2}E(X^4) + \cdots)\Big|_{s=0} \\
&\vdots \\
&= 0 + 0 + 0 + \cdots + E(X^k) + sE(X^{k+1}) + \cdots\big|_{s=0} = E(X^k).
\end{aligned} \tag{6.7}$$

This is a very useful property when it comes to calculating things like the variance of a random variable. Remember that the variance of a random variable X is defined as

$$V(X) = E[(X - E(X))^2] = E(X^2) - E(X)^2. \tag{6.8}$$

So if we were to calculate the variance of the random variable from Example 10 we just need the first and second derivative of 6.4 and then evaluate them at $s = 0$.

$$E(X) = \frac{d}{ds}M_X(s)\Big|_{s=0} \frac{d}{ds}(pe^s - p + 1)\Big|_{s=0} = pe^s|_{s=0} = p \tag{6.9}$$

$$E(X^2) = \frac{d^2}{ds^s}M_X(s)\Big|_{s=0} \frac{d}{ds}pe^s\Big|_{s=0} = pe^s|_{s=0} = p. \tag{6.10}$$

So we get

$$V(X) = p - p^2 = p(1 - p) \tag{6.11}$$

We began this chapter with a little talk regarding the first four moments of a distribution and how powerful these moments are when it comes to characterising a distribution. Now we have illustrated that calculations with moment generating functions is very efficient. It is not hard to imagine this having a lot of applications. For instance in Artificial Intelligence (A.I.), or any type of data-analysis, we analyse a lot of data and this data can be viewed as a random variable, with the moment generating function we can then calculate the moments of that variable and characterize the data and see what distribution it follows.

Chapter 7

Conclusions

We have, throughout this report, illustrated that generating functions provides a solid method of solving recurrence relations, that simplify this work, especially when it comes to recurrence relations that are a bit tougher then the linear ones.

The different kinds of generating functions have different pros. When solving recurrence relations using ops we had an easier way of determining the generating function, since it just involved an ordinary algebraic equation that we solved for the generating function, while we in the egf case often need to solve a differential equation to determine the generating function. Additionally, if the generating function is already known, we had an easier time determining an explicit formula for the sequence when we were working with the egf as oppose to the ops where we had to do some partial fraction decomposition to derive an explicit formula.

When we worked with the recurrence relation regarding branched coverings in Chapter 3 we saw that we were able to recognise that the recurrence relation had similarities with the Fibonacci recurrence, and this translated to the generating function that also had strong similarities with the generating function of the Fibonacci sequence making it possible for us to make our calculations easier by using results from the Fibonacci generating function. From this we conclude that it might be a good idea to look for similarities of generating functions that we already know to help us solve something new.

I choose to consider the derivation of the Hermite polynomials generating function as, sort of, the main result of this work. This is because that generating function generates an interesting family of functions, namely the Hermite polynomials. They are interesting since they arise as solutions to differential equations, and differential equations are what we we use to model the world. Also I wanted to show how that generating function can be obtained since it is common in literature that we already know the generating function for the Hermite polynomials and use it for calculations since those calculations are simpler.

Lastly I would like to mention another type of generating function that is a generalization of the moment generating function. It is called the *Universal Generating Function* (ugf). It is a powerful tool when working with random distributions. When we looked at moment generating functions we looked at discrete random variables, the ugf sort of "translates" this to a non-discrete case. This makes the ugf a powerful tool when it comes to simulations for instance, the interested reader should read *Universal Generating Function Based Probabilistic Production Simulation for Wind Power Integrated Power Systems* [9]. In this article they present a model that simulates the electricity demand and output in a power-system along with costs. Power-systems are quite complex and persists of multiple kinds of generators with varying outputs and capacities. Their model makes very accurate simulations using ugf which require far less expensive computations then other methods.

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