# Perron-Frobenius' Theory and Applications 

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## Abstract

This is a literature study, in linear algebra, about positive and nonnegative matrices and their special properties. We say that a matrix or a vector is positive/nonnegative if all of its entries are positive/nonnegative. First, we study some generalities and become acquainted with two types of nonnegative matrices; irreducible and reducible. After exploring their characteristics we investigate and prove the two main theorems of this subject, namely Perron's and Perron-Frobenius' theorem. In short Perron's theorem from 1907 tells us that the spectral radius of a positive matrix is a simple eigenvalue of the matrix and that its eigenvector can be taken to be positive. In 1912, Georg Frobenius generalized Perron's results also to irreducible nonnegative matrices.

The two theorems have a wide range of applications in both pure mathematics and practical matters. In real world scenarios, many measurements are nonnegative (length, time, amount, etc.) and so their mathematical formulations often relate to Perron-Frobenius theory. The theory's importance to linear dynamical systems, such as Markov chains, cannot be overstated; it determines when, and to what, an iterative process will converge. This result is in turn the underlying theory for the page-ranking algorithm developed by Google in 1998. We will see examples of all these applications in chapters four and five where we will be particularly interested in different types of Markov chains.

The theory in this thesis can be found in many books. Here, most of the material is gathered from Horn-Johnson [5, Meyer [9] and Shapiro [10]. However, all of the theorems and proofs are formulated in my own way and the examples and illustrations are concocted by myself, unless otherwise noted.

## Keywords:

Positive matrices, nonnegative matrices, Perron-Frobenius, linear dynamical systems, Leslie matrices, Markov chain, Google's PageRank algorithm

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## Sammanfattning

Det här är en litteraturstudie, inom linjär algebra, om positiva och icke-negativa matriser och deras speciella egenskaper. Vi säger att en matris eller en vektor är positiv/icke-negativ om alla dess element är positiva/icke-negativa. Inledningsvis går vi igenom några grundläggande begrepp och bekanta oss med två typer av icke-negativa matriser; irreducibla och reducibla. Efter att vi utforskat deras egenskaper så studerar vi och bevisar ämnets två huvudsatser; Perrons och Perron-Frobenius sats. Kortfattat så säger Perrons sats, från 1907, att spektralradien för en positiv matris är ett simpelt egenvärde till matrisen och att dess egenvektor kan tas positiv. År 1912 så generaliserade Georg Frobenius Perrons resultat till att gälla också för irreducibla icke-negativa matriser.

De två satserna har både många teoretiska och praktiska tillämpningar. Många verkliga scenarios har icke-negativa mått (längd, tid, mängd o.s.v) och därför relaterar dess matematiska formulering till Perron-Frobenius teori. Teorin är betydande även för linjära dynamiska system, såsom Markov-kedjor, eftersom den avgör när, och till vad, en iterativ process konvergerar. Det resultatet är i sin tur den underliggande teorin bakom algoritmen PageRank som utvecklades av Google år 1998. Vi kommer se exempel på alla dessa tillämpningar i kapitel fyra och fem, där vi speciellt intresserar oss för olika typer av Markov-kedjor.

Teorin i den här artikeln kan hittas i många böcker. Det mesta av materialet som presenteras här har hämtats från Horn-Johnson [5], Meyer (9] och Shapiro [10]. Däremot är alla satser och bevis formulerade på mitt eget sätt och alla exempel, samt illustrationer, har jag skapat själv, om inget annat sägs.

## Nyckelord:

Positiva matriser, icke-negativa matriser, Perron-Frobenius, linjära dynamiska system, Leslie matris, Markov-kedja, Google's PageRank algoritm

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## Nomenclature

For an $n \times n$ matrix $A \in \mathbb{M}_{n}(\mathbb{C})$, the following notation will be used, unless otherwise stated:

| $\sigma(A)$ | The spectrum of $A:\{\lambda \in \mathbb{C}: \lambda$ is an eigenvalue of $A\}$ |
| :---: | :--- |
| $\rho(A)$ | The spectral radius: $\max \{\|\lambda\|: \lambda \in \sigma(A)\}$ |
| $p_{A}(\lambda)$ | The secular polynomial: $\operatorname{det}(A-\lambda I)$. |
| geomult $_{A}(\lambda)$ | Geometric multiplicity: $\operatorname{dim}(N(A-\lambda I))$ |
| algmult $_{A}(\lambda)$ | Algebraic multiplicity: the multiplicity of $\lambda$ in $p_{A}(\lambda)$ |

We will consider vectors $\mathbf{x} \in \mathbb{C}^{n}$ as $n \times 1$ column matrices, so $\mathbf{x}^{T}$ is an $1 \times n$ row matrix. The number $n$ will consistently denote the dimension of the matrices and $m$ the counter of time and/or powers $A^{m}$ of a matrix $A$.

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## Chapter 1

## An Introductory Example

In the following chapter, we present an introductory example to Perron-Frobenius theory. An extended version can be found in Horn-Johnson [5, Chapter 8].

Consider a population inhabiting two cities $\mathcal{A}$ and $\mathcal{B}$. Each year, a fraction $\alpha \in(0,1)$ of the population in city $\mathcal{A}$ moves to city $\mathcal{B}$ and a fraction $\beta \in(0,1)$ of the population in city $\mathcal{B}$ moves to city $\mathcal{A}$. The yearly migration process is illustrated in fig. 1.1. We want to investigate how the population distribution, amongst the two cities, will change over time. In other words, we seek the asymptotic behaviour of this movement.


Figure 1.1: Two cities.
Towards finding an analytical solution to the problem, let $x_{m}$ and $y_{m}$ be the number of people at year $m \geq 0$ living in city $\mathcal{A}$ and $\mathcal{B}$ respectively. We can model the process using the following two recursive equations, written in matrix form:

$$
\mathbf{x}_{m+1}=\binom{x_{m+1}}{y_{m+1}}=\left(\begin{array}{cc}
1-\alpha & \beta \\
\alpha & 1-\beta
\end{array}\right)\binom{x_{m}}{y_{m}}=A \mathbf{x}_{m}, \quad m \geq 0
$$

with some initial conditions $x_{0}, y_{0} \geq 0$, where $x_{0}+y_{0}$ is the total population. Thus, we are interested in the matrix equation $\mathbf{x}_{m+1}=A \mathbf{x}_{m}$, and if we succes-
sively use the equation to express $\mathbf{x}_{m}$ in terms of the initial vector $\mathbf{x}_{0}$ we can see that

$$
\mathbf{x}_{m}=A \mathbf{x}_{m-1}=A\left(A \mathbf{x}_{m-2}\right)=A^{2}\left(A \mathbf{x}_{m-3}\right)=\cdots=A^{m} \mathbf{x}_{0}
$$

and so the asymptotic behaviour of $\mathbf{x}_{m}$ is given by the limit

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbf{x}_{m}=\lim _{m \rightarrow \infty} A^{m} \mathbf{x}_{0} \tag{1.1}
\end{equation*}
$$

if it exists. We find the eigenvalues of $A$, using its secular polynomial

$$
p_{A}(\lambda)=\lambda^{2}-\lambda(2-\alpha-\beta)+1-\alpha-\beta=(\lambda-1)(\lambda-(1-\alpha-\beta))
$$

to be $\lambda_{1}=1$ and $\lambda_{2}=1-\alpha-\beta$ Since we know that $\alpha, \beta \neq 0$ then the two eigenvalues are distinct, and so geomult $_{A}\left(\lambda_{1}\right)=\operatorname{geomult}_{A}\left(\lambda_{2}\right)=1$, because it is always true that $\operatorname{geomult}_{A}(\lambda) \leq \operatorname{algmult}_{A}(\lambda)$ for all eigenvalues $\lambda \in \sigma(A)$. Therefore, we can diagonalize $A$ via the transformation matrix $T=\left(\begin{array}{cc}\beta & -1 \\ \alpha & 1\end{array}\right)$, with inverse $T^{-1}=\frac{1}{\alpha+\beta}\left(\begin{array}{cc}1 & 1 \\ -\alpha & \beta\end{array}\right)$, a change to eigenbasis. Because both $\alpha, \beta<1$ we see that $\left|\lambda_{2}\right|<1$ and thus $\lambda_{2}^{m} \rightarrow 0$ as $m \rightarrow \infty$. Consequently, we get the following limit:

$$
\lim _{n \rightarrow \infty} A^{m}=\lim _{m \rightarrow \infty} T\left(\begin{array}{cc}
\lambda_{1}^{m} & 0 \\
0 & \lambda_{2}^{m}
\end{array}\right) T^{-1}=T\left(\begin{array}{cc}
1 & 0 \\
0 & 0
\end{array}\right) T^{-1}=\frac{1}{\alpha+\beta}\left(\begin{array}{cc}
\beta & \beta \\
\alpha & \alpha
\end{array}\right)
$$

Substituting this back into (1.1) reveals that

$$
\lim _{m \rightarrow \infty} \mathbf{x}_{m}=\lim _{m \rightarrow \infty} A^{m} \mathbf{x}_{0}=\frac{x_{0}+y_{0}}{\alpha+\beta}\binom{\beta}{\alpha}
$$

Notice that this limit is proportional to the eigenvector $\binom{\beta}{\alpha}$ corresponding to the eigenvalue $\lambda_{1}=1$ which equals the spectral radius $\rho(A)$. Further, it is clear that the limit will always be proportional to this vector, independently of the initial vector $\mathbf{x}_{0}$. These properties are true also for larger systems, even though it is not possible to analyze them in the same direct way. Instead, PerronFrobenius theory serves as useful tool for this very thing, which we will explore and exemplify in this thesis.

For an analysis of the cases $\alpha=\beta=0$ and $\alpha=\beta=1$ above, see HornJohnson [5, Chapter 8].

## Chapter 2

## Preliminaries

In the following chapter we will do some preparation work for the main theory in this thesis: Perron-Frobenius theory. We will need the concepts: positive, nonnegative, irreducible and primitive matrices. Most of this chapter is a summary and a further explanation of the material found in Horn-Johnson [5], Meyer (9) and Shapiro 10 but the information is rather elementary and can be found also elsewhere.

### 2.1 Positive and Nonnegative Matrices

Definition 2.1. We say that a matrix $A=\left(a_{i j}\right)$ is positive, denoted $A>0$ if all of its entries are positive, in other words if $a_{i j}>0$ for all $i$ and $j$. Similarly, we say that $A$ is nonnegative if $a_{i j} \geq 0$ for all $i$ and $j$.

With the definition above, we can write $A>B$ or $A \geq B$, for two matrices of equal size, meaning that the matrix $A-B$ is positive or nonnegative. We use the same notation for vectors: $\mathbf{x}>0(\mathbf{x} \geq 0)$ if $x_{i}>0\left(x_{i} \geq 0\right)$ for all $i=1, \ldots, n$. If $\mathbf{x} \neq \mathbf{0}$ is a nonnegative vector, which not the zero vector, and $A$ is a positive matrix, then the vector $A \mathrm{x}$ will be stricly positive. This is because

$$
A \mathbf{x}=\left(\begin{array}{ccc}
a_{11} & \ldots & a_{1 n} \\
\vdots & \ddots & \vdots \\
a_{n 1} & \ldots & a_{n n}
\end{array}\right)\left(\begin{array}{c}
x_{1} \\
\vdots \\
x_{n}
\end{array}\right)=\left(\begin{array}{c}
a_{11} x_{1}+\cdots+a_{1 n} x_{n} \\
\vdots \\
a_{n 1} x_{1}+\cdots+a_{n n} x_{n}
\end{array}\right)
$$

and since $\mathbf{x} \neq \mathbf{0}$ there is at least some component $x_{j}>0$ which multiplied with $a_{i j}>0$ will yield a positive product $a_{i j} x_{j}$ in every component of the vector $A \mathbf{x}$. If the situation is somewhat reversed: $\mathbf{x}>0$ and $A \geq 0$ then it is of course true
that $A \mathbf{x} \geq 0$, however in this situation, it is not generally true that $A \mathbf{x}>0$. A counterexample is easily created by letting an entire row in $A$ be zeros. Lastly, if again $\mathbf{x}>0, A \geq 0$ and it is also the case that $A \mathbf{x}=\mathbf{0}$ then $A$ must be the zero matrix, i.e. $a_{i j}=0$ for all $i, j=1, \ldots, n$.

Note that positivity and nonnegativity is preserved with matrix powers, that is if $A>0$ then $A^{2}>0, A^{3}>0, \ldots$ (same for nonnegativity) since the entries in each matrix power $A^{m}$ is the sum of products of positive (nonnegative) entries of the previous matrix power $A^{m-1}$ and $A$ itself, for all $m \geq 1$.

In terms of the spectral properties of a positive matrix, it is always the case that the eigenvalue of largest modulus (absolute value) is real and positive. This is a part of Perron's theorem, which we will investigate later. For a nonnegative matrix it is true, as for all $A \in \mathbb{M}_{n}(\mathbb{C})$, that $\rho(A) \geq 0$, however $\rho(A)$ need not to be positive if $A \geq 0$. An example of this is the matrix $\mathcal{N}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right) \geq 0$ which has the secular polynomial $p_{\mathcal{N}}(\lambda)=\left|\begin{array}{cc}-\lambda & 1 \\ 0 & -\lambda\end{array}\right|=\lambda^{2}-0=\lambda^{2}$, spectrum $\sigma(\mathcal{N})=\{0\}$ and thus spectral radius $\rho(\mathcal{N})=0$, i.e. $\mathcal{N}$ is a nonnegative matrix, not the zero matrix, with spectral radius equal to $0 \ngtr 0$. $\mathcal{N}$ is nilpotent since $\mathcal{N}^{2}=\left(\begin{array}{ll}0 & 0 \\ 0 & 0\end{array}\right)$.

If we compare the spectral radii of two nonnegative matrices, we get the following result in theorem 2.2. We can thus think about the spectral radius as a sort of 'measure' of the matrix.

Theorem 2.2. If $0 \leq A \leq B$ then $\rho(A) \leq \rho(B)$.
Proof. Suppose (towards a contradiction) that $\rho(A)>\rho(B) \geq 0$. Then scaling this inequality with some positive appropriate scalar $\gamma>0$ we get that $\gamma \rho(A)>$ $1>\gamma \rho(B) \geq 0$ and so the matrix $\gamma A$ has a spectral radius $>1$ and $\gamma B$ a spectral radius $<1$. Thus $(\gamma A)^{m}$ diverges as $m \rightarrow \infty$ whereas $(\gamma B)^{m} \rightarrow \mathbf{0}$ as $m \rightarrow \infty$. The reason behind this is easiest seen if we bring $\gamma A$ and $\gamma B$ to Jordan canonical form, where the diagonal elements will have $>1$ or $<1$ moduli, respectively. On the other hand, we know that $A \leq B$, therefore $\gamma A \leq \gamma B$ since $\gamma>0$ and so the fact that $(\gamma B)^{m} \rightarrow \mathbf{0}$ forces $(\gamma A)^{m}$ to converge to $\mathbf{0}$ as $m \rightarrow \infty$, and we have a contradiction.

### 2.2 Graphs and Matrices

Often, when studying positive or nonnegative matrices, graph theory can be used as a visual representation of the matrix and a helping hand in proofs. In this section, we present some connections between graphs and matrices.

Definition 2.3 (Directed graph). A directed graph is a graph $\mathcal{D}=(\mathcal{V}, \mathcal{E})$, on the vertex set $\mathcal{V}=\left\{\nu_{1}, \ldots, \nu_{n}\right\}$, where the edge set $\mathcal{E}$ consists of ordered pairs $\left(\nu_{i}, \nu_{j}\right)$ with $\nu_{i}, \nu_{j} \in \mathcal{V}$, called arcs or directed edges.

Two vertices linked with an arc between them (in any direction) are called adjacent. In our definition of a directed graph we allow arcs of the type $\left(\nu_{i}, \nu_{i}\right)$, called loops, and also multiple arcs between two pair of nodes. Sometimes, graphs with these extensions are called multi- or general directed graphs. However in this thesis, we will make no such distinction. Now that we have established the definition of a directed graph, we turn to matrices and construct the graph of a matrix and then the matrix of a graph.

Definition 2.4. Let $A \geq 0$ be a nonnegative matrix. $\mathcal{D}(A)=(\mathcal{V}(A), \mathcal{E}(A))$ is the (directed) graph of $A$ if

$$
\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}(A) \text { if and only if } a_{i j} \neq 0
$$

for all $i, j=1, \ldots n$. This is, we let the rows in $A$ determine the out-going vertex $\nu_{i}$ and the columns determine the in-going vertex $\nu_{j}$ of the $\operatorname{arc}\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}(A)$.

Remark 2.5. In some literature the definition above is somewhat reversed, columns determine out-going and rows in-going vertices, i.e. $\left(\nu_{j}, \nu_{i}\right) \in \mathcal{E}(A)$ if and only if $a_{i j} \neq 0$ for all $i, j=1, \ldots n$. Unless otherwise stated, we use the above stated definition 2.4 for the graph of a matrix.

Definition 2.6. Let $\mathcal{D}=(\mathcal{V}, \mathcal{E})$ be a directed graph. The nonnegative matrix $A=\left(a_{i j}\right)$ such that $a_{i j}$ is the number of $\operatorname{arcs}\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}$ from vertex $\nu_{i}$ to vertex $\nu_{j}$ in $\mathcal{D}$, for all $i, j=1, \ldots, n$, is called the adjacency matrix of $\mathcal{D}$.

To illustrate the difference between the two definitions above, let

$$
A=\left(\begin{array}{lll}
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0
\end{array}\right), \quad B=\left(\begin{array}{lll}
0 & 1 & 2 \\
0 & 3 & 4 \\
5 & 0 & 0
\end{array}\right)
$$

and let $\mathcal{D}$ be the directed graph in fig. 2.1. Then $A$ is the adjacency matrix of $\mathcal{D}$, because each entry $a_{i j}$ in $A$ is exactly the number of arcs between the vertices $\nu_{i}$ and $\nu_{j}$ for all $i, j=1,2,3$. $B$ is not the adjacency matrix of $\mathcal{D}$, for instance because $b_{31}=5$ and there is only one edge $\left(\nu_{3}, \nu_{1}\right) \in \mathcal{E}$. Yet $\mathcal{D}=\mathcal{D}(A)=\mathcal{D}(B)$, that is $\mathcal{D}$ is the graph of both matrices $A$ and $B$.


Figure 2.1: $\mathcal{D}$.
In a directed graph $\mathcal{D}$ we define a (directed) walk as an alternating sequence of vertices and arcs, denoted $\nu_{1} \rightarrow \nu_{2} \rightarrow \cdots \rightarrow \nu_{k+1}$, not necessarily all $\nu_{i}$
distinct, from an initial vertex $\nu_{1}$ to a terminal vertex $\nu_{k+1}$. The arrows in the scheme represents $\operatorname{arcs}\left(\nu_{i}, \nu_{i+1}\right) \in \mathcal{E}, i=1, \ldots, k$. We say that the walk is closed if $\nu_{1}=\nu_{k+1}$ and open otherwise. A cycle is a closed walk were all vertices are distinct. The length of a walk, $k \in \mathbb{Z}_{+}$, is the number of arcs in the sequence. We regard a single arc as a walk of length 1 . For a directed graph $\mathcal{D}$, define $\mathcal{D}^{(k)}=\left(\mathcal{V}^{(k)}, \mathcal{E}^{(k)}\right)$ on the vertex set $\mathcal{V}^{(k)}=\mathcal{V}=\left\{\nu_{1}, \ldots, \nu_{n}\right\}$ in the following way: there exists an $\operatorname{arc}\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}^{(k)}$ in $\mathcal{D}^{(k)}$ if and only if there exists a walk in $\mathcal{D}$ of length $k$ between $\nu_{i}$ and $\nu_{j}$, for $i, j=1, \ldots, n$. Note that if there is a loop $\left(\nu_{i}, \nu_{i}\right)$, for some $i$, in $\mathcal{D}$ there exists a walk between $\nu_{i}$ to itself of any length, and thus there is a loop $\left(\nu_{i}, \nu_{i}\right)$ also in $\mathcal{D}^{(k)}$ for all $k$. To relate this last definition to matrices, we give the following theorem.

Theorem 2.7. If $A \geq 0$ and $\mathcal{D}:=\mathcal{D}(A)$ is the graph of $A$, then $\mathcal{D}\left(A^{k}\right)=\mathcal{D}^{(k)}$, as defined above, for all $k \in \mathbb{Z}_{+}$.

Proof. $k=1$ is trivial, so consider the case $k=2$. Let $a_{i j}^{(2)}$ denote the entry at position $(i, j)$ in $A^{2}$. Then

$$
a_{i j}^{(2)}=\sum_{l=1}^{n} a_{i l} a_{l j} .
$$

If $a_{i j}^{(2)} \neq 0,\left(\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}\left(A^{2}\right)\right)$ then there is at least one $l$ such that the term $a_{i l} a_{l j}$ in the sum is nonzero. Then for such $l$, both $a_{i l} \neq 0$ and $a_{l j} \neq 0$, so both $\left(\nu_{i}, \nu_{l}\right) \in \mathcal{E}$ and $\left(\nu_{l}, \nu_{j}\right) \in \mathcal{E}$ so there exists a walk $\nu_{i} \rightarrow \nu_{l} \rightarrow \nu_{j}$ of length 2 in $\mathcal{D}$ and so, by the definition of $\mathcal{D}^{2},\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}^{(2)}$. On the other hand, if $a_{i j}^{(2)}=0$ then all terms in the sum are zero and there is no walk of length 2 from $\nu_{i}$ to $\nu_{j}$. If $k>2$, the same argument applies, but with a longer expression for $a_{i j}^{(k)}$.

The following definition of connectivity of a graph, also related to walks, will be especially important in the next section when we consider irreducible matrices.

Definition 2.8 (Strongly connected). A directed graph $\mathcal{D}=(\mathcal{V}, \mathcal{E})$ is strongly connected if there exists a directed path, of some length, between all pair of vertices $\nu_{i}, \nu_{j} \in \mathcal{V}$ in $\mathcal{D}$.

This is equivalent to $\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}^{(k)}$, for some $k \in \mathbb{Z}_{+}$. In the extreme case we have to visit all other vertices on the walk from $\nu_{i}$ to $\nu_{j}$, which is a walk of length $n-1$. A longer walk than that is never needed in a strongly connected directed graph, since such a walk would contain a loop, which can be removed to form a new walk of length $\leq n-1$. We conclude that for a strongly connected directed graph, and for all $i, j=1, \ldots, n$, it is true that $\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}^{(k)}$, for some $k=1, \ldots n-1$.

For further readings about the connection between graphs and matrices see Shapiro [10, Chapters 15-16].

### 2.3 Irreducible Matrices

We now introduce an important class of nonnegative matrices, the irreducible matrices. Informally, we will think about these matrices as having zeros (if there are any) in just the right place. Recall that a permutation matrix is a square binary matrix, with exactly one entry of 1 in each row and column. If we multiply a matrix $A \in \mathbb{M}_{n}(\mathbb{C})$ by a permutation matrix $P$ to the left, the resulting matrix $P A$ is the same as $A$ but with its rows permuted. Similarly, multiplication by $P$ to the right, corresponds to permuting the columns in $A$.

Definition 2.9 (Irreducibility). A nonnegative matrix $A \geq 0$ is said to be reducible if there exists a permutation matrix $P$ such that:

$$
P^{T} A P=\left(\begin{array}{cc}
X & Y  \tag{2.1}\\
\mathbf{0} & Z
\end{array}\right)
$$

where $X$ and $Z$ are square matrices and $\mathbf{0}$ is a rectangular zero block. If $A$ is not reducible, then it is called irreducible.

Remark 2.10. A positive matrix has no zeros and can thus never be brought to the form (2.1). Therefore, all positive matrices are irreducible.

Note that the matrix $P$ brings a reducible matrix to an upper block triangular form via a simultaneous permutation of the rows and columns. If we think of the matrix $A$ as being a representation of a linear map between two $n$-dimensional vector spaces, say $\mathbb{R}^{n} \longrightarrow \mathbb{R}^{n}$, we can view the action $P^{T} * P$ as a reordering of the basis in the vector space. As we will see, the graph of an irreducible matrix has a very special property. This is due to the fact that the directed graphs $\mathcal{D}(A)$ and $\mathcal{D}\left(P^{T} A P\right)$ are isomorphic $\left(\mathcal{D}(A) \simeq \mathcal{D}\left(P^{T} A P\right)\right.$ ), in the sense that we can convert one to the other by simply relabeling the vertices. Formally we say that there exists a bijective function $f$ between the vertex sets $\mathcal{V}(A)$ and $\mathcal{V}\left(P^{T} A P\right)$ which preserves adjacency, that is $\nu_{i}$ and $\nu_{j}$ in $\mathcal{D}(A)$ are adjacent if and only if $f\left(\nu_{i}\right)$ and $f\left(\nu_{j}\right)$ in $\mathcal{D}\left(P^{T} A P\right)$ are adjacent, for all $i, j=1, \ldots, n$. Now, let us use this to prove an equivalence theorem about irreducibility.

Theorem 2.11. For a matrix $A \geq 0$ the following facts are equivalent:
(i) $A$ is irreducible,
(ii) $\mathcal{D}(A)$ is strongly connected,
(iii) $(A+I)^{n-1}>0$.

Proof. (i) $\Rightarrow$ (ii): Proof by contrapositive: $\neg($ ii $) \Rightarrow \neg($ i). Suppose $\mathcal{D}(A)$ is not strongly connected, then there exists a pair of vertices $\nu_{Z}$ and $\nu_{X}$ for which there is no directed walk $\nu_{Z} \rightarrow \cdots \rightarrow \nu_{X}$. Let $\mathcal{V}_{X}:=\{\nu \in \mathcal{V}(A)$ : $\exists$ walk from $\nu$ to $\left.\nu_{X}\right\}$ and let $\mathcal{V}_{Z}:=\left\{\nu \in \mathcal{V}(A): \exists\right.$ walk from $\nu_{Z}$ to $\left.\nu\right\}$. Notice that $\nu_{X} \in \mathcal{V}_{X}$ and $\nu_{Z} \in \mathcal{V}_{Z}$ because there is a directed walk of length 0 from every vertex to itself in $\mathcal{D}(A)$, and so both $\mathcal{V}_{X}, \mathcal{V}_{Z} \neq \emptyset$. If a vertex $\omega \in \mathcal{V}_{Z}$ then it is not adjacent to any of the vertices in $\mathcal{V}_{X}$, because otherwise we could use this arc to create a walk from $\nu_{Z}$ to $\nu_{X}$ via $\omega$. Now, permute the rows and columns in $A$ simultaneously, with the permutation matrix $P$, so that the rows/columns corresponding to vertices in $\mathcal{V}_{X}$ comes before the rows/columns corresponding to vertices in $\mathcal{V}_{Z}$. The resulting matrix has the following block structure: $P^{T} A P=\left(\begin{array}{ll}A_{11} & A_{12} \\ A_{21} & A_{22}\end{array}\right)$, but since there are no arcs from vertices in $\mathcal{V}_{Z}$ to vertices in $\mathcal{V}_{X}$ the matrix $A_{21}=\mathbf{0}$. If we want, we could rename the blocks, $X:=A_{11}, Y:=A_{12}$ and $Z:=A_{22}$, to get the exact same form as 2.1) in definition 2.9. We conclude that $A$ is reducible.
(ii) $\Rightarrow$ (i): Proof by contrapositive: $\neg$ (i) $\Rightarrow \neg($ ii $)$. Suppose that $A$ is reducible and use a permutation matrix $P$ to bring $A$ to block upper triangular form, as in definition 2.9, Call this new matrix $B$. Let the matrices $X$ and $Z$ have size $r \times r$ and $(n-r) \times(n-r)$, respectively. Then $b_{i j}=0$ for all $i=r+1, \ldots, n$ and $j=1, \ldots, r$, see below.

$$
B=\left(\begin{array}{cccccc}
b_{11} & \ldots & b_{1 r} & & & \\
\vdots & \ddots & \vdots & & * & \\
b_{r 1} & \ldots & b_{r r} & & & \\
0 & \ldots & 0 & b_{(r+1)(r+1)} & \ldots & b_{(r+1) n} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \ldots & 0 & b_{n(r+1)} & \ldots & b_{n n}
\end{array}\right)
$$

Form $\mathcal{D}(B)$ and start a walk at vertex $\nu_{n}$. The only way to reach vertex $\nu_{1}$ is to at some point in the walk use an $\operatorname{arc}\left(\nu_{i}, \nu_{j}\right)$ where $\nu_{i} \in\left\{\nu_{r+1}, \ldots, \nu_{n}\right\}$ and $\nu_{j} \in\left\{\nu_{1}, \ldots, \nu_{r}\right\}$, but since $b_{i j}=0$ for such $i$ and $j$, there is no such arc, and thus there does not exists a walk between $\nu_{n}$ and $\nu_{1}$, so $\mathcal{D}(B)$ is not strongly connected. Since $\mathcal{D}(B) \simeq \mathcal{D}(A), \mathcal{D}(A)$ cannot be strongly connected either.
(ii) $\Rightarrow$ (iii): Here we will use the fact that $A$ and $I$ commute (everything commutes with the identity), that is $A I=I A$, to apply the binomial theorem to $(A+I)^{n-1}$. We get the following sum:

$$
\begin{equation*}
(A+I)^{n-1}=A^{n-1}+\binom{n}{1} A^{n-2}+\cdots+\binom{n}{n-2} A^{2}+\binom{n}{n-1} A+I \tag{2.2}
\end{equation*}
$$

We know that $\mathcal{D}:=\mathcal{D}(A)$ is strongly connected, and in section 2.2 we reasoned that for such a directed graph and for any pair of vertices it is true that $\left(\nu_{i}, \nu_{j}\right) \in$ $\mathcal{E}^{k}$ for some $k=1, \ldots, n-1$. If we let $a_{i j}^{(k)}$ be the elements in $A^{k}$, theorem 2.7 ensures that for all $i, j=1, \ldots, n$ it is true that $a_{i j}^{(k)}>0$ for (at least) one $k=1, \ldots, n-1$. But this is precisely what is needed for the sum in (2.2) to be a strictly positive matrix, so $(A+I)^{n-1}>0$.
(iii) $\Rightarrow$ (ii): Choose indices $i, j$ arbitrarily and expand $(A+I)^{n-1}$ with the binomial theorem, as in 2.2. Since $(A+I)^{n-1}>0$ there is at least one $k=1, \ldots, n-1$ such that $a_{i j}^{(k)}>0$ and so there exists an $\operatorname{arc}\left(\nu_{i}, \nu_{j}\right) \in \mathcal{E}^{(k)}(A)$ and by theorem 2.7, there exists a walk (of length $k$ ) in $\mathcal{D}(A)$. Since $i$ and $j$ were arbitrary, there exists a walk between any pair of nodes in $\mathcal{D}(A)$ and so the directed graph $\mathcal{D}(A)$ is strongly connected.

As we will see later, irreducible matrices posses some of the famous Perronproperties, inherited by positive matrices. One thing that distinguishes irreducible matrices from positive, apart from zero elements, is the number of eigenvalues on the spectral circle. This is so important that the next class of nonnegative matrices will be defined based on this distinction.

Example 2.12. As an example, consider these three nonnegative matrices:

$$
A=\left(\begin{array}{lll}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{array}\right), \quad B=\frac{1}{\sqrt{2}}\left(\begin{array}{ccc}
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & 0
\end{array}\right), \quad C=\frac{1}{3}\left(\begin{array}{lll}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{array}\right) .
$$

All three matrices are irreducible which is easiest seen using theorem 2.11 by forming their strongly connected directed graphs, see fig. 2.2. For each matrix we determine the spectrum and spectral radius in the usual way, which in this case happen to be the same $\rho=1$ for all three matrices. We find that $A$ has three eigenvalues $\left\{e^{i \frac{2 \pi}{3} k}: k=0,1,2\right\}, B$ has two eigenvalues $\{ \pm 1\}$ and $C$ has only one eigenvalue $\{1\}$ on the spectral circle $|z|=\rho=1$, see fig. 2.3.


Figure 2.2: From left to right: $\mathcal{D}(A), \mathcal{D}(B), \mathcal{D}(C)$.




Figure 2.3: From left to right: $\sigma(A), \sigma(B), \sigma(C)$.

### 2.4 Primitive Matrices

In this section, we will be able to divide the three matrices from example 2.12 into two groups; the primitive matrix $C$ and the imprimitive matrices $A$ and $B$.

Definition 2.13 (Primitivity). A nonnegative irreducible matrix $A \geq 0$ is called primitive if it has only one eigenvalue of maximum modulus and imprimitive if it has $k>1$ eigenvalues of maximum modulus. In the latter case we call the number $k$ the index of imprimitivity.

Remark 2.14. The eigenvalues with maximum modulus all lie on the spectral circle $|z|=\rho(A)$ in the complex plane $\mathbb{C}$. Also, the set of primitive matrices are a (proper) subset of the irreducible ones.

Example 2.15 (Continuation of 2.12). We conclude, see fig. 2.3 that $A$ and $B$ are imprimitive with index of imprimitivity $k=3$ and $k=2$ respectively and that matrix $C$ is primitive.

As for irreducible matrices, the graphs of primitive matrices have a special property. We first define the concept of primitive graphs and then we show that a graph of a primitive matrix is a primitive graph.

Definition 2.16. Let $\mathcal{D}=(\mathcal{V}, \mathcal{E})$ be a strongly connected directed graph. Define $k$ as the greatest common divisor of the lengths of closed walks in $\mathcal{D}$. If
$k=1, \mathcal{D}$ is primitive and otherwise $\mathcal{D}$ is imprimitive with index of imprimitivity $k>1$.

Note that if we want to prove that $\mathcal{D}$ is primitive it suffices to find two different closed walks, say $\mathcal{C}_{1}$ and $\mathcal{C}_{2}$ such that $\operatorname{gcd}\left(\left|\mathcal{C}_{1}\right|,\left|\mathcal{C}_{2}\right|\right)=1$ whereas imprimitivity must be proved by showing that all possible cycles have a common divisor which is greater than 1. The definition 2.16 is perhaps a bit hard to digest, so let us consider an example.

Example 2.17. Let $\mathcal{D}_{1}$ and $\mathcal{D}_{2}$ be the directed graphs in fig. 2.4. One can verify that in each, there is a walk between any pair of vertices, and so they are strongly connected. We will show that $\mathcal{D}_{1}$ is primitive and $\mathcal{D}_{2}$ is imprimitive with index $k=3$.


Figure 2.4: $\mathcal{D}_{1}$ (left) and $\mathcal{D}_{2}$ (right).

Consider the closed walks $\mathcal{C}_{1}: \nu_{1} \rightarrow \nu_{4} \rightarrow \nu_{2} \rightarrow \nu_{1}$ and $\mathcal{C}_{2}: \nu_{0} \rightarrow \nu_{4} \rightarrow$ $\nu_{2} \rightarrow \nu_{3} \rightarrow \nu_{0}$ in $\mathcal{D}_{1}$. Since $\left|\mathcal{C}_{1}\right|=3$ and $\left|\mathcal{C}_{2}\right|=4$ we get that $\operatorname{gcd}\left(\left|\mathcal{C}_{1}\right|\right.$, $\left.\left|\mathcal{C}_{2}\right|\right)=\operatorname{gcd}(3,4)=1$ and so $\mathcal{D}_{1}$ is a primitive graph. Because $\mathcal{D}_{2}$ is two 3-cycles joined by a common vertex $\nu_{9}$ the only closed walks in the graphs are repeated 3 -cycles. Thus the lengths of these can only be $\{0,3,6,9, \ldots\}$, i.e. multiples of 3. We conclude that the greatest common divisor of all closed walks in $\mathcal{D}_{2}$ is 3 and therefore $\mathcal{D}_{2}$ is imprimitive with index $k=3$.

Worth adding before stating the much awaited connection between primitivity of graphs and primitivity of matrices is the following facts: $A \geq 0$ is a primitive matrix if and only if $A^{m}>0$ for some power $m \in \mathbb{Z}_{+}$. Also: $A \geq 0$ is primitive matrix if and only if $\lim _{m \rightarrow \infty}\left(\frac{A}{\rho(A)}\right)^{m}$ exists. We will investigate both these facts later when we present Perron-Frobenius theorem for irreducible, primitive matrices.

Theorem 2.18. Let $A \geq 0$ be an irreducible matrix and $\mathcal{D}(A)$ its graph. Then $A$ is a primitive matrix if and only if $\mathcal{D}(A)$ is a primitive graph. Additionally, $A$ is imprimitive with index $k>1$ if and only if $\mathcal{D}(A)$ is imprimitive with the same index $k>1$.

Proof. The proof can be found in Horn-Johnson [5, Theorem 8.5.3], Shapiro [10, Theorems 17.25-17.26] or elsewhere. Both books uses the fact that $A^{m}>0$ for some power $m>0$.

Remark 2.19. As we will see later, a positive matrix $A>0$ has only one eigenvalue on its spectral circle, this is a result of Perron's theorem, and so positive matrices are also primitive. But in fact, this can be seen using only what we have seen so far. In the graph $\mathcal{D}(A)$ of a positive matrix, every possible arc between any pair of vertices exists and specifically there are loops at every vertex. Loops are closed walk of length 1 , so for any other closed walk $\mathcal{C}$ in $\mathcal{D}(A)$ we have $\operatorname{gcd}(1,|\mathcal{C}|)=1$ and thus, by theorem 2.18, $A>0$ is primitive.

As indicated above, loops have a big impact of the primitivity of a graph. Remember that a loop in a graph $\mathcal{D}(A)$ of a matrix $A \geq 0$ correspond to a positive diagonal element $a_{i i}>0$, some $i=1, \ldots, n$ in $A$. As a result, we see the following theorem.

Theorem 2.20. Let $A \geq 0$ be irreducible. If $a_{i i}>0$ for some $i=1, \ldots, n$ then $A$ is primitive.

Proof. $a_{i i}>0$ for some $i=1, \ldots, n$ implies that $\left(\nu_{i}, \nu_{i}\right) \in \mathcal{E}(A)$, i.e. $\mathcal{D}(A)$ has a loop at vertex $\nu_{i} . \mathcal{D}(A)$ is primitive by the same logic presented in remark 2.19 and so by theorem $2.18 A$ is primitive.

We will end this section by directing our attention to imprimitive matrices, which are irreducible matrices having $k>1$ eigenvalues on its spectral circle. In the middle of 20th century, the German mathematician Hermut Wielandt discovered a remarkable fact about the location of the eigenvalues on the spectral circle of imprimitive matrices. This can then be used to show that the eigenvalues also inside of the spectral circle are located in a somewhat predictable way. Here, we will present the general theorem and show complete proofs of its applications. The full proof of the following theorem, sometimes called Wielandt's Theorem, can be found in Meyer [9, pp. 675-676].

Theorem 2.21 (Wielandt's Theorem). Assume $A$ is an irreducible matrix and $B \in \mathbb{M}_{n}(\mathbb{C})$. If $\left|b_{i j}\right| \leq a_{i j}$ for all $i, j=1, \ldots, n$ then $\rho(B) \leq \rho(A)$. If equality holds, i.e. there exists some $\mu=e^{i \phi} \rho(A) \in \sigma(B)$ such that $\rho(B)=|\mu|=$ $\left|e^{i \phi} \rho(A)\right|=\rho(A)$ for some angle $\phi \in \mathbb{R}$, then we have that

$$
B=e^{i \phi} D A D^{-1} \text { for some } D=\left(\begin{array}{ccc}
e^{i \theta_{1}} & &  \tag{2.3}\\
& \ddots & \\
& & e^{i \theta_{n}}
\end{array}\right)
$$

and conversely if $B=e^{i \phi} D A D^{-1}$ then $\rho(B)=\rho(A)$.

Proof of $2^{\text {nd }}$ part: equality holds, $(\Leftarrow)$ direction. $D A D^{-1}$ is a similarity transformation. The spectrum of a matrix is invariant under similarity transformations because if $\lambda \in \rho(A)$ then there exists a vector $\mathbf{x} \in \mathbb{C}^{n}$ such that $A \mathbf{x}=\lambda \mathbf{x}$, and if we let $\mathbf{y}:=D \mathbf{x}=\left(e^{i \theta_{1}} x_{1}, \ldots, e^{i \theta_{1}} x_{n}\right)^{T}$ then

$$
\left(D A D^{-1}\right) \mathbf{y}=D A D^{-1} D \mathbf{x}=D A \mathbf{x}=D \lambda \mathbf{x}=\lambda D \mathbf{x}=\lambda \mathbf{y}
$$

which shows that $(\lambda, \mathbf{y})$ is an eigenpair of $D A D^{-1}$ and thus $\lambda \in \sigma\left(D A D^{-1}\right)$. So $\sigma(A)=\sigma\left(D A D^{-1}\right)$ and specifically $\rho(A)=\rho\left(D A D^{-1}\right)$. We are not quite finished, because $B=e^{i \phi} D A D^{-1}$. But since $e^{i \phi}$ is just a rotation (modulus 1) of numbers in the complex plane, we can view the multiplication $e^{i \phi}\left(D A D^{-1}\right)$ as rotating all the entries in $D A D^{-1}$ by the same angle, and thus also rotating the spectrum with the angle $\phi$. While $B$ can have a different eigenvalue of maximum modulus than $A$, it is only rotated around the origin, and not scaled. Therefore $\rho(B)=\rho(A)$.

We will especially be interested in the second part of the theorem, where $\rho(B)=\rho(A)$, because we want to apply Wielandt's theorem with $B=A$ in the proof of the following theorem. This is the first part about the extraordinary finds about imprimitive matrices.

Theorem 2.22 ( $k^{\text {th }}$ roots of $\rho^{k}$ ). Let $A$ be an imprimitive matrix with index $k>1$. Let $\rho:=\rho(A)$ and let $\mathcal{S}:=\left\{\lambda_{1}, \ldots, \lambda_{k}\right\}$ be the set of eigenvalues on the spectral circle $|z|=\rho, z \in \mathbb{C}$. Then:
(1) $\operatorname{algmult}_{A}\left(\lambda_{i}\right)=1$ for all $i=1, \ldots, k$
(2) $\mathcal{S}$ is the $k^{\text {th }}$ roots of $\rho^{k}$, i.e. $\mathcal{S}=\left\{\rho, \xi \rho, \xi^{2} \rho, \ldots, \xi^{k-1} \rho\right\}$ where $\xi=e^{\frac{2 \pi}{k} i}$.

Remark 2.23. The theorem 2.22 ensures that $\mathcal{S}$, as defined in the theorem, is invariant under rotation by the angle $\frac{2 \pi}{k}$.

The proof of theorem 2.22 uses one of the main results from Perron-Frobenius theorem, presented in the next chapter, namely that if $A$ is irreducible then its spectral radius $\rho(A)$ is a simple eigenvalue, often called the Perron root, i.e. $r:=\rho(A) \in \sigma(A)$ and geomult $_{A}(r)=\operatorname{algmult}_{A}(r)=1$. In many books, one sees the concept of primitive matrices presented after Perron-Frobenius theorem for irreducible matrices, but for the sake of fully investigating the primitive matrices in the preliminaries we see this "reference to the future" here. The proof also uses some results from abstract algebra regarding finite groups, the details of which can be read in for example Svensson [12, p. 79, 131].

Proof. Part (1): We start by writing the set $\mathcal{S}$ in another way. First, let $r:=\rho(A)$. By Perron-Frobenius theorem, see next chapter, $r \in \sigma(A)$, so $r \in \mathcal{S}$,
and $r$ is simple. Now we want to relate all other eigenvalues in $\mathcal{S}$ to $r$. This can be done by considering with what angle $r$ needs to be rotated with to become the considered eigenvalue. We get one angle for each element in $\mathcal{S} \backslash\{r\}$ and so we write $\mathcal{S}=\left\{r, e^{i \theta_{1}} r, \ldots, e^{i \theta_{k-1}} r\right\}$. This process is illustrated in fig. 2.5 (a). Apply Wielandt's theorem 2.21 with $B=A$ and $\rho(B)=|\mu|=\left|e^{i \theta_{j}} r\right|=r$ for some $j \in\{1, \ldots, k-1\}$, i.e. $\mu$ is some element in $\mathcal{S} \backslash\{r\}$. Then there exists a diagonal matrix $D_{j}$ such that $B=A=e^{i \theta_{j}} D_{j} A D_{j}^{-1}$ and therefore $A$ and $e^{i \theta_{j}} A$ are similar and thus they have the same spectrum. We know that $r$ is a simple eigenvalue for $A$ and so $e^{i \theta_{j}} r$ is also a simple eigenvalue. Applying the same argument for all $j=1, \ldots, k-1$ proves the first item.

(a)

(b)

(c)

Figure 2.5: (a) Rewriting $\mathcal{S}$, (b) Sum of angles, (c) $4^{\text {th }}$ roots of $\rho^{4}$.
Part (2): Take another $e^{i \theta_{s}} r \in \mathcal{S}$, by Wielandt's theorem 2.21 we have $A=$ $e^{i \theta_{s}} D_{s} A D_{s}^{-1}$ for some diagonal matrix $D_{s}$. From Part (1), we get

$$
\begin{aligned}
A & =e^{i \theta_{j}} D_{j} A D_{j}^{-1}=e^{i \theta_{j}} D_{j}\left(e^{i \theta_{s}} D_{s} A D_{s}^{-1}\right) D_{j}^{-1} \\
& =e^{i\left(\theta_{j}+\theta_{s}\right)} D_{j} D_{s} A D_{s}^{-1} D_{j}^{-1}=e^{i\left(\theta_{j}+\theta_{s}\right)}\left(D_{j} D_{s}\right) A\left(D_{j} D_{s}\right)^{-1}
\end{aligned}
$$

where $D^{\prime}:=D_{j} D_{s}$ will be a diagonal matrix of the same form as $D$ in 2.3. So applying Wielandt's theorem $2.21(\Leftarrow)$ ensures that also $\mu=e^{i\left(\theta_{j}+\theta_{s}\right)} s \in \sigma(A)$ with modulus $r$, so $\mu \in \mathcal{S}$. We now realize that the picture in fig. 2.5 (a) is wrong, since the blue point in figure (b) should also be an element in $\mathcal{S}$. If we let $\mathcal{G}=\left\{1, e^{i \theta_{1}}, \ldots, e^{i \theta_{k-1}}\right\}$ we have just shown that $\mathcal{G}$ is closed under multiplication. And since $U=\{z \in \mathbb{C}:|z|=1\}$ is an abelian group and $\mathcal{G} \subset U$ then $\mathcal{G}$ is an abelian (sub)group. A result from abstract algebra is that $g^{o(\mathcal{G})}=1_{\mathcal{G}}$ for all $g \in \mathcal{G}$ where $o(\mathcal{G})$ is the order of the group and $1_{\mathcal{G}}$ the neutral element. For us, $\mathcal{G}$ has $k$ elements, therefore $o(\mathcal{G})=k$ and so we conclude that $\left(e^{i \theta_{j}}\right)^{k}=1$ for all $j=1, \ldots, k-1$ and so $\mathcal{G}$ is the $k^{t h}$ roots of unity. Scaling $\mathcal{G}$ by $r$ yields that the set $\mathcal{S}$ is the $k^{t h}$ roots of $\rho(A)^{k}$, see the correct picture in fig. 2.5 (c).

It is not only $\mathcal{S}$ which is invariant under rotation by $\frac{2 \pi}{k}$, as the next theorem tells us. This is the second and final part of the find about imprimitive matrices.

Theorem 2.24. Let $A$ be an imprimitive matrix with index $k>1$. Then the whole spectrum $\sigma(A)$ is invariant under rotation by $\frac{2 \pi}{k}$ and no rotation less than that can preserve $\sigma(A)$.

Before we see the proof of this statement, let us stop and reflect upon what it actually means. If $A$ is imprimitive with index $k>1$, we know that it has $k$ eigenvalues on its spectral circle and theorem 2.22 assures us that these are the $k^{t h}$ roots of $\rho(A)^{k}$, so they are located in a very predictable way. So far we know nothing how the rest of the spectrum (inside the circle) looks. But with theorem 2.24 we get that these eigenvalues are located in such a way that if we rotate the whole complex plane by $\frac{2 \pi}{k}$ we get the same eigenvalues, how remarkable! We will soon see an illustrative example of this, but before we do let us prove theorem 2.24

Proof. If ( $\lambda, \mathbf{x}$ ) is an eigenpair of $A$, then $A \mathbf{x}=\lambda \mathbf{x}$ and so $e^{i \frac{2 \pi}{k}} A \mathbf{x}=e^{i \frac{2 \pi}{k}} \lambda \mathbf{x}$, i.e. ( $e^{i \frac{2 \pi}{k}} \lambda, \mathbf{x}$ ) is an eigenpair of $e^{i \frac{2 \pi}{k}} A$. By Wielandt's theorem $2.21 A$ and $e^{i \frac{2 \pi}{k}} A$ are similar and so $\sigma(A)=\sigma\left(e^{i \frac{2 \pi}{k}} A\right)$. Hence, the spectrum $\sigma\left(e^{i \frac{2 \pi}{k}} A\right)$ is the same as $\sigma(A)$ but rotated around the origin with angle $\frac{2 \pi}{k}$. No rotation less than $\frac{2 \pi}{k}$ can preserve $\sigma(A)$ since no rotation less than $\frac{2 \pi}{k}$ can preserve the $k^{t h}$ roots of $\rho(A)^{k}$.

Example 2.25. Consider the matrix

$$
A=\left(\begin{array}{llllllll}
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0
\end{array}\right)
$$

It is irreducible because its graph $\mathcal{D}(A)$ contains the red dashed cycle in fig. 2.6 visiting all the vertices $\nu_{1}, \ldots, \nu_{8}$ so $\mathcal{D}(A)$ is strongly connected. Furthermore A is imprimitive with index of imprimitivity $k=2$ because all closed paths in $\mathcal{D}(A)$ have a greatest common divisor of 2.

If we calculate the spectrum of $A$ we get the following: $\sigma(A)=\{0,1+i, 1-$ $i,-1+i,-1-i, 2,-2\}$, where 0 has algebraic multiplicity two, from which we can see that $\rho(A)=2$ and that two eigenvalues (2 and -2) lie on the spectral


Figure 2.6: Left: $\mathcal{D}(A)$, right: $\sigma(A)$
circle, thus verifying that the index is indeed $k=2$ and that they are exactly the solutions to $z^{2}=4$, i.e. the square roots of 4 . We know that they are invariant under rotation by $\pi$. Further, look at the eigenvalues inside of the circle. They are also invariant under rotation by $\pi$, in this case even $\frac{\pi}{2}$. In this way we have verified the elegant theorems 2.22 and 2.24

## Chapter 3

## Perron-Frobenius Theory

We have reached the main topic of this thesis; Perron-Frobenius theory. The theory is composed of two main results; one by Perron and one by Frobenius. In 1907 the German mathematician Oskar Perron (1880-1975) published his theorem for positive matrices. Perron was at the time not mainly studying matrices, but worked instead on Jacobi multidimensional continued fractions in which the properties of positive matrices arose in a lemma. He later realized the importance of this lemma and published it on its own. Perron's original proof, which uses induction on the size of $A>0$, can be read, along with many other proofs of the theorem, in MacCluer's article on the subject [8]. In 1912, Georg Frobenius (1849-1917), also German, generalized Perron's finding to nonnegative irreducible matrices, which he called 'unzerlegbar' matrices. From Frobenius' theorem it is possible to derive a theorem also for the primitive matrices.

The two theorems have a wide range of applications in both pure mathematics and practical matters. In real world scenarios, many measurements are nonnegative (length, time, amount, etc.) and so their mathematical formulations, which often involve matrices therefore relate to Perron-Frobenius theory. The theory's importance to linear dynamic system, such as Markov chains, cannot be overstated; it determines when, and to what, an iterative process like $\mathbf{x}_{m+1}=A \mathbf{x}_{m}$ will converge. This result is in turn the underlying theory for the page-ranking algorithm developed by Google in 1998.

The many proofs of Perron-Frobenius theory can broadly be divided into two categories; those entirely based in linear algebra and those using results from elsewhere (for example Brouwer's fixed point theorem or from complex analysis). In this chapter we will, in the first and second section, see the statement of the theorems and in the third section we will go through and illustrate a proof of Perron-Frobenius' theorem.

### 3.1 Perron's Theorem

Perron's theorem tells us about positive matrices. It is on positive matrices that this theory is as most powerful. The following version can be found in Horn-Johnson [5].

Theorem 3.1 (Perron's theorem). Let $A>0$ be a positive matrix. Let $\rho(A)$ and $\sigma(A)$ denote the spectral radius and spectrum of $A$. Then
(i) $\rho(A)>0$
(ii) $\rho(A) \in \sigma(A)$ and $\rho(A)$ is simple
(iii) there exists a unique (right) eigenvector $\boldsymbol{p}$ of $A$, which can be taken positive, such that $A \boldsymbol{p}=\rho(A) \boldsymbol{p}$ and $\|\boldsymbol{p}\|_{1}=p_{1}+\cdots+p_{n}=1$
(iv) there exists a unique (left) eigenvector $\boldsymbol{q}^{T}$ of $A$, which can be taken positive, such that $\boldsymbol{q}^{T} A=\rho(A) \boldsymbol{q}^{T}$ and $\boldsymbol{q}^{T} \boldsymbol{p}=p_{1} q_{1}+\cdots+p_{n} q_{n}=1$
(v) $|\lambda|<\rho(A)$ for all other eigenvalues $\lambda \in \sigma(A) \backslash\{\rho(A)\}$
(vi) $\lim _{m \rightarrow \infty}\left(\frac{1}{\rho(A)} A\right)^{m}=\boldsymbol{p} \boldsymbol{q}^{T}$

Remark 3.2. Note that the vectors $\mathbf{p}$ and $\mathbf{q}^{T}$ are the only eigenvectors of $A$ which can be taken positive.

Here are some explanations of the items above:
(i), (ii): Perron's theorem says that the spectral radius of a positive matrix is itself a positive eigenvalue. $\rho(A)$ is usually called the Perron root. The fact that $\rho(A)$ is simple means that $\operatorname{algmult}_{A}(\rho(A))=\operatorname{geomult}_{A}(\rho(A))=1$ or equivalently that the corresponding block in the Jordan canonical form of $A$ is $1 \times 1$ block only containing $\rho(A)$.
(iii): We can see that $(\rho(A), \mathbf{p})$ is an eigenpair of $A$. This special unique eigenvector, first taken positive and then normalized so that the sum of its entries are 1 (so that $\mathbf{p}$ is a so called probability vector) also has a special name; $\mathbf{p}$ is called the right Perron vector of $A$.
(iv): It might seem unusual to consider vector-matrix multiplication with a vector, considered as a row-matrix, to the left. Yet, it is often used when studying Markov chains. The vector $\mathbf{q}$, called the left Perron vector, is the positive eigenvector of the matrix $A^{T}$, because if $A^{T} \mathbf{q}=\rho(A) \mathbf{q}$, then transposing both sides yields $\mathbf{q}^{T} A=\rho(A) \mathbf{q}^{T}$. Note that (of course) $A>0$ implies $A^{T}>0$, so Perron's theorem holds for $A^{T}$, and that $\sigma(A)=\sigma\left(A^{T}\right)$.
$(v):$ The fact that all other eigenvalues of $A$ have (strictly) smaller modulus than $\rho(A)$, means that the only eigenvalue on the spectral circle is $\rho(A)$. Because of this property, $\rho(A)$ is sometimes called the dominant eigenvalue of $A$.
(vi): Below we give a proof of the last item of Perron's theorem. Sometimes one sees the matrix $\mathbf{p q}^{T}$, which is the outer product of the two vectors $\mathbf{p}$ and $\mathbf{q}^{T}$, being referred to as the spectral projector associated with the eigenvalue $\rho(A)$. This is because if we consider $\mathbf{p q}^{T}$ as a linear map, then for all vectors $\mathbf{x} \in \mathbb{R}^{n} \backslash\{\mathbf{0}\}$, the image of $\mathbf{x}$ is simply $\mathbf{p q}^{T} \mathbf{x}=\mathbf{p}\left(\mathbf{q}^{T} \mathbf{x}\right)=\left(\mathbf{q}^{T} \mathbf{x}\right) \mathbf{p}$ where $\mathbf{q}^{T} \mathbf{x} \in \mathbb{R} \backslash\{0\}$ is some scalar (real number) and so $\left(\mathbf{q}^{T} \mathbf{x}\right) \mathbf{p}$ is a scalar multiple of the right Perron vector. Equivalently, one can say that $\mathbf{x}$ gets projected onto the eigenspace of the eigenvalue $\rho(A)$.

Proof of (vi) in Perron's theorem 3.1: We want to bring $A$ to its Jordan canonical form. Remember that all square matrices can be brought to Jordan form via a similarity transformation $A=T J T^{-1}$ where the columns of $T$ are the eigenvectors or generalized eigenvectors of $A$ (the latter case if geomult $_{A}(\lambda)<$ $\operatorname{algmult}_{A}(\lambda)$ for the corresponding eigenvalue). The matrix $J$ is a diagonal block matrix with the eigenvalues of $A$ as diagonal element, repeated according to multiplicity. Each block correspond to distinct eigenvalues in $\sigma(A)$. If the eigenvalues of $A$ are ordered according to modulus $\rho(A)>\left|\lambda_{1}\right| \geq \cdots \geq\left|\lambda_{k}\right|>0$ for some $1 \leq k<n$, then since $\rho(A)$ is simple, we have

$$
J=([\rho(A)] \oplus B)=\left(\begin{array}{cc}
\rho(A) & \mathbf{0} \\
\mathbf{0} & B
\end{array}\right)
$$

where the $(n-1) \times(n-1)$ matrix $B$ contains the Jordan blocks of $A$ corresponding to $\lambda_{1}, \ldots, \lambda_{k}$. Since $\rho(A)$ is the dominant eigenvalue of $A$, then $\rho(A)>\rho(B)$ and so $\frac{\rho(B)}{\rho(A)}<1$ and thus the matrix $\frac{1}{\rho(A)} B$ has spectral radius less then 1 , so $\frac{1}{\rho(A)} B=D+\mathcal{N}$ where $D$ is a diagonal matrix with entries $d_{i}=\frac{\lambda_{i}}{\rho(A)}$ such that $\left|d_{i}\right|<1$ for all $i=1, \ldots, n-1$ and $\mathcal{N}$ is nilpotent, i.e. some power of $\mathcal{N}$ is the zero matrix. Thus $\left(\frac{1}{\rho(A)} B\right)^{m} \rightarrow \mathbf{0}$ as $m \rightarrow \infty$. By theorem 3.3 below, we can use $\mathbf{p}$ as the first column in $T$ and $\mathbf{q}^{T}$ as the first row in $T^{-1}$. Thus

$$
\begin{aligned}
\left(\frac{1}{\rho(A)} A\right)^{m} & =\left(\frac{1}{\rho(A)} T J T^{-1}\right)^{m}=T\left(\frac{1}{\rho(A)} J\right)^{m} T^{-1} \\
& =\left(\begin{array}{ll}
\mathbf{p} & T_{1}
\end{array}\right)\left(\begin{array}{cc}
1^{m} & \mathbf{0} \\
\mathbf{0} & \left(\frac{1}{\rho(A)} B\right)^{m}
\end{array}\right)\binom{\mathbf{q}^{T}}{Z_{1}} \\
& \rightarrow\left(\begin{array}{ll}
\mathbf{p} & T_{1}
\end{array}\right)\left(\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right)\binom{\mathbf{q}^{T}}{Z_{1}}=\left(\begin{array}{ll}
\mathbf{p} & \mathbf{0}
\end{array}\right)\binom{\mathbf{q}^{T}}{\mathbf{0}}=\mathbf{p q}^{T}
\end{aligned}
$$

as $m \rightarrow \infty$. Note that this is a rank one matrix.

In the proof above we use $\mathbf{p}$ and $\mathbf{q}^{T}$ as the first column and row in $T$ and $T^{-1}$ respectively. The fact that we can do this is motivated by the following theorem, which is a special case of the $(\Rightarrow)$ direction of part (b) in Theorem 1.4.7 from Horn-Johnson [5, p. 78]. The proof uses some nice reasoning with orthogonality. Here, we denote matrices by [ ], and linear span with span( ), to separate the partitioned matrices from vectors, still denoted by parentheses.
Theorem 3.3. If $A \geq 0$ and $\boldsymbol{p}, \boldsymbol{q}>0$ where $A \boldsymbol{p}=\rho \boldsymbol{p}$ and $\boldsymbol{q}^{T} A=\rho \boldsymbol{q}^{T}$, where $\rho \in \sigma(A)$, and $\boldsymbol{q}^{T} \boldsymbol{p}=1$ then there exists a non-singular matrix $T=\left[\begin{array}{ll}\boldsymbol{p} & T_{1}\end{array}\right]$ such that

$$
T^{-1}=\left[\begin{array}{l}
\boldsymbol{q}^{T} \\
Z_{1}
\end{array}\right]
$$

and

$$
A=T\left[\begin{array}{cc}
\rho & \boldsymbol{0} \\
\mathbf{0} & B
\end{array}\right] T^{-1}, \quad B \in \mathbb{M}_{n-1}(\mathbb{R})
$$

Proof. Choose the columns $\mathbf{t}_{1}, \ldots, \mathbf{t}_{n-1}$ in $T_{1}$ to be a basis for the orthogonal complement to $\operatorname{span}(\mathbf{q})$, i.e. $\mathbf{q}^{T} \mathbf{t}_{i}=0$ for all $i=1, \ldots, n-1$, so $\mathbf{q}^{T} T_{1}=\mathbf{0}^{T}$ Let $\mathbf{z}=\left(z_{1}, \zeta^{T}\right)^{T}$ where $\zeta \in \mathbb{R}^{n-1}$ and suppose that $T \mathbf{z}=\mathbf{0}$, i.e. $\mathbf{z} \in \mathrm{N}(T)$. We have

$$
\begin{aligned}
0 & =\mathbf{q}^{T} \mathbf{0}=\mathbf{q}^{T} T \mathbf{z}=\mathbf{q}^{T}\left[\begin{array}{ll}
\mathbf{p} & T_{1}
\end{array}\right]\binom{z_{1}}{\zeta}=\mathbf{q}^{T}\left(\mathbf{p} z_{1}+T_{1} \zeta\right) \\
& =z_{1} \underbrace{\mathbf{q}^{T} \mathbf{p}}_{=1}+\underbrace{\left(\mathbf{q}^{T} T_{1}\right)}_{=\mathbf{0}^{T}} \zeta=z_{1}+\mathbf{0}^{T} \zeta=z_{1}
\end{aligned}
$$

so $z_{1}=0$ and then $\mathbf{0}=T \mathbf{z}=\left[\begin{array}{ll}\mathbf{p} & T_{1}\end{array}\right]\binom{0}{\zeta}=T_{1} \zeta$ implies that $\zeta=\mathbf{0}$ since $T_{1}$ has full column rank (the column span is the complement of a one dimensional span). So only $\mathbf{0} \in \mathrm{N}(T)$ and so $T$ is nonsingular. Now consider $T^{-1}=\left[\begin{array}{c}\eta^{T} \\ Z_{1}\end{array}\right]$ for some $\eta \in \mathbb{R}^{n}$. We show $\eta^{T}=\mathbf{q}^{T}$ by the following computation

$$
I=T^{-1} T=\left[\begin{array}{c}
\eta^{T} \\
Z_{1}
\end{array}\right]\left[\begin{array}{ll}
\mathbf{p} & T_{1}
\end{array}\right]=\left[\begin{array}{cc}
\eta^{T} \mathbf{p} & \eta^{T} T_{1} \\
Z_{1} \mathbf{p} & Z_{1} T_{1}
\end{array}\right]=\left[\begin{array}{cc}
1 & \mathbf{0} \\
\mathbf{0} & I_{n-1}
\end{array}\right]
$$

where $\eta^{T} T_{1}=\mathbf{0}$ implies that $\eta$ is orthogonal to the orthogonal complement to $\operatorname{span}(\mathbf{q})$, i.e. $\eta=\alpha \mathbf{q}$ for some scalar $\alpha \in \mathbb{R} \backslash\{0\}$. Then $\eta^{T} \mathbf{p}=1$ implies that $\alpha \mathbf{q}^{T} \mathbf{p}=1$ so $\alpha=1$. Now, we get

$$
\begin{aligned}
T^{-1} A T & =\left[\begin{array}{l}
\mathbf{q}^{T} \\
Z_{1}
\end{array}\right] A\left[\begin{array}{ll}
\mathbf{p} & T_{1}
\end{array}\right]=\left[\begin{array}{ll}
\mathbf{q}^{T} A \mathbf{p} & \mathbf{q}^{T} A T_{1} \\
Z_{1} A \mathbf{p} & Z_{1} A T_{1}
\end{array}\right] \\
& =\left[\begin{array}{ll}
\mathbf{q}^{T} \rho \mathbf{p} & \rho \mathbf{q}^{T} T_{1} \\
Z_{1} \rho \mathbf{p} & Z_{1} A T_{1}
\end{array}\right]=\left[\begin{array}{cc}
\rho & \mathbf{0} \\
\mathbf{0} & Z_{1} A T_{1}
\end{array}\right]
\end{aligned}
$$

where $B:=Z_{1} A T_{1} \in \mathbb{M}_{n-1}(\mathbb{R})$, which was to be shown.

How much is included in Perron's theorem, and how much is left as corollaries can vary in literature. In Meyer [9, p.667] we see that also Collatz-Wielandt formula is included, which relates the Perron root to an optimization problem in the following way: $\rho(A)=\max _{\mathbf{x} \in \mathcal{R}} f(\mathbf{x})$ where

$$
\begin{equation*}
f(\mathbf{x})=\min _{\substack{1 \leq i \leq n \\ x_{i} \neq 0}} \frac{(A \mathbf{x})_{i}}{x_{i}} \quad \text { and } \quad \mathcal{R}=\left\{\mathbf{x} \in \mathbb{R}^{n}: \mathbf{x} \geq 0, \mathbf{x} \neq \mathbf{0}\right\} \tag{3.1}
\end{equation*}
$$

Also, sometimes one sees that the left Perron vector $\mathbf{q}^{T}$ is chosen such that $\left\|\mathbf{q}^{T}\right\|_{1}=q_{1}+\cdots+q_{n}=1$, i.e. the same way we choose the right Perron vector. This choice of normalization will impact the limit in (vi). Let (using the same notation as in the proof above) $\mathbf{p}$ be the first column in $T$ and $\frac{1}{\mathbf{p}^{T} \mathbf{q}} \mathbf{q}^{T}$ the first row in $T^{-1}$, which is possible by theorem 3.3 since $\frac{1}{\mathbf{p}^{T} \mathbf{q}} \mathbf{q}^{T} A=\rho(A) \frac{1}{\mathbf{p}^{T} \mathbf{q}} \mathbf{q}^{T}$, where $\rho(A) \in \sigma(A)$, and $\frac{1}{\mathbf{p}^{T} \mathbf{q}} \mathbf{q}^{T} \mathbf{p}=1$. Note that $\mathbf{p}^{T} \mathbf{q}=p_{1} q_{1}+\cdots+p_{n} q_{n}>0$ so $\frac{1}{\mathbf{p}^{T} \mathbf{q}} \mathbf{q}^{T}$ is a positive scalar multiple of $\mathbf{q}^{T}$. We get the limit

$$
\left(\frac{1}{\rho(A)} A\right)^{m}=\cdots=\left(\begin{array}{ll}
\mathbf{p} & T_{1}
\end{array}\right)\left(\begin{array}{cc}
1^{m} & \mathbf{0} \\
0 & \left(\frac{1}{\rho(A)} B\right)^{m}
\end{array}\right)\binom{\frac{1}{\mathbf{p}^{T} \mathbf{q}} \mathbf{q}^{T}}{Z_{1}} \rightarrow \frac{\mathbf{p q}^{T}}{\mathbf{p}^{T} \mathbf{q}}
$$

as $m \rightarrow \infty$. Again, note that the limit is proportional to the one stated in (vi), because $\mathbf{p}^{T} \mathbf{q}>0$ (the euclidean product between $\mathbf{p}$ and $\mathbf{q}$ ) is a positive scalar.

Before moving on to nonnegative matrices, let us see Perron's theorem in action on a concrete example.

Example 3.4. Let $A$ be the positive matrix below.

$$
A=\left(\begin{array}{lll}
1 & 1 & 2 \\
1 & 2 & 1 \\
2 & 1 & 1
\end{array}\right)
$$

The secular polynomial of $A$ is $p_{A}(\lambda)=(4-\lambda)(-1-\lambda)(1-\lambda)$ so $\sigma(A)=$ $\{-1,1,4\}$. We see that $\rho(A)=|4|=4>0$ and also $\rho(A)=4 \in \sigma(A)$ which is simple since $\lambda=4$ is a simple root of $p_{A}(\lambda)$. We conclude that $\rho(A)=4$ is the Perron root of $A$ and we have thus verified parts (i) and (ii) in Perron's theorem. Further, $| \pm 1|=1<4$ so $\rho(A)=4$ is the dominant eigenvalue, in line with ( $v$ ). Let us now find the Perron vectors of $A$. Since the rowsum is 4 for all rows, the vector $\left(\begin{array}{lll}1 & 1 & 1\end{array}\right)^{T}$ satisfies

$$
\left(\begin{array}{lll}
1 & 1 & 2 \\
1 & 2 & 1 \\
2 & 1 & 1
\end{array}\right)\left(\begin{array}{l}
1 \\
1 \\
1
\end{array}\right)=\left(\begin{array}{l}
4 \\
4 \\
4
\end{array}\right)=4\left(\begin{array}{l}
1 \\
1 \\
1
\end{array}\right)
$$

so it is a positive right eigenvector corresponding to the eigenvalue 4. Normalizing in the $\|*\|_{1}$-norm yields that

$$
\therefore \mathbf{p}=\left(\begin{array}{l}
1 / 3 \\
1 / 3 \\
1 / 3
\end{array}\right) \text { is the right Perron vector. }
$$

Since also the column sum is 4 for all columns, the vector $\left(\begin{array}{lll}1 & 1 & 1\end{array}\right)$ is a left eigenvector corresponding to the eigenvalue 4 . The euclidean product between this vector and $\mathbf{p}$ is 1 , so we conclude that

$$
\therefore \mathbf{q}^{T}=\left(\begin{array}{lll}
1 & 1 & 1
\end{array}\right) \text { is the left Perron vector. }
$$

Alternatively, we could normalize the vector in the $\|*\|_{1}$-norm and get $\mathbf{q}^{T}=$ $\frac{1}{3}\left(\begin{array}{lll}1 & 1 & 1\end{array}\right)$ as the left Perron vector. The other eigenvectors of $A$ are $(1,-2,1)^{T}$ corresponding to the eigenvalue $\lambda=1$, and $(-1,0,1)^{T}$ corresponding to the eigenvalue $\lambda=-1$, which shows that $\mathbf{p}$ and $\mathbf{q}^{T}$ are the only eigenvectors of $A$ which can be taken positive. Hence, we have verified parts (iii) and (iv) in Perron's theorem. What is left is to verify that $\left(\frac{1}{4} A\right)^{m}$ has the correct limit when $m \rightarrow \infty$. If we write $A$ in Jordan form, we get the following:

$$
\begin{aligned}
\left(\frac{1}{4} A\right)^{m} & =T\left(\frac{1}{4} J\right)^{m} T^{-1}=T\left(\begin{array}{ll}
1^{m} & \\
& (1 / 4)^{m} \\
& \\
& \rightarrow(-1 / 4)^{m}
\end{array}\right) T^{-1} \\
& \rightarrow\left(\begin{array}{ccc}
1 & 1 & -1 \\
1 & -2 & 0 \\
1 & 0 & 1
\end{array}\right)\left(\begin{array}{lll}
1 & & \\
& 0 & \\
& & 0
\end{array}\right)\left(\begin{array}{ccc}
1 / 3 & 1 / 3 & 1 / 3 \\
1 / 6 & -1 / 3 & 1 / 6 \\
-1 / 2 & 0 & 1 / 2
\end{array}\right) \\
& =\left(\begin{array}{lll}
1 / 3 & 1 / 3 & 1 / 3 \\
1 / 3 & 1 / 3 & 1 / 3 \\
1 / 3 & 1 / 3 & 1 / 3
\end{array}\right)=\left(\begin{array}{l}
1 / 3 \\
1 / 3 \\
1 / 3
\end{array}\right)\left(\begin{array}{lll}
1 & 1 & 1
\end{array}\right)=\mathbf{p q}^{T}
\end{aligned}
$$

as $m \rightarrow \infty$, i.e. the (rank one) matrix we expect from (vi).
We have not yet seen proofs for (i)-(v) of Perron's Theorem. As we will see, these all apply also to irreducible matrices, but with $\rho(A) \geq|\lambda|$ for all $\lambda \in \rho(A)$. We will go through a proof for the statements on irreducible matrices, which does not depend on Perron's theorem, and since positive matrices are irreducible, the statements in this section can be derived from that proof. For proof of Perron's theorem using linear algebra, see Bergqvist [1, lecture notes], Horn-Johnson [5], chapter 8.2], Meyer [9, chapter 8.2] or Shapiro [10, chapter 17.3].

### 3.2 Perron-Frobenius' Theorem

We now wonder how much of Perron's theorem 3.1 is true for a general nonnegative matrix. We have already seen that there are nonnegative, not zero, matrices which have spectral radius 0 . The example considered in section 2.1 was the matrix $\mathcal{N}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)$. Notice that $\mathcal{N}$ is reducible, it is already in reducible form (2.1). Nonetheless $0=\rho(\mathcal{N}) \in \sigma(\mathcal{N})$ and the vector $\mathbf{u}=\binom{1}{0}$ is a nonnegative eigenvector, not zero, which satisfies $\mathcal{N} \mathbf{u}=\left(\begin{array}{ll}0 & 1 \\ 0 & 0\end{array}\right)\binom{1}{0}=\binom{0}{0}=0\binom{1}{0}$. These observations are true also for general nonnegative matrices, as shown below.

Theorem 3.5. Let $A \geq 0$. Then $\rho(A) \in \sigma(A)$ and there exists a nonnegative vector $\boldsymbol{u} \geq 0$, such that $\boldsymbol{u} \neq \boldsymbol{O}$ and $A \boldsymbol{u}=\rho(A) \boldsymbol{u}$

To understand the proof of theorem 3.5, we first need to show that probability vectors are bounded in the $\|*\|_{2}$-norm.

Lemma 3.6. If $\boldsymbol{p}$ is a probability vector, then $\|\boldsymbol{p}\|_{2} \leq 1$.
Proof. p is a probability vector implies that $p_{i} \in[0,1]$ for all $i=1, \ldots, n$ and that $\|\mathbf{p}\|_{1}=\left|p_{1}\right|+\cdots+\left|p_{n}\right|=p_{1}+\cdots+p_{n}=1$. If some $p_{i}=1$ then all other $p_{j}=0, j \neq i$. In this case we have

$$
\|\mathbf{p}\|_{2}^{2}=\sum_{j=1}^{n}\left|p_{j}\right|^{2}=\sum_{j=1}^{n} p_{j}^{2}=\sum_{j \neq i} p_{j}^{2}+1^{2}=0+1=1
$$

Assume that no $p_{i}=1$, that is $p_{i} \in[0,1)$, then $p_{i}^{2}<p_{i}$ for all $i=1, \ldots, n$ and

$$
\|\mathbf{p}\|_{2}^{2}=\left|p_{1}\right|^{2}+\cdots+\left|p_{n}\right|^{2}<p_{1}+\cdots+p_{n}=1
$$

We conclude that $\|\mathbf{p}\|_{2} \leq 1$.
By Perron's theorem [3.1, the right Perron vector $\mathbf{p}$ of a positive matrix is a probability vector, since $\|\mathbf{p}\|_{1}=1$. Now we are ready for the proof of theorem 3.5. which uses some nice techniques from calculus in $\mathbb{R}^{n}$.

Proof of theorem 3.5. Consider the sequence of matrices $\left\{A_{k}=A+\frac{1}{k} E\right\}_{k=1}^{\infty}$ where $E$ is a matrix of all ones. Clearly $A_{k} \rightarrow A$ as $k \rightarrow \infty$. Also, $A_{k}>0$ for all $k=1,2, \ldots$, and so by Perron's theorem $0<\rho\left(A_{k}\right) \in \sigma\left(A_{k}\right)$ and there exists right Perron vectors $\mathbf{p}_{k}>0$ for all $k=1,2, \ldots$ By lemma 3.6 we know that $\left\|\mathbf{p}_{k}\right\|_{2} \leq 1$ for all $k=1,2, \ldots$ and so the sequence $\left\{\mathbf{p}_{k}\right\}_{k=1}^{\infty}$ is entirely contained in the unit sphere $\left\{\mathbf{x}:\|\mathbf{x}\|_{2} \leq 1\right\} \subseteq \mathbb{R}^{n}$ and therefore bounded. Then Bolzano-Weierstrass theorem assures that $\left\{\mathbf{p}_{k}\right\}_{k=1}^{\infty}$ has a convergent subsequence $\left\{\mathbf{p}_{k_{i}}\right\}_{i=1}^{\infty}$ converging to some vector $\mathbf{u} \in \mathbb{R}^{n}$ such that $\mathbf{u} \geq 0$, since all
$\mathbf{p}_{k_{i}}>0$, and $\mathbf{u} \neq \mathbf{0}$, since all $\left\|\mathbf{p}_{k_{i}}\right\|_{1}=1$ in the subsequence. Looking at the matrices in the sequence $\left\{A_{k}\right\}_{k=1}^{\infty}$ it is clear that $A_{1}>A_{2}>\cdots>A \geq 0$ and so by theorem 2.2 we have $\rho\left(A_{1}\right) \geq \rho\left(A_{2}\right) \geq \cdots \geq \rho(A)$ which tells us that the sequence $\left\{\rho\left(A_{k}\right)\right\}_{k=1}^{\infty}$ is a decreasing sequence bounded below by $\rho(A) \geq 0$. By the monotone convergence theorem, the sequence $\left\{\rho\left(A_{k}\right)\right\}_{k=1}^{\infty}$ converges to some limit $\rho^{*} \geq 0$ such that $\rho^{*} \geq \rho(A)$. Because subsequences of a convergent sequence converges to the same limit as the sequence, we also have $\rho\left(A_{k_{i}}\right) \rightarrow \rho^{*} \geq \rho(A)$ as $i \rightarrow \infty$, where $A_{k_{i}}$ is the positive matrix with the associated Perron vector $\mathbf{p}_{k_{i}}$ above, for all $i=1,2, \ldots$. For the same reason, we also have $A_{k_{i}} \rightarrow A$ as $i \rightarrow \infty$. So all together we have

$$
\begin{aligned}
A \mathbf{u} & =\left(\lim _{i \rightarrow \infty} A_{k_{i}}\right)\left(\lim _{i \rightarrow \infty} \mathbf{p}_{k_{i}}\right)=\lim _{i \rightarrow \infty} A_{k_{i}} \mathbf{p}_{k_{i}} \\
& =\lim _{i \rightarrow \infty} \rho\left(A_{k_{i}}\right) \mathbf{p}_{k_{i}}=\left(\lim _{i \rightarrow \infty} \rho\left(A_{k_{i}}\right)\right)\left(\lim _{i \rightarrow \infty} \mathbf{p}_{k_{i}}\right)=\rho^{*} \mathbf{u}
\end{aligned}
$$

where we use that the limit of a product is the same as the product of limits, which is true because all our limits exist finitely. We conclude that $\rho^{*} \in \sigma(A)$, so $\rho^{*} \leq \rho(A)$. From before we know $\rho^{*} \geq \rho(A)$ and so they must be equal. We have proved that $\rho^{*}=\rho(A) \in \sigma(A)$ and that there exists some $\mathbf{0} \neq \mathbf{u} \geq 0$ such that $A \mathbf{u}=\rho(A) \mathbf{u}$, which was to be shown.

If we instead consider $A^{T} \geq 0$, theorem 3.5 states that there exists some $\mathbf{w} \geq 0$ such that $\mathbf{w} \neq \mathbf{0}$ and $A^{T} \mathbf{w}=\rho\left(A^{T}\right) \mathbf{w}$. Transposing both sides and using the fact that $\rho(A)=\rho\left(A^{T}\right)$ we get $\mathbf{w}^{T} A=\rho(A) \mathbf{w}^{T}$ which shows that $\mathbf{w}^{T}$ is a left eigenvector of $A$ with the properties of the theorem. Further, the Collatz-Wielandt formula, see (3.1), also hold for all nonnegative matrices. For a proof of this, see Meyer [9, p. 670] or Shapiro [10, Corollary 17.7], which uses the same technique, with the sequence $\left\{A_{k}\right\}_{k=1}^{\infty}$, as above.

The theorem 3.5 and Collatz-Wielandt is as far as we can generalize the properties from Perron's theorem to all nonnegative matrices. Sadly, we have lost many of the desired properties, such as $\rho(A)>0$ and $\rho(A)$ being a simple eigenvalue. However, there is no reason to lose hope in Perron-Frobenius theory, because if we restrict ourselves to irreducible matrices we will see that some of the Perron-properties are restored. This is the result from Frobenius' work on the matter. We now present the version of what is called Perron-Frobenius' theorem or sometimes only Frobenius' theorem as it can be found in HornJohnson [5].

Theorem 3.7 (Perron-Frobenius' theorem). Let $A \geq 0$ be an irreducible matrix. Then
(i) $\rho(A)>0$
(ii) $\rho(A) \in \sigma(A)$ and $\rho(A)$ is simple
(iii) there exists a unique (right) eigenvector $\boldsymbol{p}$ of $A$, which can be taken positive, such that $A \boldsymbol{p}=\rho(A) \boldsymbol{p}$ and $\|\boldsymbol{p}\|_{1}=p_{1}+\cdots+p_{n}=1$
(iv) there exists a unique (left) eigenvector $\boldsymbol{q}^{T}$ of $A$, which can be taken positive, such that $\boldsymbol{q}^{T} A=\rho(A) \boldsymbol{q}^{T}$ and $\boldsymbol{q}^{T} \boldsymbol{p}=p_{1} q_{1}+\cdots+p_{n} q_{n}=1$

Comparing Perron-Frobenius' to Perron's theorem 3.1 we can see that the only things missing are items (v) and (vi) of Perron's, which states that $\rho(A)$ is the only eigenvalue with maximum modulus, and that $\left(\frac{1}{\rho(A)} A\right)^{m} \rightarrow \mathbf{p q}^{T}$ as $m \rightarrow \infty$. We have already seen examples of the first statement not being true for irreducible matrices, see matrices $A$ and $B$ in example 2.12 or matrix $A$ in example 2.25 . To illustrate that (vi) need not to be true for irreducible matrices, we consider the following example.

Example 3.8. Let $a_{12}, a_{23}, a_{34}, a_{41}>0$ and $A \geq 0$ be the following matrix

$$
A=\left(\begin{array}{cccc}
0 & a_{12} & 0 & 0 \\
0 & 0 & a_{23} & 0 \\
0 & 0 & 0 & a_{34} \\
a_{41} & 0 & 0 & 0
\end{array}\right)
$$

which has the graph $\mathcal{D}(A)$, see fig. 3.1, which is a 4-cycle. We can see that $\mathcal{D}(A)$ is strongly connected, thus $A$ is irreducible. Further, since all closed walks in $\mathcal{D}(A)$ is simply just the 4 -cycle or repeated walks around the cycle, the greatest common divisor of the lengths of all closed walks in $\mathcal{D}(A)$ is 4 , and so $A$ is an imprimtive matrix with index $k=4$. From theorem 2.22 we get that $A$ has



Figure 3.1: Left: $\mathcal{D}(A)$, right: $\sigma(A)$
four eigenvalues on its spectral circle and that these are exactly the $4^{\text {th }}$ roots of $\rho(A)^{4}$. Because $A$ is a $4 \times 4$ matrix it has only four eigenvalues (counted with multiplicities), and so all eigenvalues of $A$ lie on the spectral circle, see fig. 3.1. Let $\rho:=\rho(A)$. By Perron-Frobenius's theorem $0<\rho \in \sigma(A)$ and we can conclude that $\sigma(A)=\{\rho, i \rho,-\rho,-i \rho\}$. Since all eigenvalues are distinct they are all simple, and so if we bring $A$ to its Jordan canonical form, we get

$$
\left(\frac{1}{\rho} A\right)^{m}=T\left(\frac{1}{\rho} J\right)^{m} T^{-1}=T\left(\begin{array}{llll}
1 & & & \\
& i^{m} & & \\
& & (-1)^{m} & \\
& & & (-i)^{m}
\end{array}\right) T^{-1}
$$

which has no limit as $m \rightarrow \infty$ since $i^{m}$ and $(-1)^{m}$ does not converge to anything as $m \rightarrow \infty$. Hence, (vi) in Perron's theorem is not true for the irreducible matrix $A$ and the imprimitivity of $A$ is, as suspected, the reason behind this.

By definition 2.13, primitive matrices have only one eigenvalue of maximum modulus, and so item ( $v$ ) of Perron's Theorem is automatically satisfied. To prove that also (vi) is fulfilled, one does the same proof as for positive matrices; bringing the primitive matrix $A \geq 0$ to its Jordan canonical form will do the trick. We summarize in the following theorem, which is also by Frobenius.

Theorem 3.9 (Perron-Frobenius' theorem for primitive matrices). Let $A \geq 0$ be a primitive matrix. Then
(i) $\rho(A)>0$
(ii) $\rho(A) \in \sigma(A)$ and $\rho(A)$ is simple
(iii) there exists a unique (right) eigenvector $\boldsymbol{p}$ of $A$, which can be taken positive, such that $A \boldsymbol{p}=\rho(A) \boldsymbol{p}$ and $\|\boldsymbol{p}\|_{1}=p_{1}+\cdots+p_{n}=1$
(iv) there exists a unique (left) eigenvector $\boldsymbol{q}^{T}$ of $A$, which can be taken positive, such that $\boldsymbol{q}^{T} A=\rho(A) \boldsymbol{q}^{T}$ and $\boldsymbol{q}^{T} \boldsymbol{p}=p_{1} q_{1}+\cdots+p_{n} q_{n}=1$
(v) $|\lambda|<\rho(A)$ for all other eigenvalues $\lambda \in \sigma(A) \backslash\{\rho(A)\}$
(vi) $\lim _{m \rightarrow \infty}\left(\frac{1}{\rho(A)} A\right)^{m}=\boldsymbol{p} \boldsymbol{q}^{T}>0$

Note that the items are exactly the same as in Perron's theorem 3.1 and so in a "Perron-Frobenius sense" positive matrices and primitive matrices are equally well-behaved. Furthermore, as we noted already in section 2.4 , the power of a primitive matrix will eventually become positive. The following theorem is sometimes, for example in Meyer [9, called 'Frobenius test for primitivity'.

Theorem 3.10. $A \geq 0$ is primitive if and only if $A^{m}>0$ for some $m>0$
Proof. $(\Rightarrow): \lim _{m \rightarrow \infty}\left(\frac{1}{\rho(A)} A\right)^{m}>0$ so there exists some $m>0$ such that $\left(\frac{1}{\rho(A)} A\right)^{m}>0$ which implies $A^{m}>0$ since $\rho(A)>0$ by Perron-Frobenius' 3.9 .
$(\Leftarrow)$ : $A$ cannot be reducible since

$$
\left(\begin{array}{ll}
X & Y \\
\mathbf{0} & Z
\end{array}\right)^{m}=\left(\begin{array}{cc}
X^{m} & * \\
\mathbf{0} & Z^{m}
\end{array}\right)^{m} \ngtr 0
$$

for all $m>0$ and so $A$ is irreducible. By Perron's theorem 3.1 for positive matrices $\rho\left(A^{m}\right)$ is a simple eigenvalue of $A^{m}>0$ which is the only eigenvalue on the spectral circle. Equivalently, the only Jordan block corresponding to eigenvalue $\rho\left(A^{m}\right)$ is a $1 \times 1$, containing only $\rho\left(A^{m}\right)$, in the Jordan form of $A^{m}$. Now, suppose the eigenvalues of $A$ with maximum modulus are $\rho(A), \lambda_{1}, \ldots, \lambda_{k}$ for $k \geq 1$, where the spectral radius is one of them because of Perron-Frobenius' theorem 3.7. Because $\sigma\left(A^{m}\right)=\left\{\lambda^{m}: \lambda \in \sigma(A)\right\}$ then the numbers $\rho(A)^{m}, \lambda_{1}^{m} \ldots, \lambda_{k}^{m}$ must be the eigenvalues of the matrix $A^{m}$ with modulus $\rho\left(A^{m}\right)$. But we know that the only eigenvalue of $A^{m}$ with this property is $\rho\left(A^{m}\right)=\rho(A)^{m}$ itself. Thus $\rho(A)^{m}=\lambda_{1}^{m}=\cdots=\lambda_{k}^{m}$ and so in the Jordan form of $A^{m}$ we must have $1+k \geq 2$ blocks corresponding to the eigenvalue $\rho(A)^{m}$, which is a contradiction. Hence, only $\rho(A)$ is an eigenvalue of $A$ with maximum modulus, so $A$ is primitive.

Remark 3.11. One might wonder how large this $m$ must be for the matrix $A^{m}$ to become positive. As can be seen in Horn-Johnson [5, Chapter 8.5], the smallest such $m$, call it $\gamma(A)$ can be bounded above by the number $(n-1)^{2}+1$, so $A^{(n-1)^{2}+1}>0$. This is the best upper bound we can find if some $a_{i i}=0$. If, however exactly $d$ diagonal elements in $A$ are positive, then this upper bound on $\gamma(A)$ can be brought further down to $2 n-d-1$.

As an example of theorem 3.9, let us consider a primitive iterative system, which (among other things) we will study in depth in the following chapters. It illustrates how little one must know about the matrix of a system to apply Perron-Frobenius and thereof know very much about a systems asymptotic properties. The example is heavily inspired by Example 8.3.7 in Meyer [9, pp. 683-684].

Example 3.12. Let $\nu_{1}, \ldots, \nu_{6}$ be ponds of water and $w_{k}$ be the number of liters of water in pond $\nu_{k}$ for all $k=1, \ldots, 6$. Suppose that each day $m \geq 0$ water flows between the ponds according to the graph fig. 3.2, where vertices represent ponds and the arcs between distinct pair of vertices represent flow of water. The loops at vertices means that the water in the pond is somehow added to that
pond the next day. We can imagine a human measuring the water volume in that pond and then adding this the next day, perhaps with an appropriately large water hose.


Figure 3.2: Six ponds.
The matrix of the graph is

$$
A=\left(\begin{array}{llllll}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0
\end{array}\right)
$$

which has two positive diagonal elements ( $a_{11}=a_{44}=1>0$ ). We see in fig. 3.2 that $\mathcal{D}(A)$ is strongly connected, thus $A$ is irreducible, and so by theorem 2.20 $A$ is a primitive matrix. If we let $\mathbf{w}_{m}=\left(w_{1, m}, \ldots, w_{6, m}\right)$ be the vector with elements equal to the volume of water at each pond at the day $m \geq 0$, then $\mathbf{w}_{m+1}$ is determined via $\mathbf{w}_{m+1}=A \mathbf{w}_{m}$. We can assume that $\mathbf{w}_{0} \neq \mathbf{0}$, i.e. there is water in at least one pond when we start our study. Furthermore, define

$$
W_{k, m}:=\frac{w_{k, m}}{w_{1, m}+\cdots+w_{6, m}}=\frac{w_{k, m}}{\left\|\mathbf{w}_{m}\right\|_{1}}
$$

for $k=1, \ldots, 6$ as the proportion of the total volume of water at day $m$ in pond $\nu_{k}$. Gathering all the $W_{k, m}$ together we get that the vector

$$
\mathbf{W}_{m}:=\left(W_{1, m}, \ldots, W_{6, m}\right)^{T}=\frac{1}{\left\|\mathbf{w}_{m}\right\|_{1}}\left(w_{1, m}, \ldots, w_{6, m}\right)^{T}=\frac{\mathbf{w}_{m}}{\left\|\mathbf{w}_{m}\right\|_{1}}
$$

represents the water distribution at day $m$. We wish to calculate $\lim _{m \rightarrow \infty} \mathbf{W}_{m}$ (if it exists), i.e. the long-run water distribution in the system. Note that water gets added to the system daily, so the volume of water in each pond $w_{k, m}$ might
increase unbounded as time goes on. Therefore we are not interested in the asymptotic number of liters in each pond (which is $\lim _{m \rightarrow \infty} \mathbf{w}_{m}$ ), because this could be infinite, but instead the proportions of the total number of liters of water $W_{k, m}$ at each pond, which we will be finite, as we will see.

Now, since $\mathbf{w}_{m+1}=A \mathbf{w}_{m}$ for $m \geq 0$ we have, as usual, that $\lim _{m \rightarrow \infty} \mathbf{w}_{m}=$ $A^{m} \mathbf{w}_{0}$ for some initial conditions $\mathbf{w}_{0}$ at day 0 . Since $A$ is a primitive matrix we can apply Perron-Frobenius' theorem 3.9 to get that

$$
\lim _{m \rightarrow \infty} \frac{1}{\rho(A)^{m}} \mathbf{w}_{m}=\lim _{m \rightarrow \infty}\left(\frac{1}{\rho(A)} A\right)^{m} \mathbf{w}_{0}=\left(\mathbf{p q}^{T}\right) \mathbf{w}_{0}
$$

where $\mathbf{p}>0$ and $\mathbf{q}^{T}>0$ are the right and left Perron vectors of $A$. Moreover, we need to calculate the limit of $\left\|\mathbf{w}_{m}\right\|_{1} / \rho(A)$ to know the limit of $\mathbf{W}_{m}$ as $m \rightarrow \infty$. This is done in the following way

$$
\begin{aligned}
\lim _{m \rightarrow \infty} \frac{1}{\rho(A)^{m}}\left\|\mathbf{w}_{m}\right\|_{1} & =\lim _{m \rightarrow \infty} \frac{1}{\rho(A)^{m}}\left\|A^{m} \mathbf{w}_{0}\right\|_{1}=\lim _{m \rightarrow \infty}\left\|\left(\frac{1}{\rho(A)} A\right)^{m} \mathbf{w}_{0}\right\|_{1} \\
& =\left\|\left(\mathbf{p q}^{T}\right) \mathbf{w}_{0}\right\|_{1}=\left\|\mathbf{p}\left(\mathbf{q}^{T} \mathbf{w}_{0}\right)\right\|_{1}=\|\mathbf{p}\|_{1}\left|\mathbf{q}^{T} \mathbf{w}_{0}\right|= \\
& =\mathbf{q}^{T} \mathbf{w}_{0}>0
\end{aligned}
$$

which is a positive number since $\mathbf{q}^{T}>0$ and $\mathbf{w}_{0} \neq \mathbf{0}$. Notice that $\|\mathbf{p}\|_{1}=1$ so that term disappears. Now that we have everything we need and so we are ready to answer our question.

$$
\begin{aligned}
\lim _{m \rightarrow \infty} \mathbf{W}_{m} & =\lim _{m \rightarrow \infty} \frac{\mathbf{w}_{m}}{\left\|\mathbf{w}_{m}\right\|_{1}}=\lim _{m \rightarrow \infty} \frac{\mathbf{w}_{m} / \rho(A)^{m}}{\left\|\mathbf{w}_{m}\right\|_{1} / \rho(A)^{m}} \\
& =\frac{\lim _{m \rightarrow \infty} \frac{\mathbf{w}_{m}}{\rho(A)^{m}}}{\lim _{m \rightarrow \infty} \frac{\left\|\mathbf{w}_{m}\right\|_{1}}{\rho(A)^{m}}}=\frac{\left(\mathbf{p} \mathbf{q}^{T}\right) \mathbf{w}_{0}}{\mathbf{q}^{T} \mathbf{w}_{0}}=\frac{\mathbf{p}\left(\mathbf{q}^{T} \mathbf{w}_{0}\right)}{\mathbf{q}^{T} \mathbf{w}_{0}}=\mathbf{p}
\end{aligned}
$$

which is the Perron vector! Now, the only computation we need to do on the actual matrix $A$ is find its positive right Perron vector, which turns out to be

$$
\mathbf{p}=\frac{\varphi}{4 \varphi+3}\left(\begin{array}{c}
\varphi \\
1 \\
\varphi \\
1 \\
1 / \varphi \\
1
\end{array}\right) \quad \text { where } \quad \varphi=\frac{\sqrt{5}+1}{2} \quad \text { is the golden ratio. }
$$

We conclude that in the long run, the distribution of water amongst the six ponds will be about $0.276,0.171,0.276,0.106,0.171$ respectively. Worth mentioning is that the calculation of the eigenvector $\mathbf{p}$ can of course in itself be
cumbersome. However, the point here is that, by Perron-Frobenius', this is the only thing we need to calculate to answer our question, since we know the limit of the matrix beforehand. We can compare this to the procedure in the first example of chapter 1 Back then we had to calculate the entire spectrum of $A$ and all of its eigenvectors to be able to calculate the limit. The fact that Perron-Frobenius' theorem reduces the computations to one single eigenvector is perhaps one of the most elegant results of the theory and also why it is so useful in linear dynamical systems.

### 3.3 Proof of Perron-Frobenius' Theorem

In this section we will go through and understand the proof of Perron-Frobenius' theorem for irreducible matrices from the article A Geometric Proof of the Perron-Frobenius Theorem by Borobia and Trías [2]. For the sake of completeness we restate the theorem, as it can be found in the article.

Theorem 3.13 (Perron-Frobenius' theorem). Let $A \geq 0$ be an irreducible matrix. Then there exists a simple positive eigenvalue $\lambda>0$ of $A$ which has an associated positive eigenvector and which has the highest modulus of any other eigenvalue of $A$.

In the article, we find that a slightly different definition of irreducibility is being used. For us, the form of a reducible matrix, after simultaneous permutation of rows and columns, is $\left(\begin{array}{cc}X & Y \\ 0 & Z\end{array}\right)$ see (2.1), whereas Borobia and Trías uses the form $\left(\begin{array}{ll}Z & 0 \\ Y & X\end{array}\right)$. Notice however that we can convert one to the other using yet another simultaneous permutation matrix, namely $\left(\begin{array}{cc}0 & I \\ I & 0\end{array}\right)$, where the two $I$ are appropriately large identity matrices, because $\left(\begin{array}{cc}0 & I \\ I & 0\end{array}\right)\left(\begin{array}{cc}X & Y \\ 0 & Z\end{array}\right)\left(\begin{array}{cc}0 & I \\ I & 0\end{array}\right)=$ $\left(\begin{array}{ll}0 & I \\ I & 0\end{array}\right)\left(\begin{array}{ll}Y & X \\ Z & 0\end{array}\right)=\left(\begin{array}{ll}Z & 0 \\ Y & X\end{array}\right)$ and vice versa.

As we will see, the proof uses a result from topology called Brouwer's fixed point theorem, perhaps most often presented in functional analysis courses. For a function $f: \mathbb{X} \rightarrow \mathbb{X}$, a fixed point is a point $\mathbf{x} \in \mathbb{X}$ such that $f(\mathbf{x})=\mathbf{x}$. We do not need the most general version of Brouwer's theorem (on Banach spaces), but only its applications to certain subsets of $\mathbb{R}^{n}$ and continuous functions. The following version of the theorem can be found in [7].

Theorem 3.14 (Brouwer's fixed point theorem). Every continuous mapping $f: \mathbb{X} \rightarrow \mathbb{X}$ of a compact convex subset $\mathbb{X} \subseteq \mathbb{R}^{n}$ has a fixed point in $\mathbb{X}$.

Note that the theorem says nothing about the uniqueness of such fixed point. For example, if we let $f$ to be the identity mapping on $\mathbb{X}$ then, of course, it is continuous and every $\mathbf{x} \in \mathbb{X}$ is a fixed point. Recall that a subset $\mathbb{X} \subseteq \mathbb{R}^{n}$
is compact if it is both closed ( $\partial \mathbb{X} \subseteq \mathbb{X}$ ) and bounded. Furthermore, $\mathbb{X}$ is convex if for all $\mathbf{x}, \mathbf{y} \in \mathbb{X}$ and for all $\alpha \in(0,1)$ the line segment, called the convex combination, $\alpha \mathbf{x}+(1-\alpha) \mathbf{y} \in$ is also an element in the set $\mathbb{X}$. Finally, a continuous mapping is in this context usually defined to be the property that if $\mathbb{U} \subseteq \mathbb{X}$ is open then $f^{-1}(\mathbb{U}) \subseteq \mathbb{X}$ is also open. Linear transformations, which actions can be described in matrix form, are continuous mappings (perhaps one of the most simplest cases of such).

Let us warm up by understanding the following definition.
Definition 3.15 (Ray). A ray in the direction $\mathbf{v} \neq \mathbf{0}$ is the set $r:=r[\mathbf{v}]=$ $\left\{\alpha \mathbf{v} \in \mathbb{R}^{n}: \alpha>0\right\}$

A ray can be seen as 'half' of the linear span $[\mathbf{v}]=\{\alpha \mathbf{v}: \alpha \in \mathbb{R}\}$ for when $\alpha>0$. Below, left in fig. 3.3 we see an example in $\mathbb{R}^{2}$ where the vector $\mathbf{v}=(1,1)^{T}$. The ray $r$ is illustrated with the blue dashed line emanating from the origin stretching out towards infinity and the rest of the linear span $[\mathbf{v}]$ is the thin gray dashed line, which is not included in $r$. We can represent a ray $r$ by its direction $\mathbf{v}$, as we now have seen, but it is likewise possible to represent $r$ by its intersection with the unit hypersphere $\mathcal{S}^{n-1}:=\left\{\mathbf{x} \in \mathbb{R}^{n}:\|\mathbf{x}\|_{2}=1\right\}$. All rays will have an intersection with the hypersphere, because $\mathbf{v} \neq \mathbf{0}$ for all rays and this intersection will be unique, because the line $[\mathbf{v}] \subseteq \mathbb{R}^{n}$ intersects $\mathcal{S}^{n-1}$ at two points, one of which will have the position vector $\alpha \mathbf{v}$ for some $\alpha>0$. Returning to our example in $\mathbb{R}^{2}$ above, the intersection $r \cap \mathcal{S}^{1}$ is the point $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, as seen in the middle of fig. 3.3. We can use this point on $\mathcal{S}^{1}$ to identify the ray $r$.




Figure 3.3: Left: $r\left[(1,1)^{T}\right]$, middle: $r \cap \mathcal{S}^{1}$, right: $\mathcal{L}$.
Consequently, the set of all rays which lie in between two rays $r_{1}$ and $r_{2}$ can be represented as the corresponding set of intersection-points on $\mathcal{S}^{n-1}$. For example, if we let $r_{1}\left[(1,1)^{T}\right]$ and $r_{2}\left[(-1,1)^{T}\right]$ we get that all rays in between $r_{1}$ and $r_{2}$ can be identified by $\mathcal{L}:=\left\{(\cos \beta, \sin \beta): \beta \in\left[\frac{\pi}{4}, \frac{3 \pi}{4}\right]\right\} \subseteq \mathcal{S}^{1}$. Notice that $\mathcal{L}$ is equal to $\left\{\left(\cos \left(t \frac{\pi}{4}+(1-t) \frac{3 \pi}{4}\right), \sin \left(t \frac{\pi}{4}+(1-t) \frac{3 \pi}{4}\right): t \in[0,1]\right\}\right.$ which is the spherical convex hull of the two points $r_{1} \cap \mathcal{S}^{1}, r_{2} \cap \mathcal{S}^{1}$ on the circle $\mathcal{S}^{1}$,
as can be seen to the right in fig. 3.3. Moreover, the set $\mathcal{L} \subseteq \mathcal{S}^{1}$ is compact on the circle since its boundary $\partial \mathcal{L}$, the two points $r_{1,2} \cap \mathcal{S}^{1}$, lie in $\mathcal{L}$ and it is bounded. The same argument can be applied in higher dimensions and give rise to the following conclusion: the set of all rays which lie in between two rays can be identified with a compact convex set $\mathcal{L} \subseteq \mathcal{S}^{n-1}$. This conclusion is crucial because we need compactness and convexity to apply Brouwer's fixed point theorem.

We can let a matrix $A$ act on a ray $r$, which we will denote by $A(r)$, by defining $A(r):=A(r)[A \mathbf{v}]=\left\{\alpha A \mathbf{v} \in \mathbb{R}^{n}: \alpha>0\right\}$, i.e. the image of $r$ under $A$ is the ray with direction $A \mathbf{v}$. Note that $A(r)=r$ does not necessarily mean that $A \mathbf{v}=\mathbf{v}$ since for example $r[\mathbf{v}]=r[2 \mathbf{v}]$ and so if $A \mathbf{v}=2 \mathbf{v}$ then we also have $A(r)=r$. Instead we see that $A(r)=r$ if and only if $A \mathbf{v}=\lambda \mathbf{v}$ for some $\lambda>0$.

Now we are finally ready to begin the proof Perron-Frobenius' theorem 3.13. We will many times return to examples in $\mathbb{R}^{2}$ to help our understanding.

Proof of Perron-Frobenius' theorem 3.13. Part 1: First, we want to prove that there exists a positive eigenvalue $\lambda>0$ of $A$ and a corresponding positive eigenvector $\mathbf{v}>0$.

Let $\mathcal{R}_{+}:=\{r[\mathbf{v}]: \mathbf{v} \geq 0\}$ be all rays with a nonnegative direction vector $\mathbf{v} \geq 0$. In $\mathbb{R}^{2}$ these are the rays which lie in the first quadrant (including the axes), in $\mathbb{R}^{3}$ those in the first octant and so on.

We claim that $A(r) \neq \mathbf{0}$ for all $r \in \mathcal{R}_{+}$. Suppose that $\mathbf{v} \geq 0$ with one nonzero element (we know $\mathbf{v} \neq \mathbf{0}$ ). If $A \mathbf{v}=\mathbf{0}$, then $A$ must have (at least) one column of zeros, since $A \geq 0$. The following scheme motivates it further:

$$
\left(\begin{array}{cccc}
0 & * & \ldots & * \\
0 & * & \ldots & * \\
\vdots & \vdots & \ddots & \vdots \\
0 & * & \ldots & *
\end{array}\right)\left(\begin{array}{c}
* \\
0 \\
\vdots \\
0
\end{array}\right)=\left(\begin{array}{c}
0 \\
0 \\
\vdots \\
0
\end{array}\right)
$$

If $\mathbf{v}$ has more than one nonzero element, then that forces more columns of $A$ to be zeros. But this is impossible since if $A$ had any column of zeros at all, say $a_{i j}=0$ for all $i=1, \ldots, n$ then $\left(\nu_{i}, \nu_{j}\right) \notin \mathcal{E}(A)$ for all $i=1, \ldots, n$ and there are no in-going arcs to vertex $\nu_{j}$ in $\mathcal{D}(A)$, i.e. it cannot be reached from any other vertex. So $\mathcal{D}(A)$ is not strongly connected, and thus $A$ is reducible, which is a contradiction. We know from before that if $A \geq 0$ and $\mathbf{v} \geq 0$ then $A \mathbf{v} \geq 0$ and so together with our new finding $A \mathbf{v} \neq \mathbf{0}$ allows us to conclude that $A(r) \in \mathcal{R}_{+}$ for all $r \in \mathcal{R}_{+}$, that is

$$
\begin{equation*}
A\left(\mathcal{R}_{+}\right) \subseteq \mathcal{R}_{+} \tag{3.2}
\end{equation*}
$$

Furthermore, we claim that

$$
\begin{equation*}
A(r) \neq r \text { for all } r \in \partial \mathcal{R}_{+} \tag{3.3}
\end{equation*}
$$

That is, no ray in $\partial \mathcal{R}_{+}$is invariant under $A$. In $\mathbb{R}^{2}$ this means that neither $r\left[\left(x_{1}, 0\right)^{T}\right]$ nor $r\left[\left(0, x_{2}\right)^{T}\right]$ is invariant and in $\mathbb{R}^{3}$ neither $r\left[\left(x_{1}, x_{2}, 0\right)^{T}\right]$, $r\left[\left(x_{1}, 0, x_{3}\right)^{T}\right], r\left[\left(0, x_{2}, x_{3}\right)^{T}\right], r\left[\left(x_{1}, 0,0\right)^{T}\right], r\left[\left(0, x_{2}, 0\right)^{T}\right]$ nor $r\left[\left(0,0, x_{3}\right)^{T}\right]$ is invariant under $A$, where we assume that $x_{1}, x_{2}, x_{3}>0$. In general, $r[\mathbf{v}] \in \partial \mathcal{R}_{+}$ if $v_{i}=0$ for at least one $i=1, \ldots, n$. Suppose that $A(r)=r$ for such a ray $r \in \partial \mathcal{R}_{+}$and that $\mathbf{v}$ has exactly $0<k<n$ zero components. Then after a reordering of the basis in $\mathbb{R}^{n}$, if necessary, we can assume that the first $k$ components are zero. Remember that a reordering of the basis correspond to simultaneous permutation of the rows and columns of the matrix. The fact that $A \mathbf{v}=\lambda \mathbf{v}$ for $\lambda>0$ implies that also the $k$ first components in the vector $A \mathbf{v}$ are zero. But then $A$ has reducible form $\left(\begin{array}{cc}Z & 0 \\ Y & X\end{array}\right)$ where $Z$ is a $k \times k$ matrix, which is a contradiction. The following scheme motivates it further:

$$
\left(\begin{array}{cccccc}
z_{11} & \ldots & z_{1 k} & 0 & \ldots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
z_{k 1} & \ldots & z_{k k} & 0 & \ldots & 0 \\
& Y & & & X & \\
& & & & &
\end{array}\right)\left(\begin{array}{c}
0 \\
\vdots \\
0 \\
v_{k+1} \\
\vdots \\
v_{n}
\end{array}\right)=\left(\begin{array}{c}
0 \\
\vdots \\
0 \\
(A \mathbf{v})_{k+1} \\
\vdots \\
(A \mathbf{v})_{n}
\end{array}\right)
$$

Note that while no ray in $\partial \mathcal{R}_{+}$is invariant under $A$ it is possible for the whole set $\partial \mathcal{R}_{+}$to be invariant. Consider the example where $A=\left(\begin{array}{cc}0 & 1 \\ 1 & 0\end{array}\right)$, which has a 2 -cycle as its graph. $A$ is irreducible and $A\binom{1}{0}=\binom{0}{1}$ and $A\binom{0}{1}=\binom{1}{0}$ where the two vectors $\binom{1}{0},\binom{0}{1}$ are directions of the rays on the boundary of $\mathcal{R}_{+}$in $\mathbb{R}^{2}$. Here, the two rays are not invariant but the set $\partial \mathcal{R}_{+}$is.

To finalize part one, we now show that

$$
\begin{equation*}
A(r)=r \text { for some } r \in \mathcal{R}_{+} \backslash \partial \mathcal{R}_{+} \tag{3.4}
\end{equation*}
$$

That is, there is some positive ray $r$ which is held invariant under $A$. Let $\mathcal{L}:=\mathcal{R}_{+} \cap \mathcal{S}^{n-1}$ and identify every ray in $\mathcal{R}_{+}$with its corresponding intersection point with $\mathcal{S}^{n-1}$, see figure 3.4 for an illustration in $\mathbb{R}^{2}$.

Let $f$ be the mapping defined such that $f(\gamma)=r[A \mathbf{v}] \cap \mathcal{S}^{n-1}$ for all $\gamma=$ $r[\mathbf{v}] \cap \mathcal{S}^{n-1} \in \mathcal{L}$ where $\mathbf{v} \geq 0$. Since $A$ is a linear map, then $f$ is a continuous mapping, and from (3.2) we get that $f(\mathcal{L}) \subseteq \mathcal{L}$ because $A\left(\mathcal{R}_{+}\right) \subseteq \mathcal{R}_{+}$and so $f$ is really a mapping $f: \mathcal{L} \rightarrow \mathcal{L}$. The set $\mathcal{L}$ is not convex in $\mathbb{R}^{n}$ but its image $\psi(\mathcal{L})$ under a stereographic projection $\psi: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ from the hypersphere $\mathcal{S}^{n-1}$


Figure 3.4: $\mathcal{L}$ and $A(r)=r$.
to a hyperplane $\mathbb{E}^{n-1}$ is. Furthermore, $\mathcal{L} \subseteq \mathcal{S}^{n-1}$ and $\mathcal{S}^{n-1}$ is compact so $\mathcal{L}$ is compact and thus $\psi(\mathcal{L})$ can be taken compact as well, by necessary rotation of $\mathcal{S}^{n-1}$ if needed. Figure 3.5 illustrated the projection where $n=3$ and the purple part of $\mathcal{S}^{2}$, rotated to the bottom right, is $\mathcal{L}$. In the figure we see that the blue set $\psi(\mathcal{L})$ is convex and compact in $\mathbb{R}^{n}$.


Figure 3.5: Stereographic projection of $\mathcal{L}$
We know that $\psi$ is a homeomorphism, i.e. it is bijective, continuous and has a continuous inverse $\psi^{-1}$, so the composition $\psi \circ f \circ \psi^{-1}: \psi(\mathcal{L}) \rightarrow \psi(\mathcal{L})$ is a continuous mapping on a compact convex subset of $\mathbb{R}^{n}$. By Brouwer's fixed point theorem 3.14 there exists a $\beta:=\psi(\gamma) \in \psi(\mathcal{L})$, for some $\gamma \in \mathcal{L}$, such that

$$
\psi \circ f \circ \psi^{-1}(\beta)=\psi\left(f\left(\psi^{-1}(\beta)\right)\right)=\beta
$$

Taking the inverse of $\psi$ on both sides and expressing $\beta$ as $\psi(\gamma)$ yields

$$
f\left(\psi^{-1}(\psi(\gamma))\right)=\psi^{-1}(\psi(\gamma)) \quad \Leftrightarrow \quad f(\gamma)=\gamma
$$

Hence there exists a $\gamma=r[\mathbf{v}] \in \mathcal{L}$ such that $f(\gamma)=\gamma$, that is $r[A \mathbf{v}] \cap \mathcal{S}^{n-1}=$ $r[\mathbf{v}] \cap \mathcal{S}^{n-1}$ and so $A(r)=r$ where $r=r[\mathbf{v}]$ thanks to the one-to-one correspondence of rays in $\mathcal{R}_{+}$with the intersection points in $\mathcal{L}$ and projected points in $\psi(\mathcal{L})$. Because of (3.3), this invariant ray $r$ cannot lie in $\partial \mathcal{R}_{+}$, and so it must be strictly positive. The positive direction vector $\mathbf{v}>0$ of $r$ is hence an eigenvector of $A$ corresponding to some positive eigenvalue $\lambda>0$.

Part 2: Secondly, we want to prove that $\lambda>0$ is simple, that is

$$
\operatorname{geomult}_{A}(\lambda)=1 \quad \text { and } \quad \operatorname{algmult}_{A}(\lambda)=1
$$

Suppose that geomult $_{A}(\lambda)>1$, then there exists another eigenvector $\mathbf{u} \in$ $\mathbb{R}^{n} \backslash\{\mathbf{0}\}$ of $\lambda$ which is linearly independent to $\mathbf{v}>0$. Consider the plane $\Pi:=[\mathbf{v}, \mathbf{u}]$ spanned by the two vectors. We claim that there exists $\mathbf{w} \in \Pi$ such that $r[\mathbf{w}] \in \partial \mathcal{R}_{+}$. We construct this vector in the following way: let $\mathbf{w}:=t \mathbf{v}-\mathbf{u}$ for some sufficiently large $t>0$ such that $\mathbf{w}>0$. Now, $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, therefore $\mathbf{w} \in \Pi$ for all $t \in \mathbb{R}$. Start to decrease $t$ until (at least) one of the components in $\mathbf{w}$ becomes 0 and the remaining components are still positive. For this new value of $t, \mathbf{w}$ is a nonnegative vector with (at least) one zero component, thus $r[\mathbf{w}] \in \partial \mathcal{R}_{+}$. Additionally, $\mathbf{w}$ is an eigenvector of $\lambda>0$, because $A \mathbf{w}=A(t \mathbf{v}-\mathbf{u})=t \lambda \mathbf{v}-\lambda \mathbf{u}=\lambda \mathbf{w}$, so $A(r[\mathbf{w}])=r[\mathbf{w}]$ which contradicts (3.3). We conclude that geomult $_{A}(\lambda)=1$.

Now, suppose that $\operatorname{algmult}_{A}(\lambda)>1$. Because $\operatorname{geomult}_{A}(\lambda)=1$ we know that there must exist at least one generalized eigenvector $\mathbf{u}$ of $A$ corresponding to $\lambda$ which together with $\mathbf{v}$ (and potentially other generalized eigenvectors of $\lambda$ ) form a string basis. We can assume, after reordering if needed, that $\mathbf{u}$ is the first generalized eigenvector and so the Jordan form of $A$ has the following structure:

$$
A=T J T^{-1}=\left(\begin{array}{ccc}
\mid & \mid & \\
\mathbf{v} & \mathbf{u} & \ldots \\
\mid & \mid &
\end{array}\right)\left(\begin{array}{ccc}
\lambda & 1 & \ldots \\
0 & \lambda & \ldots \\
\vdots & \vdots & \ddots
\end{array}\right)\left(\begin{array}{ccc}
\mid & \mid & \\
\mathbf{v} & \mathbf{u} & \ldots \\
\mid & \mid &
\end{array}\right)^{-1}
$$

Remember that in such a string basis, we have $\cdots \xrightarrow{\mathcal{N}} \mathbf{u} \xrightarrow{\mathcal{N}} \mathbf{v} \xrightarrow{\mathcal{N}} \mathbf{0}$ with $\mathcal{N}=A-\lambda I$ and so we can scale the vector $\mathbf{u}$ such that $(A-\lambda I) \mathbf{u}=\mathbf{v}$. Rearranging this gives $A \mathbf{u}=\lambda \mathbf{u}+\mathbf{v}$. Let $\mathbf{w}$ be defined as above, such that $\mathbf{w} \geq 0$ and some $w_{k}=0$ for at least one $k=1, \ldots, n$. Then

$$
A \mathbf{w}=A(t \mathbf{v}-\mathbf{u})=t A \mathbf{v}-A \mathbf{u}=t \lambda \mathbf{v}-(\lambda \mathbf{u}+\mathbf{v})=\lambda(t \mathbf{v}-\mathbf{u})-\mathbf{v}=\lambda \mathbf{w}-\mathbf{v}
$$

but this is a contradiction because $A \geq 0$ and $\mathbf{w} \geq 0$ so $A \mathbf{w} \geq 0$ whereas $w_{k}=0$ and $v_{k}>0$ so $(\lambda \mathbf{w}-\mathbf{v})_{k}<0$ so $\lambda \mathbf{w}-\mathbf{v} \nsupseteq 0$. We conclude that $\operatorname{algmult}_{A}(\lambda)=1$.

Part 3: In the last part of the proof we will show that $\lambda \geq|\mu|$ for all eigenvalues $\mu \in \sigma(A)$. Suppose (towards a contradiction) that there exists some eigenvalue $\mu$ such that $|\mu|>\lambda$ and let $\mathbf{u}$ be the corresponding eigenvector. Note that $\mu \neq 0$ since $\lambda>0$. Let $r=r[\mathbf{v}]$ be the positive invariant ray corresponding to $\lambda>0$. We get two cases, depending on if $\mu \in \mathbb{R}$ or $\mu \in \mathbb{C} \backslash \mathbb{R}$. We deal with the first case in two sub-cases.
(Case 1: $\mu>0$ ) Since $\mu$ is real, we know that $\mathbf{u} \in \mathbb{R}^{n}$. First, suppose that both $r[\mathbf{u}] \notin \mathcal{R}_{+}$, and $r[-\mathbf{u}] \notin \mathcal{R}_{+}$, that is both $\mathbf{u}$ and $-\mathbf{u}$ are not nonnegative. Since $\mu$ is the dominant eigenvalue, all vectors will become gradually more parallel to $\mathbf{u}$ by repeated application of $A$. We see this because if $\mathbf{x}_{m}=A \mathbf{x}_{m-1}$ for some $\mathbf{x}_{0}$ then $\mathbf{x}_{m} \approx c \mu^{m} \mathbf{u}$, for $c \in \mathbb{R}$ constant, when $m$ is sufficiently large. Note that $\mu^{m}>0$ so the sign of $c$, which is determined by the initial vector $\mathbf{x}_{0}$, decides if $\mathbf{x}_{m}$ becomes parallel (with equal sign) to $\mathbf{u}$ or $-\mathbf{u}$. Now if we take $\mathbf{x}_{0} \neq \mathbf{v}$ such that $r\left[\mathbf{x}_{0}\right] \in \mathcal{R}_{+}$then $r\left[\mathbf{x}_{m}\right] \approx r[\mathbf{u}]$ or $r\left[\mathbf{x}_{m}\right] \approx r[-\mathbf{u}]$ for some $m$. Thus, $A^{m}\left(r\left[\mathbf{x}_{0}\right]\right)$ must eventually leave $\mathcal{R}_{+}$for large $m$, which contradicts (3.2). In the figure 3.6 we see an illustration in $\mathbb{R}^{2}$ where the black line represent a choice of $r\left[\mathbf{x}_{0}\right]$.


Figure 3.6: (Case 1: $\mu>0$ ) and $r[ \pm \mathbf{u}] \notin \mathcal{R}_{+}$.
Now assume, on the other hand, that $r[\mathbf{u}]$ lies in $\mathcal{R}_{+}$, then $r[-\mathbf{u}] \notin \mathcal{R}_{+}$(if instead $r[-\mathbf{u}] \in \mathcal{R}_{+}$then simply substitute $\mathbf{u}$ for $-\mathbf{u}$ below). We consider again $\mathbf{x}_{m}=A \mathbf{x}_{m-1}$ which we know will have the general formula

$$
\mathbf{x}_{m}=c_{1} \mu^{m} \mathbf{u}+c_{2} \lambda^{m} \mathbf{v}+\sum_{i} c_{i} \eta_{i}^{m} \mathbf{y}_{i} \quad \text { where } \quad\left|\eta_{i}\right|<|\mu|
$$

This time we will start with $\mathbf{x}_{0}$ in the plane $[\mathbf{u}, \mathbf{v}]$ such that $\mathbf{x}_{0}=\alpha \mathbf{u}+\beta \mathbf{v} \geq 0$ for some well chosen $\alpha<0$ and $\beta>0$. Plugging $m=0$ into the closed form solution we get that $c_{1}=\alpha, c_{2}=\beta$ and all other $c_{i}=0$, so $\mathbf{x}_{m}=\alpha \mu^{m} \mathbf{u}+\beta \lambda^{m} \mathbf{v}$. Since $\mu$ is the dominant eigenvalue we have $\mathbf{x}_{m} \approx \alpha \mu^{m} \mathbf{u}$ for large $m$, but because $\alpha<0$ and $\mu>0$ this means that $r\left[\mathbf{x}_{m}\right] \approx r[-\mathbf{u}] \notin \mathcal{R}_{+}$for such $m$, which again contradicts 3.2 since $r\left[\mathbf{x}_{0}\right] \in \mathcal{R}_{+}$. The figure 3.7 illustrates this in $\mathbb{R}^{3}$.
(Case 1: $\mu<0$ ) If $\mu<0$ then the asymptotic behaviour of $\mathbf{x}_{m}=A \mathbf{x}_{m-1}$ will be the following: $\mathbf{x}_{m} \approx c(-1)^{m}(-\mu)^{m} \mathbf{u}$ so the ray $r\left[\mathbf{x}_{m}\right]$ will jump between $r[\mathbf{u}]$ and $r[-\mathbf{u}]$ when $m \rightarrow \infty$. To avoid this behaviour, and to be able to conclude that all non-invariant rays are attracted to one unique ray, we consider the action of $A^{2}$. Note that $\left(\mu^{2}, \mathbf{u}\right)$ will be an eigenpair of $A^{2}$, for $A^{2} \mathbf{u}=A(A(\mathbf{u}))=$


Figure 3.7: (Case 1: $\mu>0)$ and $r[\mathbf{u}] \in \mathcal{R}_{+}$.
$A(\mu \mathbf{u})=\mu^{2} \mathbf{u}$ and since $\mu<0$ we get that $\mu^{2}>0$. Thus, we consider the system $\mathbf{x}_{m}=A^{2} \mathbf{x}_{m-1}$ where $\mathbf{x}_{m} \approx c \mu^{2 m} \mathbf{u}$, for large $m$, which will drag some $r\left[\mathbf{x}_{0}\right] \in \mathcal{R}_{+}$towards either one of $r[\mathbf{u}]$ or $r[-\mathbf{u}]$ (which one is determined by the sign of the constant $c$, which in turn is determined by $\mathbf{x}_{0}$ ) in the same way as the two instances of (Case 1: $\mu>0$ ). The contradiction will be the same since $A^{2 m}\left(r\left[\mathbf{x}_{0}\right]\right) \in \mathcal{R}_{+}$for such $r\left[\mathbf{x}_{0}\right]$ which is true again because of (3.2).
(Case 2: $\mu$ complex) If $\mu \in \mathbb{C} \backslash \mathbb{R}$ we write $\mu:=|\mu| e^{i \varphi}$ where $\varphi \in \mathbb{R} \backslash\{0\}$. Since $A$ is a matrix with real entries, it has a secular polynomial $p_{A}$ with real coefficients so then also the complex conjugate of $\mu$ will be a zero in $p_{A}$. We conclude that $\bar{\mu}=|\mu| e^{-i \varphi} \in \sigma(A)$ which has the same modulus as $\mu$. The corresponding eigenvectors, call them $\mathbf{u}$ and $\overline{\mathbf{u}}$, will be complex vectors, which can be decomposed into a real and an imaginary part. We write $\mathbf{u}=\Re(\mathbf{u})+$ $i \Im(\mathbf{u})=: \mathbf{w}+i \mathbf{y}$ and $\bar{u}=\mathbf{w}-i \mathbf{y}$ where these vectors $\mathbf{w}, \mathbf{y} \in \mathbb{R}^{n}$. Just as before we set up the system $\mathbf{x}_{m}=A \mathbf{x}_{m-1}$ for some $\mathbf{x}_{0}$ and note that if the initial vector $\mathbf{x}_{0} \in \mathbb{R}^{n}$ then $A^{m} \mathbf{x}_{0} \in \mathbb{R}^{n}$ for all $m$, since $A$ is a real matrix. In other words, by choosing a real initial vector we are guaranteed to get a real solution to the system. The vector $\mathbf{x}_{m}$ will become more and more parallel to $c_{0} \mu^{m} \mathbf{u}+c_{1} \bar{\mu} \overline{\mathbf{u}}$, $c_{0}, c_{1} \in \mathbb{C}$ constants, because both $\mu$ and $\bar{\mu}$ are dominant eigenvalues. We do the following computation

$$
\begin{align*}
& c_{0} \mu^{m} \mathbf{u}+c_{1} \bar{\mu}^{m} \overline{\mathbf{u}}=c_{0}\left(|\mu| e^{i \varphi}\right)^{m} \mathbf{u}+c_{1}\left(|\mu| e^{-i \varphi}\right)^{m} \overline{\mathbf{u}}=|\mu|^{m}\left(c_{0} e^{i \varphi m} \mathbf{u}+c_{1} e^{-i \varphi m} \overline{\mathbf{u}}\right) \\
& \quad=|\mu|^{m}\left[c_{0}(\cos \varphi m+i \sin \varphi m)(\mathbf{w}+i \mathbf{y})+c_{1}(\cos \varphi m-i \sin \varphi m)(\mathbf{w}-i \mathbf{y})\right] \\
& \quad=|\mu|^{m}[\underbrace{\left(c_{0}+c_{1}\right)}_{=: K_{1}}(\mathbf{w} \cos \varphi m-\mathbf{y} \sin \varphi m)+\underbrace{i\left(c_{0}-c_{1}\right)}_{=: K_{2}}(\mathbf{w} \sin \varphi m+\mathbf{y} \cos \varphi m)] \tag{3.5}
\end{align*}
$$

where the two constants $K_{1,2}$ will be real if $\mathbf{x}_{0}$ is a real vector. And in that
case the whole expression is real and it lies in the plane spanned by the two (real) vectors $\mathbf{w}$ and $\mathbf{y}$. Call this plane $\mathbb{P}:=[\mathbf{w}, \mathbf{y}]$. We now claim that $\mathbb{P}$ does not contain any rays in $\mathcal{R}_{+}$. Suppose that it did, then, similarly to what we have seen before, these rays lying in the plane $\mathbb{P} \subseteq \mathbb{R}^{2}$ can be identified with their intersection with $\mathcal{S}^{1}$. These intersection-points is an $\operatorname{arc} \mathcal{L}$ of $\mathcal{S}^{1}$, as illustrated in 3.8. Because of the trigonometric functions in (3.5) we see that


Figure 3.8: $\mathcal{S}^{1} \subseteq \mathbb{P}$ intersecting $\mathcal{R}_{+}$.
the action of $A$ for rays already lying in $\mathbb{P}$ is some rotation (scaling with $|\mu|>0$ does not change a ray). In particular, $A$ rotates the circle $S^{1}$ lying in $\mathbb{P}$ with some angle (depending on the value of $\varphi$ ). Thus the image of the set of rays with intersection-points in $\mathcal{L}$ under $A$ cannot lie entirely in $\mathcal{R}_{+}$anymore, since they have been rotated out of $\mathcal{R}_{+}$. This contradicts 3.2 which states that $A\left(\mathcal{R}_{+}\right) \subseteq \mathcal{R}_{+}$. Thus $\mathbb{P}$ does not contain any rays in $\mathcal{R}_{+}$.

Finally we will deduce a contradiction similar to the one in (Case 1) by using the fact that rays get attracted towards $\mathbb{P}$ (outside of $\mathcal{R}_{+}$) after repeated applications of $A$. First, consider the 3-dimensional space $\mathbb{E}^{3}:=\mathbb{P} \oplus[\mathbf{v}]$ where $\mathbf{v}>0$ is the direction of $r$. Then choose some $\mathbf{x}_{0} \in \mathbb{E}^{3}$ such that $r\left[\mathbf{x}_{0}\right] \in \mathcal{R}_{+}$ and $\mathbf{x}_{0}=\mathbf{v}+\mathbf{w}$ for some (perhaps arbitrarily small vector) $\mathbf{w} \in \mathbb{P} \backslash\{\mathbf{0}\}$. Note that we must choose $\mathbf{w} \nsupseteq 0$ such that $\mathbf{v}+\mathbf{w}$ is still a positive vector. Then

$$
A^{m}\left(\frac{\mathbf{x}}{|\mu|^{m}}\right)=\overbrace{\frac{\lambda^{m}}{|\mu|^{m}}}^{\mathrm{Co}_{0}} \mathbf{v}+\overbrace{\frac{1}{|\mu|^{m}} A^{m}(\mathbf{w})}^{\in \mathbb{P}}
$$

when $m \rightarrow \infty$. Note that the second term above is in $\mathbb{P}$, because by (3.5) vectors in $\mathbb{P}$ stay in $\mathbb{P}$ under application of $A$. So we can make $A^{m}\left(r\left[\mathbf{x}_{0}\right]\right)$ as close to $\mathbb{P}$ (which is not in $\mathcal{R}_{+}$) as we want and thus every positive ray leaves $\mathcal{R}_{+}$after
a number of applications of $A$, which again contradicts 3.2. In fig. 3.9 we see how $r\left[\mathbf{x}_{0}\right]$ gets repulsed out of $\mathcal{R}_{+}$.


Figure 3.9: $r\left[\mathbf{x}_{0}\right]$ gets retracted to $\mathbb{P}$ (outside of $\mathcal{R}_{+}$).
To summarize; in part one we proved that there exists a positive eigenvalue $\lambda>0$, which has a corresponding positive eigenvector $\mathbf{v}>0$. In part two we showed that $\lambda$ is simple and finally, in part three, that $\lambda \geq|\mu|$ for all $\mu \in \sigma(A)$. Hence, we have proved everything in theorem 3.13 .

Remark 3.16. The conventional way of proving Perron-Frobenius' in most literature on linear algebra is to first prove Perron's theorem and then finding some clever ways of applying Perron's in the proof of Perron-Frobenius', like we did in the proof of theorem 3.5 with the set $\left\{A_{k}\right\}_{k=1}^{\infty}$. These proofs are not necessarily shorter than the one we have presented, since they are often divided into many lemmas which are proved over whole chapters, but often they are a bit more comprehensible. Still, this proof uses some pretty geometry and is a good exercise in proof-techniques.

## Chapter 4

## Linear Dynamical Systems

In this chapter we will take a closer look at the world of linear dynamical systems. In the first section we will introduce, or perhaps remind the reader about, some key concepts and in the second section we consider a bigger example, the Leslie model. We will then bring with us the theory on dynamical systems to the proceeding chapter, where we narrow it down to one specific type, namely the Markov chains.

### 4.1 Introduction

The state of a linear dynamical system with constant coefficients at any given time can be represented by a vector in $\mathbb{R}^{n}$ called the state vector. We consider here both discrete time and continuous time systems.

We model a discrete time system as $\mathbf{x}_{m+1}=A \mathbf{x}_{m}+\mathbf{b}$ where $\mathbf{x}_{m}$ is the state vector at time $m \geq 0$ and $A \in \mathbb{M}_{n}(\mathbb{R}), \mathbf{b} \in \mathbb{R}^{n}$ are both constant. Sometimes, one refers to the model also as an iterative system or a recursive sequence in $\mathbb{R}^{n}$. If we restrict ourselves to integer states $\mathbb{Z}^{n}$ we recognize this model as a system of difference equations with a constant right hand side. Let us look at this famous example of a discrete time linear dynamical system.

Example 4.1 (Fibonacci numbers). Consider the famous Fibonacci numbers

$$
0,1,2,3,5,8,13,21,34,55,89,144 \ldots
$$

named after the Italian mathematician Leonardo Bonacci who introduced this sequence in 1200. Let $f_{m}$ be the $m^{t h}$ Fibonacci number, then $f_{m+1}=f_{m}+f_{m-1}$ for $m \geq 1$ where we let $f_{0}=0$ and $f_{1}=1$. We define a state vector in the
following way $\mathbf{x}_{m}:=\left(f_{m}, f_{m+1}\right)^{T}$ and use the recursive definition on $f_{m+1}$ to express $\mathbf{x}_{m+1}$ in terms of $\mathbf{x}_{m}$ as such

$$
\mathbf{x}_{m+1}=\binom{f_{m+1}}{f_{m+2}}=\binom{f_{m+1}}{f_{m}+f_{m+1}}=\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right)\binom{f_{m}}{f_{m+1}}=\left(\begin{array}{ll}
0 & 1 \\
1 & 1
\end{array}\right) \mathbf{x}_{m}
$$

and we see that this has the form $\mathbf{b}=\binom{0}{0}$ and $A=\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right)$.
Generally, we can convert any $n^{t h}$ order homogeneous difference equation with constant coefficients

$$
\begin{equation*}
f_{m+n}=a_{n-1} f_{m+n-1}+\cdots+a_{1} f_{m+1}+a_{0} f_{m} \tag{4.1}
\end{equation*}
$$

into matrix form by defining the state vector to be $\mathbf{x}_{m}:=\left(f_{m}, \ldots, f_{m+n-1}\right)^{T}$, because then

$$
\begin{align*}
\mathbf{x}_{m+1} & =\left(\begin{array}{c}
f_{m+1} \\
f_{m+2} \\
\vdots \\
f_{m+n}
\end{array}\right)=\left(\begin{array}{c}
f_{m+1} \\
f_{m+2} \\
\vdots \\
a_{0} f_{m}+\cdots+a_{n-1} f_{m+n-1}
\end{array}\right) \\
& =\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{0} & a_{1} & a_{2} & \ldots & a_{n-1}
\end{array}\right)\left(\begin{array}{c}
f_{m} \\
f_{m+1} \\
\vdots \\
f_{m+n-1}
\end{array}\right)=A \mathbf{x}_{m}+\mathbf{0} . \tag{4.2}
\end{align*}
$$

As we know, a homogeneous difference equation can be solved by finding the roots of the characteristic polynomial $\lambda^{n}-a_{n-1} \lambda^{n-1}-\cdots-a_{1} \lambda-a_{0}$. This polynomial is the same as the secular polynomial for $A$, which can be seen by expanding $\operatorname{det}(A-\lambda I)$ along the last row. The solution $f_{m}$ to the difference equation (4.1), if all roots $\lambda_{k}$ are distinct and nonzero, is $f_{m}=\sum_{k=1}^{n} c_{k} \lambda_{k}^{m}$, which can be seen in the solution $\mathbf{x}_{m}=\sum_{k=1}^{n} c_{k} \lambda_{k}^{m} \mathbf{v}_{k}$ to $\mathbf{x}_{m+1}=A \mathbf{x}_{m}$ where $\lambda_{k}$ are the eigenvalues of $A$ and $\mathbf{v}_{k}$ their eigenvectors. This corresponds to $A$ being diagonalizable and writing $A=T D T^{-1}$ for a diagonal matrix $D$ with eigenvalues as elements and a change to eigenbasis via $T$. If some $\operatorname{algmult}_{A}(\lambda)>1$, then we instead bring $A$ to its Jordan form and in this case polynomials of $m$ will appear in the solution. For a full walk-through of solving difference equations in $\mathbb{Z}^{n}$, see for example Izquierdo [6].

Example 4.2 (Continuation of Fibonacci numbers 4.1). The characteristic polynomial for the Fibonacci sequence is $\lambda^{2}-\lambda-1$ which has the irrational roots $\lambda_{1}=\frac{1+\sqrt{5}}{2}$ and $\lambda_{2}=\frac{1-\sqrt{5}}{2}$; the golden ratio $\varphi$ and its negative inverse
(conjugate) $-\varphi^{-1}=: \psi$. We find the solution to be $f_{m}=c_{1} \varphi^{m}+c_{2} \psi^{m}$ which together with $f_{0}=0, f_{1}=1$ gives us the beautiful formula

$$
f_{m}=\frac{\varphi^{m}-\psi^{m}}{\sqrt{5}}
$$

for computing the $m^{\text {th }}$ Fibonacci number. Notice that $f_{m} \rightarrow \infty$ as $m \rightarrow \infty$ and also that the spectral radius of the matrix is $\varphi>1$.

Remember that the asymptotic behaviour of the discrete time system is determined by $\lim _{m \rightarrow \infty} \mathbf{x}_{m}=\lim _{m \rightarrow \infty} A^{m} \mathbf{x}_{0}$. The Jordan form of $A$ reveals whether this limit exists or not. A formal proof of the following theorem can be found in Meyer [9, p.629-630]

Theorem 4.3. Let $A \in \mathbb{M}_{n}(\mathbb{C})$, then $\lim _{m \rightarrow \infty} A^{m}$ exists if and only if either

* $\rho(A)<1, \quad$ or else
* $\rho(A)=1 \in \sigma(A)$ is the only eigenvalue on $|z|=1$ and $\operatorname{algmult}_{A}(1)=$ geomult $_{A}(1)$, that is $\rho(A)$ is semisimple.

If $\rho(A)>1$, then $A^{m}$ grows without bound, as for the Fibonacci sequence in example4.2. Also, if there are more than one eigenvalue lying on $|z|=1$, in the case $\rho(A)=1$, then the limit of $A^{m}$ when $m \rightarrow \infty$ does not exist, however the limit of $\frac{I+A+A^{2}+\cdots+A^{m}}{m}$ when $m \rightarrow \infty$ exists finitely. This is called the Cesáro limit, which we will return to when we look at imprimitive Markov chains.

In continuous time, we use the following model $\mathbf{x}^{\prime}(t)=A \mathbf{x}(t)$ where we treat each component $x_{i}(t), i=1, \ldots, n$ as a differentiable function of the variable $t \in \mathbb{R}$. We recognize this as a system of differential equations. Just as for difference equations, every $n^{t h}$ order homogeneous differential equation with constant coefficients can be written in matrix form by the following procedure. If

$$
\begin{equation*}
f^{(n)}(t)=a_{n-1} f^{(n-1)}(t)+\cdots+a_{1} f^{\prime}(t)+a_{0} f(t) \tag{4.3}
\end{equation*}
$$

where $f \in \mathcal{C}^{n}$ is a function of $t \in \mathbb{R}$ and all $a_{i} \in \mathbb{R}$ are constants, then by letting $x_{k}(t)=f^{(k)}(t)$, for all $k=1, \ldots, n$, be the components in the state vector $\mathbf{x}(t)$ we can see that

$$
\begin{cases}x_{k}^{\prime}(t)=\left(f^{(k)}(t)\right)^{\prime}=f^{(k+1)}(t)=x_{k+1}^{\prime}(t), & k=1, \ldots, n-1 \\ x_{n}^{\prime}(t)=\left(f^{n}(t)\right)^{\prime}=a_{n-1} f^{(n-1)}(t)+\cdots+a_{0} f(t) & \end{cases}
$$

and so if we let $A$ to be the same as in 4.2 we get that $\mathbf{x}^{\prime}(t)=A \mathbf{x}(t)$. Similarly to the discrete case, if all eigenvalues of $A$ are distinct and nonzero we have the solution function $f(t)=\sum_{k=1}^{n} c_{k} e^{\lambda_{k} t}$ to the differential equation 4.3.

Example 4.4 (Fibonacci Functions). Consider the second order differential equation $f^{\prime \prime}(t)=f^{\prime}(t)+f(t)$ with initial conditions $f(0)=0$ and $f^{\prime}(0)=1$ and note that this function $f(t)$ has the same property as Fibonacci numbers, but in terms of its derivatives. By letting $\mathbf{x}(t)=\left(f(t), f^{\prime}(t)\right)^{T}$ we can write the equation as $\mathbf{x}^{\prime}(t)=\left(\begin{array}{ll}0 & 1 \\ 1 & 1\end{array}\right) \mathbf{x}(t)$. The matrix is the same as for the Fibonacci sequence and so with eigenvalues $\varphi$ and $\psi$ we get $f(t)=c_{0} e^{\varphi t}+c_{1} e^{\psi}$ which together with initial conditions yields

$$
f(t)=\frac{e^{\varphi t}-e^{\psi t}}{\sqrt{5}}, \quad t \in \mathbb{R}
$$

Notice the similarity between $f(t)$ and the Fibonacci numbers. It is easy to find $f$ :s derivatives: $f^{(m)}(t)=\left(\varphi^{m} e^{\varphi t}-\psi^{m} e^{\psi t}\right) / \sqrt{5}$ and by plugging $t=0$ into $f^{m}(t)$ we find that

$$
f^{(m)}(0)=\frac{\varphi^{m}-\psi^{m}}{\sqrt{5}}=f_{m}
$$

the Fibonacci numbers! For further connections between the two, and also to Lucas numbers, see Elmore [3].

### 4.2 Leslie Population Model

Now we consider a bigger example of a discrete time linear dynamical system, inspired by Chapters 19.2-19.6 in Shapiro [10].

Assume that we have a population $\mathcal{P}$ with a finite lifespan $L$. Divide the population into $n$ groups $\nu_{1}, \ldots, \nu_{n}$, each with an age span of $\frac{L}{n}$. For example, if we are dealing with a human population, we could take $L=120$ and $n=$ 12 and thus consider the twelve age groups $0-9,10-19, \ldots, 110-119$. We let the variable $m \in \mathbb{N}$ count how many time periods $\frac{L}{n}$ have passed. In our example, $m=1$ means that twelve years has passed since we began our study. Now, let $x_{i, m} \in[0,|\mathcal{P}|]$ be the number of individuals in the group $\nu_{i}$ at time $m$. Note that the surviving individuals in age groups $\nu_{i}$ at time $m$ will be in group $\nu_{i+1}$ at time $m+1$. Also, the individuals in the first group $\nu_{1}$ are simply those who where born during the time period $m$. We assume that an individual in age group $\nu_{n}$ at time $m$ will not survive until the next time period $m+1$. To model the survival of the population, let $\alpha_{i} \in[0,1]$ denote the proportion of age group $\nu_{i}$ that survive to reach age group $\nu_{i+1}$. Note that $\alpha_{n}=0$ and that it is reasonable to assume that survival rates for younger age groups are higher than the survival rates for older age groups. We get the following relationship: $x_{i+1, m+1}=\alpha_{i} x_{i, m}$ for all $i=1, \ldots, n-1$ and $m=$ $0,1,2, \ldots$ To go one step further, we can model the reproduction in $\mathcal{P}$ by
introducing another parameter $\beta_{i} \geq 0$, which is equal to the average number of offspring produced per individual in age group $\nu_{i}$ during one single time period. These offspring will, in the following time period belong to age group $\nu_{1}$. Thus we get the following relation: $x_{1, m+1}=\sum_{i=1}^{n} \beta_{i} x_{i, m}$ for all $m=0,1,2, \ldots$. We will assume that these numbers $\alpha_{1}, \ldots, \alpha_{n}, \beta_{1}, \ldots, \beta_{n}$ are the same for all time periods $m$. Define a state vector to be $\mathbf{x}_{m}=\left(x_{1, m}, \ldots, x_{n, m}\right)^{T}$, then the relations above give

$$
\begin{aligned}
\mathbf{x}_{m+1} & =\left(\begin{array}{c}
x_{1, m+1} \\
x_{2, m+1} \\
\vdots \\
x_{n, m+1}
\end{array}\right)=\left(\begin{array}{c}
\sum_{i=1}^{n} \beta_{i} x_{i, m} \\
\alpha_{1} x_{1, m} \\
\vdots \\
\alpha_{n-1} x_{n-1, m}
\end{array}\right) \\
& =\underbrace{\left(\begin{array}{ccccc}
\beta_{1} & \beta_{2} & \ldots & \beta_{n-1} & \beta_{n} \\
\alpha_{1} & 0 & \ldots & 0 & 0 \\
0 & \alpha_{2} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \alpha_{n-1} & 0
\end{array}\right)}_{=: A}\left(\begin{array}{c}
x_{1, m} \\
x_{2, m} \\
x_{3, m} \\
\vdots \\
x_{n, m}
\end{array}\right)=A \mathbf{x}_{m}
\end{aligned}
$$

and so we see that this is a discrete time linear dynamical system. This is called the Leslie population model, named after the British ecologist Patrick Holt Leslie who developed the model for a population of rodents in 1945. The nonnegative matrix $A$ is called a Leslie matrix. We will start by assuming that the system is homogeneous, that is we have no term $\mathbf{b}$ and thus no migration in and out of $\mathcal{P}$. We will later introduce this term to get $\mathbf{x}_{m+1}=A \mathbf{x}_{m}+\mathbf{b}$ and see how it changes the behaviour of the system. First, let us practice using the model in a somewhat degenerated case.

Example 4.5. Assume $\alpha_{i}>0$ for all $i=1, \ldots, n-1$, i.e. there are at least one individual who survives in each age group at each time period. Assume also and that $\beta_{i}=0$ for all $i=1, \ldots, n$, i.e. no reproduction is taking place in the population. We expect $\mathcal{P}$ to be completely extinct after $L$ years. We verify this using the model. We see

$$
\mathbf{x}_{n}=\underbrace{\left(\begin{array}{ccccc}
0 & 0 & \ldots & 0 & 0 \\
\alpha_{1} & 0 & \ldots & 0 & 0 \\
0 & \alpha_{2} & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \ldots & \alpha_{n-1} & 0
\end{array}\right)}_{=: \mathcal{N}} \mathbf{x}_{n-1}=\mathcal{N}^{n} \mathbf{x}_{0}=\mathbf{0}
$$

because $\mathcal{N}$ is a nilpotent matrix such that $\mathcal{N}^{n}=\mathbf{0}$. Thus, each age group will contain no individuals after exactly $n \frac{L}{n}=L$ years.

We call the example 4.5 degenerate, for it is not really a population without any reproduction. Similarly, the case where at least one $\alpha_{i}=0$ for $i<n$ can also be consider as non-realistic, for in that case no one in age group $\nu_{i}$ survives to $\nu_{i+1}$ at time $m+1$, but then age group $\nu_{i}$ has the maximum lifespan $\frac{L}{n} i<L$. Henceforth, we may therefore assume that $\alpha_{i}>0$ for all $i=1, \ldots, n$.


Figure 4.1: The graph of a Leslie matrix
In fig. 4.1 we see the graph $\mathcal{D}(A)$ of a Leslie matrix $A$.
Remark 4.6. Observe that for $\mathcal{D}(A)$ to make intuitive sense, where we interpret arcs as individuals aging over time, we let columns determine the out-going vertices and rows in-going vertices of the arcs. This is the reversed way of defining the graph of a matrix, see remark 2.5 .

By using properties of $\mathcal{D}(A)$ we can determine for which values of $\alpha_{i}$ and $\beta_{i}$ $A$ is irreducible or even primitive. Since $\alpha_{i}>0$ for all $i$ we get that $\mathcal{D}(A)$ is strongly connected if and only if $\beta_{n}>0$, because in that case $\left(\nu_{n}, \nu_{1}\right) \in \mathcal{E}(A)$ and the $n$-cycle in $\mathcal{D}(A)$ creates paths between any pair of vertices. Recall that the number $n$ determines the partition of $\mathcal{P}$. If we are using a very fine model (large $n$ ), then $\beta_{n}=0$ is probable because the oldest individuals of a species tend not to reproduce as much. At the same time, we do not want to choose an $n$ too small because then we might miss out on important generational traits. In applications, one might even take $\beta_{n}$ to be some small positive number $\delta>0$ just to make $A$ irreducible and inherit the coveted Perron-Frobenius-properties in 3.7. We go one step further and determine when $A$ is a primitive Leslie matrix. All closed paths in $\mathcal{D}(A)$ contain the first node $\nu_{1}$. It is not reasonable to assume $\beta_{1}>0$ (which would automatically make $A$ primitive by theorem 2.20) as the first group should represent the individuals early life. However, it is realistic to think that if some age group $\nu_{i}$ has a fertile cohort, then the groups $\nu_{i+1}$ (and/or perhaps $\nu_{i-1}$ ) might also contribute to reproduction. In this case $\nu_{1} \rightarrow \cdots \rightarrow \nu_{i} \rightarrow \nu_{1}$ and $\nu_{1} \rightarrow \cdots \rightarrow \nu_{i+1} \rightarrow \nu_{1}$ are two closed paths of lengths $i$
and $i+1$. Since $\operatorname{gcd}(i, i+1)=1$ then $\mathcal{D}(A)$ is primitive and thus $A$ is primitive too. As such, it suffices that two consecutive $\beta_{i}, \beta_{i+1}$ are positive for $A$ to become primitive. By Perron-Frobenius' theorem 3.9 we have $\lim _{m \rightarrow \infty}\left(\frac{1}{\rho(A)} A\right)^{m}=\mathbf{p q}^{T}$ where $\mathbf{p}$ and $\mathbf{q}^{T}$ are the Perron vectors of the primitive Leslie matrix $A$. If $\rho(A)$ happens to equal 1 , then in the long run, the number of individuals in each age group will be proportional to the left Perron vector $\mathbf{p}$. If $\rho(A)>1$ the number of individuals in each age group will tend towards infinity by theorem 4.3 and so the population will grow unbounded. However the result of Perron-Frobenius' is still interesting for when we consider the matrix $\frac{1}{\rho(A)} A$ instead of $A$ in the system it means that we can look for the long-term distribution of individuals amongst the age groups, which again will be proportional to $\mathbf{p}$. We have seen a similar example of this, see example 3.12 with the six ponds, where the litres of water tends towards infinity but not the distribution. Observe however that $A$ in that example is not a Leslie matrix. Lastly, if $\rho(A)<1$ then $A^{m}$ will tend towards $\mathbf{0}$ which means that the population will become extinct. One could continue on and consider the meaning, reason and consequences also for reducible and imprimitive Leslie models, but we will save these type of discussions for the next chapter when we look at Markov chains.

Now we are ready to introduce migration in or out of the population. Assume that during each time period, $b_{i}$ individuals will migrate (on average) out of age group $\nu_{i}$, if $b_{i}<0$, or into age group $\nu_{i}$, if $b_{i}>0$ for all $i=1, \ldots n$. Let $\mathbf{b}=\left(b_{1}, \ldots, b_{n}\right)$ and note that if $\mathbf{b}=\mathbf{0}$ then we have the homogeneous model that we are already familiar with. It is not immediately clear how some nonzero $\mathbf{b}$ changes the dynamics of the system. Here, we will explain the results in terms of the Leslie model because it is a nice tool for explanation, but the results can be applied to other discrete time nonnegative linear dynamical systems as well. First, we make a computation to show how the model propagates.

$$
\begin{aligned}
& \mathbf{x}_{1}=A \mathbf{x}_{0}+\mathbf{b} \\
& \mathbf{x}_{2}=A \mathbf{x}_{1}+\mathbf{b}=A^{2} \mathbf{x}_{0}+A \mathbf{b}+\mathbf{b} \\
& \mathbf{x}_{3}=A \mathbf{x}_{2}+\mathbf{b}=A^{3} \mathbf{x}_{0}+A^{2} \mathbf{b}+A \mathbf{b}+\mathbf{b} \\
& \vdots \\
& \mathbf{x}_{m}=A \mathbf{x}_{m-1}+\mathbf{b}=A^{m} \mathbf{x}_{0}+\left(A^{m-1}+\cdots+A^{2}+A+I\right) \mathbf{b}
\end{aligned}
$$

Then, since the following sum is telescoping

$$
\begin{aligned}
& (A-I)\left(A^{m-1}+\cdots+A+I\right) \\
& =A^{m}+A^{m-1} \cdots+A^{2}+A-A^{m-1}-\cdots-A^{2}-A-I \\
& =A^{m}+\left(A^{m-1}-A^{m-1}\right)+\cdots+\left(A^{2}-A^{2}\right)+(A-A)-I \\
& =A^{m}-I
\end{aligned}
$$

we can conclude that whenever $A-I$ is nonsingular we have

$$
\begin{equation*}
\mathbf{x}_{m}=A^{m} \mathbf{x}_{0}+(A-I)^{-1}\left(A^{m}-I\right) \mathbf{b} \tag{4.4}
\end{equation*}
$$

Now, the population $\mathcal{P}$ does not need to become extinct if $\rho(A)<1$ anymore since if $\mathbf{b} \geq 0$ is sufficiently large, this will make up for $A^{m} \rightarrow \mathbf{0}$ as $m \rightarrow \infty$. We want to know what the second term $(A-I)^{-1}\left(A^{m}-I\right) \mathbf{b}$ in 4.4 will tend towards when $m \rightarrow \infty$. Notice that this term is independent of the initial vector $\mathbf{x}_{0}$. To find out, we first consider an alternative approach of finding a formula for $\mathbf{x}_{m+1}=A \mathbf{x}_{m}+\mathbf{b}$; first solving the homogeneous system $\mathbf{x}_{m+1}=A \mathbf{x}_{m}$ and then adding a particular solution, which is in some ways specialized just for the vector $\mathbf{b}$. The particular solution will be an equilibrium, as defined below.

Definition 4.7 (Equilibrium point). $\mathrm{x}^{*} \in \mathbb{R}^{n}$ is an equilibrium point or fixed point of the system $\mathbf{x}_{m+1}=A \mathbf{x}_{m}+\mathbf{b}$ if $\mathbf{x}^{*}=A \mathbf{x}^{*}+\mathbf{b}$

If the system is homogeneous, $\mathbf{b}=\mathbf{0}$, then an equilibrium point is vector such that $\mathbf{x}^{*}=A \mathbf{x}^{*}$, i.e. an eigenvector of $A$ with the eigenvalue 1 . For now we will allow the equilibrium point to be in $\mathbb{R}^{n}$, so not necessarily nonnegative. Whenever $A-I$ is nonsingular, then

$$
\mathbf{x}^{*}=(I-A)^{-1} \mathbf{b}=-(A-I)^{-1} \mathbf{b}
$$

and in this case $\mathbf{x}^{*}$ is very easy to find given $A$ and $\mathbf{b}$. If $A-I$ is singular there are either infinitely many equilibrium points or none. Note that $A-I$ is singular if $0 \in \sigma(A-I)$ and since $\sigma(A-I)=\sigma(A)-1$ this is equivalent to $1 \in \sigma(A)$. We can think of nonnegative equilibrium points of a Leslie model as some initial sizes of each age group which are optimally suited for the particular model. As time passes, the number of individuals in each age group will neither decrease nor increase in size, they remain invariant. Now suppose that every initial state $\mathbf{x}_{0}$ of $\mathcal{P}$ become more and more equal to the equilibrium point $\mathbf{x}^{*}$ as time goes one, then such $\mathbf{x}^{*}$ is called asymptotically stable. In this case the Leslie model forces all initial age group sizes to become that of $\mathbf{x}^{*}$ in the long run. Note that, if it exists, such $\mathbf{x}^{*}$ must be unique. The following definition describes this exact situation.

Definition 4.8. The equilibrium point $\mathbf{x}^{*}$ is called asymptotically stable if $\lim _{m \rightarrow \infty} \mathbf{x}_{m}=\mathbf{x}^{*}$ for all choices of the initial vector $\mathbf{x}_{0}$.

In the general case $\mathbf{x}_{m+1}=A \mathbf{x}_{m}+\mathbf{b}$, the solution to the homogeneous system $\mathbf{x}_{m+1}=A \mathbf{x}_{m}$ is $\mathbf{x}_{m}=A^{m} \mathbf{x}_{0}$ and so adding the particular solution $\mathbf{x}^{*}$ yields $\mathbf{x}_{m}=A^{m} \mathbf{x}_{0}+\mathbf{x}^{*}$. If we instead substitute $\mathbf{x}^{*}=-(A-I)^{-1} \mathbf{b}$ into 4.4 we get $\mathbf{x}_{m}=A^{m} \mathbf{x}_{0}-\left(A^{m}-I\right) \mathbf{x}^{*}$ since the matrices $A^{m}-I$ and $(A-I)^{-1}$ commute. We see that for $\mathbf{x}^{*}$ to be an asymptotically stable equilibrium point for the model, we need $A^{m} \rightarrow \mathbf{0}$ as $m \rightarrow \infty$ in both expressions, which is equivalent to $\rho(A)<1$. Conversely, if $\rho(A)<1$ then in particular $1 \notin \sigma(A)$ and so $A-I$ is invertible which together with $A^{m} \rightarrow \mathbf{0}$ as $m \rightarrow \infty$ in (4.4) yield

$$
\mathbf{x}_{m} \rightarrow \mathbf{0} \mathbf{x}_{0}+(A-I)^{-1}(\mathbf{0}-I) \mathbf{b}=-(A-I)^{-1} \mathbf{b}
$$

as $m \rightarrow \infty$ for all $\mathbf{x}_{0}$. Because $-(A-I)^{-1} \mathbf{b}$ is an equilibrium point, it must be asymptotically stable. We have thus proved the following theorem.

Theorem 4.9. $\boldsymbol{x}_{m+1}=A \boldsymbol{x}_{m}+\boldsymbol{b}$ has an (unique) asymptotically stable equilibrium point if and only if $\rho(A)<1$

To tie up the loose ends, let us see when a Leslie model, which is a nonnegative model, has an asymptotically stable nonnegative equilibrium point. It is almost the same as theorem 4.9 but with one extra condition. The proof uses the fact that $\sum_{m=0}^{\infty} A^{m}=(I-A)^{-1}$ if $\rho(A)<1$, i.e. the matrix-version of the geometric series formula, often called the Neumann series for matrices. For a proof of this fact, see for example Meyer [9, p. 618, Chapter 7]

Theorem 4.10. $\boldsymbol{x}_{m+1}=A \boldsymbol{x}_{m}+\boldsymbol{b}$, where $A \geq 0$ and $\boldsymbol{b}>0$, has an (unique) asymptotically stable equilibrium point if and only if $\rho(A)<1$

Proof. $(\Rightarrow)$ : Suppose $\mathbf{x}^{*}=A \mathbf{x}^{*}+\mathbf{b} \geq 0 . A \geq 0$ so theorem 3.5 implies that $0 \leq \rho(A) \in \sigma(A)$ and there exists a left eigenvector $\mathbf{0} \neq \mathbf{v}^{T} \geq 0$. Thus

$$
\mathbf{v}^{T} \mathbf{x}^{*}=\mathbf{v}^{T}\left(A \mathbf{x}^{*}+\mathbf{b}\right)=\mathbf{v}^{T} A \mathbf{x}^{*}+\mathbf{v}^{T} \mathbf{b}=\rho(A) \mathbf{v}^{T} \mathbf{x}^{*}+\mathbf{v}^{T} \mathbf{b}
$$

which we rearrange to be $(1-\rho(A)) \mathbf{v}^{T} \mathbf{x}^{*}=\mathbf{v}^{T} \mathbf{b}$ So $1-\rho(A)>0$, because $\mathbf{v}^{T} \mathbf{x}^{*} \geq 0$ and $\mathbf{v}^{T} \mathbf{b}>0$ since $\mathbf{b}>0$. Therefore $\rho(A)<1$.
$(\Leftarrow)$ : Suppose $\rho(A)<1$, then by theorem $4.9 \mathbf{x}^{*}=(I-A)^{-1} \mathbf{b}$ exists. We show $\mathbf{x}^{*} \geq 0$ by writing $(I-A)^{-1}=\sum_{m=0}^{\infty} A^{m}$ so $\mathbf{x}^{*}=\sum_{m=0}^{\infty} A^{m} \mathbf{b}$ and since $A \geq 0$ and $\mathbf{b}>0$ we have $\mathbf{x}^{*} \geq 0$.

We finish this section with an example.

Example 4.11. Consider the following Leslie population model.

$$
\mathbf{x}_{m+1}=\underbrace{\left(\begin{array}{cccccc}
0 & 0 & 0.5 & 1 & 0.5 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.75 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.25 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.125 & 0
\end{array}\right)}_{=: A} \mathbf{x}_{m}+\underbrace{\left(\begin{array}{l}
1 \\
1 \\
1 \\
1 \\
1 \\
1
\end{array}\right)}_{=: \mathbf{b}}
$$

We determine $\mathbf{x}^{*}$ easily be solving the equation $(I-A) \mathbf{x}^{*}=\mathbf{b}$. The resulting equilibrium point is approximately

$$
\mathbf{x}^{*}=\left(\begin{array}{llllll}
22.0769 & 23.0769 & 18.3077 & 10.1538 & 3.5385 & 1.8846
\end{array}\right)^{T} \geq 0
$$

To see if $\mathbf{x}^{*}$ is an asymptotically stable equilibrium we calculate the spectral radius, which turns out to be approximately $\rho(A)=0.94<1$. By theorem 4.10 all initial vectors $\mathbf{x}_{0}$ in the Leslie model will therefore become $\mathbf{x}^{*}$ in the long run. We can empirically verify this by setting $\mathbf{x}_{0}=\left(\begin{array}{llllll}0 & 0 & 0 & 1 & 0 & 0\end{array}\right)^{T}$, i.e. only one individual in age group $\nu_{4}$, and seeing what happens. Over $m=100$ time periods we get

$$
\begin{aligned}
& \mathbf{x}_{1}=A \mathbf{x}_{0}+\mathbf{b}=\left(\begin{array}{lllllll}
2 & 1 & 1 & 1 & 1.25 & 1
\end{array}\right)^{T} \\
& \mathbf{x}_{2}=A \mathbf{x}_{1}+\mathbf{b}=\left(\begin{array}{lllllll}
3.125 & 3 & 1.75 & 1.5 & 1.25 & 1.3125
\end{array}\right)^{T} \\
& \vdots \\
& \mathbf{x}_{100}=A \mathbf{x}_{99}+\mathbf{b}=\left(\begin{array}{llllll}
22.0375 & 23.0349 & 18.2741 & 10.1360 & 3.5337 & 1.8833
\end{array}\right)^{T}
\end{aligned}
$$

where $\mathbf{x}_{100}$ is quite close to the equilibrium since $\left\|\mathbf{x}^{*}-\mathbf{x}_{100}\right\|_{2}=0.0692$. After yet another 100 time periods, the distance between $\mathbf{x}_{200}$ and $\mathbf{x}^{*}$ is basically zero, thus verifying that $\mathbf{x}_{m}$ comes closer and closer to $\mathbf{x}^{*}$.

As an ending note on equilibrium points; it is possible to derive similar theorems as 4.9 and 4.10 also for continuous time systems, see Shapiro [10]. Worth mentioning there is that the requirements on $A$ and $\mathbf{b}$ for preservation of nonnegativity in continuous time differs from those in the discrete time case, where if $\mathbf{b} \geq 0$ and $\mathbf{x}_{0} \geq 0$ then all entries in $A$ must be nonnegative for $\mathbf{x}_{m}$ to be nonnegative for all $m$. However in continuous time it suffices that only off-diagonal elements in $A$ are nonnegative, that is $a_{i j} \geq 0$ if $i \neq j$. These types of matrices are called Metzler matrices and the reason for the looser requirement ties into the fact that the matrix $e^{A}$ is nonnegative for Metzler matrices.

## Chapter 5

## Markov Chains

For the remaining part of this thesis we will investigate Markov chains, which is one of the best example of an application of Perron-Frobenius theory. We will denote the matrices of Markov chains consistently by $\mathbf{P}$, as in Probability, and other matrices by $A$. In the case that we have to use also permutation matrices, they will be denoted by $Q$. The matrix properties we have looked at previously (irreducible, reducible, primitive, imprimitive) in chapter two, will now be used to also categorize the different Markov chains. The following presentation of the subject is inspired by Chapter 8.4 in Meyer [9]

### 5.1 Definition

A Markov chain, named after the Russian mathematician Andrey Markov (18561922), is a stochastic model which describes sequences of moves, called steps, between different states. We will assume that the state space $\mathcal{S}:=\left\{\nu_{1}, \nu_{2}, \ldots, \nu_{n}\right\}$ is of finite dimension $n$ and that the steps occur in discrete time. Let the random variable $X_{m}$ be the state of the process at time step $m \in \mathbb{N}$, so all $X_{m} \in \mathcal{S}$. Then, the special thing that distinguishes Markov chains from other stochastic processes is the Markov property, also called memorylessness, as defined below. Remember that $P(\mathcal{E} \mid \mathcal{F})$ is the conditional probability of the event $\mathcal{E}$ happening, given that the event $\mathcal{F}$ has already happened.

Definition 5.1 (Markov property). The set of random variables $\left\{X_{m}\right\}_{m=0}^{\infty}$ satisfies the Markov property if

$$
\begin{equation*}
P\left(X_{m+1}=\nu_{i_{m+1}} \mid X_{m}=\nu_{i_{m}}, \ldots, X_{0}=\nu_{i_{0}}\right)=P\left(X_{m+1}=\nu_{i_{m+1}} \mid X_{m}=\nu_{i_{m}}\right) \tag{5.1}
\end{equation*}
$$

for all $m=0,1,2, \ldots$ where $\nu_{i_{m+1}}, \nu_{i_{m}}, \ldots, \nu_{i_{0}} \in \mathcal{S}$. In other words, the probability of the chain being in state $\nu_{i_{m+1}}$ at the time $m+1$ is only determined by which state the chain was in at the previous time step $m$ and not the ones before that.

We see that a Markov chain only 'remembers' the most previous state, thus having the memory of a goldfish, hence the term memorylessness. Furthermore, we will assume that the probability of the chain stepping from $\nu_{i}$ to $\nu_{j}$ is the same at all times $m$. These types of Markov chains are called stationary or homogeneous. For those we can define what is called the transition probability as being $p_{i j}=P\left(X_{m+1}=\nu_{j} \mid X_{m}=\nu_{i}\right)$ which is a number between 0 and 1 such that $\sum_{j} p_{i j}=1$ for all $i$ since the sum of all outcomes must equal 1. The matrix $\mathbf{P}=\left(p_{i j}\right)$ is called the transition matrix of the Markov chain and it is a stochastic matrix, as defined below.

Definition 5.2 (Stochastic matrix). $A \geq 0$ is a (row) stochastic matrix, or sometimes a probability matrix, if

$$
\sum_{j=1}^{n} a_{i j}=1 \quad \text { for all } i=1, \ldots, n
$$

that is $A \mathbf{e}=\mathbf{e}$ where $\mathbf{e}=(1, \ldots, 1)^{T}$. If instead all columns sum to 1 , that is $\mathbf{e}^{T} A=\mathbf{e}^{T}$, then $A$ is called column stochastic and if $A$ is both row and column stochastic then $A$ is doubly stochastic.

All Markov chains define a stochastic matrix and we can read every stochastic matrix as a transition matrix by associating each row with a state. Now, define the state vector of the Markov chain, in this case called $m^{t h}$ step probability distribution vector, to be $\mathbf{p}_{m}=\left(p_{1, m}, \ldots, p_{n, m}\right)^{T}$ where $p_{j, m}=P\left(X_{m}=\nu_{j}\right)$ for all $j=1, \ldots, n$. In other words $p_{j, m}$ is the probability that the chain will be in state $\nu_{j}$ after $m$ time steps. The vector $\mathbf{p}_{0}$ is the initial probability distribution vector and it tells us how likely it is that the chain will start in the different states. For example, if $\mathbf{p}_{0}=(1 / 2,0,1 / 2)^{T}$ then there is a $50-50$ chance for the chain to start in the states $\nu_{1}$ and $\nu_{3}$ and if, two time steps later, $\mathbf{p}_{2}=(1,0,0)^{T}$ then this tells us that the probability of the chain now being in $\nu_{1}$ is equal to 1, i.e. we know for certain that the chain is in state $\nu_{1}$ after two time steps.

We suspect that a Markov chain can be modeled by a matrix multiplication between the matrix $\mathbf{P}$ and the vector $\mathbf{p}$. We show that this is the case by using the definition and the memorylessness of this system. Recall that $P(\mathcal{E} \vee$ $\mathcal{F})=P(\mathcal{E})+P(\mathcal{F})$ if $\mathcal{E}$ and $\mathcal{F}$ are two disjoint events, where $\vee$ is 'or'. Also $P(\mathcal{E} \wedge \mathcal{F})=P(\mathcal{F}) P(\mathcal{E} \mid \mathcal{F})$ where $\wedge$ is 'and'. Suppose $\mathbf{p}_{0}$ is known, then the $j^{t h}$
component in $\mathbf{p}_{1}$ is

$$
\begin{aligned}
p_{j, 1} & =P\left(X_{1}=\nu_{i}\right)=P\left(X_{1}=\nu_{i} \wedge\left(X_{0}=\nu_{1} \vee \cdots \vee X_{0}=\nu_{n}\right)\right) \\
& =P\left(\left(X_{1}=\nu_{j} \wedge X_{0}=\nu_{1}\right) \vee \cdots \vee\left(X_{1}=\nu_{j} \wedge X_{0}=\nu_{n}\right)\right) \\
& =\sum_{i=1}^{n} P\left(X_{1}=\nu_{j} \wedge X_{0}=\nu_{i}\right) \\
& =\sum_{i=1}^{n} P\left(X_{0}=\nu_{i}\right) P\left(X_{1}=\nu_{j} \mid X_{0}=\nu_{i}\right) \\
& =\sum_{i=1}^{n} p_{i, 0} p_{i j}
\end{aligned}
$$

for all $j=1, \ldots, n$, so $\mathbf{p}_{1}^{T}=\mathbf{p}_{0}^{T} \mathbf{P}$. Because of the Markov property the same arguments also apply from time step 1 to 2 and 2 to 3 etc. If we want we could show this by letting $\mathbf{p}_{0}^{\prime}=\mathbf{p}_{1}$ and then the conclusion would be as above. We have just proved the following theorem on the definition of a Markov chain.

Theorem 5.3. An n-state Markov chain can be modelled as $\boldsymbol{p}_{m+1}^{T}=\boldsymbol{p}_{m}^{T} \boldsymbol{P}$ where $\boldsymbol{p}_{m}^{T}$ is the $m^{\text {th }}$ step probability distribution vector and $\boldsymbol{P}$ the transition matrix. The closed form solution of an Markov chain is consequently $\boldsymbol{p}_{m}^{T}=\boldsymbol{p}_{0}^{T} \boldsymbol{P}^{m}$ where $\boldsymbol{p}_{0}^{T}$ is the initial probability distribution vector.

Remark 5.4. If we let $\mathbf{p}_{0}^{T}=\mathbf{e}_{i}^{T}$ where $\mathbf{e}_{i}^{T}$ is the $i^{t h}$ base vector in the standard basis of $\mathbb{R}^{n}$ we get that the $j^{t h}$ component in $\mathbf{p}_{m}^{T}$ is the $(i, j)^{t h}$ element in $\mathbf{P}^{m}$ for all $m$. Thus the element $p_{i j}^{(m)}$ in $\mathbf{P}^{m}$ is the probability of the chain moving from state $\nu_{i}$ to $\nu_{j}$ in exactly $m$ time steps.

Before we use the definitions in an examples we note that, just as for all nonnegative matrices, we can form the graph $\mathcal{D}(\mathbf{P})$ of a transition matrix $\mathbf{P}$ in which the states $\nu_{1}, \ldots, \nu_{n}$ are the vertices $\mathcal{V}(\mathbf{P})$ and there exists an arc $\left(\nu_{i}, \nu_{j}\right) \in \mathcal{V}(\mathbf{P})$ if and only if $p_{i j}=P\left(X_{m+1}=\nu_{j} \mid X_{m}=\nu_{i}\right)>0$. One often sees the positive probabilities $p_{i j}$ as labels on each arc, making $\mathcal{D}(\mathbf{P})$ a weighted directed graph. Now, we take on this classical example below.

Example 5.5 (A weather forecast). We wonder what the probability is that the day after tomorrow will be a sunny day if we have the following estimate: a sunny day is followed by another sunny day nine out of ten times and a rainy day is followed by yet another rainy day half of the times. We assume that there are no other weather conditions and that today is a rainy day.

The situation can be modeled as a Markov chain. Let $\nu_{1}$ and $\nu_{2}$ be the states sunny and rainy and let the random variable $X_{m}$ be the weather condition at day $m \geq 0$. If today is day $m=0$, then the initial probability distribution vector $\mathbf{p}_{0}^{T}=\left(\begin{array}{ll}0 & 1\end{array}\right)$ because we know ( $100 \%$ probability) that $X_{0}=\nu_{2}$. The given estimate can be translated into a transition matrix with the following
values:

$$
\begin{array}{lll}
p_{11}=P\left(X_{m+1}=\nu_{1} \mid X_{m}=\nu_{1}\right)=0.9 \quad & \Rightarrow \quad p_{12}=1-p_{11}=0.1 \\
p_{22}=P\left(X_{m+1}=\nu_{2} \mid X_{m}=\nu_{2}\right)=0.5 \quad \Rightarrow \quad p_{21}=1-p_{22}=0.5
\end{array}
$$

where we make use of the fact that a transition matrix is stochastic. Thus $\mathbf{P}=\left(\begin{array}{ccc}0.9 & 0.1 \\ 0.5 & 0.5\end{array}\right)$ and by theorem 5.3, $\mathbf{p}_{2}^{T}$ is given by

$$
\mathbf{p}_{2}^{T}=\mathbf{p}_{0}^{T} \mathbf{P}^{2}=\left(\begin{array}{ll}
0 & 1
\end{array}\right)\left(\begin{array}{cc}
0.86 & 0.14  \tag{0.3}\\
0.7 & 0.3
\end{array}\right)=(0.7
$$

Since $p_{1,2}=P\left(X_{2}=\nu_{1}\right)=0.7$ there is $70 \%$ chance of sunny weather the day after tomorrow. In figure 5.1 we see the graph $\mathcal{D}(\mathbf{P})$ where the number 0.7 can also be calculated using the multiplication and addition principle by considering the two ways of walking from $\nu_{2}$ to $\nu_{1}$ using exactly two arcs (either $\nu_{2} \rightarrow \nu_{2} \rightarrow \nu_{1}$ or $\left.\nu_{2} \rightarrow \nu_{1} \rightarrow \nu_{1}\right)$. The computation is $0.5 \cdot 0.5+0.5 \cdot 0.9=0.7$.


Figure 5.1: $\mathcal{D}(\mathbf{P})$
Let us precede the theory a little bit by also considering the weather in the long run. That is, given that it rains today and that $\mathbf{P}$ has the values above, how many days will be sunny in the long run. As usual, we are interested in $\lim _{m \rightarrow \infty} \mathbf{p}_{m}^{T}=\lim _{m \rightarrow \infty} \mathbf{p}_{0}^{T} \mathbf{P}^{m}$ and without using any Perron-Frobenius-tricks this time, we can find the limit the 'old way' by diagonalizing $\mathbf{P}$ and calculating $D^{m}$ as $m \rightarrow \infty$.

$$
\begin{aligned}
\lim _{m \rightarrow \infty} \mathbf{p}_{0}^{T} \mathbf{P}^{m} & =\lim _{m \rightarrow \infty} \mathbf{p}_{0}^{T} T D^{m} T^{-1}=\lim _{m \rightarrow \infty} \mathbf{p}_{0}^{T} T\left(\begin{array}{cc}
1^{m} & 0 \\
0 & (2 / 5)^{m}
\end{array}\right) T^{-1} \\
& =\left(\begin{array}{ll}
0 & 1
\end{array}\right)\left(\begin{array}{cc}
1 & 1 \\
1 & -5
\end{array}\right)\left(\begin{array}{ll}
1 & 0 \\
0 & 0
\end{array}\right) \frac{1}{6}\left(\begin{array}{cc}
5 & 1 \\
1 & -1
\end{array}\right)=\left(\begin{array}{ll}
5 / 6 & 1 / 6
\end{array}\right)
\end{aligned}
$$

So in the long run, five days out of six will be sunny. Two noteworthy facts from this example: (1): $\rho(A)=1 \in \sigma(A)$ with $(1,1)^{T}$ as its eigenvector. (2): $\mathbf{P}$ is primitive and $(5 / 6 \quad 1 / 6) \mathbf{P}=\left(\begin{array}{ll}5 / 6 & 1 / 6\end{array}\right)$.

### 5.2 Properties

In this section we prove some useful results regarding Markov chains and their transition matrices $\mathbf{P}$. As we will see, Perron-Frobenius theory is in some ways Markov chains best friend.

For starters, the elements in any stochastic matrix $A$ are nonnegative so $\left|a_{i j}\right|=a_{i j} \geq 0$ and thus

$$
\|A\|_{\infty}=\max _{i} \sum_{j}\left|a_{i j}\right|=\max _{i} \sum_{j} a_{i j}=1
$$

Moreover, for all valid matrix norms and all matrices $A \in \mathbb{M}_{n}(\mathbb{C})$ it is true that $\rho(A) \leq\|A\|$ because if $A \mathbf{x}=\lambda \mathbf{x}$ where $|\lambda|=\rho(A)$ and we let $M$ be a matrix with $\mathbf{x}$ in every column, then we get

$$
\begin{aligned}
\rho(A)\|M\| & =|\lambda|\|M\| \stackrel{(*)}{=}\|\lambda M\|=\left\|\left(\begin{array}{ccc}
\mid & & \mid \\
\lambda \mathbf{x} & \cdots & \lambda \mathbf{x} \\
\mid & & \mid
\end{array}\right)\right\| \\
& =\left\|\left(\begin{array}{ccc}
\mid & & \mid \\
A \mathbf{x} & \cdots & A \mathbf{x} \\
\mid & & \mid
\end{array}\right)\right\|=\|A M\| \stackrel{(* *)}{\leq}\|A\|\|M\|
\end{aligned}
$$

where $(*)$ is true for all norms and $(* *)$ is an extra requirement for matrix norms specifically, see for example Meyer [9, p. 280]. If $A$ is stochastic, this argument implies that $\rho(A) \leq\|A\|_{\infty}=1$. At the same time, we know from definition 5.2 that $(1, \mathbf{e})$ is an eigenpair of $A$, where $\mathbf{e}=(1, \ldots, 1)^{T}$ and since $1 \in \sigma(A)$ we have $1 \leq \rho(A)$ so we conclude that $\rho(A)=1$. Hence we have proved the following theorem. The corollary follows immediately because all transition matrices are stochastic.

Theorem 5.6. If $A \geq 0$ is stochastic then $\rho(A)=1$.
Corollary 5.7. All transition matrices $\boldsymbol{P}$ have a spectral radius of 1 .
Remark 5.8. Note that this does not imply that we can call the vector $\frac{1}{n} \mathbf{e}$ the right Perron vector, because (at least in this thesis) we use this term exclusively for irreducible matrices. As an example we could take $\left(\begin{array}{ll}0 & 1 \\ 0 & 1\end{array}\right)$ which is stochastic, has spectral radius 1 and even has eigenvector $\frac{1}{2}\binom{1}{1}$ but is not irreducible. However, whenever $A$ is irreducible, then $\frac{1}{n} \mathbf{e}$ will of course be the positive right Perron vector.

We devote the remainder of this section to the following two questions:

* When does $\lim _{m \rightarrow \infty} \mathbf{p}_{m}$ exists and, when it does, what is its value?
* If $\lim _{m \rightarrow \infty} \mathbf{p}_{m}$ does not exists, can we say anything, and if so what, about the long term behaviour of the system?

First, we divide the set of transition matrices into four categories, as shown below in 5.2. Note that reducible matrices for which $\lim _{m \rightarrow \infty} \mathbf{P}^{m} \exists$ or $\nexists$ have no special name, so we will simply refer to these as case (3) and (4).


Figure 5.2: The four types of transition matrices.

### 5.2.1 Primitive Markov Chains

Perron-Frobenius' theorem for primitive matrices 3.9 tells us that there exists a right Perron vector, which we know is $\frac{1}{n} \mathbf{e}$ by 5.8 , and a left Perron vector, call it $\boldsymbol{\pi}^{T}>0$. If we want $\boldsymbol{\pi}^{T}$ to be a probability vector (i.e. $\sum_{j} \pi_{j}=1$ ), which we do want it to be, then it is most natural to choose $\boldsymbol{\pi}^{T}$ be normalized in the $\|*\|_{1}$-norm and not so that $\frac{1}{n} \boldsymbol{\pi}^{T} \mathbf{e}=1$, see section 3.1. The only difference this will have on the limit is whether the factor $\frac{1}{n}$ appears in the result or not. In the case $\frac{1}{n} \boldsymbol{\pi}^{T} \mathbf{e}=1$ we get that $\boldsymbol{\pi}^{T}$ itself is not a probability vector, but $\frac{1}{n} \boldsymbol{\pi}^{T}$ is. Henceforth, assume $\left\|\boldsymbol{\pi}^{T}\right\|_{1}=1$. We get

$$
\begin{equation*}
\lim _{m \rightarrow \infty} \mathbf{p}_{m}^{T}=\lim _{m \rightarrow \infty} \mathbf{p}_{0}^{T} \mathbf{P}^{m}=\mathbf{p}_{0}^{T} \frac{(\mathbf{e} / n) \boldsymbol{\pi}^{T}}{\boldsymbol{\pi}^{T}(\mathbf{e} / n)}=\mathbf{p}_{0}^{T} \frac{\mathbf{e} \boldsymbol{\pi}^{T}}{\boldsymbol{\pi}^{T} \mathbf{e}} \stackrel{(*)}{=} \mathbf{p}_{0}^{T} \mathbf{e} \boldsymbol{\pi}^{T} \stackrel{(* *)}{=} \boldsymbol{\pi}^{T} \tag{5.2}
\end{equation*}
$$

where we in $(*)$ and $(* *)$ use the fact that $\mathbf{p}^{T} \mathbf{e}=p_{1}+\cdots+p_{n}=1$ for all probability vectors $\mathbf{p}$. Notice how the limit is independent on the initial probability distribution vector, indicating that $\boldsymbol{\pi}^{T}$ could in fact be an asymptotically stable equilibrium point, defined in 4.8. To confirm that this is the truth, we show that $\boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T} \mathbf{P}$ by

$$
\mathbf{e} \boldsymbol{\pi}^{T}=\lim _{m \rightarrow \infty} \mathbf{P}^{m+1}=\left(\lim _{m \rightarrow \infty} \mathbf{P}^{m}\right) \mathbf{P}=\left(\mathbf{e} \boldsymbol{\pi}^{T}\right) \mathbf{P}
$$

where

$$
\mathbf{e} \boldsymbol{\pi}^{T}=\left(\begin{array}{c}
1 \\
\vdots \\
1
\end{array}\right)\left(\begin{array}{lll}
\pi_{1} & \ldots & \pi_{n}
\end{array}\right)=\left(\begin{array}{ccc}
\pi_{1} & \ldots & \pi_{n} \\
\vdots & \ddots & \vdots \\
\pi_{1} & \ldots & \pi_{n}
\end{array}\right)=\left(\begin{array}{ccc}
- & \boldsymbol{\pi}^{T} & - \\
& \vdots & \\
- & \boldsymbol{\pi}^{T} & -
\end{array}\right)
$$

so

$$
\left(\mathbf{e} \boldsymbol{\pi}^{T}\right) \mathbf{P}=\left(\begin{array}{ccc}
- & \boldsymbol{\pi}^{T} \mathbf{P} & - \\
\vdots & \\
- & \boldsymbol{\pi}^{T} \mathbf{P} & -
\end{array}\right) \quad \Rightarrow \quad \boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T} \mathbf{P}
$$

We summarize what we have found in a theorem.
Theorem 5.9 (Properties of primitive Markov chains). If $\boldsymbol{P}$ is a primitive transition matrix then

$$
\lim _{m \rightarrow \infty} \boldsymbol{P}^{m}=\boldsymbol{e} \boldsymbol{\pi}^{T}, \quad \text { so } \quad \lim _{m \rightarrow \infty} \boldsymbol{p}_{m}^{T}=\boldsymbol{\pi}^{T}
$$

where $\boldsymbol{\pi}^{T}$ is an asymptotically stable equilibrium point given by $\boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T} \boldsymbol{P}$.
The vector $\boldsymbol{\pi}^{T}$ is therefore often called the steady state of a Markov chain. The example on a weather forecast 5.5 is a primitive Markov chain, and the ending two notes verifies this theorem. Let us see another example, which is a modified version of the 'Ehrenfest's chain', see [11, Chapter 15.8].

Example 5.10 (Six molecules in a box: the primitive version). Let there be six molecules in a box which is partitioned into two halves: $\mathcal{A}, \mathcal{B}$. Let $X_{m}$ be the number of molecules in $\mathcal{A}$ after time step $m$, so $X_{m} \in\{0,1, \ldots, 6\}=\mathcal{S}$. At every time step we first choose a molecule from the box, and then we choose which half-box this molecule gets moved to. The choices are made at random. For example, if five molecules are in $\mathcal{A}$ after some time step $m$, then after time step $m+1$ we either have two molecules in $\mathcal{B}$ and four left in $\mathcal{A}$ or five molecules still in $\mathcal{A}$. In fig. 5.3, we see an illustration of the first case. We wonder how many molecules will eventually, on average, be in $\mathcal{A}$.


Figure 5.3: Six molecules in a box: the primitive version
At each time step, we have transition probabilities:

* $p_{i i}=P\left(X_{m+1}=i \mid X_{m}=i\right)=\frac{1}{2}$ because this is the probability that the number of molecules in $\mathcal{A}$ does not change from $m$ to $m+1$, which
happens if either a molecule in $\mathcal{A}$ stays in $\mathcal{A}$ (with $\frac{i}{6} \cdot \frac{1}{2}$ chance), or if a molecule in $\mathcal{B}$ stays in $\mathcal{B}$ (with $\frac{6-i}{6} \cdot \frac{1}{2}$ chance). The addition principle yields $\frac{1}{2}\left(\frac{i}{6}+\frac{6-i}{6}\right)=\frac{1}{2}$.
* $p_{i(i+1)}=P\left(X_{m+1}=i+1 \mid X_{m}=i\right)=\frac{6-i}{12}$, this is the probability of $\mathcal{A}$ gaining one molecule at time step $m+1$ which happens if we choose a molecule from $\mathcal{B}$ and choose $\mathcal{A}$ as its new half-box (with $\frac{6-i}{6} \cdot \frac{1}{2}$ chance).
* $p_{i(i-1)}=P\left(X_{m+1}=i-1 \mid X_{m}=i\right)=\frac{i}{12}$, this is $\mathcal{A}$ loosing one molecule which happens if we choose a molecule from $\mathcal{A}$ and move it to $\mathcal{B}$ (with $\frac{i}{6} \cdot \frac{1}{2}$ chance).

Thus, the transition matrix (of dimension $7 \times 7$ ) for this Markov chain is

$$
\mathbf{P}:=\frac{1}{12}\left(\begin{array}{ccccccc}
6 & 6 & 0 & 0 & 0 & 0 & 0 \\
1 & 6 & 5 & 0 & 0 & 0 & 0 \\
0 & 2 & 6 & 4 & 0 & 0 & 0 \\
0 & 0 & 3 & 6 & 3 & 0 & 0 \\
0 & 0 & 0 & 4 & 6 & 2 & 0 \\
0 & 0 & 0 & 0 & 5 & 6 & 1 \\
0 & 0 & 0 & 0 & 0 & 6 & 6
\end{array}\right)
$$

where we number the rows and columns as $0,1, \ldots, 6$. Below, fig. 5.4 we see that the graph of $\mathbf{P}$ is strongly connected, so $\mathbf{P}$ is irreducible. Because $p_{i i}>0$ for all $i=0, \ldots, 6$ then $\mathbf{P}$ is also primitive by theorem 2.20


Figure 5.4: $\mathcal{D}(\mathbf{P})$ of the primitive Ehrenfest's chain
By theorem 5.9 we know that $\boldsymbol{\pi}^{T}$ is the steady-state vector which can be calculated as the solution to $\boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T} \mathbf{P}$, i.e. $\mathbf{0}^{T}=\boldsymbol{\pi}^{T}(\mathbf{P}-I)$. For convenience purposes, we could transpose both sides and write $(\mathbf{P}-I)^{T} \boldsymbol{\pi}=\mathbf{0}$ and find that the solution-space is $\boldsymbol{\pi}^{T}=t\left(\begin{array}{lllllll}1 & 6 & 15 & 20 & 15 & 6 & 1\end{array}\right), t \in \mathbb{R}$, which if $t=1$ we recognise as the binomial coefficients for $(x+y)^{6}$. Normalizing the vector with $t=\frac{1}{2^{6}}$ yields

$$
\pi_{j}=\binom{6}{j}\left(\frac{1}{2}\right)^{6}, \quad \text { for } \quad j=0, \ldots, 6
$$

This is the probability mass function for the binomial distribution of six independent Bernoulli experiments with a success rate of $1 / 2$. We know that this has the mean $6 \cdot 1 / 2=3$ and so we conclude that in the long run there will be, on average, three molecules in half-box $\mathcal{A}$, i.e. half of the molecules will be in $\mathcal{A}$ and half in $\mathcal{B}$ as perhaps expected.

### 5.2.2 Imprimitive Markov Chains

We now move on to category (2) in fig. 5.2 in which $\mathbf{P}$ is an imprimitive matrix. Remember from theorem 2.22 that an imprimitive matrix $\mathbf{P}$ with index $k$ has the $k^{t h}$ roots of $\rho(\mathbf{P})^{k}$, which in this case is $\rho(\mathbf{P})=1$, as eigenvalues on the spectral circle, each of which is simple. Then, by theorem 4.3 we see that $\lim _{m \rightarrow \infty} \mathbf{P}^{m}$ cannot exist. Here we will reason how, despite of this, the limit of $\frac{I+\mathbf{P}+\ldots+\mathbf{P}^{m}}{m}$ as $m \rightarrow \infty$ can exist and interpret it in a nice way for the chain.

First, if $\left\{\alpha_{m}\right\}_{m=1}^{\infty}$ is a convergent sequence of scalars in $\mathbb{R}$, then its so called Cesáro sum or Cesáro mean (named after the Italian mathematician Ernesto Cesáro (1859-1906)) which is the sequence $\left\{\beta_{m}\right\}_{m=1}^{\infty}$ where $\beta_{m}=\frac{\alpha_{1}+\cdots+\alpha_{m}}{m}$, will converge to the same limit. We refer the reader elsewhere for a proof of this. The same is true for matrices: if $A^{m}$ is convergent when $m \rightarrow \infty$ then its Cesáro sum $\frac{I+A+A^{2}+\cdots+A^{m}}{m}$ will converge to the same limit. Returning to scalars, it turns out that the sequence $\left\{\beta_{m}\right\}_{m=1}^{\infty}$ can converge even if the original sequence $\left\{\alpha_{m}\right\}_{m=1}^{\infty}$ does not. As an example, consider $\alpha_{m}=(-1)^{m}$, which lack a limit as $m \rightarrow \infty$. Then $\beta_{1}=-1, \beta_{2}=\frac{-1+1}{2}=0, \beta_{3}=\frac{-1+1-1}{3}=\frac{-1}{3}, \ldots$ so in general $\beta_{m}=\frac{-1}{m}$ if $m$ is odd and $\beta_{m}=0$ if $m$ is even. We conclude that the sequence $\left\{\beta_{m}\right\}_{m=1}^{\infty}$ converges to zero.

Now we will explain how an imprimitive transition matrix can be Cesáro summable. We will do this in the case that $k=2$, that is $\mathbf{P}$ has the eigenvalues 1 and -1 of maximum modulus. For a full walk-through of Cesáro summability, see Meyer [9, pp. 630-633]. We start by bringing $\mathbf{P}$ to its Jordan form which after reordering of the eigenvalues, if necessary, will have the following appearance

$$
\mathbf{P}=T J T^{-1}=T\left(\begin{array}{ccc}
1 & 0 & \mathbf{0} \\
0 & -1 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & B
\end{array}\right) T^{-1}
$$

where $B$ is the Jordan blocks for all eigenvalues with modulus less than 1 , and
$\mathbf{0}$ are zero matrices. We get

$$
\begin{aligned}
& \frac{I+\mathbf{P}+\mathbf{P}^{2} \cdots+\mathbf{P}^{m}}{m}=\frac{T T^{-1}+T J T^{-1}+T J^{2} T^{-1} \cdots+T J^{m} T^{-1}}{m} \\
& =T\left(\frac{I+J+J^{2} \cdots+J^{m}}{m}\right) T^{-1} \\
& =T\left(\begin{array}{ccc}
\frac{1+1+1^{2}+\cdots+1^{m}}{m} & 0 & \mathbf{0} \\
0 & \frac{1+(-1)+(-1)^{2}+\cdots+(-1)^{m}}{m} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \frac{I+B+B^{2}+\cdots+B^{m}}{m}
\end{array}\right) T^{-1} \\
& \rightarrow T\left(\begin{array}{lll}
1 & 0 & \mathbf{0} \\
0 & 0 & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0}
\end{array}\right) T^{-1}, \quad \text { when } m \rightarrow \infty
\end{aligned}
$$

because

$$
\begin{aligned}
& \frac{1+1+1^{2}+\cdots+1^{m}}{m}=\frac{m}{m}=1 \longrightarrow 1 \\
& \frac{1-1+1-\cdots+(-1)^{m}}{m}=\beta_{m}= \begin{cases}0, & m \text { even } \\
\frac{-1}{m}, & m \text { odd }\end{cases}
\end{aligned}
$$

as $m \rightarrow \infty$. And since $\rho(B)<\rho(\mathbf{P})=1$ then $\lim _{m \rightarrow \infty} B^{m} \rightarrow \mathbf{0}$ and because Cesáro sums converge to the same limit as the matrix powers, when it exists, we get that $\frac{I+B+B^{2}+\cdots+B^{m}}{m} \rightarrow \mathbf{0}$ when $m \rightarrow \infty$ as well. Furthermore, we can take $\frac{1}{n} \mathbf{e}$ as the first column in $T$ and $\frac{\boldsymbol{\pi}^{T}}{\boldsymbol{\pi}^{T}(\mathbf{e} / n)}$ as the first row in $T^{-1}$ because of theorem 3.3 and so the Cesáro limit of $\mathbf{P}$ has the value

$$
\lim _{m \rightarrow \infty} \frac{I+\mathbf{P}+\mathbf{P}^{2} \cdots+\mathbf{P}^{m}}{m}=\left(\begin{array}{ll}
\mathbf{e} / n & T_{1}
\end{array}\right)\left(\begin{array}{ll}
1 & \mathbf{0} \\
\mathbf{0} & \mathbf{0}
\end{array}\right)\binom{\frac{\boldsymbol{\pi}^{T}}{\boldsymbol{\pi}^{T}(\mathbf{e} / n)}}{Z_{1}}=\mathbf{e} \boldsymbol{\pi}^{T}
$$

If we also consider the Cesáro limit of the probability distribution vector we get

$$
\begin{aligned}
\lim _{m \rightarrow \infty} \frac{\mathbf{p}_{0}^{T}+\mathbf{p}_{1}^{T}+\mathbf{p}_{2}^{T} \cdots+\mathbf{p}_{m}^{T}}{m} & =\lim _{m \rightarrow \infty} \frac{\mathbf{p}_{0}^{T}+\mathbf{p}_{0}^{T} \mathbf{P}+\mathbf{p}_{0}^{T} \mathbf{P}^{2} \cdots+\mathbf{p}_{0}^{T} \mathbf{P}^{m}}{m} \\
& =\mathbf{p}_{0}^{T}\left(\lim _{m \rightarrow \infty} \frac{I+\mathbf{P}+\mathbf{P}^{2} \cdots+\mathbf{P}^{m}}{m}\right) \\
& =\mathbf{p}_{0}^{T} \mathbf{e} \boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T}
\end{aligned}
$$

i.e. the same limit as $\lim _{m \rightarrow \infty} \mathbf{p}_{m}$ for primitive matrices. This time, however, we interpret this vector $\boldsymbol{\pi}^{T}$ a bit differently.

Fix a state $\nu_{j} \in \mathcal{S}$ and define a sequence of random variables $\left\{Y_{m}\right\}_{m=0}^{\infty}$ where we let $Y_{m}$ be 1 if the Markov chain is in state $\nu_{j}$ after the $m^{t h}$ step and 0 otherwise. $Y_{0}=1$ if the chain starts in state $\nu_{j}$ and the sum $Y_{0}+Y_{1}+\cdots+Y_{m}$ is the number of times that the chain is in state $\nu_{j}$ before time step $m+1$. Thus, $\frac{Y_{0}+Y_{1}+\cdots+Y_{m}}{m}$ is the fraction of the total time steps that the chain spends in state $\nu_{j}$. We calculate that

$$
E\left(Y_{m}\right)=1 \cdot P\left(Y_{m}=1\right)+0 \cdot P\left(Y_{m}=0\right)=P\left(Y_{m}=1\right)=P\left(X_{m}=\nu_{j}\right)=p_{j, m}
$$

where $E$ is the expected value, $X_{m}$ the state of the chain at time step $m$ and $p_{j, m}$ is the $j^{t h}$ component of $\mathbf{p}_{m}$ and so

$$
\begin{aligned}
& \lim _{m \rightarrow \infty} E\left(\frac{Y_{0}+Y_{1}+\cdots+Y_{m}}{m}\right)=\lim _{m \rightarrow \infty} \\
& \quad=\lim _{m \rightarrow \infty} \frac{E\left(Y_{0}\right)+E\left(Y_{1}\right)+\cdots+E\left(Y_{m}\right)}{m} \\
& m p_{j, 0}+p_{j, 1}+\cdots+p_{j, m} \\
& m \lim _{m \rightarrow \infty}\left(\frac{\mathbf{p}_{0}^{T}+\mathbf{p}_{1}^{T}+\cdots+\mathbf{p}_{m}^{T}}{m}\right)_{j}=\pi_{j}
\end{aligned}
$$

Thus, we interpret the components in $\boldsymbol{\pi}^{T}$ as the long term expected fraction of time steps that the chain spends in the states in $\mathcal{S}$. We summarize our findings in the following theorem.

Theorem 5.11 (Properties of imprimitive Markov chains). If $\boldsymbol{P}$ is an imprimitive transition matrix then

$$
\lim _{m \rightarrow \infty} \frac{I+\boldsymbol{P}+\cdots+\boldsymbol{P}^{m}}{m}=\boldsymbol{e} \boldsymbol{\pi}^{T}, \quad \text { so } \quad \lim _{m \rightarrow \infty} \frac{\boldsymbol{p}_{0}+\boldsymbol{p}_{1}+\cdots+\boldsymbol{p}_{m}}{m}=\boldsymbol{\pi}^{T}
$$

which components $\pi_{j}$ are interpreted as the long term expected proportion of time steps that the Markov chains spends in state $\nu_{j}$ for all $j=1, \ldots, n$.

Remark 5.12. (i): $\boldsymbol{\pi}^{T}$ is still calculated as the $\|*\|_{1}$-normalized vector to the equation $\boldsymbol{\pi}^{T}=\boldsymbol{\pi}^{T} \mathbf{P}$ because it is the left Perron vector of $\mathbf{P}$, however it cannot be considered as a steady-state vector/asymptotically stable equilibrium point since we need $\mathbf{P}^{m}$ to be convergent as $m \rightarrow \infty$ for this.
(ii): If we consider the Cesáro limit of $\mathbf{p}_{m}$ for a primitive Markov chain, this will also be equal to $\boldsymbol{\pi}^{T}$ and have the same interpretation. Thus, theorem 5.11 is more of a generalization than it is a specialization to imprimitive matrices.

Now, let us consider what is sometimes known as the 'basic Ehrenfest's chain' for it is the imprimitive version of example 5.10 which is constructed by dropping one of the events at each time step.

Example 5.13 (Six molecules in a box: the imprimitive version). As before, let there be six molecules in a box partitioned into two halves $\mathcal{A}$ and $\mathcal{B}$ and let $X_{m}$ be the number of molecules in $\mathcal{A}$ after time step $m$. At every time step we now only choose a molecule at random in the box and then we have to move it to the other half-box. For example, if the situation is as in fig. 5.5 and if we choose a molecule in $\mathcal{B}$, then we have to move it to $\mathcal{A}$. We want to know how much of the time we have the same number of molecules in $\mathcal{A}$ as in $\mathcal{B}$, in the long run.


Figure 5.5: Six molecules in a box: the imprimitive version
We have the following transition probabilities:

* $p_{01}=P\left(X_{m+1}=1 \mid X_{m}=0\right)=1$ because if there are no molecules in $\mathcal{A}$, we must choose one from $\mathcal{B}$ which will be moved to $\mathcal{A}$.
* $p_{65}=P\left(X_{m+1}=5 \mid X_{m}=6\right)=1$ for the same reason as above but for $\mathcal{B}$ instead of $\mathcal{A}$.
* $p_{i(i+1)}=P\left(X_{m+1}=i+1 \mid X_{m}=i\right)=1-\frac{i}{6}$, this is $\mathcal{A}$ gaining one molecule given that it has $0<i<6$ molecules, which happens if we choose a molecule from $\mathcal{B}$ with $\frac{6-i}{6}$ chance.
* $p_{i(i-1)}=P\left(X_{m+1}=i-1 \mid X_{m}=i\right)=\frac{i}{6}$, this is $\mathcal{A}$ loosing one molecule, which happens if we choose a molecule from $\mathcal{A}$ with $\frac{i}{6}$ chance.

Thus, we get the following transition matrix

$$
\mathbf{P}:=\frac{1}{6}\left(\begin{array}{lllllll}
0 & 6 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 5 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 4 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 4 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 5 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 6 & 0
\end{array}\right)
$$

which is irreducible because its graph is strongly connected, see fig. 5.6. Also, we see that all closed walks in $\mathcal{D}(\mathbf{P})$ have even length, so the index of imprimitivity is $k=2$. By theorem 5.11 we need to find the Cesáro limit of the probability distribution vector, which will is equal to the left Perron vector. This is calculated as the solution to $(\mathbf{P}-I)^{T} \boldsymbol{\pi}=\mathbf{0}$ normalized such that $\left\|\boldsymbol{\pi}^{T}\right\|_{1}=1$, which we find to be

$$
\pi_{j}=\binom{6}{j}\left(\frac{1}{2}\right)^{6}, \quad \text { for } \quad j=0, \ldots, 6
$$

where $\pi_{3}=\binom{6}{3}\left(\frac{1}{2}\right)^{6}=0.3125$, so roughly three out of ten time steps will be spent in the chain with equal number of molecules in $\mathcal{A}$ and $\mathcal{B}$ in the long run. Notice how the formula for $\boldsymbol{\pi}^{T}$ is exactly the same as in example 5.10 but the interpretation is different.


Figure 5.6: $\mathcal{D}(\mathbf{P})$ of the imprimitive Ehrenfest's chain

### 5.2.3 Reducible Markov Chains

Let us examine the reducible Markov chains, i.e. categories (3) and (4) in fig. 5.2 This is where things get a bit complicated. Remember that the theorem 3.5 is as far as we can generalize Perron-Frobenius' theorem, thus we have not so much to work with. Yet, we can draw many conclusions by dividing a reducible chain into smaller, irreducible, ones. We begin by a simple example.

Example 5.14 (Throwing a ball, part one). Suppose two people $\nu_{1}, \nu_{2}$ are throwing a ball back and forth and a third person $\nu_{3}$ stands in the middle of the two, trying to catch the ball mid air, see fig. 55.7. Suppose there is a $70 \%$ chance of $\nu_{3}$ catching the ball from either $\nu_{1}$ or $\nu_{2}$ and that the game ends if $\nu_{3}$ catches the ball. We want to know the expected number of throws per game.

Let $X_{m}$ be the position of the ball before throw $m+1$, so $X_{m} \in\left\{\nu_{1}, \nu_{2}, \nu_{3}\right\}$. Then $P\left(X_{m+1}=\nu_{3} \mid X_{m}=\nu_{k}\right)=0.7$ for $k=1,2$ and we let let $P\left(X_{m+1}=\right.$ $\left.\nu_{3} \mid X_{m}=\nu_{3}\right)=1$ for if $\nu_{3}$ catches it, the ball 'never leaves' the middles persons hand, i.e. the game is over. It is clear that $\left\{X_{m}\right\}_{m=0}^{\infty}$ satisfies the Markov property since the position of the ball after each throw is only determined by the probabilities of the thrower itself.


Figure 5.7: Throwing a ball

Thus, we get the following transition matrix

$$
\mathbf{P}=\left(\begin{array}{ccc}
0 & 0.3 & 0.7 \\
0.3 & 0 & 0.7 \\
0 & 0 & 1
\end{array}\right)
$$

which is reducible because it has reducible form 2.1) with $X=\left(\begin{array}{cc}0 & 0.3 \\ 0.3 & 0\end{array}\right), Y=$ $\binom{0.7}{0.7}$ and $Z=(1)$. Remember that the $j^{t h}$ entry in the $m^{t h}$ step probability distribution vector $\mathbf{p}_{m}^{T}$ is $P\left(X_{m}=\nu_{j}\right)$ which in this case is the probability that person $\nu_{j}$ has the ball after the $m^{t h}$ throw for $j=1,2,3$. We find the general formula for $\mathbf{p}_{m}^{T}=\mathbf{p}_{0}^{T} \mathbf{P}^{m}$ by finding the eigenvalues and eigenvectors of $\mathbf{P}^{T}$. Suppose that $\mathbf{p}_{0}=(1,0,0)^{T}$, i.e. $\nu_{1}$ starts to throw the ball. Then

$$
\mathbf{p}_{m}^{T}=1^{m}\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)+\overbrace{\frac{1}{2}\left(\frac{3}{10}\right)^{m}}^{\rightarrow 0}\left(\begin{array}{c}
1 \\
1 \\
-2
\end{array}\right)-\overbrace{\frac{1}{2}\left(-\frac{3}{10}\right)^{m}}^{\rightarrow 0}\left(\begin{array}{c}
-1 \\
1 \\
0
\end{array}\right) \rightarrow\left(\begin{array}{l}
0 \\
0 \\
1
\end{array}\right)
$$

as $m \rightarrow \infty$, so all games will eventually end with $\nu_{3}$ catching the ball. Now we calculate the expected number of time steps before this happens. Let $Y$ be a random variable equal to the number of throws in the game.

$$
\begin{aligned}
& P(Y=1)=P\left(X_{1}=\nu_{3}\right)=0.7 \\
& P(Y=2)=P\left(X_{2}=\nu_{3}\right)=0.3 \cdot 0.7 \\
& P(Y=3)=P\left(X_{2}=\nu_{3}\right)=0.3^{2} \cdot 0.7 \\
& \vdots \\
& P(Y=m)=P\left(X_{m}=\nu_{3}\right)=0.3^{m} \cdot 0.7
\end{aligned}
$$

so generally, the probability of the game-length being $m$ is equal to $P\left(X_{m}=\right.$ $\nu_{3}$ ) which in turn is the probability that $\nu_{1}$ and $\nu_{2}$ throws the ball amongst themselves $m-1$ times, until finally $\nu_{3}$ catches it, i.e. $0.3^{m-1} \cdot 0.7$. The expected value of $Y$ is

$$
E(Y)=\sum_{m=1}^{\infty} m \cdot P(Y=m)=\sum_{m=1}^{\infty} m \cdot 0.3^{m-1} \cdot 0.7=0.7 \sum_{m=1}^{\infty} m \cdot 0.3^{m-1}
$$

which we can calculate using the fact that

$$
\sum_{m=1}^{\infty} m x^{m-1}=\sum_{m=1}^{\infty} \frac{d}{d x} x^{m}=\frac{d}{d x} \sum_{m=0}^{\infty} x^{m+1}=\frac{d}{d x} \frac{x}{1-x}=\frac{1}{(1-x)^{2}}
$$

if $|x|<1$ and then plugging $x=0.3$ into the expression.

$$
E(Y)=0.7 \frac{1}{(1-0.3)^{2}}=0.7 \frac{1}{0.7^{2}}=\frac{1}{0.7}=\frac{10}{7} \approx 1.428
$$

So the expected game-length is almost one and a half throw. The fact that we could calculate this, using some neat calculus, says more about the properties of this particular example than it does for reducible Markov chains in general. It is easy to imagine scenarios where the expected value will not be this easy to calculate and we might then wonder if there some other general method for this. Luckily, there is one! The method, which we will see down below, works for all reducible Markov chains we could possibly imagine.

Since $\mathbf{P}$ is reducible, there exists a permutation matrix $Q$ such that $Q^{T} \mathbf{P} Q=$ $\left(\begin{array}{ll}X & Y \\ 0 & Z\end{array}\right)$. If either $X$ or $Z$ themselves are reducible, then there exist a permutation matrix $Q^{\prime}$ such that

$$
Q^{\prime}\left(Q \mathbf{P} Q^{T}\right) Q^{\prime T}=\left(\begin{array}{ccc}
R & S & T  \tag{5.3}\\
\mathbf{0} & U & V \\
\mathbf{0} & \mathbf{0} & W
\end{array}\right)
$$

and because the product of two permutation matrices is still a permutation matrix, then (5.3) is a just another simultaneous permutation of $\mathbf{P}$. For convenience purposes, write $\mathbf{P} \sim \mathbf{P}^{\prime}$ if $\mathbf{P}^{\prime}$ can be obtained from $\mathbf{P}$ via repeated simultaneous permutations. Repeat the process of bringing the reducible square matrices to reducible form, until we arrive at the following upper triangular block matrix

$$
\mathbf{P} \sim\left(\begin{array}{cccc}
P_{11} & P_{12} & \ldots & P_{1 k}  \tag{5.4}\\
\mathbf{0} & P_{22} & \ldots & P_{2 k} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \ldots & P_{k k}
\end{array}\right)
$$

where each diagonal block $P_{i i}, i=1, \ldots, k$ is either irreducible or a zero block. Finally, permute potential rows with positive elements only in the diagonal block $P_{i i}$ down to the bottom. The resulting form is the so called canonical form for
reducible matrices, as seen below.

$$
\mathbf{P} \sim\left(\begin{array}{cccc|cccc}
P_{11} & P_{12} & \ldots & P_{1 r} & P_{1, r+1} & P_{1, r+2} & \ldots & P_{1, s}  \tag{5.5}\\
\mathbf{0} & P_{22} & \ldots & P_{2 r} & P_{2, r+1} & P_{2, r+2} & \ldots & P_{2, s} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \ldots & P_{r r} & P_{r, r+1} & P_{r, r+2} & \ldots & P_{r, s} \\
\hline \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & P_{r+1, r+1} & \mathbf{0} & \ldots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & P_{r+2, r+2} & \ldots & \mathbf{0} \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & P_{s s}
\end{array}\right)
$$

where $P_{11}, \ldots, P_{r r}$ are irreducible or zero blocks and $P_{r+1, r+1}, \ldots, P_{s s}$ are irreducible (they cannot be zero because the row-sums are equal to 1 for all rows). Now we find out what the canonical form means for the chain by studying subgraphs of $\mathcal{D}(\mathbf{P})$.

We first consider the states corresponding to the rows in a block $P_{i i}$ for $i=$ $1, \ldots, r$ which lies somewhere in the upper left quadrant. Call the corresponding states $\nu_{i_{1}}, \ldots, \nu_{i_{k}}$. In the induced subgraph $\mathcal{D}\left(P_{i i}\right)$ by the vertices $\left\{\nu_{i_{1}}, \ldots, \nu_{i_{k}}\right\}$ there are either no arcs, if $P_{i i}=\mathbf{0}$, or $\mathcal{D}\left(P_{i i}\right)$ is strongly connected, if $P_{i i}$ is irreducible. In the latter case, since $P_{i+1, i}=\mathbf{0}$, i.e. the block below is zero, this means that if the chain moves from some states in $\mathcal{D}\left(P_{i i}\right)$ and moves to some other state $\omega \notin\left\{\nu_{i_{1}}, \ldots, \nu_{i_{k}}\right\}$ then the chain will never enter $\mathcal{D}\left(P_{i i}\right)$ again. Because of this fleeting behaviour, each state $\nu_{i_{1}}, \ldots, \nu_{i_{k}}$ is called transient and the set $\left\{\nu_{i_{1}}, \ldots, \nu_{i_{k}}\right\}$ is called the $i^{\text {th }}$ transient class which we will denote by $\mathcal{T}_{i}$ for $i=1, \ldots, r$. Note that it is of course possible for a chain to move from the $i^{t h}$ transient class to any other state in the chain, if there are arcs to allow it. We can get out of, but not return to $\mathcal{T}_{i}$.

Onto to the blocks in the bottom right quadrant: $P_{r+j, r+j}$ where $1 \leq j \leq$ $s-r$. The induced subgraph $\mathcal{D}\left(P_{r+j, r+j}\right)$ is strongly connected since $P_{r+j, r+j}$ is irreducible, and since there are zero blocks both to the right and left side of the block $P_{r+j, r+j}$ in $\mathbf{P}$, then if the chain enters $\mathcal{D}\left(P_{r+j, r+j}\right)$ it can never leave the states in $\mathcal{D}\left(P_{r+j, r+j}\right)$, i.e. its trapped inside the smaller irreducible Markov chain, corresponding to $\mathcal{D}\left(P_{r+j, r+j}\right)$, forever. We say that a state in $\mathcal{D}\left(P_{r+j, r+j}\right)$ are absorbing or ergodic and the set of the states, $\mathcal{V}\left(P_{r+j, r+j}\right)$, is called the $j^{\text {th }}$ absorbing/ergodic class which we denote by $\mathcal{E}_{j}$ for $j=1, \ldots, r$ (not to be confused with the edge-set of a graph $\mathcal{E}$ ).

Before we go any further, we see the form (5.5) in two examples.

Example 5.15 (Throwing a ball, part two). The matrix $\mathbf{P}$ is already in canonical form, since

$$
\mathbf{P}=\left(\begin{array}{cc|c}
0 & 0.3 & 0.7 \\
0.3 & 0 & 0.7 \\
\hline 0 & 0 & 1
\end{array}\right)=\left(\begin{array}{c|c}
P_{11} & P_{12} \\
\hline \mathbf{0} & P_{22}
\end{array}\right)
$$

where both $P_{11}=\left(\begin{array}{cc}0 & 0.3 \\ 0.3 & 0\end{array}\right)$ and $P_{22}=(1)$ are irreducible matrices. We classify the states $\mathcal{T}_{1}=\left\{\nu_{1}, \nu_{2}\right\}$ and $\mathcal{E}_{1}=\left\{\nu_{3}\right\}$ as the first and only transient and ergodic class, respectively. We recognize the behaviour of the ergodic state from the game-definition: once $\nu_{3}$ catches the ball, then the ball (of this particular game) will never leave $\nu_{3}$. We can see the partition also in $\mathcal{D}(\mathbf{P})$, fig. 5.8.


Figure 5.8: Transient states inside orange ellipse, ergodic inside blue

Example 5.16. In this example we will write a transition matrix $\mathbf{P}$ in canonical form, by considering different subgraphs in $\mathcal{D}(\mathbf{P})$. Suppose we have the matrix

$$
\mathbf{P}=\left(\begin{array}{ccccccc}
0 & 0 & 1 / 2 & 1 / 2 & 0 & 0 & 0 \\
1 / 2 & 0 & 1 / 2 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 / 2 & 1 / 2 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 / 3 & 0 & 1 / 3 & 1 / 3 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array}\right)
$$

We find the graph of $\mathbf{P}$ to be the following, 5.9. For convenience purposes we do not label the arcs in this example.
$\mathcal{D}(\mathbf{P})$ is not strongly connected and in particular that there is only one arc between the set of vertices $\left\{\nu_{5}, \nu_{6}, \nu_{7}\right\}$ and $\left\{\nu_{1}, \nu_{2}, \nu_{3}, \nu_{4}\right\}$. Relabel the vertices such that $\nu_{i^{\prime}}=\nu_{8-i}$, see the red labelling in fig. 5.9 and the rows and columns


Figure 5.9: $\mathcal{D}(\mathbf{P})$
in $\mathbf{P}$ accordingly. The resulting matrix has the reducible form 2.1

$$
\mathbf{P} \sim\left(\begin{array}{ccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 / 3 & 1 / 3 & 0 & 1 / 3 & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 / 2 & 1 / 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 / 2 & 0 & 1 / 2 \\
0 & 0 & 0 & 1 / 2 & 1 / 2 & 0 & 0
\end{array}\right)
$$

Now, for the relabelled vertices in $\mathcal{D}(\mathbf{P})$ we look at the subgraphs $\mathcal{D}_{1}$ induced by $\left\{\nu_{1}, \nu_{2}, \nu_{3}\right\}$ and $\mathcal{D}_{2}$ by $\left\{\nu_{4}, \nu_{5}, \nu_{6}, \nu_{7}\right\}$ respectively, see fig. 5.10. We see that $\mathcal{D}_{2}$ is strongly connected since it contains a 4 -cycle, so the corresponding bottom right block in $\mathbf{P}$ is irreducible. The subgraph $\mathcal{D}_{1}$ is however not strongly


Figure 5.10: Subgraphs $\mathcal{D}_{1}$ (right) and $\mathcal{D}_{2}$ (left).
connected, and by permuting the indices according to (1 $\left.\begin{array}{lll}1 & 2 & 3\end{array}\right)$ and keeping the rest invariant, see the red labelling in fig. 5.10, and also permuting the rows and columns in $\mathbf{P}$ accordingly we get that

$$
\mathbf{P} \sim\left(\begin{array}{cc|c:cccc}
0 & 1 / 3 & 1 / 3 & 1 / 3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hdashline 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 / 2 & 1 / 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 / 2 & 0 & 1 / 2 \\
0 & 0 & 0 & 1 / 2 & 1 / 2 & 0 & 0
\end{array}\right)=\left(\begin{array}{ccccc}
P_{11} & P_{12} & P_{13} & P_{14} \\
\mathbf{0} & P_{22} & P_{23} & P_{24} \\
\hline \mathbf{0} & \mathbf{0} & P_{33} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & P_{44}
\end{array}\right)
$$

in which $P_{11}, P_{22}$ are zero and $P_{33}, P_{44}$ are irreducible. Thus, $\mathbf{P}$ has been brought to its canonical form and we can conclude that the chain has two transit states and two ergodic classes, one of which is a single absorbing state and the other one contains four ergodic states.

From now on we will assume that all transition matrices $\mathbf{P}$ have been brought to canonical form. Now, remember the wonderful theorem 2.22 which for an irreducible transition matrix with index of imprimitivity $k>0$ implies that the eigenvalues of modulus 1 are precisely the $k^{t h}$ unit roots and that each such eigenvalue is simple. Below, we prove a generalization of this also to reducible transition matrices. Remember that an eigenvalue $\lambda$ is semisimple if $\operatorname{algmult}_{A}(\lambda)=$ geomult $_{A}(\lambda)$.

Theorem 5.17. Let $\boldsymbol{P}$ be a reducible transition matrix. Then all $\lambda \in \sigma(\boldsymbol{P})$ such that $|\lambda|=1$ are semisimple.

Proof. The Jordan form of $\mathbf{P}$ is obtained from the Jordan forms of all diagonal irreducible (or zero) blocks $P_{k k}, k=1, \ldots, r, r+1, \ldots, s$ in the canonical form.

Suppose that $P_{k k}$ corresponds to a transient class, that is $k=1, \ldots, r$. Then there is at least one arc from this class to some other transient or ergodic class because otherwise $P_{k k}$ corresponds to an ergodic class. Thus, there must be other nonzero elements, than those elements in $P_{k k}$, in the rows corresponding to the block $P_{k k}$. We conclude that $\mathbf{e}^{T} P_{k k} \leq \mathbf{e}^{T}$, i.e. $\mathbf{e}^{T}\left(P_{k k}-I\right) \leq 0$. If it were the case that $\rho\left(P_{k k}\right)=1$, then by Perron-Frobenius' theorem 3.7. we would get that $P_{k k} \mathbf{p}=\mathbf{p}$ with the right Perron vector $\mathbf{p}>0$. But then

$$
\underbrace{\mathbf{e}^{T}\left(P_{k k}-I\right)}_{\leq 0} \mathbf{p}<0
$$

contradicts that $\left(P_{k k}-I\right) \mathbf{p}=\mathbf{0}$. Thus, $\rho\left(P_{k k}\right)<1$ and so the blocks of transient classes do not contribute with any eigenvalue of modulus 1 .

Suppose instead that $P_{k k}$ corresponds to an ergodic class, that is $k=r+$ $1, \ldots, s$. Then $\rho\left(P_{k k}\right)=1$ and by theorem 2.22 we know that all eigenvalues of
$P_{k k}$ with modulus 1 are simple. So all the blocks of ergodic classes contribute with simple eigenvalues of modulus 1 . The values themselves can be the same for different blocks and thus be repeated in the Jordan form of $\mathbf{P}$. Nevertheless, the eigenvalues $\lambda$ such that $|\lambda|=1$ will have a Jordan block with no 1:s above the diagonal in the Jordan form of $\mathbf{P}$, i.e. $\lambda$ is semisimple.

Just as we did for imprimitive transition matrices, in section 5.2.2, one could prove that also reducible transition matrices are Cesáro summable, using theorem 5.17 and a similar argument as we saw back then. Therefore, all stochastic matrices are Cesáro summable. To find out the value of the Cesáro limit, we will do some (tedious) calculations.

We let $\mathbf{P}=\left(\begin{array}{cc}T_{11} & T_{12} \\ 0 & E_{22}\end{array}\right)$ be the canonical form of $\mathbf{P}$ where $T_{11}$ is the upper left, $T_{12}$ the upper right and $E_{22}$ the bottom right quadrant in 5.5). The classes corresponding to blocks in $T_{11}$ and $E_{22}$ are transient and ergodic, respectively. The Cesáro limit of $\mathbf{P}$ equals

$$
\lim _{m \rightarrow \infty} \frac{I+\mathbf{P}+\cdots+\mathbf{P}^{m}}{m}=\lim _{m \rightarrow \infty}\left(\begin{array}{cc}
\frac{I+T_{11}+\cdots+T_{11}^{m}}{m} & \frac{I+T_{12}+\cdots+T_{12}^{m}}{m} \\
\mathbf{0} & \frac{I+E_{22}+\cdots+E_{22}^{m}}{m}
\end{array}\right)=\left(\begin{array}{ll}
\mathbf{0} & \mathbf{T} \\
\mathbf{0} & \mathbf{E}
\end{array}\right)
$$

where $\frac{I+T_{11}+\cdots+T_{11}^{m}}{m} \longrightarrow \mathbf{0}$ as $m \rightarrow \infty$ because $\rho\left(T_{11}\right)<1$ so $T_{11}^{m} \rightarrow \mathbf{0}$ and the Cesáro limit is equal to the power limit if it exists. Furthermore, we calculate $\mathbf{T}$ and $\mathbf{E}$ accordingly

$$
\begin{aligned}
\mathbf{E} & =\lim _{m \rightarrow \infty} \frac{I+E_{22}+\cdots+E_{22}^{m}}{m} \\
& =\lim _{m \rightarrow \infty}\left(\begin{array}{lll}
\left(\frac{I+P_{r+1, r+1}+\cdots+P_{r+1, r+1}^{m}}{m}\right. & & \\
& \ddots & \\
& =\left(\begin{array}{lll}
\mathbf{e} \boldsymbol{\pi}_{r+1}^{T} & & \\
& \ddots & \\
& & \mathbf{e} \boldsymbol{\pi}_{s}^{T}
\end{array}\right)
\end{array}>. \begin{array}{l}
\frac{I+P_{s s}+\cdots+P_{s s}^{m}}{m}
\end{array}\right)
\end{aligned}
$$

by theorem 5.11 since each $P_{r+j}$ is irreducible. Here, $\boldsymbol{\pi}_{r+j}^{T}$ is the left Perron vector of $P_{r+j}$ for $j=1, \ldots, s-r$. To find the matrix $\mathbf{T}$ we first calculate the Cesáro limit in another way. As we have seen many times, after renumbering if necessary, we can assume that the first $k \geq 1$ diagonal elements in the Jordan
form of $\mathbf{P}$ are 1:s.

$$
\begin{aligned}
\lim _{m \rightarrow \infty} & \frac{I+\mathbf{P}+\cdots+\mathbf{P}^{m}}{m}=T\left(\lim _{m \rightarrow \infty} \frac{I+J+\cdots+J^{m}}{m}\right) T^{-1} \\
& =\left(\begin{array}{cccc}
\mid & & \mid & \\
\mathbf{v}_{1} & \ldots & \mathbf{v}_{k} & T_{1} \\
\mid & & \mid
\end{array}\right)\left(\begin{array}{cccc}
1 & & & \\
& \ddots & & \\
& & 1 & \\
& & & \mathbf{0}
\end{array}\right)\left(\begin{array}{ccc}
- & \mathbf{w}_{1}^{T} & - \\
\vdots & \\
- & \mathbf{w}_{k}^{T} & - \\
Z_{1}
\end{array}\right) \\
& =\left(\begin{array}{cccc}
\mid & & \mid & \\
\mathbf{v}_{1} & \ldots & \mathbf{v}_{k} & \mathbf{0} \\
\mid & & \mid
\end{array}\right)\left(\begin{array}{ccc}
- & \mathbf{w}_{1}^{T} & - \\
& \vdots & \\
- & \mathbf{w}_{k}^{T} & - \\
& \mathbf{0}
\end{array}\right)= \\
& =\mathbf{v}_{1} \mathbf{w}_{1}^{T}+\cdots+\mathbf{v}_{k} \mathbf{w}_{k}^{T}:=\mathbf{G}
\end{aligned}
$$

and $\mathbf{G}$ is the spectral projector onto the eigenspace of 1 because for all $\mathbf{u} \in \mathbb{R}^{n}$

$$
\mathbf{G u}=\mathbf{v}_{1} \overbrace{\mathbf{w}_{1}^{T} \mathbf{u}}^{=: \beta_{1} \in \mathbb{R}}+\cdots+\mathbf{v}_{k} \overbrace{\mathbf{w}_{k}^{T} \mathbf{u}}^{=: \beta_{k} \in \mathbb{R}}=\beta_{1} \mathbf{v}_{1}+\cdots+\beta_{k} \mathbf{v}_{k} \in\left[\mathbf{v}_{1}, \ldots, \mathbf{v}_{k}\right]
$$

This is almost the same as we have seen earlier, in the comment about (vi) below Perron's theorem 3.1, but this time the eigenspace is $k$-dimensional instead of one-dimensional. We know that the eigenspace is the same as the nullspace of the matrix $\mathbf{P}-I$, i.e. $\mathbf{G u} \in \mathrm{N}(\mathbf{P}-I)$ for all $\mathbf{u} \in \mathbb{R}^{n}$ and so $(\mathbf{P}-I) \mathbf{G u}=\mathbf{0}$ for all $\mathbf{u} \in \mathbb{R}^{n}$ and the only matrix which has the entire $\mathbb{R}^{n}$ as its nullspace is the zero matrix, hence $(\mathbf{P}-I) \mathbf{G}=\mathbf{0}$. At the same time, we know from above that $\mathbf{G}=\left(\begin{array}{lll}\mathbf{0} & \mathbf{T} \\ \mathbf{0} & \mathrm{E}\end{array}\right)$ so

$$
(\mathbf{P}-I) \mathbf{G}=\left(\begin{array}{cc}
T_{11}-I & T_{12} \\
\mathbf{0} & E_{22}-I
\end{array}\right)\left(\begin{array}{cc}
\mathbf{0} & \mathbf{T} \\
\mathbf{0} & \mathbf{E}
\end{array}\right)=\left(\begin{array}{cc}
\mathbf{0} & \left(T_{11}-I\right) \mathbf{T}+T_{12} \mathbf{E} \\
\mathbf{0} & \left(E_{22}-I\right) \mathbf{E}
\end{array}\right)=\mathbf{0}
$$

and thus

$$
\left(T_{11}-I\right) \mathbf{T}=-T_{12} \mathbf{E} \quad \Leftrightarrow \quad \mathbf{T}=\left(I-T_{11}\right)^{-1} T_{12} \mathbf{E}
$$

where the inverse exists because $\rho\left(T_{11}\right)<1$ so $1 \notin \sigma\left(T_{11}\right)$ and $0 \notin \sigma\left(T_{11}-I\right)$. We have completed the task of calculating the Cesáro limit of $\mathbf{P}$ and summarize what we have found in a theorem.

Theorem 5.18 (Properties of reducible Markov chains). Let $\boldsymbol{P}=\left(\begin{array}{cc}T_{11} & T_{12} \\ 0 & E_{22}\end{array}\right)$ be a reducible transition matrix brought to canonical form (5.5). Then

$$
\lim _{m \rightarrow \infty} \frac{I+\boldsymbol{P}+\cdots+\boldsymbol{P}^{m}}{m}=\left(\begin{array}{cc}
\boldsymbol{O} & \left(I-T_{11}\right)^{-1} T_{12} \boldsymbol{E}  \tag{5.6}\\
\boldsymbol{O} & \boldsymbol{E}
\end{array}\right)=\boldsymbol{G}
$$

where

$$
\boldsymbol{E}=\left(\begin{array}{ccc}
e \boldsymbol{\pi}_{r+1}^{T} & & \\
& \ddots & \\
& & e \boldsymbol{\pi}_{s}^{T}
\end{array}\right)
$$

and $\boldsymbol{\pi}_{r+j}$ is the left Perron vector of the irreducible block matrix $P_{r+j}$ in $E_{22}$ for all $j=1, \ldots, s-r$. Furthermore, $\lim _{m \rightarrow \infty} \boldsymbol{P}^{m}$ exists if and only if each $P_{r+j}, j=1, \ldots, s-r$ is primitive, in which case the value of the limit is the same as 5.6.

The furthermore part of the theorem above is again a consequence of Cesáro sums converging to the same limit as matrix powers, provided it exists.

The theorem above is of little use if we do not know how to interpret the limit. Every reducible Markov chain will eventually be absorbed into one of its ergodic classes $\mathcal{E}_{j}$, corresponding to an irreducible matrix $P_{r+j}$ and subgraph $\mathcal{D}\left(P_{r+j}\right)$, where $j=1, \ldots, s-r$. Once the chain has entered $\mathcal{E}_{j}$ it will either settle down into a steady state vector inside of the class, if $P_{r+j}$ is primitive, or oscillate between different distribution vectors inside of the class, if $P_{r+j}$ is an imprimitive matrix. The next step is to answer these two interesting questions:
(i): Which ergodic class $\mathcal{E}_{j}$ will the reducible Markov chain eventually get absorbed into?
(ii): How long is it expected to take before the Markov chain gets absorbed into any of the ergodic classes?
The answers to both of these questions will, unlike for irreducible chains, depend on the initial probability distribution vector $\mathbf{p}_{0}^{T}$.
(i): Let $\mathcal{T}_{i}$ and $\mathcal{E}_{j}$ be the $i^{t h}$ transient and $j^{\text {th }}$ ergodic class, respectively. Since we only want to know which ergodic class will absorb the chain and not what happens after absorption, we can convert every ergodic state into a trap by letting $p_{t t}=1$ for all ergodic states $\nu_{t}$. Thus $P_{r+j}=I$ for $j=1, \ldots, s-r$. The modified matrix is $\tilde{\mathbf{P}}=\left(\begin{array}{cc}T_{11} & T_{22} \\ 0 & I\end{array}\right)$ and theorem 5.18 yields

$$
\lim _{m \rightarrow \infty} \tilde{\mathbf{P}}=\left(\begin{array}{cc}
\mathbf{0} & \left(I-T_{11}\right)^{-1} T_{22} \\
\mathbf{0} & I
\end{array}\right)=\left(\right)
$$

The $(p, q)^{t h}$ element in the block matrix $L_{i j}$ (of size $\left.\left|\mathcal{T}_{i}\right| \times\left|\mathcal{E}_{j}\right|\right)$ in the bigger matrix $\left(I-T_{11}\right)^{-1} T_{22}$ is the limiting probability that the chain will be in the $q^{t h}$ state of $\mathcal{E}_{j}$ given that it starts in the $p^{t h}$ state of $\mathcal{T}_{i}$. If the chain is in the $q^{\text {th }}$ state of $\mathcal{E}_{j}$ it has been absorbed by $\mathcal{E}_{j}$ so

$$
\begin{aligned}
& \lim _{m \rightarrow \infty} P\left(X_{m} \in \mathcal{E}_{j} \mid X_{0}=\nu_{p} \in \mathcal{T}_{i}\right)=\sum_{q: \nu_{q} \in \mathcal{E}_{j}}\left(L_{i j}\right)_{p q}=\left(L_{i j} \mathbf{e}\right)_{p} \\
& \lim _{m \rightarrow \infty} P\left(X_{m} \in \mathcal{E}_{j} \mid \mathbf{p}_{0, i}^{T}\right)=\mathbf{p}_{0, i}^{T} L_{i j} \mathbf{e}
\end{aligned}
$$

where the second one is the limiting probability that the chain gets absorbed by the ergodic class $\mathcal{E}_{j}$ given that the probability of starting somewhere in $\mathcal{T}_{i}$ is given by $\mathbf{p}_{0, i}$ (a vector of length $\left.\left|\mathcal{T}_{i}\right|\right)$.
(ii): Next we want to find out how long the chain is in transient states before absorption. If we find the expected number of time steps that the chains spends in any transient state if it starts in any other transient state, this value will be equal to the expected time steps before absorption into any ergodic state. Let $\nu_{i}$ and $\nu_{k}$ be two transient states (not necessarily in the same transient class) and assume that $X_{0}=\nu_{i}$. Define the random variable

$$
Y_{0}=\left\{\begin{array}{ll}
1 & \text { if } X_{0}=\nu_{i}=\nu_{k} \\
0 & \text { otherwise }
\end{array}, \quad Y_{m}= \begin{cases}1 & \text { if } X_{m}=\nu_{k} \\
0 & \text { otherwise }\end{cases}\right.
$$

The sequence $\left\{Y_{m}\right\}_{m=0}^{\infty}$ records the times that the chain is in $\nu_{k}$ given that it starts in $\nu_{i}$. The expected value of this variable is

$$
\begin{aligned}
E\left(Y_{m}\right) & =1 \cdot P\left(Y_{m}=1\right)+0 \cdot P\left(Y_{m}=0\right)=P\left(Y_{m}=1\right) \\
& =P\left(X_{m}=\nu_{k}\right)=\left(T_{11}^{m}\right)_{i k}
\end{aligned}
$$

since $\left(T_{11}^{m}\right)_{i k}$ is the probability that the chain will be in state $\nu_{k}$ given that it starts in state $\nu_{i}$ after exactly $m$ time steps. Remember that $\mathbf{P}=\left(\begin{array}{cc}T_{11} & T_{12} \\ 0 & E_{22}\end{array}\right)$ so $T_{11}$ contains probabilities of moving between transient states. Using the Neumann series $\sum_{m=0}^{\infty} T_{11}^{m}=\left(I-T_{11}\right)^{-1}$ (where $\rho\left(T_{11}\right)<1$ ) we can calculate the expected value of $\sum_{m=0}^{\infty} Y_{m}$, the limiting number of times that the chain is in state $\nu_{k}$ given that it starts in state $\nu_{i}$, as done below.

$$
E\left(\sum_{m=0}^{\infty} Y_{m}\right)=\sum_{m=0}^{\infty} E\left(Y_{m}\right)=\sum_{m=0}^{\infty}\left(T_{11}^{m}\right)_{i k}=\left(\left(I-T_{11}\right)^{-1}\right)_{i k}
$$

Finally if we sum this number over all transient states $\nu_{k}$ in the chain we get the expected number of times that the chain is in any transient state (before
getting absorbed into an ergodic state) given that it starts in state $\nu_{i}$ :

$$
\sum_{k}\left(\left(I-T_{11}\right)^{-1}\right)_{i k}=\left(\left(I-T_{11}\right)^{-1} \mathbf{e}\right)_{i}
$$

so the vector $\left(I-T_{11}\right)^{-1} \mathbf{e}$ tells us how long it will take for the chain to get absorbed into any ergodic state given that it starts in different transient states in the chain. We summarize our answers to question (i) and (ii) in the following theorem.

Theorem 5.19. Assume that $\boldsymbol{P}=\left(\begin{array}{cc}T_{11} & T_{12} \\ 0 & E_{22}\end{array}\right)$ is a reducible transition matrix which has been brought to canonical form (5.5).
(i): The probability that the reducible Markov chain gets absorbed into the particular ergodic class $\mathcal{E}_{j}, j=1, \ldots, s-r$ given that the chain starts somewhere in the transient class $\mathcal{T}_{i}, i=1, \ldots, r$ is determined by

$$
\boldsymbol{p}_{0, i}^{T} L_{i j} \boldsymbol{e}
$$

where $\boldsymbol{p}_{0, i}^{T}$ (of length $\left|\mathcal{T}_{i}\right|$ ) is the part of $\boldsymbol{p}_{0}^{T}$ which indices correspond to states in $\mathcal{T}_{i}$ and $L_{i j}$ is the $(i, j)^{\text {th }}$ block (of size $\left|\mathcal{T}_{i}\right| \times\left|\mathcal{E}_{j}\right|$ ) in the matrix $\left(I-T_{11}\right)^{-1} T_{12}$.
(ii): The expected number of time steps before the chain gets absorbed into any ergodic class, given that the chain starts in the transient state $\nu_{i}$ is determined by the number

$$
\left(\left(I-T_{11}\right)^{-1} \boldsymbol{e}\right)_{i}
$$

All of the values above can be calculated quite easily once we know the canonical form of $\mathbf{P}$. Now we verify the answer in example 5.14 using the general method in the theorem above.

Example 5.20 (Throwing a ball, part three). Remember the transition matrix, in canonical form:

$$
\mathbf{P}=\left(\begin{array}{cc|c}
0 & 0.3 & 0.7 \\
0.3 & 0 & 0.7 \\
\hline 0 & 0 & 1
\end{array}\right)=\left(\begin{array}{c|c}
P_{11} & P_{12} \\
\hline \mathbf{0} & P_{22}
\end{array}\right)=\left(\begin{array}{c|c}
T_{11} & T_{12} \\
\hline \mathbf{0} & E_{22}
\end{array}\right)
$$

Here we have one transient class $\mathcal{T}_{1}$ containing the two states $\nu_{1}, \nu_{2}$ and one ergodic class $\mathcal{E}_{1}$ containing the state $\nu_{3}$. We already know that eventually the state of the ball will be $\nu_{3}$. Still, it is nice to see that our theory is consistent with the previous calculation. We let $\mathbf{p}_{0}^{T}=\left(\begin{array}{lll}1 & 0 & 0\end{array}\right)$ and thus $\mathbf{p}_{0,1}^{T}=\left(\begin{array}{ll}1 & 0\end{array}\right)$ which we multiply to the left of the $(1,1)^{\text {th }}$ block of size $2 \times 1$ in the matrix

$$
\left(I-T_{11}\right)^{-1} T_{22}=\left(\begin{array}{cc}
1 & -0.3 \\
-0.3 & 1
\end{array}\right)^{-1}\binom{0.7}{0.7}=\frac{1}{91}\left(\begin{array}{cc}
100 & 30 \\
30 & 100
\end{array}\right)\binom{0.7}{0.7}=\binom{1}{1}
$$

i.e. the whole matrix itself, together with $\mathbf{e}=(1,1)^{T}$ to the right, giving the product

$$
\mathbf{p}_{0,1}^{T} L_{12} \mathbf{e}=\left(\begin{array}{ll}
1 & 0
\end{array}\right)\binom{1}{1}\binom{1}{1}=\binom{1}{1}
$$

So the probability of this Markov chain getting absorbed by $\mathcal{E}_{1}$ (state $\nu_{3}$ ) both if we start in $\nu_{1}$ or $\nu_{2}$. Let us also calculate the estimated time before absorption, which equals the expected game-length, using theorem 5.19. Suppose that $\nu_{1}$ (the first transient state) starts throwing the ball. By

$$
\left(\left(I-T_{11}\right)^{-1} \mathbf{e}\right)_{1}=\left(\frac{1}{91}\left(\begin{array}{cc}
100 & 30 \\
30 & 100
\end{array}\right)\binom{1}{1}\right)_{1}=\binom{10 / 7}{10 / 7}_{1}=\frac{10}{7} \approx 1.428
$$

we see that the expected game length is almost one and a half throw, which is the same answer we got before.

Now let us take this example one step further and generalize it to arbitrary size. Intuitively it should not make any difference if there are two or $n-1$ people throwing the ball back and forth, if the probability of the middle person catching it is the same for every thrower, since every throw in the game can be seen as an interaction between three agents; the thrower, the catcher and the middle person. Thus we suspect the expected number of throws per game to still equal $\frac{10}{7}$. Once again, we can show that this is the case using theorem 5.19 .

Example 5.21 (Throwing a ball, generalized version). Assume that there are $n \geq 3$ people throwing a ball, one of which in middle trying to catch it mid-air. Suppose that at every throw there is chance $\varphi \in(0,1)$ of the middle person catching the ball and with that ending the game. Let $X_{m} \in\left\{\nu_{1}, \ldots, \nu_{n}\right\}$ be the position of the ball after the $m^{t h}$ throw and suppose that the middle person is state $\nu_{n}$. Then we get the following matrix, already in canonical form, and graph below, see fig. 5.11.
$\mathbf{P}=\left(\begin{array}{ccccc|c}0 & \frac{1-\varphi}{n-2} & \frac{1-\varphi}{n-2} & \ldots & \frac{1-\varphi}{n-2} & \varphi \\ \frac{1-\varphi}{n-2} & 0 & \frac{1-\varphi}{n-2} & \ldots & \frac{1-\varphi}{n-2} & \varphi \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ \frac{1-\varphi}{n-2} & \frac{1-\varphi}{n-2} & \frac{1-\varphi}{n-2} & \ldots & 0 & \varphi \\ \hline 0 & 0 & 0 & \ldots & 0 & 1\end{array}\right)=\left(\begin{array}{c|c}P_{11} & P_{12} \\ \hline \mathbf{0} & P_{22}\end{array}\right)=\left(\begin{array}{c|c}T_{11} & T_{12} \\ \hline \mathbf{0} & E_{22}\end{array}\right)$
There is only one transient class $\mathcal{T}_{1}$ containing the states $\nu_{1}, \ldots, \nu_{n-1}$, which corresponds to the irreducible matrix $T_{11}$. The only ergodic class $\mathcal{E}_{1}$ is person


Figure 5.11: Throwing a ball, generalized version
$\nu_{n}$, a single absorbing state. For convenience purpose let $\phi:=\frac{1-\varphi}{n-2}$. The matrix $I-T_{11}$ and its inverse will be

$$
\begin{gather*}
I-T_{11}=\left(\begin{array}{cccc}
1 & -\phi & \ldots & -\phi \\
-\phi & 1 & \ldots & -\phi \\
\vdots & \vdots & \ddots & \vdots \\
-\phi & -\phi & \vdots & 1
\end{array}\right) \\
\left(I-T_{11}\right)^{-1}=\frac{1}{(n-2) \phi^{2}+(n-3) \phi-1}\left(\begin{array}{ccccc}
(n-3) \phi-1 & -\phi & \ldots & -\phi \\
-\phi & (n-3) \phi-1 & \ldots & -\phi \\
\vdots & \vdots & \ddots & \vdots \\
-\phi & -\phi & \vdots & (n-3) \phi-1
\end{array}\right) \tag{5.7}
\end{gather*}
$$

Note that both matrices are of dimension $(n-1) \times(n-1)$. For example, the result of multiplying the first row in $I-T_{11}$ with the first column in $\left(I-T_{11}\right)^{-1}$ is

$$
\frac{(n-3) \phi-1+\overbrace{\phi^{2}+\cdots+\phi^{2}}^{(n-2) \text { many }}}{(n-2) \phi^{2}+(n-3) \phi-1}=\frac{(n-3) \phi-1+(n-2) \phi^{2}}{(n-2) \phi^{2}+(n-3) \phi-1}=1
$$

and multiplication of the first row in $I-T_{11}$ and the second column in $\left(I-T_{11}\right)^{-1}$ yields

$$
\frac{-\phi-(n-3) \phi^{2}+\phi+\overbrace{\phi^{2}+\cdots+\phi^{2}}^{(n-3) \text { many }}}{(n-2) \phi^{2}+(n-3) \phi-1}=\frac{-(n-3) \phi^{2}+(n-3) \phi^{2}}{(n-2) \phi^{2}+(n-3) \phi-1}=0
$$

which is hopefully enough to convince the reader that 5.7 ) is in fact the inverse. Every component of $\left(I-T_{11}\right)^{-1} \mathbf{e}$ is the row sum of the inverse, which is equal to

$$
\begin{align*}
& \frac{(n-3) \phi-1+\overbrace{-\phi-\cdots-\phi}^{(n-2) \text { many }}}{(n-2) \phi^{2}+(n-3) \phi-1}=\frac{(n-3) \phi-1-(n-2) \phi}{(n-2) \phi^{2}+(n-3) \phi-1} \\
& =\frac{-\phi-1}{(n-2) \phi^{2}+(n-3) \phi-1}=\frac{1}{\varphi} \tag{5.8}
\end{align*}
$$

if we substitute in $\phi:=\frac{1-\varphi}{n-2}$, so the number $\frac{1}{\varphi}$ is the expected game-length no matter how many people there are in the game. Just as we thought! If we have a probability of $\varphi=0.7$ we get the answer $\frac{1}{0.7}=\frac{10}{7} \approx 1.428$, the same as before. While this example is a nice exercise in the properties of reducible Markov chains, the answer could be calculated just using example 5.14 and the logic before this example. Suppose however that the probability of $\nu_{n}$ catching the ball was different for each thrower $\nu_{1}, \ldots, \nu_{n-1}$ or that there were to be more than one middle person, then the expected game-length would probably be difficult to find using other methods than the one in theorem 5.19 ,

Example 5.22 5.16, continued). To finish of this section, let us return to the second reducible Markov chain that we have seen. Here, we will illustrate that the results in theorem 5.19 can be different depending on what the initial probability distribution vector is. We found that, after a couple of simultaneous permutations we could write

$$
\mathbf{P}=\left(\begin{array}{cc|ccccc}
0 & 1 / 3 & 1 / 3 & 1 / 3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 / 2 & 1 / 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 / 2 & 0 & 1 / 2 \\
0 & 0 & 0 & 1 / 2 & 1 / 2 & 0 & 0
\end{array}\right)=\left(\begin{array}{cc|cc}
P_{11} & P_{12} & P_{13} & P_{14} \\
\mathbf{0} & P_{22} & P_{23} & P_{24} \\
\hline \mathbf{0} & \mathbf{0} & P_{33} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & P_{44}
\end{array}\right)
$$

which is equal to $\mathbf{P}=\left(\begin{array}{cc}T_{11} & T_{12} \\ 0 & E_{22}\end{array}\right)$ where $T_{11}=\left(\begin{array}{cc}P_{11} & P_{12} \\ 0 & P_{22}\end{array}\right), T_{12}=\left(\begin{array}{ll}P_{13} & P_{14} \\ P_{23} & P_{24}\end{array}\right)$ and $E_{22}=\left(\begin{array}{cc}P_{33} & \mathbf{0} \\ \mathbf{0} & P_{44}\end{array}\right)$. There are two transient classes $\mathcal{T}_{1}, \mathcal{T}_{2}$, with one state in each, and two ergodic classes $\mathcal{E}_{1}$ and $\mathcal{E}_{2}$ containing one and four states respectively, see fig. 5.12. First, suppose that the chain starts in the transient state $\nu_{1}$ so $\mathbf{p}_{0,1}=(1)$. We show that it is two times more likely that the chain will be absorbed into the ergodic class $\mathcal{E}_{1}$ than class $\mathcal{E}_{2}$, which can also be seen in


Figure 5.12: Transient classes in orange, ergodic classes in blue.
fig. 5.12 since there are two paths from $\nu_{1}$ to $\mathcal{E}_{1}$, one via $\nu_{2}$, and only one to $\mathcal{E}_{2}$.

$$
\begin{aligned}
\left(I-T_{11}\right)^{-1} T_{12} & =\left(\begin{array}{cc}
1 & -1 / 3 \\
0 & 1
\end{array}\right)^{-1}\left(\begin{array}{ccccc}
1 / 3 & 1 / 3 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0
\end{array}\right) \\
& =\left(\begin{array}{c|cccc}
2 / 3 & 1 / 3 & 0 & 0 & 0 \\
\hline 1 & 0 & 0 & 0 & 0
\end{array}\right)=\left(\begin{array}{cc|c}
L_{11} & L_{12} \\
\hline L_{21} & L_{22}
\end{array}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
& \mathbf{p}_{0,1}^{T} L_{11} \mathbf{e}=1(2 / 3) 1=2 / 3 \\
& \mathbf{p}_{0,1}^{T} L_{12} \mathbf{e}=1\left(\begin{array}{llll}
1 / 3 & 0 & 0 & 0
\end{array}\right)\left(\begin{array}{c}
1 \\
1 \\
1 \\
1
\end{array}\right)=1 / 3
\end{aligned}
$$

where $\frac{2}{3}=2 \frac{1}{3}$ which was to be shown. Now, suppose instead that the chain starts in the transient state $\nu_{2}$. In the graph we see that the class $\mathcal{E}_{2}$ cannot be reached from $\nu_{2}$. This is verified by the probabilities below

$$
\begin{aligned}
& \mathbf{p}_{0,2}^{T} L_{21} \mathbf{e}=1(1) 1=1 \\
& \mathbf{p}_{0,2}^{T} L_{22} \mathbf{e}=1\left(\begin{array}{llll}
0 & 0 & 0 & 0
\end{array}\right)\left(\begin{array}{c}
1 \\
1 \\
1 \\
1
\end{array}\right)=0
\end{aligned}
$$

the first one being the probability of absorption into $\mathcal{E}_{1}$. We might wonder how long it will take for the chain to be absorbed into either $\mathcal{E}_{1}$ or $\mathcal{E}_{2}$. This is calculated below.

$$
\left(I-T_{11}\right)^{-1} \mathbf{e}=\left(\begin{array}{cc}
1 & -1 / 3 \\
0 & 1
\end{array}\right)^{-1}\binom{1}{1}=\binom{4 / 3}{1}
$$

The expected number of time steps before absorption is therefore $4 / 3$ if we start in $\nu_{1}$ and 1 if we start in $\nu_{2}$ (of course, since there are nowhere else to go).

Finally, if the chain starts in $\nu_{1}$ and happen to get absorbed by $\mathcal{E}_{2}$, it would be nice to know what happens to the chain inside of the ergodic class. Because the chain will stay in $\mathcal{E}_{2}$, we can ignore all of the other states and consider the 4 -state irreducible Markov chain defined by the matrix

$$
P_{44}=\left(\begin{array}{cccc}
0 & 0 & 1 & 0 \\
1 / 2 & 1 / 2 & 0 & 0 \\
0 & 1 / 2 & 0 & 1 / 2 \\
1 / 2 & 1 / 2 & 0 & 0
\end{array}\right)
$$

which has the graph in fig. 5.13. This smaller Markov chain is also primitive


Figure 5.13: $\mathcal{D}\left(P_{44}\right)$
since the second diagonal element in $P_{44}$ is positive, so we know by theorem 5.9 that the chain will eventually settle down into a steady-state determined by the left Perron vector $\boldsymbol{\pi}_{4}^{T}$. We find this vector to be

$$
\boldsymbol{\pi}_{4}^{T}=\frac{1}{8}\left(\begin{array}{llll}
2 & 3 & 2 & 1
\end{array}\right)
$$

with the interpretation that in the long-run we will most likely find the chain in state $\nu_{2}$ in fig. 5.13 and least likely in $\nu_{4}$ given that the chain has sometime entered class $\mathcal{E}_{2}$.

### 5.3 Google's PageRank Algorithm

We have reached the final section of this thesis where we present one of the most, if not the most, famous modern application of Perron-Frobenius theory: the PageRank algorithm. The material below is a summary of that in the article "Google's secret and Linear Algebra" [4] along with one concrete example.

In the late 1900s, there was a need for structure in the early days of search engines on the internet and specifically for an efficient way of ranking suitable webpages to display for the surfer given some search terms. Two computer science doctorates at Stanford, Sergei Brin and Lawrence Page, developed an
algorithm for this very thing in 1998. Their algorithm, called PageRank, was implemented (and is perhaps used to this day) in the Google search engine.

The basic idea is this: suppose a surfer wants to find a good recipe for lasagna. Then he/she might put in the words 'recipe for lasagne' in a search box. The algorithms job is to determine in which order the websites are displayed to the surfer, it ranks the pages from most to least relevant. The way we might accomplish this is to assign some value or significance, call it $x_{i}$ to each of the pages $i=1, \ldots, n$ on the web (or a subset of them). We assume that the webpages form a network, in which each page is connected to other pages via internet links, see fig. 5.14. We draw an arrow $(i, j)$ if there is an internet link on page $i$ leading to page $j$ and we see that this network is a directed graph $\mathcal{D}$ with the pages as vertices $\mathcal{V}$ and links as arcs $\mathcal{E}$. A first attempt at defining the significance $x_{i}$ is to let it be proportional to the number of in-going arcs to each vertex, like so: $x_{i}=\frac{1}{\lambda}|\{(j, i) \in \mathcal{E}\}|$ where $\lambda$ is some constant.


Figure 5.14: Network (graph) of webpages

This might seem like a reasonable definition, but is however not the way to go, since it could lead to some trouble. Namely, this definition grants websites which are linked to from many other websites high significance, no matter which these other websites are. Consider the search for the best lasagna recipe again. One might have set up a website about pizza, i.e. completely irrelevant to our search, along with many other websites about lasagna, each of which containing one or several links to the pizza-page. The pizza-page would consequently have many in-going arcs in the graph and thus be granted a high significance. This does not seem reasonable. Somehow we need to account for, not only how many but also, which websites link to page $i$. We want to assign a high significance to a page which are linked to from other pages with high significance. Let therefore
$x_{i}$ be proportional to the sum of all values of the pages linking to $i$, as such

$$
x_{i}=\frac{1}{\lambda}\left(\sum_{j:(j, i) \in \mathcal{E}} x_{j}\right)
$$

where $\lambda \neq 0$ again is some constant. We have a system of $n$ linear equations where if we let $\mathbf{x}$ be the vector of the significances we have

$$
\frac{1}{\lambda} A \mathbf{x}=\mathbf{x} \quad \Leftrightarrow \quad A \mathbf{x}=\lambda \mathbf{x}
$$

in which the binary matrix $A$ is the adjacency matrix of $\mathcal{D}$, as defined in 2.6. We recognize this as an eigenvalue problem for the matrix $A$. But we do not just want to solve this for any eigenvalue and eigenvector, we demand two extra requirements for the algorithm to make sense: $\mathbf{x}$ must be nonnegative, since the significance $x_{i} \geq 0$ for all pages $i$ and $\mathbf{x}$ must be unique, for if we have more than one ranking, which one would we choose?

Google's approach of solving this problem is to modify $A$ so that we can model the algorithm as a Markov chain. First, suppose each page has a total importance of 1 and that this is importance is uniformly distributed as weights amongst all outgoing arcs from the page. If the page has no outgoing arcs the we draw a loop at the page. Notice that this also counteracts the potential problem with empty websites just linking to many other pages for these will no longer contribute as much to the linked pages significances.


Figure 5.15: Weighted graph of webpages

Label each arc in the graph $\mathcal{D}$ with these weights, see fig. 5.15 and let $\mathbf{P}$ be the corresponding matrix. The matrix $\mathbf{P}$ will be row stochastic and therefore we can interpret it as a transition matrix for a Markov chain. We let $X_{m}$ be the page
at the time $m$ and let the chain move randomly between all websites through the links with the corresponding probabilities. We let, as usual, $p_{i, m}=P\left(X_{m}=i\right)$ and $\mathbf{p}_{m}^{T}=\left(p_{1, m}, \ldots, p_{n, m}\right)$ be the $m^{t h}$ probability distribution vector. Each time we move from one page to another we update the vector according to $\mathbf{p}_{m+1}^{T}=\mathbf{p}_{m}^{T} \mathbf{P}$. We want to find the limiting probability distribution vector of this chain, for this tells us how likely a random surfer is to visit the pages in our network. We assign the significance $x_{i}$ according to

$$
x_{i}=\lim _{m \rightarrow \infty} p_{i, m} \quad \text { for all } \quad i=1, \ldots, n
$$

By Perron-Frobenius' theory, if this limit exists then is given by the left Perron vector $\boldsymbol{\pi}^{T}$, which is nonnegative and unique (if normalized in the $\|*\|_{1}$-norm) thus satisfying our demands.

Remark 5.23. In practice, one makes sure that $\mathbf{P}$ will be positive by introducing something called a damping factor, which we will discuss below. However, it is entirely possible that the chain is imprimitive or even reducible in which case the limit of powers does not exist. From a mathematical point of view it might be interesting to consider the properties of those rankings either way with the help of the results in the sections above 5.11 and 5.18 but in applications it is easier to adjust the matrix and interpret the modification as a random factor which models a humans impulsive behaviour in some way. For now, we will assume that the matrix $\mathbf{P}$ is primitive.

We know that the vector $\boldsymbol{\pi}^{T}$ is an asymptotically stable equilibrium point, so by mere repeated application of $\mathbf{P}$, on any initial probability distribution, will get us closer and closer to the correct ranking values. Since this is a particularly important point in this algorithm, we remind the reader about why this is the case by the following

$$
\mathbf{p}_{m}^{T}=\mathbf{p}_{0}^{T} \mathbf{P}^{m} \Rightarrow \mathbf{p}_{m}^{T}=c_{1} \boldsymbol{\pi}^{T}+\overbrace{\sum_{k} c_{k} \lambda_{k} \mathbf{w}_{k}^{T}}^{\rightarrow \mathbf{0}} \longrightarrow c_{1} \boldsymbol{\pi}^{T}
$$

as $m \rightarrow \infty$ since all $\lambda_{k} \in \sigma(\mathbf{P}) \backslash\{1\}$ has a modulus smaller than 1 . This way of finding the limit, sometimes called the power method, is often way to compute $\boldsymbol{\pi}^{T}$ in practice because the matrix $\mathbf{P}$ is so big (there are roughly a billion webpages out there) so either spectral analysis or inverse computations would take to long, or be impossible.

This algorithm is dependent, as brought up in the remark, of $\mathbf{P}$ being at least primitive. Now, suppose $\mathbf{P}$ is imprimitive, or even reducible. The workaround that Google did is to take a convex combination of $\mathbf{P}$ and a matrix with certain
probabilities for the pages. The constant parameter $c \in(0,1)$ is sometimes called a damping factor, taken to be 0.85 by Google (at least back in 1998), and the process makes the new matrix $\mathbf{P}^{\prime}=c \mathbf{P}+(1-c) \mathbf{p e}^{T}$ strictly positive. The probabilities in $\mathbf{p}=\left(p_{1}, \ldots, p_{n}\right)^{T}$ is interpreted as giving the random surfer the ability to jump to any webpage, which is not necessarily linked to, from page $i$ (with the probability $1-c$ ) obeying a certain probability distribution. For example, one could let $p_{i}=\frac{1}{n}$ for all $i$, i.e. uniform distribution. Since $\mathbf{P}^{\prime}>0$, it will have all of the desired properties in Perron's theorem 3.1, one of which being that the limit of powers exists. We end this section, and thus the entire thesis by finding the ranking values for the webpages in fig. 5.15.

Example 5.24. We let $p_{i j}$ be the probabilities in the figure and get the following matrix.

$$
\mathbf{P}=\left(\begin{array}{ccccccccc}
0 & 1 / 5 & 1 / 5 & 1 / 5 & 0 & 0 & 1 / 5 & 1 / 5 & 0 \\
1 / 5 & 0 & 0 & 1 / 5 & 1 / 5 & 0 & 1 / 5 & 1 / 5 & 0 \\
0 & 0 & 0 & 1 / 3 & 0 & 1 / 3 & 0 & 0 & 1 / 3 \\
1 / 5 & 0 & 0 & 0 & 1 / 5 & 1 / 5 & 0 & 1 / 5 & 1 / 5 \\
0 & 0 & 0 & 1 / 2 & 0 & 0 & 1 / 2 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 / 2 & 0 & 0 & 0 & 1 / 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 / 2 & 0 & 1 / 2 & 0 & 0
\end{array}\right)
$$

It is reducible, since there are no out-going arcs from page 8, so we do like Google does and form the new matrix: $\mathbf{P}^{\prime}=0.85 \mathbf{P}+0.15 \mathbf{p e}^{T}$ where $\mathbf{p}=$ $(1 / 9,1 / 9, \ldots, 1 / 9)^{T}$ follows uniform distribution. Note that if we would have calculated the left eigenvector corresponding to 1 of the matrix $\mathbf{P}$ alone, this would give us $(0,0,0,0,0,0,0,1,0)^{T}$, and the result based of this vector would be that only page 8 shows up. Perhaps is for the best that we modify the matrix, in this case, after all. We find $\boldsymbol{\pi}^{T}$ to be

$$
\left(\begin{array}{lllllllll}
0.0738 & 0.0292 & 0.0292 & 0.0985 & 0.0561 & 0.0417 & 0.0757 & 0.5540 & 0.0417
\end{array}\right)
$$

and assigning the values $x_{i}$ accordingly gives us the following ranking, from highest to lowest score: $8,4,7,1,5,6,9,2,3$.

## Chapter 6

## Conclusion and Discussion

In this thesis we have shown many general properties of positive and nonnegative matrices. For the latter type, we have considered irreducible matrices, which have strongly connected associated graphs, and reducible matrices, which do not have such graphs. We have also considered primitive and imprimitive matrices, subclasses of irreducible matrices, and we have seen that primitive matrices have only one eigenvalue of maximum modulus and that this eigenvalue is the spectral radius. In chapter three we saw the two main theorems of the theory, namely Perron's and Perron-Frobenius' theorem. We have shown that positive and primitive matrices possess the complete set of Perron-properties, labeled (i)-(vi) in this thesis, imprimitive matrices possesses fewer ones and that not much of the properties hold for reducible matrices. We have also seen many applications of Perron-Frobenius' theory, most of which in the world of Markov chains. These chains can model many different stochastic processes and we have just looked at few of the many interesting examples.

In conclusion, the two, more than a hundred year old, theorems by Oskar Perron and Georg Frobenius are to this day used in a variety of mathematical subjects. It plays an important role in determining the long-term behaviour of nonnegative systems, which is unprecedented by any other theory.

Now we discuss the material in the thesis and present some areas of interest for future studies. There is much more that can be said about the different types of matrices. Here we consider three examples. First, an imprimitive matrix $A$ can be written brought, via a simultaneous permutation, to the following canonical form, sometimes called Frobenius form:

$$
\left(\begin{array}{cccccc}
\mathbf{0} & A_{12} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & A_{23} & \ldots & \mathbf{0} & \mathbf{0} \\
\vdots & \vdots & \ddots & \ddots & \vdots & \\
\mathbf{0} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & A_{k-1, k} \\
A_{k, 1} & \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} & \mathbf{0}
\end{array}\right)
$$

where these zero block matrices on the diagonal are square and the number $k$ is the index of imprimitivity. This form indicates that imprimitive matrices are somehow 'block-similar' to a $k$-cycle graph.

Secondly, there are nice proofs, which can be read for example in HornJohnson [5] Chapter 8.5], on the upper bound for the number $\gamma(A)$ which is the smallest exponent $t$ such that $A^{t}>0$ if $A$ is primitive. These proofs are quite interesting since they rely on finding some property in the graph, relating to some partition of $A$, which can be used to show positivity. In the remark 3.11, we saw that $A^{(n-1)^{2}+1}>0$ but it can even be proved that $A^{(m-1)^{2}+1}>0$ where $m$ is the degree of the minimal polynomial of $A$, which often times will be a smaller number than $n$, see Horn-Johnson [5, p. 545, Further Reading]

Thirdly, as we have shown, the spectrum of an imprimitive matrix is invariant under a rotation by $\frac{2 \pi}{k}$, where $k$ is the index. This means that for a primitive matrix, the spectrum is only invariant under a full $2 \pi$ rotation. We could use to construct interesting patterns by disturbing say a primitive matrix into an imprimitive one and tracing the eigenvalues in $\mathbb{C}$ during the transformation. In fig. 6.1 we see an example of this where the transformation from the primitive matrix (when $\delta=0$ ) to the imprimitive matrix (when $\delta=1$ ) happens by letting $\delta: 0 \rightarrow 1$. I bet that it is possible to create many such interesting and beautiful curves by applying this principle between two different types of matrices.

$$
\left(\begin{array}{ccccc}
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1
\end{array}\right)-\left(\begin{array}{ccccc}
\delta & -4 \delta & \delta & \delta & \delta \\
\delta & \delta & -4 \delta & \delta & \delta \\
\delta & \delta & \delta & -5 \delta & \delta \\
\delta & \delta & \delta & \delta & -4 \delta \\
-4 \delta & \delta & \delta & \delta & \delta
\end{array}\right)
$$

There is also further connection between Markov chains and Perron-Frobenius theory, namely Perron complementation. A full walk-through of this topic can be found in Meyer [9, Chapter 8.5], but the general idea is to obtain the Perron vector of an irreducible transition matrix by gluing together Perron vectors of smaller components in a partition of the matrix. This corresponds to considering certain subsets of vertices in the graph of the matrix and recording the visits to only these states by the chain. These smaller Markov chains, that sit inside the larger one, are called Censored Markov chains. This could perhaps


Figure 6.1: $\delta: 0 \rightarrow 1$
be used as a modification of the PageRank-algorithm, where one could consider certain subsets of the webpages, in which they are easily ranked, and then glue these ranking vectors together using the so called Coupling theorem.

Another application of Perron-Frobenius theory, which we have not talked about, is in economic modelling. For example, the 1973 Nobel prize laureate Wassily Leontief used Perron-Frobenius theory in his demand and supply model.

The interested reader is referred to the following literature for further theorems in and applications of Perron-Frobenius' theory: Horn-Johnson [5], MacCluer [8], Meyer [9] and Shapiro [10].

## Bibliography

[1] G. Bergqvist. Lecture 13. Perron-Frobenius theory. https://courses. mai.liu.se/GU/TATA53/, 2021. Lecture notes. Linear Algebra honours course. TATA53.
[2] A. Borobia and U. R. Trías. A Geometric Proof of the Perron-Frobenius Theorem. Revista Matemática Complutense, 5(1):57-63, 1992.
[3] M. Elmore. Fibonacci Functions. https://www.fq.math.ca/Scanned/ 5-4/elmore.pdf, 1967.
[4] P. F. Gallardo. Google's secret and Linear Algebra. EMS Newsletter, March 2007.
[5] R. A. Horn and C. R. Johnson. Matrix Analysis. Cambridge University Press (Cambridge), second edition, 2013.
[6] M. Izquierdo. Lecture 6. Solving Linear Difference Equations with Constant Coefficients. https://courses.mai.liu.se/GU/TATA82/Dokument/ Lecture6.pdf, 2022. Didactic Materials. Discrete Mathematics. TATA82.
[7] P. Kumlin. 1 A Note on Fixed Point Theory. http://www.math. chalmers. se/Math/Grundutb/CTH/tma401/0304/fixedpointtheory.pdf, 2003/2004. TMA 401/MAN 670 Functional Analysis (Chalmers \& GU).
[8] C. R. MacCluer. The Many Proofs and Applications of Perron's Theorem. SIAM Review, 42(3):487-498, 2000.
[9] C. D. Meyer. Matrix Analysis and Applied Linear Algebra. SIAM, first edition, 2010.
[10] H. Shapiro. Linear Algebra and Matrices: Topics for a Second Course. American Mathematical Society (Swarthmore), 2015.
[11] K. Siegrist. Random Probability, Mathematical Statistics, Stochastic Processes. http://www.randomservices.org/random/. Accessed: 2023-0528.
[12] P. A. Svensson. Abstrakt Algebra. Studentlitteratur (Lund), 2001.

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