Hyperbolic fillings of bounded metric spaces

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Abstract

The aim of this thesis is to expand on parts of the work of Björn–Björn–Shanmugalingam [2] and in particular on the construction and properties of hyperbolic fillings of nonempty bounded metric spaces. In light of [2], we introduce two new parameters λ and ξ to the construction while relaxing a specific maximal-condition. With these modifications we obtain a slightly more flexible model that generates a larger family of hyperbolic fillings. We then show that every hyperbolic filling in this family possess the desired property of being Gromov hyperbolic. Next, we uniformize an arbitrary hyperbolic filling of this type and show that, under fairly weak conditions, the boundary of the uniformization is snowflake-equivalent to the completion of the metric space it corresponds to. Finally, we show that this unifomized hyperbolic filling is a uniform space.

In summary, our construction generates hyperbolic fillings which satisfy the necessary conditions for it to serve its intended purpose of an analytical tool for further studies in [2, Chapters 9-13] or similar. As such, it can be regarded as an improvement to the reference model.

Keywords:

biLipschitz equivalent, Gromov hyperbolic space, hyperbolic filling, metric space, snowflake-equivalent.

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Finally, I would like to thank my fellow students and friends at the department of mathematics for these past few years. Some of you will dive deeper in the academics along with me, and others will seek wisdom elsewhere. I wish you all well on your respective journeys. Special thanks to my friend Kalle Eriksson for his work as my opponent.

I have decided to dedicate this thesis to Klara, whom I hold the dearest.

Nomenclature

$\mathbb{N}$	The set of natural numbers $\{0, 1, 2, ...\}$
$\mathbb{N}^*$	$\mathbb{N}\backslash\{0\}$
$\overline{X}$	Completion of the metric space X (not the closure)
log	Refers to the natural logarithm
$\subset$	Subset, allows equality
$\subsetneq$	Proper subset
α, τ, λ	Parameters that the hyperbolic filling X depends on
ε	Additional parameter that the uniformized hyperbolic filling X_ε depends on

Contents

Chapter 1

Introduction

As far as this thesis is concerned, a *hyperbolic filling* is a particular metric graph with edges considered as unit intervals, constructed from a bounded metric space of at least two points. While metric graphs are subject to independent studies by their own merits, a hyperbolic filling of this kind has shown to be a helpful analytical tool in the study of function spaces on general metric spaces.

There have been many successful attempts at constructing a hyperbolic filling with desirable properties (see e.g Buyalo–Schroeder [3] or Björn–Björn–Shanmugalingam [2]). However, with different assumptions on the reference space and constraints in the method, these models come in varying degree of flexibility. Fewer assumptions on the reference space make the construction more general and among the constructions we have seen, [2] appear to have the least.

We are initially interested in modifying the construction from [2] which is based on a bounded metric space of at least two points, denoted Z, and is monitored by a set of parameters. This set of parameters includes – but is not limited to – the parameters in [2, Chapter 3]. With new conditions unique to this thesis, we can find a vertex set V from Z, and an edge set E, that together define a graph (V, E). By recognizing the edges as unit intervals, the set $X := (V, E)$ equipped with the metric

$$d_X(x, y) = \inf_{\gamma} \ell(\gamma)$$

defines a metric space (X, d_X), and in particular an infinite metric graph. This metric graph is a hyperbolic filling and an arbitrary one of many – due to the freedom in choosing V and E, and the flexibility offered by the parameters, the construction generates a large family of them. From here, we are interested in what relevant properties these possess and how our results relate to [2, Chapters 3-5].

Having deduced some structural properties of (X, d_X), we show the first of three main results of this thesis. It is a nonlocal but global property shared with spaces of constant negative curvature, forcing the sides of geodesic triangles in (X, d_X) to not bend too much outwards.

Theorem 4.6. *X is Gromov hyperbolic.*

We then uniformize X by equipping it with the uniformized metric

$$d_\varepsilon(x, y) = \inf_\gamma \int_\gamma \rho_\varepsilon \, ds,$$

where $\rho_\varepsilon(x) = e^{-\varepsilon d_X(x,v_0)}$, and $\varepsilon > 0$ is a parameter, which yields the metric space $(X, d_\varepsilon) =: X_\varepsilon$. This changes the structure of X and in particular we show that $\operatorname{diam} X_\varepsilon \leq \frac{2}{\varepsilon}$. Moreover, we arrive at the following result.

Proposition 5.4. *Fix $0 < \varepsilon \leq \log \alpha$. Then $\overline{Z}$ and $\partial_\varepsilon X$ are snowflake-equivalent.*

Note that α is one of the parameters that govern the construction of X. Specifically, it means that the two spaces are homeomorphic and that they are in the same equivalence class by the equivalence relation $\simeq$ on $\partial_\varepsilon X$.

Theorem 6.1. *Fix $0 < \varepsilon \leq \log \alpha$. Then X_ε is a uniform domain.*

Finally, we show that X_ε is uniform. In particular, there exists a uniformity constant $A \geq 1$ such that, for every two points $x, y \in X_\varepsilon$, there is a curve in X_ε joining them with bounds on its length depending on A, $d(x, y)$ and the distance from the curve to the boundary of X_ε.

The layout of the thesis is as follows. In Chapter 2, preliminaries that are necessary for this thesis are introduced as definitions, remarks on notations and relevant background results, along with a few examples to illustrate important concepts. Chapter 3 is dedicated to the construction of the family of hyperbolic fillings and concludes with a comparison of the method of construction to our main reference [2]. In Chapter 4 we eventually show the Gromov hyperbolicity of X. The uniformization of X takes place in Chapter 5, where we also show that $\overline{Z}$ is snowflake-equivalent to the boundary of X_ε. Chapter 6 is dedicated to the uniformity of the unifomization of X and concludes with a discussion on the constraint $0 < \varepsilon \leq \log \alpha$ and why it is necessary for the results of Chapters 5 and 6.

Chapter 2

Preliminaries

In this chapter we outline the preliminaries required for this thesis. Definitions and results of greater significance to later chapters or which are less known to the intended audience are introduced with extra care. Note that there do appear definitions in later chapters which are not included here.

2.1 Set Theory

In this thesis, the subset symbol $\subset$ allows equality between sets while $\subsetneq$ denotes the proper subset relation, which is notably different from $\not\subset$. Additionally, we let $\mathbb{N}$ be the set of natural numbers including 0, and $\mathbb{N}^* := \mathbb{N} \backslash \{0\}$.

Briefly on the relevant set theory, let $\mathcal{M}$ be a collection of sets and take $A \in \mathcal{M}$. Then A is *maximal* if $A \not\subset A'$ for every $A' \in \mathcal{M}$ with $A \neq A'$. Furthermore, with $\mathcal{M}$ in mind, a special case of Zorn's lemma states the following:

Lemma 2.1 (Zorn's Lemma)**.** *Consider the partial order $\subset$ on $\mathcal{M}$. If every totally ordered subset of $\mathcal{M}$ has an upper bound in $\mathcal{M}$, then there exists at least one maximal element of $\mathcal{M}$.*

2.2 Metric Spaces

Consider a set X. By first defining what a metric is, we will soon explore what it means for X to be a metric space and what properties (of interest to this thesis) it can possess.

Definition 2.2 (Metric)**.** A function $d : X \times X \to [0, \infty)$ is a metric on X if

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) (Symmetry) $d(x, y) = d(y, x)$,

(iii) (Triangle inequality) $d(x, y) \leq d(x, z) + d(z, y)$,

for all $x, y, z \in X$.

Definition 2.3 (Metric Space)**.** Let d be a metric on X, then the ordered pair (X, d) is a *metric space.*

By definition, two distinct metrics defined on the same set generate two different metric spaces. Depending on the specifics of the metrics, these spaces may not share certain properties when the properties rely on the metric.

For the remainder of this chapter we let (X, d) be a metric space with respect to the metric d, and we let E be a subset of X. Denoting a metric space by the set it is generated from is customary whenever the metric is unambiguous, so we will refer to (X, d) as X.

We denote the open ball in X centred at $x \in X$ and with radius r by $B(x, r)$. It can be scaled by some $a \in \mathbb{R}$ to obtain a radius of ar, in which case we let $aB(x, r)$ denote the scaled ball $B(x, ar)$. We use the notation B_X in later chapters to specify the space to consider the ball in, so for instance

$$B_X(x, r) = \{y \in X : d(y, x) < r\}.$$

E° denotes the *interior* of E and is the set of all inner points of the set, where $x \in E$ is an *inner point* of E if there exists an $r > 0$ such that $B(x, r) \subset E$. The *closure* of E with respect to X is the union of itself and all its limit points in X, where $x \in X$ is a *limit point* to E if for every $B(x, r)$, $r > 0$, the intersection $B(x, r) \cap E$ contains some point other than x. E is open in X if and only if $E = E^\circ$, and is closed in X if and only if E is equal to its closure. However, E is both open and closed *in itself* since on the basis of itself we do not consider points that are not in E. This is the general perspective on metric spaces, so X is both open and closed and as such, the closure of X is itself.

X is *disconnected* if there are two nonempty (relatively) open subsets A and A' of X such that $A \cup A' = X$ while $A \cap A' = \varnothing$. X is *connected* if it is not disconnected. Continuing on with open and closed sets, X is connected if and only if the empty set and X itself are the only two subsets of X that both are open and closed in X. We discuss connectedness further below.

The *diameter* of X, denoted $\operatorname{diam} X$, is given by $\sup_{x,y \in X} d(x, y)$. X is *bounded* if $\operatorname{diam} X$ is finite, otherwise it is *unbounded.* X is *totally bounded* if for every $r > 0$, there exists a finite collection of open balls

$$\{B(x_i, r) : x_i \in X,\, i = 1, 2, \ldots N\}$$

so that $X = \bigcup_{i=1}^{N} B(x_i, r)$. Clearly X is bounded whenever it is totally bounded.

A sequence $\{x_n\}_{n\in\mathbb{N}}$ in X is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ whenever $m, n \geq N$. Since the constant sequence $\{x_n\}_{n\in\mathbb{N}}$ with $x_n = x \in X$ for all $n \in \mathbb{N}$ is a Cauchy sequence, there is at least one Cauchy sequence in X converging to x for each $x \in X$. The converse is not necessarily true.

Example 2.4. Suppose X is the set of rationals, denoted $\mathbb{Q}$, and consider the Cauchy sequence $\{x_n\}_{n\in\mathbb{N}}$ where

$$x_n = \left(1 + \frac{1}{n}\right)^n .$$

Then x_n is rational for all $n \in \mathbb{N}$ so the sequence is in X, but $x_n \to e$ which is irrational so the sequence does not converge in $X = \mathbb{Q}$.

If every Cauchy sequence in X converges in X then the space is *complete.* If X is not complete, we can construct a complete set $\overline{X} \supset X$ such that X is a dense subset of $\overline{X}$. Then $\overline{X}$ is called the *completion* of X and $\partial X := \overline{X} \backslash X$ is the *boundary* of X in its completion. The subset E is *dense* in X if the closure of E with respect to X is X.

Note that while this notation for completeness is otherwise common for denoting the closure, we will only use it for completeness in this thesis. Comparing the two, we see that the closure of $\mathbb{Q}$ with respect to itself is still $\mathbb{Q}$ while the completion is not, evident from Example 2.4. In fact, $\overline{\mathbb{Q}} = \mathbb{R}$.

In the following we construct the completion of a metric space.

Example 2.5. Consider the metric space $X = (X, d)$ and take any two Cauchy sequences $\{x_j\}_{j\in\mathbb{N}}$ and $\{y_j\}_{j\in\mathbb{N}}$ in X with limits x and y respectively, not necessarily in X. Set $\overline{d}(x, y) := \lim_{j\to\infty} d(x_j, y_j)$ and let $\{x_j\}_{n\in\mathbb{N}} \sim \{y_j\}_{n\in\mathbb{N}}$ whenever $\overline{d}(x, y) = 0$. Then $\sim$ defines an equivalence relation (a reflexive, symmetric and transitive relation). Now let C be the set of all equivalence classes induced by this equivalence relation on Cauchy sequences in X. We identify $x \in X$ with the equivalence class $\{x_n\}_{n\in\mathbb{N}}$, where $x_n = x$ for all $n \in \mathbb{N}$. It follows that C is complete and that $C \supset X$. Moreover, by the construction of C, every $x \in C$ is a limit point of X with respect to C so the closure of X is C. Thus X is dense in C and as such, we have managed to construct the completion of X with $\overline{X} = C$.

In the example we do not make any assumptions on the completeness of X. In fact, if X is already complete then $X = \overline{X}$ by the construction of $\overline{X}$.

Furthermore, the metric d on X is induced by the metric $\overline{d}$ on $\overline{X}$ and is the restriction of $\overline{d}$ to X.

X is a *compact* space if for every collection of open sets $\{O_\mu : \mu \in \Gamma\}$ with $X \subset \bigcup_{\mu\in\Gamma} O_\mu$ there exists a finite index set $\Gamma_2 \subset \Gamma$ such that $X \subset \bigcup_{\mu\in\Gamma_2} O_\mu$. The collection $\{O_\mu : \mu \in \Gamma\}$ is an *open cover* of X and the subcollection $\{O_\mu : \mu \in \Gamma_2\}$ is a *finite subcover* of the open cover. It follows that X is totally bounded whenever it is compact. In the case of subsets of $\mathbb{R}^n$, the Heine–Borel theorem serves as an excellent analytical tool on the topic of compactness:

Theorem 2.6 (Heine–Borel theorem)**.** *Let $X \subset \mathbb{R}^n$, then X is compact if and only if X is closed and bounded.*

For general metric spaces it is still the case that a compact metric space is closed and bounded, but the converse does not hold. A generalization of the Heine-Borel Theorem states that a metric space is compact if and only if it is complete and totally bounded. X is *locally compact* if every $x \in X$ has a compact neighbourhood.

The interested reader is primarily directed to Erickson-Andersson–Wiman [4, Chapters 2-4] but also Abbott [1, Chapters 3.2-3.5, 8.2] for more material on the topic of metric spaces as well as proofs of the claims and theorems above.

2.3 Curves In a Metric Space X

A *curve* γ in X is a continuous mapping from an interval $\mathcal{J} \subset \mathbb{R}$ into X. In this thesis we almost exclusively work with $\mathcal{J} = [a, b]$ where $a, b \in \mathbb{R}$, with the exception of geodesic rays (see below). Thus assuming $\mathcal{J} = [a, b]$, the length of γ is given by

$$\ell(\gamma) := \sup_P \sum_{j=1}^{n} d\left(\gamma(x_j), \gamma(x_{j-1})\right)$$

where the supremum is taken over all partitions

$$P = \{x_j : j = 0, 1, 2, \dots n,\ a \leq x_j \leq b\},$$

with $a = x_0 < x_1 < \cdots < x_n = b$, of $[a, b]$. In later chapters when multiple metrics have been defined, we will index the length of a curve with the metric space where the metric that the length is with respect to resides – e.g $\ell_X(\gamma)$. If $\ell(\gamma)$ is finite then γ is *rectifiable*, in which case it can be parameterized by arc length ds so that $\mathcal{J} = [0, \ell(\gamma)]$ and $\ell(\gamma|_{[s,t]}) = t - s$ for any two $s, t \in \mathcal{J}$ with $t \geq s$. Given a function f on X and an arc length parameterized curve

$\gamma : [a, b] \to X$ we define

$$\int_\gamma f\,ds = \int_a^b f\big(\gamma(t)\big)\,dt$$

as the *curve integral* of f along γ. Note that if f is continuous then so is $f\big(\gamma(t)\big)$.

Since most results in this thesis involves curves and rely on them being arc length parameterized, we will from here on always assume that a given curve is arc length parameterized. Färm [5, Chapter 3] expands much further and in more detail on curves and their arc length parameterization. To some extent so does Haefliger–Bridson [6, Part 1, pp. 12-14].

A *geodesic* $\gamma : [0, \ell(\gamma)] \to X$ in $X = (X, d)$ from x to y is a length minimizing curve with $\gamma(0) = x$ and $\gamma(\ell(\gamma)) = y$ such that $d(\gamma(t), \gamma(t')) = |t - t'|$ for all $t, t' \in \mathcal{J}$. In particular, $d(x, y) = \ell(\gamma)$. Hence, if there does not exist a curve γ in X with endpoints x and y such that $\ell(\gamma) = d(x, y)$ then there is no geodesic in X joining the points. A *geodesic ray* is a curve $\gamma : [0, \infty) \to X$ with infinite length such that the restriction $\gamma|_{[0,t]}$ of γ to $[0, t]$ is a geodesic for each $t > 0$.

If the metric $d(x, y) := \inf_\gamma \ell(\gamma)$, where the infimum is taken over all rectifiable curves γ with endpoints x and y, then d is a *length metric* and the metric space is a *length space*. Moreover, X *itself* is *geodesic* if for every $x, y \in X$, there is a curve γ with endpoints x and y such that $\ell(\gamma) = d(x, y)$. Thus, X is geodesic if and only if there is at least one geodesic between every pair of points in X. Furthermore, X is a length space whenever it is geodesic, and conversely the Hopf-Rinow theorem states that a length space X is geodesic whenever it is complete and locally compact. Geodesics and geodesic metric spaces are treated in-depth in [6, Part 1, pp. 2-12, 32-39].

X is *pathconnected* if there is a curve in X, not necessarily rectifiable, between any two points in X. It can be shown that this implies connectedness, but the converse is not true – see e.g [4, Example 6.14] for a counterexample.

When speaking of distance between a point $x \in X$ and a set $E \subset X$ the ordinary metric is insufficient alone, so we introduce the distance function

$$\operatorname{dist}(x, E) := \inf_{y \in E} d(x, y)$$

which gives the distance in X between x and the point in E closest to x (which exists whenever E is compact). For instance, suppose that E is the image of a curve $\gamma : \mathcal{J} \to X$. Then $E = \{\gamma(t) : t \in \mathcal{J}\}$ so $\operatorname{dist}(x, \gamma) := \operatorname{dist}(x, E)$ gives the distance in X from x to the point on the curve closest to x. By $y \in \gamma$ we mean that there is some $t \in \mathcal{J}$ such that $\gamma(t) = y$, thus e.g. $\operatorname{dist}(x, \gamma) = \inf_{y \in \gamma} d(x, y)$.

Definition 2.7 (Roughly Starlike)**.** An unbounded metric space X is *roughly starlike* if there are $x_0 \in X$ and $M > 0$ such that, for any $x \in X$, there is a geodesic ray γ in X starting from x_0 with $\operatorname{dist}(x, \gamma) \le M$.

Example 2.8. Consider an equilateral triangle in $\mathbb{R}^2$ with unit intervals as sides and one of its corners positioned at the origin, denoted x_0. We now take one of the edges that are attached to x_0 and extend it indefinitely in the direction away from the point, see Figure 2.1. Let X be the set given by this altered triangle and equip it with the metric

$$d(x, y) := \inf_{\gamma} \ell(\gamma),$$

where the infimum is taken over all rectifiable curves γ in X joining $x, y \in X$. Then $X = (X, d)$ defines a geodesic metric space which contains a single geodesic ray (the extended edge). Take the corner which does not lie on the extended edge and call it x_1. Clearly $\sup_{x \in X} \operatorname{dist}(x, \gamma) = \operatorname{dist}(x_1, \gamma) = 1$ so for any $x \in X$, $\operatorname{dist}(x, \gamma) \leq 1$. As such, X is roughly starlike with constant $M = 1$.

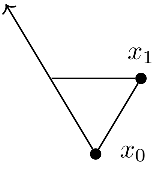

Figure 2.1: Equilateral triangle with extended side.

2.4 Hyperbolic Geometry

Definition 2.9 (Geodesic Triangle)**.** A *geodesic triangle* $\Delta(x, y, z)$ is a triangle in a metric space X with points $x, y, z \in X$ as vertices and geodesics $[x, y]$, $[y, z]$ and $[z, x]$ joining the points as sides.

Figure 2.2: Examples of geodesic triangles.

Definition 2.10 (Slim Triangles)**.** A geodesic triangle $\Delta(x, y, z)$ in a metric space, with $E_1 := [x, y]$, $E_2 := [y, z]$ and $E_3 := [z, x]$, is *δ-slim* if there exists a $\delta \geq 0$ such that for every $i, j = 1, 2, 3$ with $i \neq j$,

$$\operatorname{dist}(w, E_i \cup E_j) \leq \delta \quad \text{for each } w \in E_k \text{ with } k \neq i, j.$$

Definition 2.11 (Gromov Hyperbolicity)**.** A complete unbounded geodesic metric space X is *Gromov hyperbolic* if there is a constant $\delta \geq 0$ for which every geodesic triangle is δ-slim.

Gromov hyperbolicity of a metric space is a global property reliant on the metric and does not prevent local positive curvature, whereas general hyperbolic metric spaces have constant negative curvature as they are spaces of hyperbolic geometry. The *hyperbolic plane* is a common example of a space of hyperbolic geometry and it has curvature -1. The *Poincaré disc* is a model of the hyperbolic plane, representing it as a unit disc where hyperbolic straight lines appear as arcs on the disc orthogonal to the boundary at the points of intersection, or as diameters of the disc. If the Euclidean distance from a point to the origin is r then the hyperbolic distance on the Poincaré model is $2\operatorname{arctanh} r$.

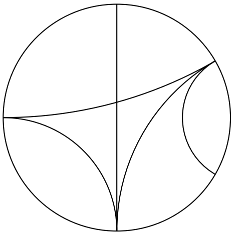

Figure 2.3: The Poincaré disc model.

Due to the negative curvature, the angle sum of triangles in hyperbolic spaces is less than π radians. Since triangle (a) in Figure 2.2 clearly has an angle sum of less than π radians we can imagine it on the Poincaré disc corresponding to a triangle in the hyperbolic plane. Triangle (b) however does not correspond to a triangle of such a space, but it is still possible for it to be contained in a Gromov hyperbolic space assuming the triangle is small enough. Cederberg [7, Chapters 2.4-2.8] provides a gentle introduction to non-Euclidean geometry and in particular hyperbolic geometry. Haefliger–Bridson [6, Part 1, Chapters 2 and 6; Part 2; Part 3] takes a more rigorous approach to hyperbolic spaces and treats the notion of negative curvature and metric spaces of such, before introducing and exploring Gromov hyperbolicity of metric spaces.

There are several different but equivalent ways of defining Gromov hyperbolicity of a metric space. Another relevant definition makes use of the Gromov product as follows.

Definition 2.12 (Gromov Product)**.** Let X be a metric space and consider three points $p, q, s \in X$. Then

$$(p|q)_s = \frac{1}{2}[d_X(s,p) + d_X(s,q) - d_X(p,q)]$$

is the *Gromov product* of p and r with respect to s.

Definition 2.13 ((δ')-hyperbolicity)**.** Let $\delta' > 0$. A metric space X is (δ')-*hyperbolic* if

$$(p|r)_s \geq \min\{(p|q)_s, (q|r)_s\} - \delta'$$

for all $p, q, r, s \in X$.

Note that the constants from each of the definitions of hyperbolicity are not necessarily equal to one another.

The equivalence between the definitions is elaborated on in Theorem 4.5. In the same chapter yet another equivalent definition is mentioned. The reason we recognize all these different definitions instead of deciding on one is due to the unique advantages each of them possess – the first is intuitive and easy to illustrate while the two latter are evidently more suitable analytically.

2.5 Topology and Uniformity

Definition 2.14 (Homeomorphism)**.** Let Y_1 and Y_2 be two metric spaces. A function $\Psi : Y_1 \to Y_2$ is a *homeomorphism* if

(i) Ψ is a bijection,

(ii) Ψ is continuous,

(iii) The inverse function Ψ^{-1} is continuous.

Whenever a homeomorphism $\Psi : Y_1 \to Y_2$ exists, Y_1 and Y_2 are *homeomorphic*, in which case they share the same *topological properties*; connectedness and compactness are such properties, while boundedness and completeness are not. To confirm the validity of the last claim, consider the function $h : (0, 1] \to [1, \infty)$ given by $h(x) = \frac{1}{x}$. It is continuous, is a bijection and has a continuous inverse, so it is a homeomorphism. However, $(0, 1]$ is bounded and incomplete while $[1, \infty)$ is unbounded and complete.

Definition 2.15 (Comparable)**.** Two functions $f : F_1 \to F_2$ and $g : G_1 \to G_2$ are *comparable*, denoted $f \simeq g$, if there are *comparison constants* $C, D > 0$ such that

$$f(x) \leq Cg(y) \quad \text{and} \quad g(y) \leq Df(x),$$

for all $x \in F_1$ and $y \in G_1$, where C and D are independent of x and y.

We will use this relation many times throughout the thesis and whenever two comparison constants are mentioned without explicit reference to any inequalities, we let the order that they are mentioned determine which one of C and D they correspond to (first to C, second to D).

Definition 2.16 (Snowflake-equivalence)**.** Two metric spaces X and Z are *snowflake-equivalent* if there is a homeomorphism $\Psi : Z \to X$ such that for every $z, y \in Z$,

$$d_X(\Psi(z), \Psi(y)) \simeq d_Z(z, y)^\sigma \quad \text{with } \sigma > 0$$

and $d(z, y) := d_Z(z, y)^\sigma$ defines a metric.

Note that $d(z, y) := d_Z(z, y)^\sigma$ always defines a metric whenever $\sigma \leq 1$ but not necessarily otherwise.

Definition 2.17 (biLipschitz equivalent)**.** Two metric spaces X and Z are *biLipschitz equivalent* if there is a bijection $\Psi : Z \to X$ such that for every $z, y \in Z$,

$$d_X(\Psi(z), \Psi(y)) \simeq d_Z(z, y)$$

with the same comparison constants both ways.

Definition 2.18 (Uniform Domain)**.** A nonempty open subset $\Omega \subsetneq X$ of a metric space is an *A-uniform domain*, with $A \geq 1$, if for every pair $x, y \in \Omega$ there is a rectifiable arc length parameterized curve $\gamma : [0, \ell(\gamma)] \to \Omega$ with $\gamma(0) = x$ and $\gamma(\ell(\gamma)) = y$ such that

(i) (Quasiconvex) $\ell(\gamma) \leq Ad(x, y)$,

(ii) (Twisted cone)

$$\operatorname{dist}(\gamma(t), X \backslash \Omega) \geq \frac{1}{A} \min\{t, \ell(\gamma) - t\} \quad \text{for } 0 \leq t \leq \ell(\gamma).$$

The curve γ is an *A-uniform curve* and a noncomplete metric space (Ω, d) is *A-uniform* if it is an A-uniform domain in its completion $\overline{\Omega}$, in which case we may simply call it *uniform* as well.

Chapter 3

Constructing a Hyperbolic Filling

In this chapter we are constructing a hyperbolic filling of a bounded metric space $Z = (Z, d)$ containing at least two points. First fix parameters $\alpha > 1$, $\xi > 0$, $\lambda \geq 1$, $\tau \geq 1$ and $\zeta = \max\{\lambda, \tau\} > 1$. By scaling with some factor $k > 0$ so that $0 < k \operatorname{diam} Z < 1$ we can assume $0 < \operatorname{diam} Z < 1$. Now, take $z_0 \in Z$ and set $A_0 = \{z_0\}$. For each $n \in \mathbb{N}^*$, choose a set $A_n \supset A_{n-1}$ such that

$$B_Z(x, \xi\alpha^{-n}) \cap B_Z(y, \xi\alpha^{-n}) = \varnothing \tag{3.1}$$

for any two points $x, y \in A_n$ with $x \neq y$, and

$$Z = \bigcup_{x \in A_n} B_Z(x, \alpha^{-n}). \tag{3.2}$$

Thus if Z is connected then for every $n \in \mathbb{N}^*$, $2\xi\alpha^{-n} \leq d_Z(x, y) \leq \operatorname{diam} Z$ for all $x, y \in A_n$ with $x \neq y$. If Z is disconnected then there can be exceptions where two points in A_n for one or multiple $n \in \mathbb{N}^*$ are separated by less than $2\xi\alpha^{-n}$. In either case, x and y are always separated by at least $\xi\alpha^{-n}$. Furthermore, depending on Z, it can be possible to choose A_n uniquely for each $n \in \mathbb{N}^*$, or it may be necessary to have $A_n = A_1$ for all $n \in \mathbb{N}^*$.

Example 3.1. Consider the sets $Z_1 = (0, 1)$ and $Z_2 = \{0, \frac{1}{2}\}$ equipped with the metric $d(x, y) = |x - y|$. For the metric space Z_1 there are many ways of choosing A_n for each $n \in \mathbb{N}^*$, but with Z_2 we either have $A_0 = \{0\}$ or $A_0 = \{\frac{1}{2}\}$, and then necessarily $A_n = Z$ for all $n \in \mathbb{N}^*$.

While we allow $\xi > 0$ it is possible that there is a specific set of parameters with $\xi > \frac{1}{2}$ for which we cannot choose A_n for some $n \in \mathbb{N}^*$ such that both (3.1) and (3.2) are satisfied. On the contrary, for any set of parameters with $\xi \leq \frac{1}{2}$ we can always find such a subset of Z for each $n \in \mathbb{N}^*$. We discuss this further in Examples 3.2 and 3.3 below.

Next we define the vertex set $V = \bigcup_{n=0}^{\infty} V_n$, where $V_n = \{(x,n) : x \in A_n\}$. If $m \geq n$ then $x \in A_m$ whenever $x \in A_n$ so if $x \notin A_j,\ j = 0, 1, ..., n-1$, but $x \in A_n$ then it is the first coordinate for points $(x,m) \in V_m \subset V$ for every $m \geq n$ and not for any other point in V. We let two distinct vertices $(x,n), (y,m) \in V$ be *neighbours* if and only if they satisfy the *neighbour conditions*

(i) $|n - m| \leq 1$

(ii)

$$\tau B_Z(x, \alpha^{-n}) \cap \tau B_Z(y, \alpha^{-m}) \neq \varnothing \quad \text{if } m = n, \tag{3.3}$$

$$\lambda B_Z(x, \alpha^{-n}) \cap \lambda B_Z(y, \alpha^{-m}) \neq \varnothing \quad \text{if } m = n \pm 1, \tag{3.4}$$

in which case we denote their relation by $(x,n) \sim (y,m)$. Along with the vertex set we introduce an accompanying edge set E, which contains edges that correspond to the neighbour relations satisfying the above conditions. We consider these edges to be unit intervals. Finally we let the graph $X := (V, E)$ be a *hyperbolic filling* of Z and recognize it as a metric space equipped with the metric $d_X(x,y) = \inf_\gamma \ell(\gamma)$, where the infimum is taken over all rectifiable curves γ in X between $x, y \in X$. As such, X is a *metric graph*. We call a rectifiable curve in X with vertices as endpoints, or a geodesic ray starting from a vertex, a *path*. Due to the similarities between curves and paths the two terms are used somewhat interchangeably throughout the thesis but with the subtle difference that the term path emphasizes the structure of the curve from the perspective of a graph – a connected set of vertices and edges.

A vertex $(x,n) \in V$ is a *parent* of $(y,m) \in V$ if $(x,n) \sim (y,m)$ and $n = m-1$, in which case (y,m) is a *child* of (x,n). Every vertex has at least one child since $(x,n) \sim (x,n+1)$ for any $(x,n) \in V$ as per (3.4). We call $(z_0, 0)$ the *root* of the graph as it is the only vertex with no parent. An edge that is connecting a child and parent is said to be *vertical*, otherwise *horizontal*. As we may already take note of, greater values of τ and λ result in more neighbour relations following the larger radii of the balls in (3.3) and (3.4). In particular, for a fixed τ we get more vertical edges by increasing λ, while the converse yields more horizontal edges.

The construction of a hyperbolic filling of Z is now complete and we conclude this chapter with a comparison of the method and parameters to our main reference Björn–Björn–Shanmugalingam [2]. The construction of $A_n, n \in \mathbb{N}$, in [2] is the special case of the somewhat more flexible construction treated in this thesis, where $\xi = \frac{1}{2}$ and requirement (3.2) is substituted for a maximal-condition:

Given some $n \in \mathbb{N}^*$, assume the set A_{n-1} is defined and consider the collection M_n of α^{-n}-separated subsets ($d(x,y) \geq \alpha^{-n}$ for every point x and y in the set with $x \neq y$) of Z, each of which contains A_{n-1}. $(M_n, \subset)$ is a partially ordered set. Let S be a totally ordered subset of M_n, then the union of the elements in S is an upper bound to S contained in Z. By Zorn's lemma, there is at least one maximal element $A_n \in M_n$. Thus, we can recursively choose a *maximal* α^{-n}-separated set A_n for each $n \in \mathbb{N}$ such that $A_n \subset A_m$ whenever $m \geq n \geq 0$. Since A_n is maximal, there is no $z \in Z$ satisfying $z \notin B_Z(x, \alpha^{-n}) \cap B_Z(y, \alpha^{-n})$ for any $x, y \in A_n$, so by construction $Z = \bigcup_{x \in Z} B_Z(x, \alpha^{-n})$. Notice that the maximal condition reduces the possible choices of A_n for each $n \in \mathbb{N}$ with respect to $A_0 = \{z_0\}$. No such condition is forced on our construction and so we allow a wider range of hyperbolic fillings of Z for every set of fixed parameters and root.

Comparing the neighbour conditions, [2, Equation (3.2), (3.3)] treats the special case of (3.3) and (3.4) where $\lambda = 1$ and $\tau > 1$. As [2, Example 8.8] has shown, it is possible by their construction to find a "hyperbolic" filling that is not Gromov hyperbolic when $\lambda = \tau = 1$, but as is evident from Chapter 4 and in particular Lemma 4.3 with Theorem 4.6, this issue does not concern our construction as it takes the necessary precaution of forcing $\lambda > 1$ through $\zeta = \max\{\lambda, \tau\} > 1$. However, in the case that $\lambda = 1$, we require $\tau > 1$.

Regarding α it has the same purpose here as in [2] with the constraint $\alpha > 1$ being obvious. As for the constraint (or lack thereof) set on ξ in this thesis, the following two examples illustrate the comparative freedom we have in choosing ξ depending on the connectedness of Z.

Example 3.2. Let $Z = \{\frac{x}{10} : x = 0, 1, \ldots, 5\}$ where $\operatorname{diam} Z = \frac{1}{2}$ and set $A_0 = \{0\}$ and $A_n = Z$ for all $n \in \mathbb{N}^*$. Then clearly (3.2) is satisfied for any $\alpha > 1$, and A_n contains A_{n-1} for each $n \in \mathbb{N}^*$. Set $\alpha = 10$, then $\xi\alpha^{-n} < \frac{1}{10}$ for every $n \in \mathbb{N}^*$ and any $\xi < 1$, thus satisfying (3.1) whenever $\xi < 1$ since $\inf_{x,y \in Z} d(x,y) = \frac{1}{10}$ whenever $x \neq y$. As such, the set $\{A_n\}_{n \in \mathbb{N}}$ can generate a well-posed hyperbolic filling (by our standards) for choices of ξ greater than $\frac{1}{2}$. In fact, we can allow any $\xi < 10^k$, $k \in \mathbb{N}$, by setting $\alpha = 10^{k+1}$.

Since ξ is fixed to $\frac{1}{2}$ in [2] it follows by the similarities of our constructions that any set of parameters with $\xi \leq \frac{1}{2}$ yields a satisfactory hyperbolic filling. With Example 3.2 we have seen cases where much greater choices of ξ are allowed, but the following shows that this is an exception.

Example 3.3. Let Z be a connected metric space and suppose A_0 has been chosen. For $A_1 \supset A_0$ to satisfy (3.1) it is necessary that we choose A_1 so that $d_Z(x, y) \geq \xi \frac{1}{\alpha}$ for any two $x, y \in A_1$ with $x \neq y$. By (3.2) it is also necessary that $d_Z(x, y) < \frac{2}{\alpha}$ whenever x and y are adjacent in A_1. As such, we need $\xi \frac{1}{\alpha} \leq d_Z(x, y) < \frac{2}{\alpha}$ for adjacent x and y in A_1, but this can only happen if $\xi < 2$.

A greater value of ξ yields larger radii of the balls in (3.1), which results in fewer points in A_n for each $n \in \mathbb{N}$ and therefore fewer vertices in X.

Chapter 4

Properties of The Hyperbolic Filling

For the remainder of this thesis we let Z be a metric space such that $0 < \operatorname{diam} Z < 1$ and X be an arbitrary hyperbolic filling of Z with fixed parameters and root z_0 in accordance with the construction in Chapter 3. Recall that the metric we equipped X with is $d_X(x,y) = \inf_\gamma \ell(\gamma)$.

In this chapter we take a closer look at the properties of X by ultimately showing that it is Gromov hyperbolic. To begin with, consider the mapping $\pi : V \to \mathbb{N}$ defined by $\pi((x,n)) = n$ and set $v_0 := (z_0, 0)$ as the root of the metric graph X.

Lemma 4.1. *For every $v \in V$ there exists a geodesic γ between v and v_0 corresponding to a path of only vertical edges such that $d_X(v,v_0) = \ell(\gamma) = \pi(v)$. Moreover, X is connected.*

Proof. We begin by proving the first claim. If $v = v_0$ then $d_X(v,v_0) = d_X(v_0,v_0) = 0$ and $\pi(v_0) = \pi(z_0,0) = 0$, so suppose $v = (x,n) \in V$ with $x \in A_n$, $n \in \mathbb{N}$, such that $v \neq v_0$. By construction, $\bigcup_{z \in A_j} B_Z(z, \alpha^{-j})$ covers Z for each $j \in \mathbb{N}$, so there exists a sequence $\{x_j\}_{j=0}^{n-1}$ with $x_j \in A_j$ such that $x \in B_Z(x_j, \alpha^{-j})$ for $j = 0,1,\dots,n-1$. Hence for each such $j \neq 0$,

$$x \in B_Z(x_{j-1}, \alpha^{-(j-1)}) \cap B_Z(x_j, \alpha^{-j})$$

and therefore

$$\lambda B_Z(x_{j-1}, \alpha^{-(j-1)}) \cap \lambda B_Z(x_j, \alpha^{-j}) \neq \varnothing.$$

Set $v_j = (x_j, j) \in V$. By the neighbour condition (3.4) it follows that $v_0 \sim v_1 \sim \cdots \sim v_{n-1}$, which defines a path of vertical edges from v_0 to v_{n-1} of length $n-1$. But

$$x \in B_Z(x_{n-1}, \alpha^{-(n-1)}) \cap B_Z(x, \alpha^{-n})$$

so there is a vertical edge from v_{n-1} to v as well. Thus, $d_X(v, v_0) \leq n$. As all edges in this path from v_0 to v are vertical and necessary to reach v, it is the shortest possible path and thus defines a geodesic in X. By definition of d_X we arrive at $d_X(v, v_0) = \inf_\gamma \ell(\gamma) = n$. Since $\pi(v) = \pi((x, n)) = n$ we finally get $d_X(v, v_0) = \pi(v)$.

It now follows from the above that, since there is a geodesic from v_0 to any $v \in V$, there is a path between any two vertices in V passing through v_0. Therefore, V is connected as a graph and X is pathconnected and thus connected as a metric space. □

While the following corollary is interesting in itself as it tells us more about the structure of the hyperbolic filling, it is also relevant for further studies on X in the upcoming chapters.

Corollary 4.2.

(a) *X is a geodesic space.*

(b) *For every $v \in V$, there is a geodesic ray starting at v_0 and containing v.*

(c) *Every geodesic ray starting at v_0 consists only of vertical edges.*

(d) *Any geodesic from any $x \in X$ to the root v_0 contains at most a half of a horizontal edge.*

(e) *X is roughly starlike with $M = \frac{1}{2}$.*

Proof.

(a) Take $x, y \in X$ with $x \neq y$, we want to show that there exists a shortest curve γ in X so that $\ell(\gamma) = d_X(x, y)$. Since edges in X are unit intervals it is clear that there exists a shortest curve from any point on an edge to any of its vertices, so it will suffice to show the existence of a shortest curve joining x and y whenever they are vertices. Let $\mathcal{D}$ be the set of the lengths of every curve in X with endpoints x and y. By Lemma 4.1 we know that $\mathcal{D}$ is nonempty and since $d_X(x, y) \in \mathbb{N}^*$ whenever $x, y \in V$ it follows that $\mathcal{D} \subset \mathbb{N}^*$. But then there is a minimal element of $\mathcal{D}$, so there is a shortest curve in X joining x and y.

(b) Let $v \in V$ be fixed. From Lemma 4.1 we know there is a geodesic γ from v_0 to v, so by construction of X a curve from v_0 to any $w \in \gamma$ along γ is also a geodesic. Since every vertex in V has at least one child we can easily extend γ indefinitely so that the curve along γ from v to any $w \in \gamma$ that is further away from v_0 than v is a geodesic. Thus, γ is a geodesic ray.

(c) We will show the contrapositive of the statement. Take a ray γ in X and suppose it contains a horizontal edge between vertices $v := (x, n)$ and $w := (y, n)$. Then $\ell(\gamma|_{[0,t]}) \geq n$ for t such that $\gamma(t) = v$, so $\ell(\gamma|_{[0,t+1]}) \geq n + 1$ where $\gamma(t + 1) = w$. By Lemma 4.1 we have $d_X(v_0, w) = \pi(w) = n$ and so $\gamma|_{[0,w]}$ is not geodesic and thus γ is not a geodesic ray.

(d) If $x \in V$ then there are no horizontal edges in any geodesic from x to v_0 according to (b) and (c). Similarly, if x is on a vertical edge (recall that we consider edges to be unit intervals) then any geodesic from x to v_0 is the extension of a geodesic from v_0 to the upper vertex of the vertical edge that x in on, together with the part of the edge that is between x and the vertex. There are no horizontal edges in such a geodesic. Suppose instead that x is on a horizontal edge, then x is between two vertices which are on equal distance from v_0. Therefore, any geodesic from x to v_0 is the extension of a geodesic from v_0 to the vertex closest to x together with the part of the horizontal edge that is between x and the vertex. Thus, for any $x \in X$, any geodesic from x to v_0 contains at most half a horizontal edge.

(e) It follows from (b), (c) and (d) that, for any $x \in X$, there is a geodesic ray γ in X starting from the root v_0 with $\mathrm{dist}(x, \gamma) \leq \frac{1}{2}$. By definition, X is roughly starlike with $M = \frac{1}{2}$. □

Lemma 4.3. *Let $v = (z, n)$ and $w = (y, m)$ be two vertices in X. Then*

$$\alpha^{-(v|w)_{v_0}} \simeq d_Z(z, y) + \alpha^{-n} + \alpha^{-m}$$

with comparison constants α^{l+2} and $\frac{4\zeta\alpha}{\alpha-1}$, where l is the smallest nonnegative integer such that $\alpha^{-l} \leq \zeta - 1$.

Proof. Without loss of generality, assume $n \leq m$. If $z = y$ then there are $m-n$ vertical edges between v and w so $d_X(v, w) = \inf_\gamma \ell(\gamma) = m-n$, which yields the Gromov product $(v|w)_{v_0} = \frac{1}{2}(\pi_2(v)+\pi_2(w)-(m-n)) = \frac{1}{2}(n+m-(m-n)) = n$. Moreover, $d_Z(z, y) = 0$ and $\alpha^{-m} \leq \alpha^{-n}$. Thus,

$$\alpha^{-(v|w)_{v_0}} = \alpha^{-n} \leq d_Z(z, y) + \alpha^{-n} + \alpha^{-m}$$

and

$$d_Z(z,y) + \alpha^{-n} + \alpha^{-m} \leq \alpha^{-n} + \alpha^{-n} = 2\alpha^{-(v|w)_{v_0}}$$

so $\alpha^{-(v|w)_{v_0}} \simeq d_Z(z,y) + \alpha^{-n} + \alpha^{-m}$ with comparison constants 1 and 2, respectively.

Now suppose that $z \neq y$ and let $w_0 \sim w_1 \sim \cdots \sim w_k$ with $w_i = (y_i, n_i),\ i = 0, 1, \ldots, k$, be a geodesic in X between $v = w_0$ and $w = w_k$. Then $d_X(v,w) = k$, so

$$(v|w)_{v_0} = \frac{1}{2}(d_X(v,v_0) + d_X(w,v_0) - d_X(v,w)) = \frac{1}{2}(n + m - k).$$

Recall that $\zeta = \max\{\lambda, \tau\}$. It follows from the neighbour conditions that

$$\zeta B_Z(y_i, \alpha^{-n_i}) \cap \zeta B_Z(y_{i+1}, \alpha^{-n_{i+1}}) \neq \varnothing \quad i = 0, 1, \ldots, k-1,$$

so

$$d_Z(y_i, y_{i+1}) < \zeta(\alpha^{-n_i} + \alpha^{-n_{i+1}})$$

and thus by the triangle inequality

$$d_Z(z,y) = d_Z(y_0, y_k) \leq \sum_{i=0}^{k-1} d_Z(y_i, y_{i+1}) < \sum_{i=0}^{k-1} \zeta(\alpha^{-n_i} + \alpha^{-n_{i+1}}). \tag{4.1}$$

Since $n = n_0$ and $m = n_k$, and therefore $\alpha^{-n} < \zeta\alpha^{-n_0}$ and $\alpha^{-m} < \zeta\alpha^{-n_k}$, we then get

$$\begin{aligned} d_Z(z,y) + \alpha^{-n} + \alpha^{-m} &< \alpha^{-n} + \alpha^{-m} + \sum_{i=0}^{k-1} \zeta(\alpha^{-n_i} + \alpha^{-n_{i+1}}) \\ &= \alpha^{-m} + \sum_{i=0}^{k-1} \zeta\alpha^{-n_i} + \alpha^{-n} + \sum_{i=1}^{k} \zeta\alpha^{-n_i} \\ &< 2\zeta \sum_{i=0}^{k} \alpha^{-n_i} \\ &= 2\zeta \left(\sum_{i=0}^{N-1} \alpha^{-n_i} + \sum_{j=N}^{k} \alpha^{-n_j} \right) \end{aligned}$$

for every $N = 0, 1, \ldots, k, k+1$ with $\sum_{i=0}^{N-1} \alpha^{-n_i}$ and $\sum_{j=N}^{k} \alpha^{-n_j}$ empty when $N = 0$ and $N = k+1$ respectively. We have that

$$\sum_{j=N}^{k} \alpha^{-n_j} = \alpha^{-n_N} + \alpha^{-n_{N+1}} + \cdots + \alpha^{-n_{k-1}} + \alpha^{-n_k} = \sum_{j=0}^{k-N} \alpha^{-n_{k-j}}$$

so with $n_i \geq n_0 - i = n - i$ and $n_{k-j} \geq n_k - j = m - j$ it follows that

$$\begin{aligned}
2\zeta\left(\sum_{i=0}^{N-1} \alpha^{-n_i} + \sum_{j=N}^{k} \alpha^{-n_j}\right) &\leq 2\zeta\left(\sum_{i=0}^{N-1} \alpha^{-n}\alpha^{i} + \sum_{j=0}^{k-N} \alpha^{-m}\alpha^{j}\right) \\
&= 2\zeta\left(\alpha^{-n}\frac{\alpha^{N}-1}{\alpha-1} + \alpha^{-m}\frac{\alpha^{k-N+1}-1}{\alpha-1}\right) \\
&< \frac{2\zeta}{\alpha-1}\left(\alpha^{N-n} + \alpha^{k-N-m+1}\right) \\
&\leq \frac{2\zeta}{\alpha-1}\left(\alpha^{\frac{1}{2}(k-m+n)+1-n} + \alpha^{k-\frac{1}{2}(k-m+n)-m+1}\right) \\
&= \frac{2\zeta}{\alpha-1}\left(\alpha^{\frac{1}{2}(k-m-n)+1} + \alpha^{\frac{1}{2}(k-m-n)+1}\right) \\
&= \frac{4\zeta\alpha}{\alpha-1}\alpha^{-(v|w)_{v_0}}
\end{aligned}$$

whenever $\frac{1}{2}(k-m+n) \leq N \leq \frac{1}{2}(k-m+n)+1$, which shows that $d_Z(z,y)+\alpha^{-n}+\alpha^{-m} \lesssim \alpha^{-(v|w)_{v_0}}$ with comparison constant $\frac{4\zeta\alpha}{\alpha-1}$. The estimations $\frac{1}{2}(k-m+n) \leq k+1$ and $\frac{1}{2}(k-m+n)+1 \geq 0$ shows that we can indeed choose such an N with $N \in \{0,1,\ldots,k,k+1\}$.

Next we show $\alpha^{-(v|w)_{v_0}} \lesssim d_Z(z,y) + \alpha^{-n} + \alpha^{-m}$. By Lemma 4.1 there are geodesics

$$v_0 \sim v_1 \sim \cdots \sim v_n \quad \text{and} \quad w_0 \sim w_1 \sim \cdots \sim w_m \tag{4.2}$$

in X from the root $v_0 = w_0$ to $v_n = (z,n)$ and $w_m = (y,m)$ respectively, where $v_j := (z_j, j)$ and $w_i := (y_i, i)$. By construction,

$$d_Z(z,z_j) < \alpha^{-j} \quad \text{and} \quad d_Z(y,y_i) < \alpha^{-i}$$

for each $j = 0,1\ldots,n$ and $i = 0,1\ldots,m$ respectively. Note that since $(z,n) \sim (z,n+1) \sim \ldots$ there exist $z_j \in A_j$ with $d_X(z,z_j) < \alpha^{-j}$ (e.g $z_j = z$) such that $(z_j,j) \sim (z_{j+1},j+1)$ for $j \geq n$ as well. The same applies to the other path.

Let k be the smallest nonnegative integer such that and $\alpha^{-k-1} < d_Z(z,y)$. Set $k_0 := \min\{k-l,n\}$, where l is the smallest nonnegative integer such that

$\alpha^{-l} \leq \zeta - 1$, and suppose that $k_0 \geq 0$ to begin with. Then $\alpha^{k_0-k} \leq \alpha^{-l} \leq \zeta - 1$ from which it follows that

$$\begin{aligned} d_Z(z, y_{k_0}) &\leq d_Z(z, y) + d_Z(y, y_{k_0}) \\ &< \alpha^{-k} + \alpha^{-k_0} = \alpha^{-k_0}\left(\alpha^{k_0-k} + 1\right) \leq \zeta\alpha^{-k_0}, \end{aligned}$$

so

$$d_Z(z, y_{k_0}) < \zeta\alpha^{-k_0}. \tag{4.3}$$

If $\zeta = \tau$ then $d_Z(z, y_{k_0}) < \tau\alpha^{-k_0}$ so $z \in \tau B_Z(y_{k_0}, \alpha^{-k_0})$. Moreover, $z \in \tau B_Z(z_{k_0}, \alpha^{-k_0})$ since $d_Z(z, z_{k_0}) < \alpha^{-k_0} < \tau\alpha^{-k_0}$. Thus

$$\tau B_Z(z_{k_0}, \alpha^{-k_0}) \cap \tau B_Z(y_{k_0}, \alpha^{-k_0}) \neq \varnothing$$

and therefore $(z_{k_0}, k_0) \sim (y_{k_0}, k_0)$ which with (4.2) yields

$$\begin{aligned} (z, n) \sim (z_{n-1}, n-1) \sim \cdots \sim (z_{k_0}, k_0) \sim (y_{k_0}, k_0) \sim \\ \sim \cdots \sim (y_{m-1}, m-1) \sim (y_m, m), \end{aligned} \tag{4.4}$$

where $(z_{k_0}, k_0) \sim (y_{k_0}, k_0)$ collapses into a single vertex if $z_{k_0} = y_{k_0}$. It follows that $d_X(v, w) \leq (n - k_0) + (m - k_0) + 1 = n + m + 1 - 2k_0$ and as such,

$$(v|w)_{v_0} = \frac{1}{2}(n + m - d_X(v, w)) \geq k_0 - \frac{1}{2}.$$

If $\zeta = \lambda$ then by (4.3) we have $d_Z(z, y_{k_0}) < \lambda\alpha^{-k_0}$ so $z \in \lambda B_Z(y_{k_0}, \alpha^{-k_0})$. Moreover, $z \in \lambda B_Z(z_{k_0+1}, \alpha^{-(k_0+1)})$ since $d_Z(z, z_{k_0+1}) < \alpha^{-(k_0+1)} < \lambda\alpha^{-(k_0+1)}$. Thus

$$\lambda B_Z(z_{k_0+1}, \alpha^{-(k_0+1)}) \cap \lambda B_Z(y_{k_0}, \alpha^{-k_0}) \neq \varnothing$$

and therefore $(z_{k_0+1}, k_0 + 1) \sim (y_{k_0}, k_0)$ which with (4.2) yields

$$\begin{aligned} (z, n) \sim (z_{n-1}, n-1) \sim \cdots \sim (z_{k_0+1}, k_0+1) \sim (y_{k_0}, k_0) \sim \\ \sim \cdots \sim (y_{m-1}, m-1) \sim (y_m, m). \end{aligned} \tag{4.5}$$

In the special case where $k_0 = n$ this gives us the path

$$(z, n) \sim (z_{n+1}, n+1) \sim (y_{k_0}, k_0) \sim \ \sim \cdots \sim (y_{m-1}, m-1) \sim (y_m, m).$$

When $k_0 \neq n$, (4.5) yields $d_X(v, w) \leq (n-(k_0+1))+(m-k_0)+1 = n+m-2k_0$, whereas the special case results in $d_X(v, w) \leq 2+(m-k_0)$. Either way we arrive at

$$(v|w)_{v_0} = \frac{1}{2}(n + m - d_X(v, w)) \geq k_0 - 1.$$

In summary, $(v|w)_{v_0} \geq k_0 - 1$ whenever $k_0 = \min\{k-l, n\} \geq 0$. If $l > k$ so that $k_0 < 0$ then $(v|w)_{v_0} \geq 0 > k_0 - 1$, where the first inequality follows from the fact that $d_X(v,w) \leq m+n$ for any two $v, w \in V$. In both cases we get

$$\begin{aligned}
\alpha^{-(v|w)_{v_0}} &\leq \alpha^{-(k_0-1)} = \alpha\alpha^{-k_0} \\
&< \alpha\left(\alpha^{-(k-l)} + \alpha^{-n}\right) = \alpha^{l+1}\left(\alpha^{-k} + \alpha^{-n-l}\right) \\
&\leq \alpha^{l+1}\left(\alpha d_Z(z,y) + \alpha^{-n-l}\right) = \alpha^{l+2}\left(d_Z(z,y) + \alpha^{-n-l-1}\right) \\
&\leq \alpha^{l+2}\left(d_Z(z,y) + \alpha^{-n} + \alpha^{-m}\right),
\end{aligned}$$

which finally shows that $\alpha^{-(v|w)_{v_0}} \lesssim d_Z(z,y) + \alpha^{-n} + \alpha^{-m}$ with comparison constant α^{l+2}, where l only depends on α and ζ. $\square$

The following corollary designs a specific path in X between two vertices corresponding to two points in Z and it is an immediate result of the method for reaching (4.4) and (4.5) from Lemma 4.1. The length of the path is the least of the length of each case, which are estimated analogously to the lengths of (4.4) and (4.5).

Corollary 4.4. *Take two vertices (z,n) and (y,m) in X. Let k be the greatest nonnegative integer such that $d_Z(z,y) \leq \alpha^{-k}$ and let l be the smallest nonnegative integer such that $\alpha^{-l} \leq \zeta - 1$. Then, whenever $m, n \geq h := \max\{k-l, 0\}$, there exists a curve γ corresponding to the path*

$$\begin{cases} (z_n, n) \sim \cdots \sim (z_h, h) \sim (y_h, h) \sim \cdots \sim (y_m, m), & \text{if } \zeta = \tau, \\ (z_n, n) \sim \cdots \sim (z_{h+1}, h+1) \sim (y_h, h) \sim \cdots \sim (y_m, m), & \text{if } \zeta = \lambda, \end{cases} \tag{4.6}$$

where $z = z_n$ and $y = y_m$, and where $(z_h, h) \sim (y_h, h)$ collapses into a single vertex if $z_h = y_h$. Moreover, the length of γ with respect to d_X satisfies

$$d_X((z_n, n), (y_m, m)) \leq \ell(\gamma) \leq n + m + 2 - 2h.$$

As mentioned in Chapter 2 there are several different but equivalent definitions of Gromov hyperbolicity and we will now see how they go together to prove the hyperbolicity of X.

Theorem 4.5. *Definitions 2.11 and 2.13 are equivalent. In particular, if X is (δ')-hyperbolic then it is Gromov hyperbolic with constant $\delta \leq 6\delta'$.*

Sketch of proof. The proof relies heavily on the results of Haefliger–Bridson [6] so the interested reader is encouraged to search there for more details. Note that it is necessary for the space of interest to be geodesic, which X is by Corollary 4.2 (a). The outline of the proof is as follows.

By [6, Proposition III.H.1.17], Definition 2.11 is satisfied if and only if there exists a $\delta_2 > 0$ such that $\operatorname{insize}\Delta \le \delta_2$ for all geodesic triangles Δ in X, where [6, Definition III.H.1.16] provides the definition of $\operatorname{insize}\Delta$. In particular, if $\operatorname{insize}\Delta \le \delta$ for all geodesic triangles in X then X is Gromov hyperbolic with constant δ.

It is then shown by [6, Proposition III.H.1.22] that Definition 2.13 is equivalent to there existing a $\delta_2 > 0$ such that $\operatorname{insize}\Delta \le \delta_2$ for all geodesic triangles Δ in X. Specifically, X being (δ)-hyperbolic is shown to imply that $\operatorname{insize}\Delta \le 6\delta$ for all geodesic triangles Δ in X, which concludes the proof. $\square$

Theorem 4.6. *X is Gromov hyperbolic.*

Proof. By Theorem 4.5 it will suffice to show that X is (δ')-hyperbolic by Definition 2.13 to show that it is Gromov hyperbolic by Definition 2.11. First let $v = (z, n)$, $w = (y, m)$ and $u = (x, k)$, then Lemma 4.3 yields

$$\begin{aligned}
\alpha^{-(v|w)_{v_0}} &\le \alpha^{l+2}\left(d_Z(z,y) + \alpha^{-n} + \alpha^{-m}\right) \\
&\le \alpha^{l+2}\left(\left(d_Z(z,x) + \alpha^{-n} + \alpha^{-k}\right) + \left(d_Z(x,y) + \alpha^{-k} + \alpha^{-m}\right)\right) \\
&\le \alpha^{l+2}\left(\frac{4\zeta\alpha}{\alpha-1}\alpha^{-(v|u)_{v_0}} + \frac{4\zeta\alpha}{\alpha-1}\alpha^{-(u|w)_{v_0}}\right) \\
&\le \frac{8\zeta\alpha^{l+3}}{\alpha-1}\alpha^{-\min\{(v|u)_{v_0},(u|w)_{v_0}\}},
\end{aligned}$$

where l is the smallest nonnegative integer such that $\alpha^{-l} \le \zeta - 1$. Hence

$$-(v|w)_{v_0}\log\alpha \le -\min\{(v|u)_{v_0}, (u|w)_{v_0}\}\log\alpha + \log\left(\frac{8\zeta\alpha^{l+3}}{\alpha-1}\right)$$

which is equivalent to

$$(v|w)_{v_0} \ge \min\{(v|u)_{v_0}, (u|w)_{v_0}\} - \log\left(\frac{8\zeta\alpha^{l+3}}{\alpha-1}\right)\Big/\log\alpha.$$

Notice the similarities of the above and the inequality in Definition 2.13, but here with $s = v_0$ fixed and $p, q, r \in V$. The next step is to show that the inequality is satisfied whenever $p, q, r \in X$, still with $s = v_0$ fixed.

Take $p, q, r \in X$ and let $[v', v]$, $[u', u]$ and $[w', w]$ be the edges of X which contain one of these points each, in the enumerated order. Then by the above

$$(v|w)_{v_0} \geq \min\{(v|u)_{v_0}, (u|w)_{v_0}\} - \delta''$$

where

$$\delta'' := \log\left(\frac{8\zeta\alpha^{l+3}}{\alpha - 1}\right) \Big/ \log \alpha.$$

Assuming $(v|u)_{v_0} \leq (u|w)_{v_0}$, this yields

$$d_X(v_0, v) + d_X(v_0, w) - d_X(v, w) \geq d_X(v_0, v) + d_X(v_0, u) - d_X(v, u) - 2\delta'' \quad (4.7)$$

upon expanding the Gromov products. Straightforward calculations yield

$$\begin{aligned} d_X(v_0, w) &\leq d_X(v_0, r) + 1, \\ d_X(v, w) &\geq d_X(p, r) - 2, \\ d_X(v_0, u) &\geq d_X(v_0, q) - 1, \\ d_X(v, u) &\leq d_X(p, q) + 2. \end{aligned}$$

With the first two inequalities we get

$$d_X(v_0, w) - d_X(v, w) \leq (d_X(v_0, r) + 1) - (d_X(p, r) - 2)$$

while the two latter give

$$d_X(v_0, u) - d_X(v, u) \geq (d_X(v_0, q) - 1) - (d_X(p, q) + 2).$$

Thus, by cancelling $d_X(v_0, v)$ and adding $d_X(v_0, p)$ on each side of (4.7), we arrive at

$$d_X(v_0, p) + d_X(v_0, r) - d_X(p, r) + 3 \geq d_X(v_0, p) + d_X(v_0, q) - d_X(p, q) - 3 - 2\delta''$$

which is equivalent to

$$(p|r)_{v_0} \geq (p|q)_{v_0} - \frac{2\delta'' + 6}{2}.$$

In the case $(u|w)_{v_0} \leq (v|u)_{v_0}$ it can be shown similarly that

$$(p|r)_{v_0} \geq (q|r)_{v_0} - \frac{2\delta'' + 6}{2}.$$

Thus,

$$(p|r)_{v_0} \geq \min\{(p|q)_{v_0}, (q|r)_{v_0}\} - (\delta'' + 3)$$

for all $p, q, r \in X$.

By Haeflinger–Bridson [6, Remark III.H.1.21], the inequality in Definition 2.13 holds for all $p, q, r, s \in X$ with double the constant for which it holds when $s = v_0$ is fixed, so

$$(p|r)_s \geq \min\{(p|q)_s, (q|r)_s\} - 2(\delta'' + 3)$$

for all $p, q, r, s \in X$. Hence, X is (δ')-hyperbolic with

$$\delta' := 2(\delta'' + 3) = 2 \log \left(\frac{8\zeta\alpha^{l+3}}{\alpha - 1} \right) \Big/ \log \alpha + 6 > 0$$

and therefore Gromov hyperbolic with constant $\delta = 6\delta'$ by Theorem 4.5. $\square$

As such, the hyperbolic filling X is a roughly starlike Gromov hyperbolic space, similar to the results of Björn–Björn–Shanmugalingam [2]. By Definitions 2.11 and 2.13 it follows that X is roughly starlike for all choices of $x_0 \in X$ and not exclusively for the root v_0 which was used in the proof of Corollary 4.2 (e). However, as noted in [2, p. 202], the constant M may change depending on the choice of x_0.

Chapter 5

The Uniformized Boundary of The Hyperbolic Filling

In this chapter we are investigating the relation between the bounded metric space Z and the boundary $\partial_\varepsilon X$ of the uniformization of its hyperbolic filling by ultimately showing that $\partial_\varepsilon X$ is snowflake equivalent to $\overline{Z}$. Whether X_ε actually is uniform is the topic of the next chapter, where we also address and expand on potential issues with $\varepsilon > \log \alpha$ in regard to results involving the parameter – evidently we often restrict ourselves to $0 < \varepsilon \leq \log \alpha$.

Fix $\varepsilon > 0$ and consider the *uniformized metric*

$$d_\varepsilon(x, y) = \inf_\gamma \int_\gamma \rho_\varepsilon \, ds \quad \text{with } \rho_\varepsilon(x) = e^{-\varepsilon d_X(x, v_0)}$$

on X, where the infimum is taken over all rectifiable curves in X with endpoints x and y. Then $X_\varepsilon = (X, d_\varepsilon)$ is the *uniformization* of X with the root $v_0 = (z_0, 0)$ as its centre and

$$ds_\varepsilon = \rho_\varepsilon \, ds,$$

where ds_ε is the arc length with respect to the metric d_ε. Since X is a length space this makes X_ε and therefore also $\overline{X_\varepsilon}$ a length space. Moreover, $\overline{X_\varepsilon}$ is geodesic whenever it is compact, and it is compact if and only if Z is totally bounded; see Proposition 5.7 below for more.

The impact d_ε has on the hyperbolic filling in comparison to d_X is made explicit by $\operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) \simeq \frac{1}{\varepsilon}\rho_\varepsilon(x)$. In particular, X_ε is bounded. In the following we will prove these two claims and in doing so also get familiar with X_ε. First note that for any $x \in [v, w]$, where $[v, w]$ is an arbitrary edge of X with $\pi(v) \leq \pi(w)$, we have

$$e^{-\varepsilon\left(d_X(v,v_0)+1\right)} \leq e^{-\varepsilon d_X(x,v_0)} \leq e^{-\varepsilon\left(d_X(v,v_0)-1\right)},$$

or equivalently,

$$\rho_\varepsilon(x) \simeq \rho_\varepsilon(v) \tag{5.1}$$

with comparison constant e^ε both ways. By Corollary 4.2 there exists an arc length parameterized (with respect to d_X) geodesic ray $\gamma : [0,\infty) \to X$ from v_0 and through v so that

$$\operatorname{dist}_\varepsilon(v, \partial_\varepsilon X) = \int_{\gamma|_{[r,\infty)}} ds_\varepsilon,$$

where $\gamma|_{[r,\infty)}$ is the restriction of γ to $[r,\infty)$ with $\gamma|_{[r,\infty)}(r) = \gamma(r) := v$ such that $r = \ell(\gamma|_{[0,r]})$. Further, for every $r' \geq r$, $r' = \ell(\gamma|_{[0,r']})$. Since $\gamma|_{[0,r']}$ is geodesic, $\ell(\gamma|_{[0,r']}) = d_X(\gamma|_{[0,r']}(r'), \gamma|_{[0,r']}(0))$, and so

$$r' = d_X(\gamma(r'), \gamma(0)) = d_X(\gamma(r'), v_0).$$

With $r' = r$ we thus get $r = d_X(v, v_0)$. As the exception of being the first integral of this chapter we will treat every step of the computation with care, hence the details on $\gamma|_{[r,\infty)}$. It follows that

$$\begin{aligned}
\int_{\gamma|_{[r,\infty)}} ds_\varepsilon &= \int_{\gamma|_{[r,\infty)}} \rho_\varepsilon \, ds \\
&= \lim_{r'\to\infty} \int_r^{r'} e^{-\varepsilon d_X(\gamma|_{[r,r']}(t), v_0)} \, dt \\
&= \lim_{r'\to\infty} \int_{d_X(v,v_0)}^{d_X(\gamma(r'),v_0)} e^{-\varepsilon d_X(\gamma(t),v_0)} \, dt \\
&= \lim_{r'\to\infty} \int_{d_X(v,v_0)}^{d_X(\gamma(r'),v_0)} e^{-\varepsilon t} \, dt \\
&= \lim_{r'\to\infty} \left[-\frac{1}{\varepsilon} e^{-\varepsilon t}\right]_{d_X(v,v_0)}^{d_X(\gamma(r'),v_0)} \\
&= \frac{1}{\varepsilon} e^{-\varepsilon d_X(v,v_0)} = \frac{1}{\varepsilon}\rho_\varepsilon(v),
\end{aligned}$$

where the arc length parameterization of $\gamma|_{[r,\infty)}$ and the fact that $r' = d_X(\gamma(r'), v_0)$ for all $r' \geq r$ yields $d_X(\gamma(t), v_0) = t$ for $t \in [r, \infty)$ and thus the fourth equality. Therefore,

$$\operatorname{dist}_\varepsilon(v, \partial_\varepsilon X) = \frac{1}{\varepsilon}\rho_\varepsilon(v) \tag{5.2}$$

and so with (5.1) we get

$$\operatorname{dist}_\varepsilon(v, \partial_\varepsilon X) \simeq \rho_\varepsilon(x) \tag{5.3}$$

with comparison constants $e^\varepsilon/\varepsilon$ and $\varepsilon e^\varepsilon$.

Since $\pi(v) \leq \pi(w)$ we can estimate $\operatorname{dist}_\varepsilon(x, \partial_\varepsilon X)$ by

$$\operatorname{dist}_\varepsilon(w, \partial_\varepsilon X) \leq \operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) \leq \operatorname{dist}_\varepsilon(v, \partial_\varepsilon X) + d_\varepsilon(v, w). \tag{5.4}$$

If $[v, w]$ is vertical then

$$\begin{aligned}
\operatorname{dist}_\varepsilon(w, \partial_\varepsilon X) &= \frac{1}{\varepsilon}\rho_\varepsilon(w) = \frac{1}{\varepsilon}e^{-\varepsilon d_X(w,v_0)} = \frac{1}{\varepsilon}e^{-\varepsilon(d_X(v,v_0)+1)} \\
&= \frac{1}{\varepsilon}e^{-\varepsilon d_X(v,v_0)}e^{-\varepsilon} = \frac{e^{-\varepsilon}}{\varepsilon}\rho_\varepsilon(v) = e^{-\varepsilon}\operatorname{dist}_\varepsilon(v, \partial_\varepsilon X)
\end{aligned}$$

and

$$d_\varepsilon(v, w) = \int_{\gamma_{vw}} ds_\varepsilon = \int_{d_X(v,v_0)}^{d_X(w,v_0)} e^{-\varepsilon t}\, dt \leq \frac{1}{\varepsilon}\rho_\varepsilon(v), \tag{5.5}$$

where γ_{vw} is the arc length parameterized (with respect to d_X) geodesic from v to w which happens to be the edge itself. If $[v, w]$ is horizontal we take the midpoint m of $[v, w]$ and since the edge defines a geodesic between the vertices even in this case it follows that $d_\varepsilon(v, w) = 2d_\varepsilon(v, m)$, where

$$d_\varepsilon(v, m) = \int_{[v,m]} ds_\varepsilon = \int_{d_X(v,v_0)}^{d_X(v,v_0)+\frac{1}{2}} e^{-\varepsilon t}\, dt \leq \frac{1}{\varepsilon}\rho_\varepsilon(v). \tag{5.6}$$

The reason we compute the curve integral along a horizontal edge differently follows from Corollary 4.2 (d); the curve defining $d_X(x, v_0)$ only goes through v when $d_X(x, v) \leq \frac{1}{2}$ and so $d_X(x, v_0) \neq d_X(v, v_0) + d_X(x, v)$ whenever $d_X(x, v) > \frac{1}{2}$. Nevertheless, with $[v, w]$ horizontal, we have $\operatorname{dist}_\varepsilon(w, \partial_\varepsilon X) = \operatorname{dist}_\varepsilon(v, \partial_\varepsilon X)$. Thus, in both the vertical and horizontal case,

$$\operatorname{dist}_\varepsilon(v, \partial_\varepsilon X) \leq e^\varepsilon \operatorname{dist}_\varepsilon(w, \partial_\varepsilon X)$$

and

$$\begin{aligned}
\operatorname{dist}_\varepsilon(v, \partial_\varepsilon X) + d_\varepsilon(v, w) &\leq \frac{1}{\varepsilon}\rho_\varepsilon(v) + \frac{2}{\varepsilon}\rho_\varepsilon(v) \\
&= \frac{3}{\varepsilon}\rho_\varepsilon(v) \\
&= 3\operatorname{dist}_\varepsilon(v, \partial_\varepsilon X),
\end{aligned}$$

which together with (5.4) yields

$$\begin{cases} \operatorname{dist}_\varepsilon(v, \partial_\varepsilon X) \le e^\varepsilon \operatorname{dist}_\varepsilon(x, \partial_\varepsilon X), \\ \operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) \le 3 \operatorname{dist}_\varepsilon(v, \partial_\varepsilon X). \end{cases}$$

As such,

$$\operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) \simeq \operatorname{dist}_\varepsilon(v, \partial_\varepsilon X)$$

with comparison constants 3 and e^ε. Hence, by (5.3),

$$\operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) \simeq \rho_\varepsilon(x) \quad \text{for all } x \in X_\varepsilon,$$

with comparison constants $\frac{3}{\varepsilon}e^\varepsilon$ and $\varepsilon e^{2\varepsilon}$. In particular, with (5.2) we obtain $\operatorname{dist}_\varepsilon(v_0, \partial_\varepsilon X) = \frac{1}{\varepsilon}$, from which it follows that

$$\operatorname{diam}_\varepsilon \overline{X}_\varepsilon \le \sup_{x,y \in \overline{X}_\varepsilon} \big(d_\varepsilon(x, v_0) + d_\varepsilon(y, v_0)\big) \le 2 \operatorname{dist}_\varepsilon(v_0, \partial_\varepsilon X) = \frac{2}{\varepsilon},$$

and so $\frac{1}{\varepsilon} \le \operatorname{diam}_\varepsilon \overline{X_\varepsilon} \le \frac{2}{\varepsilon}$.

Proposition 5.1. *The diameter of the completion $\overline{X_\varepsilon}$ of the uniformization X_ε is finite and bounded by ε, with $\frac{1}{\varepsilon} \le \operatorname{diam}_\varepsilon \overline{X_\varepsilon} \le \frac{2}{\varepsilon}$. Moreover,*

$$\operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) \simeq \rho_\varepsilon(x) \tag{5.7}$$

for all $x \in X_\varepsilon$, with comparison constants $\frac{3}{\varepsilon}e^\varepsilon$ and $\varepsilon e^{2\varepsilon}$.

In the discussion on vertical and horizontal edges we also arrived at the following, which is quite a powerful tool for estimating the length of an edge whenever the orientation of the edge is unknown.

Lemma 5.2. *Let $[v, w]$ be an edge in X_ε with $\pi(v) \le \pi(w)$. Then*

$$\ell_\varepsilon([v, w]) \le \frac{2}{\varepsilon}\rho_\varepsilon(v) = \frac{2}{\varepsilon}e^{-\varepsilon\pi(v)}.$$

Proof. See (5.5) and (5.6). □

Ahead of this chapter's main result we introduce the mapping $\phi : V \to Z$, defined by $\phi((x, n)) = x$, along with two additional lemmas.

Lemma 5.3. *Let γ be the curve defined in Corollary 4.4. Then $\ell_\varepsilon(\gamma)$, with respect to d_ε, satisfies*

$$d_\varepsilon((z_n, n), (y_m, m)) \le \ell_\varepsilon(\gamma) \le \frac{4}{\varepsilon}e^{-\varepsilon h} \le \frac{4}{\varepsilon}e^{-\varepsilon(k-l)}, \tag{5.8}$$

where k and l are the greatest and smallest nonnegative integers such that $d_Z(z_n, y_m) \le \alpha^{-k}$ and $\alpha^{-l} \le \zeta - 1$ respectively, and $m, n \ge h = \max\{k - l, 0\}$.

Proof. Consider the arc length parameterized (with respect to d_X) curve $\gamma_2 : [0, \ell_X(\gamma_2)] \to X$ defined by the path $(z_n, n) \sim \cdots \sim (z_h, h)$. Then, as these are all vertical edges, γ_2 defines a geodesic and so

$$\ell_\varepsilon(\gamma_2) = \int_{\gamma_2} ds_\varepsilon = \int_h^n e^{-\varepsilon t}\, dt \leq \frac{1}{\varepsilon} e^{-\varepsilon h}.$$

Extrapolating this idea onto γ yields

$$\ell_\varepsilon(\gamma) \leq \int_h^n e^{-\varepsilon t}\, dt + \int_h^m e^{-\varepsilon t}\, dt + 2\int_h^{h+\frac{1}{2}} e^{-\varepsilon t}\, dt \leq \frac{4}{\varepsilon} e^{-\varepsilon h}$$

whenever $\zeta = \tau$, where

$$2\int_h^{h+\frac{1}{2}} e^{-\varepsilon t}\, dt \leq \frac{2}{\varepsilon} e^{-\varepsilon h}$$

estimates the d_ε length of $(z_h, h) \sim (y_h, h)$. Meanwhile, when $\zeta = \lambda$, we distinguish between two cases: if $h < n$ then

$$\ell_\varepsilon(\gamma) \leq \int_h^n e^{-\varepsilon t}\, dt + \int_h^m e^{-\varepsilon t}\, dt + \int_h^{h+1} e^{-\varepsilon t}\, dt \leq \frac{3}{\varepsilon} e^{-\varepsilon h}$$

where

$$\int_h^{h+1} e^{-\varepsilon t}\, dt \leq \frac{1}{\varepsilon} e^{-\varepsilon h}$$

estimates the d_ε length of $(z_{h+1}, h+1) \sim (y_h, h)$. If $h = n$ then

$$\ell_\varepsilon(\gamma) \leq \int_h^m e^{-\varepsilon t}\, dt + 2\int_h^{h+1} e^{-\varepsilon t}\, dt \leq \frac{3}{\varepsilon} e^{-\varepsilon h},$$

where

$$2\int_h^{h+1} e^{-\varepsilon t}\, dt \leq \frac{2}{\varepsilon} e^{-\varepsilon h}$$

estimates the d_ε length of $(z_n, n) \sim (z_{n+1}, n+1) \sim (y_n, n)$.

Thus, we conclude that

$$\ell_\varepsilon(\gamma) \leq \frac{4}{\varepsilon} e^{-\varepsilon h} \leq \frac{4}{\varepsilon} e^{-\varepsilon(k-l)}.$$

□

Lemma 5.4. *Fix $0 < \varepsilon \leq \log \alpha$. Then for all vertices $v, w \in V$,*

$$d_Z(\phi(v), \phi(w))^\sigma \leq D d_\varepsilon(v, w) \quad \textit{with } \sigma = \frac{\varepsilon}{\log \alpha},$$

where $D = (2\zeta\alpha)^{-\sigma}$.

Proof. Let

$$w_0 \sim w_1 \sim \cdots \sim w_k$$

be a path γ in X_ε with $w_0 = v$ and $w_k = w$. Without loss of generality we can assume $\pi(v) \leq \pi(w)$. Then, similar to how we arrived at (4.1) with the use of the triangle inequality, we get

$$d_Z(\phi(v), \phi(w)) < \sum_{i=0}^{k-1} \zeta(\alpha^{-\pi(w_i)} + \alpha^{\pi(w_{i+1})}) \leq 2\zeta \sum_{i=0}^{k-1} \alpha^{-\pi(w_i)},$$

where the last inequality follows from $\alpha^{-\pi(w_0)} \geq \alpha^{-\pi(w_k)}$ and is easily seen upon expanding the sum.

With $\sigma = \frac{\varepsilon}{\log \alpha}$ we have $e^\varepsilon = \alpha^\sigma$, so then $e^{-\varepsilon(\pi(w_i)+1)} = \alpha^{-\sigma}\alpha^{-\pi(w_i)\sigma}$. Since

$$\ell_\varepsilon(\gamma) = \sum_{i=0}^{k-1} \ell_\varepsilon([w_i, w_{i+1}]),$$

where

$$\ell_\varepsilon([w_i, w_{i+1}]) \geq \int_{\pi(w_i)}^{\pi(w_i)+1} e^{-\varepsilon t}\, dt = e^{-\varepsilon(\pi(w_i)+1)} \left(\frac{e^\varepsilon - 1}{\varepsilon}\right) > e^{-\varepsilon(\pi(w_i)+1)}$$

regardless of the orientation of $[w_i, w_{i+1}]$, we thus get

$$\begin{aligned} \ell_\varepsilon(\gamma) &\geq \sum_{i=0}^{k-1} \alpha^{-\sigma}\alpha^{-\pi(w_i)\sigma} = \alpha^{-\sigma} \sum_{i=0}^{k-1} \alpha^{-\pi(w_i)\sigma} \\ &\geq \alpha^{-\sigma} \left(\sum_{i=0}^{k-1} \alpha^{-\pi(w_i)}\right)^\sigma \geq \alpha^{-\sigma} \left(\frac{d_Z(\phi(v), \phi(w))}{2\zeta}\right)^\sigma, \end{aligned}$$

where the second to last inequality follows from the fact that $\sigma \leq 1$. By taking the infimum over all curves in X_ε from v to w we arrive at

$$d_\varepsilon(v, w) \geq (2\zeta\alpha)^{-\sigma} d_Z(\phi(v), \phi(w))^\sigma. \qquad \square$$

Proposition 5.5. *Fix $0 < \varepsilon \leq \log \alpha$, then $\overline{Z}$ and $\partial_\varepsilon X$ are snowflake-equivalent with $\sigma = \frac{\varepsilon}{\log \alpha} \leq 1$ and comparison constants $\frac{4}{\varepsilon}\alpha^{\sigma(l+1)}$ and $(2\zeta\alpha)^\sigma$, where l is the smallest nonnegative integer such that $\alpha^{-l} \leq \zeta - 1$.*

Proof. First we show that there exists a well-defined mapping $\Psi : \overline{Z} \to \partial_\varepsilon X$. Let $z \in \overline{Z}$, then there exists a Cauchy sequence $\{z_j\}_{j\in\mathbb{N}}$ in Z converging to

z. Fix $j \in \mathbb{N}$, then by Lemma 4.1 there exists a sequence $\{z_m^j\}_{m\in\mathbb{N}} \subset Z$ with $z_m^j \in A_m$ and $z_m^j \to z_j$ so that

$$(z_0^j, 0) \sim (z_1^j, 1) \sim \cdots \sim (z_m^j, m) \sim (z_{m+1}^j, m+1) \sim \ldots$$

and $z_j \in B_Z(z_m^j, \alpha^{-m})$ for each $m \in \mathbb{N}$ by construction. Note that for each $j \in \mathbb{N}$, $z_j \in B_Z(z_m^j, \alpha^{-m})$ implies $d_Z(z_j, z_m^j) < \alpha^{-m}$ for all $m \in \mathbb{N}$.

Let $\varepsilon' > 0$ be arbitrary. Since $\{z_j\}_{j\in\mathbb{N}}$ is a Cauchy sequence, there exists an $N \in \mathbb{N}$ such that $d_Z(z_m, z_n) < \frac{\varepsilon'}{3}$ whenever $m, n \geq N$. Moreover, with $M \in \mathbb{N}$ chosen so that $\alpha^{-M} < \frac{\varepsilon'}{3}$, it follows from the above that for each (fixed) $j \in \mathbb{N}$, $d_Z(z_j, z_m^j) < \frac{\varepsilon'}{3}$ whenever $m \geq M$. Thus, $d_Z(z_m, z_m^m) < \frac{\varepsilon'}{3}$ whenever $m \geq M$, where z_m^m is the m'th element of the sequence $\{z_i^m\}_{i\in\mathbb{N}} \subset Z$ corresponding to $z_m \in Z$, and we arrive at

$$d_Z(z_m^m, z_n^n) \leq d_Z(z_m^m, z_m) + d_Z(z_m, z_n) + d_Z(z_n, z_n^n) < \frac{\varepsilon'}{3} + \frac{\varepsilon'}{3} + \frac{\varepsilon'}{3} = \varepsilon'$$

whenever $m, n \geq \max\{N, M\}$. Hence, $\{z_m^m\}_{m\in\mathbb{N}}$ is a Cauchy sequence in Z.

Consider the sequence $\{(z_m^m, m)\}_{m\in\mathbb{N}}$ in X_ε and take (z_m^m, m) and (z_n^n, n) for $m, n \in \mathbb{N}$. Since X_ε is connected, there is a path γ_{nm} between these two vertices. Let k be the greatest nonnegative integer such that $k \leq \min\{m, n\}$ and $d_Z(z_m^m, z_n^n) \leq \alpha^{-k}$. Also, l is the smallest nonnegative integer such that $\alpha^{-l} \leq \zeta - 1$, which will be the case for the remainder of the proof. Then we can assume γ_{nm} is the path given by Corollary 4.4, in which case it follows from Lemma 5.3 that

$$d_\varepsilon((z_m^m, m), (z_n^n, n)) \leq \ell_\varepsilon(\gamma) \leq \frac{4}{\varepsilon} e^{-\varepsilon(k-l)}. \tag{5.9}$$

Since the sequence $\{z_m^m\}_{m\in\mathbb{N}}$ is Cauchy, we can choose $K \in \mathbb{N}$ such that $d_Z(z_m^m, z_n^n)$ is small enough and $\min\{m, n\}$ large enough to make k large enough for

$$\frac{4}{\varepsilon} e^{-\varepsilon(k-l)} < \varepsilon'.$$

Thus,

$$d_\varepsilon((z_m^m, m), (z_n^n, n)) < \varepsilon' \quad \text{whenever} \quad m, n \geq K,$$

so $\{(z_m^m, m)\}_{m\in\mathbb{N}}$ is a Cauchy sequence with $\lim_{m\to\infty}(z_m^m, m) \in \partial_\varepsilon X$.

Set $\Psi(z) = \lim_{m\to\infty}(z_m^m, m)$. Evidently the sequence $\{(z_m^m, m)\}_{m\in\mathbb{N}}$ is Cauchy in X_ε so the limit exists in $\overline{X_\varepsilon}$ and in particular it is located on the boundary. To show that Ψ is well-defined, let $\{\hat{z}_j\}_{j\in\mathbb{N}}$ be another Cauchy sequence in

Z which also converges to z, and consider its corresponding Cauchy sequence $\{(\hat{z}_m^m, m)\}_{m\in\mathbb{N}}$ in X_ε. By Corollary 4.4 and Lemma 5.3,

$$d_\varepsilon((z_m^m, m), (\hat{z}_m^m, m)) \leq \frac{4}{\varepsilon} e^{-\varepsilon(k-l)} \tag{5.10}$$

where we now let k be the greatest nonnegative integer such that

$$k \leq m \quad \text{and} \quad d_Z(z_m^m, \hat{z}_m^m) \leq \alpha^{-k}.$$

Since

$$d_\varepsilon(z_m^m, \hat{z}_m^m) \leq d_\varepsilon(z_m^m, z_m) + d_\varepsilon(z_m, \hat{z}_m) + d_\varepsilon(\hat{z}_m, \hat{z}_m^m),$$

where $d_\varepsilon(z_m^m, z_m) < \alpha^{-m} \to 0$, $d_\varepsilon(\hat{z}_m, \hat{z}_m^m) < \alpha^{-m} \to 0$ and $d_\varepsilon(z_m, \hat{z}_m) \to 0$ as $m \to \infty$, it follows that $k \to \infty$ and thus by (5.10) that $d_\varepsilon((z_m^m, m), (\hat{z}_m^m, m)) \to 0$ as $m \to \infty$. We arrive at $\lim_{m\to\infty}(\hat{z}_m^m, m) = \lim_{m\to\infty}(z_m^m, m) = \Psi(z)$, so then Ψ is well-defined and gives the map $\Psi : \overline{Z} \to \partial_\varepsilon X$.

In order to show that Ψ is a bijection we will first show the equivalence $d_\varepsilon(\Psi(z), \Psi(y)) \simeq d_Z(z, y)^\sigma$ since the injective and surjective property of Ψ follows fairly easily from there. Furthermore, it is a necessary condition for the snowflake-equivalence between Ψ and $\overline{Z}$, so it needs to be shown regardless.

Take $z, y \in \overline{Z}$ with $z \neq y$ and let k be the greatest nonnegative integer such that $k \leq m$ and $d_Z(z, y) \leq \alpha^{-k}$. Then by Lemma 5.3,

$$d_\varepsilon(\Psi(z), \Psi(y)) = \lim_{m\to\infty} d_\varepsilon((z_m^m, m), (y_m^m, m)) \leq \frac{4}{\varepsilon} e^{-\varepsilon(k-l)}.$$

With $\sigma = \frac{\varepsilon}{\log \alpha}$ we have $e^\varepsilon = \alpha^\sigma$, which yields

$$\frac{4}{\varepsilon} e^{-\varepsilon(k-l)} = \frac{4}{\varepsilon} \alpha^{-\sigma(k-l)} = \frac{4}{\varepsilon} \alpha^{\sigma(l+1)} \alpha^{(-k-1)\sigma} < \frac{4}{\varepsilon} \alpha^{\sigma(l+1)} d_Z(z, y)^\sigma,$$

and so

$$d_\varepsilon(\Psi(z), \Psi(y)) < \frac{4}{\varepsilon} \alpha^{\sigma(l+1)} d_Z(z, y)^\sigma.$$

Moreover, from Lemma 5.4 it also follows that

$$\begin{aligned} d_\varepsilon(\Psi(z), \Psi(y)) &= \lim_{m\to\infty} d_\varepsilon((z_m^m, m), (y_m^m, m)) \\ &\geq \lim_{m\to\infty} \frac{1}{D} d_Z\left(\phi((z_m^m, m)), \phi((y_m^m, m))\right)^\sigma \\ &= \lim_{m\to\infty} \frac{1}{D} d_Z(z_m^m, y_m^m)^\sigma = \frac{1}{D} d_Z(z, y)^\sigma. \end{aligned}$$

Hence, $d_\varepsilon(\Psi(z), \Psi(y)) \simeq d_Z(z, y)^\sigma$ with comparison constants $\frac{4}{\varepsilon}\alpha^{\sigma(l+1)}$ and $\frac{1}{D} = (2\zeta\alpha)^\sigma$.

To show that Ψ is injective we show the contrapositive. Take $z, y \in \overline{Z}$ such that $\Psi(z) = \Psi(y)$, then $d_\varepsilon(\Psi(z), \Psi(y)) = 0$ but $d_\varepsilon(\Psi(z), \Psi(y)) \geq \frac{1}{D}\, d_Z(z, y)^\sigma$ so $d_Z(z, y) = 0$ and therefore $z = y$. Thus, Ψ is injective.

To show that Ψ is surjective, consider a Cauchy sequence $\{x_j\}_{j\in\mathbb{N}}$ in X_ε with $x := \lim_{j\to\infty} x_j \in \partial_\varepsilon X$ and let ε' be arbitrary. Then there exists an $N \in \mathbb{N}$ such that $d_\varepsilon(x_n, x_m) < \frac{\varepsilon'}{2}$ whenever $n, m \geq N$. Note that for each $j \in \mathbb{N}$ there is a vertex v_j satisfying $d_X(v_j, x_j) \leq \frac{1}{2}$ and so $d_\varepsilon(v_j, x_j) \leq \frac{1}{\varepsilon} e^{-\varepsilon\pi(v_j)}$. Thus

$$\begin{aligned} d_\varepsilon(v_n, v_m) &\leq d_\varepsilon(v_n, x_n) + d_\varepsilon(x_n, x_m) + d_\varepsilon(x_m, v_m) \\ &\leq \frac{1}{\varepsilon}\left(e^{-\varepsilon\pi(v_n)} + e^{-\varepsilon\pi(v_m)}\right) + \frac{\varepsilon'}{2} \end{aligned}$$

whenever $n, m \geq N$. Take $M \in \mathbb{N}$ such that $\frac{2}{\varepsilon} e^{-\varepsilon\pi(v_M)} < \frac{\varepsilon'}{2}$ (which exists since $\pi(x_j) \to \infty$ and therefore $\pi(v_j) \to \infty$ as $j \to \infty$), then

$$d_\varepsilon(v_n, v_m) < \varepsilon' \quad \text{whenever} \quad n, m \geq \max\{N, M\},$$

so $\{v_j\}_{j\in\mathbb{N}}$ is also a Cauchy sequence in X_ε with $\lim_{j\to\infty} v_j = \lim_{j\to\infty} x_j = x$.

Now set $z_j := \phi(v_j) \in Z$. It follows immediately from Lemma 5.4 that $\{z_j\}_{j\in\mathbb{N}}$ is Cauchy in Z with $z_\infty := \lim_{j\to\infty} z_j \in \overline{Z}$. Take v_j, $j \in \mathbb{N}$, and $\Psi(z_\infty) \in \partial_\varepsilon X$, then there exists a greatest nonnegative integer k such that

$$k \leq \pi(v_j) \quad \text{and} \quad d_{\overline{Z}}(z_j, z_\infty) < \alpha^{-k}$$

so by Corollary 4.4 and Lemma 5.3 it follows that $d_\varepsilon(v_j, \Psi(z_\infty)) \leq \frac{4}{\varepsilon} e^{-\varepsilon(k-l)}$. But

$$\lim_{j\to\infty} d_{\overline{Z}}(z_j, z_\infty) = 0 \quad \text{and} \quad \pi(v_j) \xrightarrow{j\to\infty} \infty$$

so $k \to \infty$ as $j \to \infty$, which yields $\lim_{j\to\infty} d_\varepsilon(v_j, \Psi(z_\infty)) = 0$ and thus $\Psi(z_\infty) = \lim_{j\to\infty} v_j = x$. ☐

Corollary 5.6. *$\overline{Z}$ and $\partial_\varepsilon X$ are biLipschitz equivalent when $\varepsilon = \log\alpha$, with comparison constant $4\zeta\alpha^{2l}$, where l is the smallest nonnegative integer such that $\alpha^{-l} \leq \zeta - 1$.*

Proof. By Proposition 5.5 there is a well-defined homeomorphism $\Psi : \overline{Z} \to \partial_\varepsilon X$ such that

$$d_\varepsilon(\Psi(z), \Psi(y)) \leq \frac{4}{\varepsilon}\alpha^{\sigma(l+1)} d_Z(z, y)^\sigma$$

and

$$d_Z(z, y)^\sigma \leq (2\zeta\alpha)^\sigma d_\varepsilon(\Psi(z), \Psi(y)). \tag{5.11}$$

If $\varepsilon = \log\alpha$ then $\sigma = \frac{\varepsilon}{\log\alpha} = 1$ and

$$\frac{4}{\varepsilon}\alpha^{\sigma(l+1)} = \frac{4}{\log\alpha}\alpha^{l+1} < 4\alpha^{l},$$

since $\alpha > \log\alpha$. Further, $\alpha^{-l} \leq \zeta - 1 < \zeta$ so $4\alpha^{l} = 4\alpha^{-l}\alpha^{2l} < 4\zeta\alpha^{2l}$, which yields

$$\frac{4}{\varepsilon}\alpha^{\sigma(l+1)} < 4\zeta\alpha^{2l}.$$

Moreover, since l is a nonnegative integer, we get $(2\zeta\alpha)^{\sigma} = 2\zeta\alpha \leq 4\zeta\alpha^{2l}$. Thus,

$$d_{\varepsilon}(\Psi(z), \Psi(y)) \leq 4\zeta\alpha^{2l} d_Z(z, y)$$

and

$$d_Z(z, y) \leq 4\zeta\alpha^{2l} d_{\varepsilon}(\Psi(z), \Psi(y)), \tag{5.12}$$

and therefore $d_{\varepsilon}(\Psi(z), \Psi(y)) \simeq d_Z(z, y)$ with comparison constant $4\zeta\alpha^{2l}$ both ways, which concludes the proof. □

We bring this chapter to a close with a collection of statements which elaborates on the structure of X_{ε} and $\partial_{\varepsilon} X$ and how it depends on properties of Z and the construction of X. Note that V_n may be referred to as *vertex layer* n due to the structure of X, appearing as a graph of several layers of vertices and with edges within a vertex layer or across adjacent vertex layers. Moreover, the *degree* of a vertex specifies the number of neighbours it has.

Proposition 5.7. *The following are equivalent:*

(a) *Z is totally bounded,*

(b) *each vertex layer V_n is finite,*

(c) *every vertex in X has finite degree,*

(d) *X and X_{ε} are locally compact,*

(e) *$\overline{X_{\varepsilon}}$ is compact.*

Additionally, $\overline{X_{\varepsilon}}$ is geodesic whenever any of these hold, and if $\varepsilon \leq \log\alpha$ then $\partial_{\varepsilon} X$ is compact if and only if any and all of (a)-(e) *holds.*

Sketch of proof. The proposition is identical to Björn–Björn–Shanmugalingam [2, Proposition 4.6] and the proof with respect to their construction of X and X_{ε} works well with ours in this case. In particular, [2] shows that (a) holds whenever $\overline{X_{\varepsilon}}$ is compact by using the fact that compactness is a topological property and that $\overline{X_{\varepsilon}}$ and $\overline{Z}$ are homeomorphic as per Proposition 5.5. Ascoli's theorem is then used to show that $\overline{X_{\varepsilon}}$ is geodesic when $\overline{X_{\varepsilon}}$ is compact. □

Chapter 6

Uniformity of X_ε

We now make the claim that the uniformization X_ε of the hyperbolic filling X is uniform whenever $\varepsilon \le \log \alpha$. This chapter is dedicated to prove this statement.

Theorem 6.1. *For every ε satisfying $0 < \varepsilon \le \log \alpha$, X_ε is uniform with the constant*

$$A = \max \left\{ \frac{8}{\varepsilon} e^{\varepsilon(l+3)}, 8e^{4\varepsilon} \right\},$$

where l is the smallest nonnegative integer such that $\alpha^{-l} \le \zeta - 1$.

Proof. We want to show that X_ε is an A-uniform domain in its completion $\overline{X_\varepsilon}$, with $A \ge 1$. To this end, we determine a quasiconvex curve between an arbitrary pair of points in X_ε and show that it satisfies the twisted cone condition. As usual, l is the smallest nonnegative integer such that $\alpha^{-l} \le \zeta - 1$. Also recall that $\sigma = \frac{\varepsilon}{\log \alpha}$ and therefore $\alpha^\sigma = e^\varepsilon$.

Take $x, y \in X_\varepsilon$, $x \ne y$, with $x \in [v, v']$ and $y \in [w, w']$, where $[v, v']$ and $[w, w']$ are edges defined by vertices $v, v', w, w' \in X_\varepsilon$ such that $v \ne v'$ and $w \ne w'$. Set $\pi(v) := n$ and $\pi(w) := m$. Without loss of generality, we can assume that

$$\operatorname{dist}_\varepsilon(x, \partial_\varepsilon X) \ge \operatorname{dist}_\varepsilon(y, \partial_\varepsilon X),$$

which implies $n \le m + 1$. Assume further that v and w are two (not necessarily unique) vertices of $[v, v']$ and $[w, w']$ which are the closest to one another with respect to d_ε. We distinguish between two main cases: either $v \ne w$ or $v = w$. In particular, in the first case every vertex is distinct while in the second either $v' \sim v \sim w'$ with $v = w$, or $[v, v'] = [w, w']$.

Case 1: $v \ne w$. Let k be the greatest nonnegative integer such that

$$k \le \min\{n, m\} \quad \text{and} \quad d_Z(\phi(v), \phi(w)) \le \alpha^{-k}, \tag{6.1}$$

and consider the curve γ from v to w thus given by Corollary 4.4. By Lemma 5.3,

$$\ell_\varepsilon(\gamma) \leq \frac{4}{\varepsilon}e^{-\varepsilon h} \leq \frac{4}{\varepsilon}e^{-\varepsilon(k-l)},$$

where $h = \max\{k-l, 0\}$. Set

$$\gamma_{xy} := [x, v] \cup \gamma \cup [w, y],$$

which defines a curve from x to y with $\gamma_{xy} \supset \gamma$. Then, by Lemma 5.2,

$$\begin{aligned}
\ell_\varepsilon(\gamma_{xy}) &= d_\varepsilon(x, v) + \ell_\varepsilon(\gamma) + d_\varepsilon(w, y) \\
&\leq \frac{2}{\varepsilon}e^{-\varepsilon(n-1)} + \frac{4}{\varepsilon}e^{-\varepsilon h} + \frac{2}{\varepsilon}e^{-\varepsilon(m-1)} \\
&\leq \frac{4}{\varepsilon}e^{-\varepsilon h} + \frac{4}{\varepsilon}e^{-\varepsilon(\min\{n,m\}-1)} \\
&= \frac{4}{\varepsilon}e^{-\varepsilon h}\left(1 + e^{-\varepsilon((\min\{n,m\}-1)-h)}\right) \\
&\leq 4\frac{1+e^\varepsilon}{\varepsilon}e^{-\varepsilon h} \\
&\leq \frac{8e^\varepsilon}{\varepsilon}e^{-\varepsilon h} \qquad (6.2)\\
&= \frac{8}{\varepsilon}e^{-\varepsilon(h-1)} \\
&\leq \frac{8}{\varepsilon}e^{-\varepsilon(k-l-1)} \\
&= \frac{8}{\varepsilon}e^{-\varepsilon(k+1-(l+2))} \\
&= \frac{8}{\varepsilon}e^{\varepsilon(l+2)}e^{-\varepsilon(k+1)} \\
&\leq \frac{8}{\varepsilon}e^{\varepsilon(l+2)}(\alpha^{-k-1})^\sigma. \qquad (6.3)
\end{aligned}$$

Note that by Equation (6.1),

$$\alpha^{-k-1} < d_Z(\phi(v), \phi(w)) \qquad (6.4)$$

whenever $k < \min\{n, m\}$, so if $k < \min\{n, m\}$ then

$$\begin{aligned}
\frac{8}{\varepsilon}e^{\varepsilon(l+2)}(\alpha^{-k-1})^\sigma &< \frac{8}{\varepsilon}e^{\varepsilon(l+2)}d_Z(\phi(v), \phi(w))^\sigma \\
&\leq \frac{8}{\varepsilon}e^{\varepsilon(l+2)}(2\zeta\alpha)^{-\sigma}d_\varepsilon(v, w),
\end{aligned}$$

where the last inequality follows from Lemma 5.4. Since $2\zeta\alpha > 1$ we have $(2\zeta\alpha)^{-\sigma} < 1$, and $d_\varepsilon(v, w) \leq d_\varepsilon(x, y)$ by assumption, so $k < \min\{n, m\}$ yields

$$\ell_\varepsilon(\gamma_{xy}) \leq \frac{8}{\varepsilon} e^{\varepsilon(l+2)} d_\varepsilon(x, y) \leq A d_\varepsilon(x, y).$$

If instead $k = \min\{n, m\} \geq n - 1$ then (6.4) does not necessarily hold. However, since $v \neq w$, there is an edge $v \sim u$ with $u \in V$ such that

$$n - 1 \leq \pi(u) \leq n + 1$$

on the curve which defines $d_\varepsilon(x, y)$. As such,

$$\begin{aligned} d_\varepsilon(x, y) &\geq \int_{[v,u]} ds_\varepsilon \geq \int_n^{n+1} e^{-\varepsilon t}\, dt = e^{-\varepsilon(n+1)} \left(\frac{e^\varepsilon - 1}{\varepsilon} \right) \\ &\geq e^{-\varepsilon(n+1)} \geq e^{-\varepsilon(k+2)} = (\alpha^{-k-2})^\sigma = \alpha^{-\sigma}(\alpha^{-k-1})^\sigma = e^{-\varepsilon}(\alpha^{-k-1})^\sigma. \end{aligned}$$

As shown leading up to (6.3), $\frac{8}{\varepsilon} e^{\varepsilon(l+2)} (\alpha^{-k-1})^\sigma \geq \ell_\varepsilon(\gamma_{xy})$, so then

$$e^\varepsilon \frac{8}{\varepsilon} e^{\varepsilon(l+2)} d_\varepsilon(x, y) \geq \frac{8}{\varepsilon} e^{\varepsilon(l+2)} (\alpha^{-k-1})^\sigma \geq \ell_\varepsilon(\gamma_{xy}).$$

Thus, even when $k = \min\{n, m\}$, we arrive at

$$\ell_\varepsilon(\gamma_{xy}) \leq A d_\varepsilon(x, y).$$

What remains is to show that $\mathrm{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X) \geq \frac{1}{A} \min\{t, \ell_\varepsilon(\gamma_{xy}) - t\}$, for all $t \in [0, \ell_\varepsilon(\gamma_{xy})]$, where γ_{xy} and γ are parameterized by arc length with respect to d_ε. Our approach is to divide into cases by first assuming $h < \min\{n, m\}$, then $h = n$ and finally $h = m$.

Suppose first that $h < \min\{n, m\}$, then γ consists of two vertical segments connected by either a horizontal (possibly collapsed) edge (if $\zeta = \tau$) or another vertical edge (if $\zeta = \lambda$). Take $t \in [0, \ell(\gamma_{xy})]$. We shall consider the cases where $\gamma_{xy}(t)$ is on $[x, v]$, the vertical segment connecting to $[x, v]$, or the (possibly collapsed) horizontal/vertical edge. If $\gamma_{xy}(t) \in [x, v]$, then by Lemma 5.2,

$$\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) \leq \int_{[v,v']} ds_\varepsilon \leq \frac{1}{\varepsilon} e^{-\varepsilon \min\{\pi(v), \pi(v')\}} \leq \frac{1}{\varepsilon} e^{-\varepsilon(n-1)}.$$

Moreover,

$$\mathrm{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X) \geq \frac{1}{\varepsilon} e^{-2\varepsilon} e^{-\varepsilon d_X(\gamma_{xy}(t), v_0)} \geq \frac{1}{\varepsilon} e^{-2\varepsilon} e^{-\varepsilon(n+1)} = \frac{1}{\varepsilon} e^{-4\varepsilon} e^{-\varepsilon(n-1)}$$

by Proposition 5.1, so

$$e^{-4\varepsilon}\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) \le \operatorname{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X). \tag{6.5}$$

Next, if $\gamma_{xy}(t)$ is somewhere on the vertical segment of γ connecting to $[x, v]$, then $d_X(\gamma_{xy}(t), v_0) \le \pi(v) = n$ and so

$$\begin{aligned}
\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) &\le \int_{[v,v']} ds_\varepsilon + \int_{d_X(\gamma_{xy}(t),v_0)}^{n} ds_\varepsilon \\
&\le \frac{1}{\varepsilon}e^{-\varepsilon(n-1)} + \int_{d_X(\gamma_{xy}(t),v_0)}^{\infty} ds_\varepsilon \\
&\le \frac{1}{\varepsilon}e^{\varepsilon}e^{-\varepsilon d_X(\gamma_{xy}(t),v_0)} + \int_{d_X(\gamma_{xy}(t),v_0)}^{\infty} ds_\varepsilon,
\end{aligned}$$

where $\int_{d_X(\gamma_{xy}(t),v_0)}^{\infty} ds_\varepsilon = \operatorname{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X)$. Further, by Proposition 5.1,

$$e^{-\varepsilon d_X(\gamma_{xy}(t),v_0)} \le \varepsilon e^{2\varepsilon} \operatorname{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X). \tag{6.6}$$

As such,

$$\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) \le (e^{3\varepsilon} + 1) \operatorname{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X),$$

or equivalently,

$$\frac{1}{e^{3\varepsilon}+1}\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) \le \operatorname{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X).$$

Finally, if $\gamma_{xy}(t)$ is on the horizontal/vertical edge connecting the two vertical segments of γ then $d_X(\gamma_{xy}(t), v_0) \le h + 1$. Recall that $\ell_\varepsilon(\gamma_{xy}) \le \frac{8e^\varepsilon}{\varepsilon}e^{-\varepsilon h}$ by (6.2), so then

$$\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) \le \frac{8e^\varepsilon}{\varepsilon}e^{-\varepsilon h} \le \frac{8e^\varepsilon}{\varepsilon}e^{\varepsilon}e^{-\varepsilon d_X(\gamma_{xy}(t),v_0)}.$$

With (6.6) we thus get

$$\frac{e^{-4\varepsilon}}{8}\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) \le \operatorname{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X). \tag{6.7}$$

In summary, since $A \ge \max\{8e^{4\varepsilon}, e^{3\varepsilon} + 1, e^{4\varepsilon}\}$,

$$\operatorname{dist}_\varepsilon(\gamma_{xy}(t), \partial_\varepsilon X) \ge \frac{1}{A}\ell_\varepsilon(\gamma_{xy}\big|_{[0,t]}) = \frac{1}{A}t$$

for every $t \in [0, \ell_\varepsilon(\gamma_{xy})]$ such that $\gamma_{xy}(t)$ is on $[x, v]$, the vertical segment connecting to $[x, v]$, or the (possibly collapsed) horizontal/vertical edge. Let γ_{yx} be

γ_{xy} but with reverse orientation. Then, by symmetry, the same result applies to γ_{yx} when $\gamma_{xy}(t)$ is on $[y,w]$, the vertical segment connecting to $[y,w]$ or the (possibly collapsed) horizontal/vertical edge. Thus,

$$\operatorname{dist}_\varepsilon(\gamma_{xy}(t),\partial_\varepsilon X) \geq \frac{1}{A}\min\{t,\ell_\varepsilon(\gamma_{xy})-t\} \quad \text{for all } t\in[0,\ell_\varepsilon(\gamma_{xy})],$$

when $h<\min\{n,m\}$.

Now suppose that $h=n$. The only relevant difference from when $h<\min\{n,m\}$ is when $\zeta=\lambda$ since we then need to take the vertical edge

$$(z_n,n)\sim(z_{n+1},n+1),$$

where $z_n,z_{n+1}\in Z$, into account. Therefore, we again consider the reverse oriented curve γ_{yx} and take $t\in[0,\ell_\varepsilon(\gamma_{yx})]$ such that $\gamma_{yx}(t)$ is on this vertical edge. But then $d_X(\gamma_{yx}(t),v_0)\leq h+1$ and we obtain (6.7). Thus,

$$\operatorname{dist}_\varepsilon(\gamma_{xy}(t),\partial_\varepsilon X) \geq \frac{1}{A}\min\{t,\ell_\varepsilon(\gamma_{xy})-t\} \quad \text{for all } t\in[0,\ell_\varepsilon(\gamma_{xy})],$$

in this case as well.

Finally, suppose that $h=m$ and note that by construction of γ, we have $n\geq m$ and

$$\operatorname{dist}_\varepsilon(\gamma_{xy}(t),\partial_\varepsilon X)\geq \operatorname{dist}_\varepsilon(u,\partial_\varepsilon X)$$

for all such t, where $u\in V$ such that $\pi(u)=n+1$. But then

$$\pi(u)\leq m+2=h+2,$$

so with (5.2) we get

$$\operatorname{dist}_\varepsilon(u,\partial_\varepsilon X)=\frac{1}{\varepsilon}e^{-\varepsilon(n+1)}\geq\frac{1}{\varepsilon}e^{-\varepsilon(h+2)}=\frac{e^{-2\varepsilon}}{\varepsilon}e^{-\varepsilon h}.$$

Since $\ell_\varepsilon(\gamma_{xy})\leq\frac{8e^\varepsilon}{\varepsilon}e^{-\varepsilon h}$ by (6.2) it follows that

$$\operatorname{dist}_\varepsilon(u,\partial_\varepsilon X)\geq\frac{e^{-3\varepsilon}}{8}\ell_\varepsilon(\gamma_{xy})\geq\frac{e^{-3\varepsilon}}{8}t\geq\frac{1}{A}t, \quad t\in[0,\ell_\varepsilon(\gamma_{xy})],$$

so $\operatorname{dist}_\varepsilon(\gamma_{xy}(t),\partial_\varepsilon X)\geq\frac{1}{A}\min\{t,\ell_\varepsilon(\gamma_{xy})-t\}$ for all $t\in[0,\ell_\varepsilon(\gamma_{xy})]$.

In conclusion, γ_{xy} is a quasiconvex curve which satisfies the twisted cone condition independently of h. Hence, γ_{xy} is an A-uniform curve.

Case 2: $v=w$. In addition to $v=w$ we either have $v'\neq w'$ or $v'=w'$ so there is at most one vertex, assuming to be v, between x and y. Thus, by Lemma 5.2,

$$d_\varepsilon(x,y)\leq\int_{[x,v]}ds_\varepsilon+\int_{[v,y]}ds_\varepsilon\leq\frac{2}{\varepsilon}e^{-\varepsilon(n-1)}.$$

Let $\hat{\gamma}_{xy}$ be the curve defining $d_\varepsilon(x,y)$ (which immediately makes it quasiconvex with constant 1), and let u be a vertex with $\pi(u) \geq n+1$. Then with (5.7) we get

$$\operatorname{dist}_\varepsilon(\hat{\gamma}_{xy}(t), \partial_\varepsilon X) \geq \operatorname{dist}_\varepsilon(u, \partial_\varepsilon X) \quad \text{for all } t \in [0, \ell_\varepsilon(\hat{\gamma}_{xy})],$$

where

$$\operatorname{dist}_\varepsilon(u, \partial_\varepsilon X) = \frac{1}{\varepsilon} e^{-\varepsilon(n+1)} = \frac{e^{-2\varepsilon}}{\varepsilon} e^{-\varepsilon(n-1)},$$

so

$$\operatorname{dist}_\varepsilon(\hat{\gamma}_{xy}(t), \partial_\varepsilon X) \geq \frac{e^{-2\varepsilon}}{2} d_\varepsilon(x,y) = \frac{e^{-2\varepsilon}}{2} \ell_\varepsilon(\hat{\gamma}_{xy}) \geq \frac{1}{2e^{2\varepsilon}} \min\{t, \ell_\varepsilon(\gamma_{xy}) - t\}$$

for all $t \in [0, \ell_\varepsilon(\hat{\gamma}_{xy})]$. Notice that $2e^{2\varepsilon} \leq 8e^{4\varepsilon} \leq A$, so $\hat{\gamma}_{xy}$ is an A-uniform curve, thus concluding the proof. $\square$

As shown by Björn–Björn–Shanmugalingam [2, Proposition 4.1], it can happen that the boundary of X_ε only consists of one point if $\varepsilon > \log \alpha$. The result specifically concerns pathconnected metric spaces Z where there exists an $L < \infty$ such that $\ell_{\overline{Z}}(\gamma_Z) \leq L$ for some path γ_Z joining arbitrary points $x, y \in \overline{Z}$. In particular, it means that $\overline{Z}$ and $\partial_\varepsilon X$ are not homeomorphic and that X_ε is not uniform, see Rogovin–Shibahara–Zhou [8, Corollary 4.4] for more on the latter claim. Hence, we set the constraint $0 < \varepsilon \leq \log \alpha$ whenever we work with arbitrary metric spaces Z.

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