Admissibility and A_p classes for radial weights in $\mathbf{R}^n$

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Abstract

In this thesis we study radial weights on $\mathbf{R}^n$. We study two radial weights with different exponent sets. We show that they are both 1-admissible by utilizing a previously shown sufficient condition, for radial weights to be 1-admissible, together with some results connecting exponent sets and A_p weights. Furthermore applying a similar method on a more general radial weight, we manage to improve the previously shown sufficient condition for radial weights to be 1-admissible. Finally we show for one of these two weights that even though it is 1-admissible, whether or not it belongs to some class A_p depends both on the value of p and on the dimension n. Additionally, both of these weights as well as another simple weight are, at least in some dimensions n, not A_1 even though they are 1-admissible.

Keywords:

Doubling measure, exponent sets, p-admissible weight, A_p-weight, p-Poincaré-inequality, radial weight

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Nomenclature

$\mathbf{R}^n$	Set of all ordered real n-tuples.
$B(x,r)$	Open ball in $\mathbf{R}^n$, $B(x,r) = \{y \in \mathbf{R}^n : \|y - x\| < r\}$.
B_r	Open ball with center at the origin, $B_r = B(0,r)$.
$w(x)$	Weight function on $\mathbf{R}^n$.
$w(\|x\|)$	Radial weight on $\mathbf{R}^n$, we write $w(x) = w(\|x\|)$ or $w(x) = w(\rho)$.
μ	Positive measure on $\mathbf{R}^n$.
$A_p(\mu)$	Class of weights on $\mathbf{R}^n$, see Definition 2.2.
$\mathrm{cap}_{p,\mu}^{\mathbf{R}^n}$	Variational p-capacity.
ω_{n-1}	Surface measure of the unit sphere in $\mathbf{R}^n$.

Chapter 1

Introduction

The purpose of this thesis is to study weight functions. Weight functions are of interest for example if one chooses to study the so called weighted Laplace equation

$$\operatorname{div}(w\nabla u) = 0,$$

or more generally in the study of nonlinear equations like the weighted p-Laplace equation, $1 < p < \infty$,

$$\operatorname{div}(w\nabla u|\nabla u|^{p-2}) = 0.$$

In 1993 Heinonen–Kilpeläinen–Martio [5] studied such equations and imposed four conditions on the weight w, for the solutions to behave somewhat regularly, and they called such weights p-admissible. Later on two of the four conditions have been shown to be redundant, the remaining two conditions are that the measure given by $d\mu = w\,dx$ should be doubling and support a p-Poincaré inequality (see Definitions 2.3 and 2.4). These two conditions have also later on in Björn–Björn [1] been used to develop a rich potential theory for so called p-harmonic functions on metric spaces.

Björn–Björn–Lehrbäck [3], Sofia Svensson [7] and Hanna Svensson [8] gave examples of various so called radial weights with different exponent sets. The exponent sets describe the local dimension of $\mathbf{R}^n$ equipped with such a weight function. Moreover these exponent sets play an important role in how capacity of annuli with respect to these weights behaves. Additionally as we will also see in this thesis the exponent sets are important when determining admissibility of the weight functions. In this thesis we will therefore study such radial weight functions on $\mathbf{R}^n$, and investigate whether or not they are 1-admissible. To do this we will make use of Proposition 10.5 from [3] (given as Theorem 2.6 here) which gives a sufficient conditions for radial weights to be 1-admissible. By combining this condition with some results from Jonsson [6] we show that two of the radial weights defined in [7] are 1-admissible. With the same methods we also prove Corollary 4.1 which is an improvement of [3, Proposition 10.5].

Finally we are also interested in a class of p-admissible weights known as A_p-weights. We show that for some of the weights which we have already shown to be 1-admissible whether or not they belong to some class A_p depends both on the value of p and on the dimension n. In particular, some of these weights are not A_1 even though they are 1-admissible.

Chapter 2

Preliminaries

In this chapter we will state most of the different definitions and theorems that we will use in this thesis.

Definition 2.1. (Weight functions on $\mathbf{R}^n$). A function $w : \mathbf{R}^n \to [0, \infty)$ is called a *weight* on $\mathbf{R}^n$ if $w > 0$ almost everywhere. If $w(x) = \widetilde{w}(|x|)$ for some $\widetilde{w} : [0, \infty) \to [0, \infty)$ then w is called a *radial* weight function. For radial weights we will make use of an abuse of notation and write $w(x) = w(|x|)$ or $w(x) = w(\rho)$.

Definition 2.2. (Measure corresponding to a weight). For a given weight w on $\mathbf{R}^n$ the corresponding measure μ is defined as

$$\mu(A) = \int_A w\, dx = \int_A d\mu$$

and we write $d\mu = w\, dx$. The measure is defined for any set A where the integral is defined, such sets are called measurable.

Measures can also be defined more generally without weight functions. In this thesis however all our measures will be given by weight functions as in Definition 2.2.

Definition 2.3. (Doubling measure). A measure μ on $\mathbf{R}^n$ is said to be *doubling* if there exists a $C > 0$ such that for all open balls $B(x, r) \subset \mathbf{R}^n$ we have that

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Definition 2.4. (p-Poincaré inequality). For $1 \leq p < \infty$, we say that a measure μ on $\mathbf{R}^n$ supports a *p-Poincaré inequality* if there exists a $C > 0$ such that for every function $f \in C^\infty(\mathbf{R}^n)$ and every ball $B = B(x,r) \subset \mathbf{R}^n$ the following inequality holds

$$\frac{1}{\mu(B)} \int_B \left| f - \frac{1}{\mu(B)} \int_B f \, d\mu \right| d\mu \le Cr \left(\frac{1}{\mu(B)} \int_B |\nabla f|^p \, d\mu \right)^{\frac{1}{p}}.$$

Definition 2.5. (p-admissible weights). A weight w is said to be p-*admissible* if the corresponding measure $d\mu = w\,dx$ is doubling and supports a p-Poincaré inequality.

The following result from Björn–Björn–Lehrbäck [3, Proposition 10.5] is the main tool we use to investigate 1-admissibility of radial weights, see also Corollary 4.4.

Theorem 2.6. *Assume the radial weight $w(\rho)$ on $\mathbf{R}^n, n \ge 2$, is locally absolutely continuous on $(0,\infty)$ and that for some $\gamma_1 < n-1$ and $0 < M < \infty$ it holds for almost every $\rho > 0$ that*

$$-\gamma_1 \le \frac{\rho w'(\rho)}{w(\rho)} \le M.$$

Then the weight w is 1-admissible.

Remark 2.7. (Locally absolutely continuous). All the weight functions we study in this thesis will be continuous and piecewise differentiable which is a stronger condition than locally absolutely continuous.

Example 2.8. Take $c > 0$ and let $w(\rho) = \rho^{c-n}$ be a radial weight on $\mathbf{R}^n$. Clearly w is continuous and differentiable, so we can apply Theorem 2.6. We get that

$$\frac{\rho w'(\rho)}{w(\rho)} = c - n$$

and thus the weight satisfies the condition in Theorem 2.6 and is 1-admissible when $c > 1$. In fact after improving Theorem 2.6 we will see that w is 1-admissible for any $c > 0$ (see Theorem 6.1).

The next theorem is well known and follows easily from Hölder's inequality, for a proof see Jonsson [6, Proposition 2.15].

Theorem 2.9. *If the weight w is 1-admissible then w is also p-admissible for any $p \ge 1$.*

Definition 2.10. (Comparable). If for two functions $w(x)$ and $v(x)$ there exists a constant $C > 0$ (independent of x) such that $w(x) \le Cv(x)$ for every x, we say that $w(x) \lesssim v(x)$ and $v(x) \gtrsim w(x)$. Additionally if $w(x) \lesssim v(x)$ and $w(x) \gtrsim v(x)$ we say that $w(x)$ is *comparable* to $v(x)$ and write $w(x) \simeq v(x)$.

Theorem 2.11. *Assume that $w(\rho) \simeq v(\rho)$ are radial weights on $\mathbf{R}^n$. If v is p-admissible then w is also p-admissible.*

Proof. Since $w(\rho) \simeq v(\rho)$ there exists an $M > 0$ such that

$$\frac{1}{M} v(\rho) \leq w(\rho) \leq M v(\rho).$$

And similarly for the corresponding measures $d\mu = w\,dx$ and $d\nu = v\,dx$

$$\frac{1}{M} \nu(B) \leq \mu(B) \leq M \nu(B)$$

for every ball B in $\mathbf{R}^n$. If ν is doubling we get that

$$\mu(B(x,2r)) \leq M\nu(B(x,2r)) \leq MC\nu(B(x,r)) \leq M^2 C\mu(B(x,r))$$

for some $C > 0$ and thus μ is also doubling.

Now for any ball $B = B(x,r) \subset \mathbf{R}^n$ and any $f \in C^\infty(\mathbf{R}^n)$ we let

$$f_{B,\mu} = \frac{1}{\mu(B)} \int_B f\,d\mu$$

and

$$f_{B,\nu} = \frac{1}{\nu(B)} \int_B f\,d\nu.$$

We get that

$$\frac{1}{\mu(B)} \int_B |f_{B,\mu} - f_{B,\nu}|\,d\mu = |f_{B,\mu} - f_{B,\nu}| \leq \frac{1}{\mu(B)} \int_B |f - f_{B,\nu}|\,d\mu.$$

Hence,

$$\begin{aligned}\frac{1}{\mu(B)} \int_B |f - f_{B,\mu}|\,d\mu &\leq \frac{1}{\mu(B)} \int_B |f - f_{B,\nu}|\,d\mu + \frac{1}{\mu(B)} \int_B |f_{B,\mu} - f_{B,\nu}|\,d\mu \\ &\leq \frac{2}{\mu(B)} \int_B |f - f_{B,\nu}|\,d\mu.\end{aligned}$$

Now if ν supports a p-Poincaré inequality we get for some $C > 0$ that

$$\begin{aligned}
\frac{1}{\mu(B)}\int_B |f - f_{B,\mu}|\,d\mu &\leq \frac{2}{\mu(B)}\int_B |f - f_{B,\nu}|\,d\mu \\
&\leq \frac{2M^2}{\nu(B)}\int_B |f - f_{B,\nu}|\,d\nu \\
&\leq 2M^2 Cr\left(\frac{1}{\nu(B)}\int_B |\nabla f|^p\,d\nu\right)^{\frac{1}{p}} \\
&\leq 2M^2 Cr\left(\frac{M^2}{\mu(B)}\int_B |\nabla f|^p\,d\mu\right)^{\frac{1}{p}} \\
&= 2M^{2+\frac{2}{p}} Cr\left(\frac{1}{\mu(B)}\int_B |\nabla f|^p\,d\mu\right)^{\frac{1}{p}}
\end{aligned}$$

and thus μ also supports a p-Poincaré inequality and w is thus p-admissible. □

Definition 2.12. (Exponent sets). In this thesis we let $B_r = B(0,r)$. We define the exponent sets for the measure μ on $\mathbf{R}^n$ as

$$\begin{aligned}
\underline{Q}_0(\mu) &:= \left\{q > 0 : \text{there is } C_q \text{ so that } \frac{\mu(B_r)}{\mu(B_R)} \leq C_q\left(\frac{r}{R}\right)^q \text{ for } 0 < r < R \leq 1\right\}, \\
\underline{S}_0(\mu) &:= \{q > 0 : \text{there is } C_q \text{ so that } \mu(B_r) \leq C_q r^q \text{ for } 0 < r \leq 1\}, \\
\overline{S}_0(\mu) &:= \{q > 0 : \text{there is } C_q \text{ so that } \mu(B_r) \geq C_q r^q \text{ for } 0 < r \leq 1\}, \\
\overline{Q}_0(\mu) &:= \left\{q > 0 : \text{there is } C_q \text{ so that } \frac{\mu(B_r)}{\mu(B_R)} \geq C_q\left(\frac{r}{R}\right)^q \text{ for } 0 < r < R \leq 1\right\}, \\
\underline{Q}(\mu) &:= \left\{q > 0 : \text{there is } C_q \text{ so that } \frac{\mu(B_r)}{\mu(B_R)} \leq C_q\left(\frac{r}{R}\right)^q \text{ for } 0 < r < R\right\}, \\
\overline{Q}(\mu) &:= \left\{q > 0 : \text{there is } C_q \text{ so that } \frac{\mu(B_r)}{\mu(B_R)} \geq C_q\left(\frac{r}{R}\right)^q \text{ for } 0 < r < R\right\}.
\end{aligned}$$

If μ is given by $d\mu = w\,dx$ where w is a weight on $\mathbf{R}^n$ we write $\underline{Q}_0(w) = \underline{Q}_0(\mu)$ and we say that $\underline{Q}_0(w)$ and so on, are the exponent sets for w.

Next we will define A_p-weights. The definition makes use of the essential infimum of the weight w over a ball, denoted $\operatorname*{ess\,inf}_B w$, which is a generalization of the usual infimum. Note that if w is continuous then $\operatorname*{ess\,inf}_B w = \inf_B w$, and in this thesis all the weights we will consider are continuous.

Definition 2.13. (A_p-weights). A weight w that is locally intergrable with respect to a measure μ is said to be of *class* A_p with respect to μ, if one of two inequalities is satisfied, depending on the value of p. If $p = 1$, there must exist a $C > 0$ such that for every ball B we have that

$$\int_B w\,d\mu \le C\left(\operatorname*{ess\,inf}_B w\right)\mu(B).$$

If $1 < p < \infty$, there should exist $C > 0$ such that for all balls B

$$\left(\int_B w\,d\mu\right)\left(\int_B w^{\frac{1}{1-p}}\,d\mu\right)^{p-1} \le C\mu(B)^p.$$

We write $w \in A_p(\mu)$ and if μ is the Lebesgue measure we write $w \in A_p$.

The next theorem was proved in Jonsson [6, Theorem 4.6].

Theorem 2.14. *Let μ be a doubling measure on $\mathbf{R}^n$ and $w(x) = |x|^\alpha$ where $-\sup \underline{Q}(\mu) < \alpha \le 0$. Then $w \in A_1(\mu)$.*

A result similar to Theorem 2.9 is also true for A_p-weights. It also follows from Hölder's inequality, for a proof see Jonsson [6, Proposition 2.18].

Theorem 2.15. *Let $p \ge 1$ and $w \in A_p(\mu)$. Then $w \in A_s(\mu)$ for every $s > p$.*

The following result connects p-admissible weights to A_p-weights. It was proved by J. Björn in [4, Theorem 4].

Theorem 2.16. *Let v be an s-admissible weight and let $w \in A_p(v)$. Then the weight vw is ps-admissible.*

In particular since the Lebesgue measure dx is known to be 1-admissible, it follows from Theorem 2.16 (taking $v = 1$) that A_p-weights are p-admissible.

The two following theorems from Björn–Björn–Christensen [2] are useful for showing that certain radial 1-admissible weights are also A_p-weights. Theorem 2.17 follows from [2, Theorem 1.2 and Corollary 5.4]. Both Theorem 2.17 and 2.18 make statements about capacities, in these cases $\operatorname{cap}_{p,\mu}^{\mathbf{R}^n}(\{0\}, B_r)$ refers to the variational p-capacity of $\{0\}$ with respect to B_r. A precise definition of what that is will for our purposes not be necessary and will therefore be left out.

Theorem 2.17. *Let w be a radial weight function on $\mathbf{R}^n$ such that $d\mu = w\,dx$ is a doubling measure. Assume that*

$$\operatorname{cap}_{p,\mu}^{\mathbf{R}^n}(\{0\}, B_r) \simeq r^{-p}\mu(B_r) \quad \text{for all } r > 0.$$

Then μ supports a p-Poincaré inequality on $\mathbf{R}^n$ if and only if w is of class A_p.

The next theorem is from [2, Theorem 1.3].

Theorem 2.18. *Assume that μ is a doubling measure supporting a p-Poincaré inequality on $\mathbf{R}^n$, where $p > 1$. Then*

$$\operatorname{cap}_{p,\mu}^{\mathbf{R}^n}(\{0\}, B_r) \simeq r^{-p}\mu(B_r) \quad \textit{for all } r > 0$$

if and only if $p > \inf \overline{Q}(\mu)$.

Chapter 3

Introductory example

We study the weight $w(\rho)$ defined in Chapter 3 in Svensson [7]. We will later modify the weight and see that the modified weight is 1-admissible and maintains the same structure for the exponent sets.

The weight is defined with the following variables. For $k = 2, 3, ...$, let

$$\alpha_k = 2^{-2^k},$$

and let

$$\beta_k = \alpha_k^{\frac{4+k}{3+k}}.$$

These are the same β_k as in [7], but written in a different way. Now define the weight by

$$w(\rho) = \begin{cases} \alpha_k^{\frac{1}{k}} \rho^{1-\frac{1}{k}-n}, & \text{if } \alpha_k < \rho < \beta_{k-1}, \\ \alpha_{k-1}^{-1-\frac{1}{k}} \rho^{2+\frac{1}{k}-n}, & \text{if } \beta_{k-1} \le \rho \le \alpha_{k-1}, \end{cases} \qquad \text{for } k = 3, 4, ...$$

Note that w is continuous since for $k = 3, 4, ...$,

$$w(\beta_{k-1}) = \alpha_{k-1}^{-1-\frac{1}{k}} \beta_{k-1}^{2+\frac{1}{k}-n} = \alpha_k^{\frac{1}{k}} \beta_{k-1}^{1-\frac{1}{k}-n},$$

which we can see is true since, using $\alpha_k = \alpha_{k-1}^2$ we get that

$$\alpha_k^{\frac{1}{k}} \beta_{k-1}^{-\frac{1}{k}} = \alpha_{k-1}^{\frac{2}{k}} \alpha_{k-1}^{-\frac{3+k}{k(2+k)}} = \alpha_{k-1}^{\frac{1+k}{k(2+k)}} = \alpha_{k-1}^{-1-\frac{1}{k}} \alpha_{k-1}^{\left(1+\frac{1}{k}\right)\left(\frac{3+k}{2+k}\right)} = \alpha_{k-1}^{-1-\frac{1}{k}} \beta_{k-1}^{1+\frac{1}{k}},$$

and

$$w(\alpha_k) = \alpha_k^{\frac{1}{k}} \alpha_k^{1-\frac{1}{k}-n} = \alpha_k^{1-n} = \alpha_k^{-1-\frac{1}{k+1}} \alpha_k^{2+\frac{1}{k+1}-n}.$$

Svensson [7, Chapter 3] also showed that the measure defined on $B(0,\alpha_2)$ by $d\mu = w(\rho)\,dx$ has the following exponent sets

$$\underline{Q}_0(\mu) = (0,1), \quad \underline{S}_0(\mu) = (0,1], \quad \overline{S}_0(\mu) = (1,\infty), \quad \overline{Q}_0(\mu) = (2,\infty).$$

Now to investigate whether or not the weight is 1-admissible on $\mathbf{R}^n$ we first need to extend the domain for the weight to $\rho > \alpha_2$. We let

$$w(\rho) = \begin{cases} \alpha_k^{\frac{1}{k}} \rho^{1-\frac{1}{k}-n}, & \text{if } \alpha_k < \rho < \beta_{k-1}, \\ \alpha_{k-1}^{-1-\frac{1}{k}} \rho^{2+\frac{1}{k}-n}, & \text{if } \beta_{k-1} \le \rho \le \alpha_{k-1}, \\ \rho^{1-n}, & \text{if } \rho > \alpha_2. \end{cases} \quad \text{for } k = 3, 4, \dots \tag{3.1}$$

Note that since $w(\alpha_2) = \alpha_2^{1-n}$, our weight with the extended domain is still continuous.

Now since $w(\rho)$ is continuous we can try to apply Theorem 2.6 to see if the weight is 1-admissible. Note that Theorem 2.6 holds only for $n \ge 2$ thus we investigate our weight on $\mathbf{R}^n, n \ge 2$. We get

$$\frac{\rho w'(\rho)}{w(\rho)} = \begin{cases} 1 - \frac{1}{k} - n, & \text{if } \alpha_k < \rho < \beta_{k-1}, \\ 2 + \frac{1}{k} - n, & \text{if } \beta_{k-1} < \rho < \alpha_{k-1}, \\ 1 - n, & \text{if } \rho > \alpha_2, \end{cases} \quad \text{for } k = 3, 4, \dots,$$

where we now see that in the first case the quotient is less than $1 - n$, so Theorem 2.6 does not imply that w is 1-admissible. One of our aims is therefore to show this by other means, see Corollary 3.5. We will therefore create a new auxiliary weight by multiplying w with ρ^α for some appropriate α. With $v(\rho) = \rho^\alpha w(\rho)$ we will get new weights which are indeed 1-admissible. The following is our first result in this direction.

Theorem 3.1. *Let w be the weight defined on $\mathbf{R}^n, n \ge 2$, by* (3.1). *For $\alpha > 0$, the weight $v(\rho) = \rho^\alpha w(\rho)$ is* 1*-admissible.*

Proof. Let $m \ge 2$ be a integer such that $m > \frac{1}{\alpha}$, and let

$$w_0(\rho) = \begin{cases} w(\rho), & \text{if } \rho \le \alpha_m, \\ \rho^{1-n}, & \text{if } \rho > \alpha_m. \end{cases}$$

Now with $v_0(\rho) = \rho^\alpha w_0(\rho)$ we get that

$$\frac{\rho v_0'(\rho)}{v_0(\rho)} = \begin{cases} 1 + \alpha - \frac{1}{k} - n, & \text{if } \alpha_k < \rho < \beta_{k-1}, \\ 2 + \alpha + \frac{1}{k} - n, & \text{if } \beta_{k-1} < \rho < \alpha_{k-1}, \\ 1 + \alpha - n, & \text{if } \rho > \alpha_m, \end{cases} \quad \text{for } k = m+1, m+2, \dots$$

hence

$$\frac{\rho v_0'(\rho)}{v_0(\rho)} > 1-n \quad \text{for all } \rho > 0,$$

so Theorem 2.6 implies that $v_0(\rho)$ is 1-admissible. Furthermore we have that $v_0(\rho) = v(\rho)$ when $\rho < \alpha_m$ and when $\rho > \alpha_2$. Since both weights are continuous we get for $\alpha_m \le \rho \le \alpha_2$ that $v_0(\rho) \simeq v(\rho)$. Hence $v_0(\rho) \simeq v(\rho)$ for all $\rho > 0$, with comparison constants independent of ρ, which implies by Theorem 2.11 that v is also 1-admissible. □

We shall now also see what the exponent sets for the new weight v look like. Roughly speaking the endpoints of the exponent sets are shifted forward by α.

Theorem 3.2. *Let μ be a doubling measure on $\mathbf{R}^n$ and let ν be the measure defined as $d\nu = \rho^\alpha \, d\mu$ where $\alpha > -\sup \underline{Q}(\mu)$. Then ν has the following exponent sets*

$$\begin{aligned}
\underline{Q}_0(\nu) &= \{q > 0 : q - \alpha \le \sigma \text{ for some } \sigma \in \underline{Q}_0(\mu)\},\\
\underline{S}_0(\nu) &= \{q > 0 : q - \alpha \le \sigma \text{ for some } \sigma \in \underline{S}_0(\mu)\},\\
\overline{S}_0(\nu) &= \{q > 0 : q - \alpha \ge \sigma \text{ for some } \sigma \in \overline{S}_0(\mu)\},\\
\overline{Q}_0(\nu) &= \{q > 0 : q - \alpha \ge \sigma \text{ for some } \sigma \in \overline{Q}_0(\mu)\},\\
\underline{Q}(\nu) &= \{q > 0 : q - \alpha \le \sigma \text{ for some } \sigma \in \underline{Q}(\mu)\},\\
\overline{Q}(\nu) &= \{q > 0 : q - \alpha \ge \sigma \text{ for some } \sigma \in \overline{Q}(\mu)\}.
\end{aligned}$$

Proof. For the proof we utilize Theorem 3.5 from Jonsson [6] which states that for every ball $B_r = B(0,r) \subset \mathbf{R}^n$ and with $\alpha > -\sup \underline{Q}(\mu)$, we have

$$\nu(B_r) = \int_{B_r} |x|^\alpha d\mu \simeq r^\alpha \mu(B_r). \tag{3.2}$$

Now let $q > 0$, then $q \in \underline{Q}_0(\nu)$ if and only if

$$\frac{\nu(B_r)}{\nu(B_R)} \lesssim \left(\frac{r}{R}\right)^q$$

for all $0 < r < R \le 1$. By (3.2), this is equivalent to

$$\frac{\mu(B_r)}{\mu(B_R)} \lesssim \left(\frac{r}{R}\right)^{q-\alpha} \quad \text{for all } 0 < r < R \le 1.$$

This inequality is satisfied if and only if $q - \alpha \le \sigma$ for some $\sigma \in \underline{Q}_0(\mu)$. The statement for the other exponent sets can be shown similarly, where we for $\underline{Q}(\nu)$ and $\overline{Q}(\nu)$ consider all $0 < r < R < \infty$. □

Theorems 3.1 and 3.2 prove the following corollary.

Corollary 3.3. *Let w be the weight defined on $\mathbf{R}^n, n \geq 2$, by* (3.1)*. Then for any $\alpha > 0$, the weights $v(\rho) = \rho^\alpha w(\rho)$ are 1-admissible and have the following exponent sets*

$$\underline{Q}_0(\nu) = (0, 1+\alpha), \quad \underline{S}_0(\nu) = (0, 1+\alpha], \quad \overline{S}_0(\nu) = (1+\alpha, \infty), \quad \overline{Q}_0(\nu) = (2+\alpha, \infty).$$

We shall now see that we can generalize further and find 1-admissible weights with exponent sets of the form

$$\underline{Q}_0(\mu) = (0, c), \quad \underline{S}_0(\mu) = (0, c], \quad \overline{S}_0(\mu) = (c, \infty), \quad \overline{Q}_0(\mu) = (c+1, \infty)$$

for any $c > 0$.

Theorem 3.4. *Let $c > 0$ and let w be the weight defined on $\mathbf{R}^n, n \geq 2$, by* (3.1)*. Then the weight $\widetilde{w}(\rho) = \rho^{c-1} w(\rho)$ is 1-admissible and has the following exponent sets*

$$\underline{Q}_0(\widetilde{w}) = (0, c), \quad \underline{S}_0(\widetilde{w}) = (0, c], \quad \overline{S}_0(\widetilde{w}) = (c, \infty), \quad \overline{Q}_0(\widetilde{w}) = (c+1, \infty).$$

Proof. For the proof we will start with the weight $v(\rho) = \rho^\alpha w(\rho)$ where w is the weight in (3.1) and then use Theorem 2.14. In addition to Theorem 3.1, we will also need to know what the exponent set $\underline{Q}(v)$ looks like. We can by Theorem 3.2 get that $\underline{Q}(v)$ has its endpoint shifted up by α compared to $\underline{Q}(\mu)$, where $d\mu = w\,dx$. So we need to determine how $\underline{Q}(\mu)$ looks like.

Clearly $\underline{Q}(\mu) \subset \underline{Q}_0(\mu) = (0, 1)$. To see that $(0, 1) \subset \underline{Q}(\mu)$ we need to estimate $\mu(B_R)$ for $R \geq \alpha_2$. We get

$$\mu(B_R) = \mu(B_{\alpha_2}) + \mu(B_R \setminus B_{\alpha_2}).$$

The measure $\mu(B_{\alpha_2})$ is constant and thus comparable to 1. The other measure $\mu(B_R \setminus B_{\alpha_2})$ can be calculated using polar coordinates. We get

$$\mu(B_R \setminus B_{\alpha_2}) \simeq \int_{\alpha_2}^{R} \rho^{n-1} \rho^{1-n} d\rho = R - \alpha_2$$

and thus

$$\mu(B_R) \simeq 1 + R - \alpha_2.$$

Furthermore we get that

$$1 + R - \alpha_2 \leq 16R \quad \text{since } R \geq \alpha_2 = 2^{-2^2} = \frac{1}{16}.$$

Additionally,

$$1 + R - \alpha_2 \geq R$$

and thus

$$\mu(B_R) \simeq 1 + R - \alpha_2 \simeq R.$$

Let $0 < q < 1$ be arbitrary. For $0 < r < R < \alpha_2$ we now get

$$\frac{\mu(B_r)}{\mu(B_R)} \lesssim \left(\frac{r}{R}\right)^q,$$

from $\underline{Q}_0(w) = (0, 1)$. For $0 < r < \alpha_2 < R$ we also get that

$$\frac{\mu(B_r)}{\mu(B_R)} = \frac{\mu(B_r)}{\mu(B_{\alpha_2})} \frac{\mu(B_{\alpha_2})}{\mu(B_R)} \lesssim \left(\frac{r}{\alpha_2}\right)^q \left(\frac{\alpha_2}{R}\right) \leq \left(\frac{r}{R}\right)^q.$$

Finally for $R > r \geq \alpha_2$ we have that

$$\frac{\mu(B_r)}{\mu(B_R)} \simeq \left(\frac{r}{R}\right) \leq \left(\frac{r}{R}\right)^q.$$

This shows that $(0, 1) \subset \underline{Q}(\mu)$ and thus that $\underline{Q}(\mu) = (0, 1)$. Hence with $v(\rho) = \rho^\alpha w(\rho)$, we get that $\underline{Q}(v) = (0, 1 + \alpha)$.

We now for any $c > 0$ fix $\alpha > 0$ and let $\beta = 1 + \alpha - c$. Recall that

$$\widetilde{w}(\rho) = \rho^{c-1} w(\rho) = \rho^{-\beta} v(\rho).$$

Theorem 2.14 gives that $\rho^{-\beta} \in A_1(v)$ since

$$-\beta = -(1 + \alpha - c) > -(1 + \alpha) = -\sup \underline{Q}(v).$$

Since v is 1-admissible by Theorem 3.1, it follows from Theorem 2.16 that $\widetilde{w}$ is 1-admissible. Finally we get from Theorem 3.2 that the exponent sets for $\widetilde{w}$ are

$$\underline{Q}_0(\widetilde{w}) = (0, c), \quad \underline{S}_0(\widetilde{w}) = (0, c], \quad \overline{S}_0(\widetilde{w}) = (c, \infty), \quad \overline{Q}_0(\widetilde{w}) = (c + 1, \infty).$$

□

With $c = 1$ we get that the original weight is 1-admissible.

Corollary 3.5. *The weight w defined in* (3.1) *is* 1*-admissible.*

Chapter 4

Admissibility of more general weights

In this chapter we apply some of the methods used in Chapter 3 to more general radial weights and prove an improvement of Theorem 2.6.

Theorem 4.1. *Assume that the radial weight $w(\rho)$ on $\mathbf{R}^n, n \geq 2$, is locally absolutely continuous on $(0, \infty)$ and that for some $0 < M < \infty$ it holds for almost every $\rho > 0$ that*

$$-M < \frac{\rho w'(\rho)}{w(\rho)} < M.$$

If the exponent set $\underline{Q}(w)$ is non-empty, then w is 1-admissible.

To prove Theorem 4.1 we will need the following lemma.

Lemma 4.2. *Assume that the radial weight $w(\rho)$ on $\mathbf{R}^n, n \geq 2$, is locally absolutely continuous on $(0, \infty)$ and that for some $0 < M < \infty$ it holds for almost every $\rho > 0$ that*

$$-M < \frac{\rho w'(\rho)}{w(\rho)} < M.$$

Then the weight $v(\rho) = \rho^\alpha w(\rho)$ is 1-admissible for any $\alpha > M + 1 - n$.

Proof. Take $\alpha > M + 1 - n$. Then we get that

$$\frac{\rho v'(\rho)}{v(\rho)} = \frac{\rho(\rho^\alpha w(\rho))'}{\rho^\alpha w(\rho)} = \alpha + \frac{\rho w'(\rho)}{w(\rho)} > 1 - n$$

and also

$$\frac{\rho v'(\rho)}{v(\rho)} < M + \alpha < \infty$$

so Theorem 2.6 implies that v is 1-admissible. □

Proof of Theorem 4.1. Let $v(\rho) = \rho^\alpha w(\rho)$ for some $\alpha > 0$ large enough such that Lemma 4.2 implies that the weight v is 1-admissible. Since

$$-\alpha > -(\alpha + \sup \underline{Q}(\mu)) = -\sup \underline{Q}(\nu),$$

where the equality $\alpha + \sup \underline{Q}(\mu) = \sup \underline{Q}(\nu)$ follows from Theorem 3.2, Theorem 2.14 shows that $\rho^{-\alpha} \in A_1(v)$. Finally since $w(\rho) = \rho^{-\alpha} v(\rho)$ and v is 1-admissible, Theorem 2.16 implies that w is 1-admissible. □

Next we will show a sufficient condition for the exponent set $\underline{Q}(\mu)$ to be non-empty.

Lemma 4.3. *If there exists* $0 < \theta < 1$ *such that*

$$\mu(B_r) \leq \theta \mu(B_{2r}) \quad \textit{for all } r > 0. \tag{4.1}$$

Then $\underline{Q}(\mu)$ *is non-empty.*

The condition (4.1) is sometimes called reverse doubling.

Proof. Take $0 < r < R < \infty$. Then $r \leq 2^k r \leq R$ for some integer $k \geq 0$. We take the largest such k and then get using (4.1) that

$$\frac{\mu(B_r)}{\mu(B_R)} = \frac{\mu(B_r)}{\mu(B_{2r})} \frac{\mu(B_{2r})}{\mu(B_{4r})} \cdots \frac{\mu(B_{2^k r})}{\mu(B_R)} \leq \theta^k.$$

Now we want to show that

$$\theta^k \lesssim \left(\frac{r}{R}\right)^q \quad \text{for some } q > 0.$$

We have that

$$\left(\frac{r}{R}\right)^q = \left(\left(\frac{r}{2r}\right) \left(\frac{2r}{4r}\right) \cdots \left(\frac{2^k r}{R}\right) \right)^q \geq \left(\frac{1}{2}\right)^{(k+1)q}.$$

Now

$$\theta^k \leq \left(\frac{1}{2}\right)^{kq}$$

is equivalent to

$$k \log \theta \leq kq \log \frac{1}{2}$$

which can be rewritten as

$$q \leq \frac{\log \theta}{\log \frac{1}{2}}.$$

Hence we have that

$$\frac{\mu(B_r)}{\mu(B_R)} \leq \theta^k \leq \left(\frac{1}{2}\right)^{kq} \leq 2^q \left(\frac{r}{R}\right)^q \quad \text{for } 0 < q \leq \frac{\log \theta}{\log \frac{1}{2}},$$

and thus $\underline{Q}(\mu)$ is non-empty. □

With this lemma we can prove the following corollary which is an improvement of Theorem 2.6.

Corollary 4.4. *Assume the radial weight $w(\rho)$ on $\mathbf{R}^n, n \geq 2$, is locally absolutely continuous on $(0, \infty)$ and that for some $m < n$ and $0 < M < \infty$ it holds for almost every $\rho > 0$ that*

$$-m \leq \frac{\rho w'(\rho)}{w(\rho)} \leq M. \tag{4.2}$$

Then the weight w is 1-admissible.

Proof. The inequality (4.2) gives

$$\frac{-m}{\rho} \leq \frac{w'(\rho)}{w(\rho)} = (\log w(\rho))'.$$

Integrating the right-hand side from r to $2r$ we get

$$\int_r^{2r} (\log w(\rho))' \, d\rho = \log w(2r) - \log w(r) = \log \frac{w(2r)}{w(r)}$$

and integrating the left-hand side gives

$$\int_r^{2r} \frac{-m}{\rho} \, d\rho = -m(\log 2r - \log r) = -m \log 2 = \log(2^{-m}).$$

So we get the following inequality

$$\log \frac{w(2r)}{w(r)} \geq \log(2^{-m})$$

or equivalently

$$w(r) \leq 2^m w(2r). \tag{4.3}$$

Calculating the measures of B_r and B_{2r} with polar coordinates and using (4.3) we get the following estimate (with ω_{n-1} being the surface measure of the unit sphere in $\mathbf{R}^n$)

$$\begin{aligned}\frac{\mu(B_r)}{\omega_{n-1}} &= \int_0^r w(\rho)\rho^{n-1}\,d\rho = \sum_{j=0}^{\infty}\int_{2^{-j-1}r}^{2^{-j}r} w(\rho)\rho^{n-1}\,d\rho \\ &\leq 2^m \sum_{j=0}^{\infty}\int_{2^{-j-1}r}^{2^{-j}r} w(2\rho)\rho^{n-1}\,d\rho = 2^m \sum_{j=0}^{\infty}\int_{2^{-j}r}^{2^{1-j}r} w(t)\left(\frac{t}{2}\right)^{n-1}\frac{1}{2}\,dt \\ &= 2^{m-n}\int_0^{2r} w(t)t^{n-1}\,dt = 2^{m-n}\frac{\mu(B_{2r})}{\omega_{n-1}}.\end{aligned}$$

Since $m < n$ and thus $0 < 2^{m-n} < 1$, Lemma 4.3 implies that $\underline{Q}(w)$ is non-empty and w is therefore 1-admissible by Theorem 4.1. □

Chapter 5

Admissibility of another weight

In this section we will use Theorem 4.1 to show admissibility of one more weight.

We study another weight from Svensson [7, Chapter 2]. Just as before in Chapter 3 we need to extend the domain of the weight, so we define the weight as follows. Take any $c > 0$, fix an integer $k_0 > 1/c$ and let for positive integers $k \geq k_0$,

$$\alpha_k = 2^{-2^k} \text{ and } \quad \beta_k = \alpha_k^{\frac{3}{2}}.$$

We then define the weight as

$$w(\rho) = \begin{cases} \alpha_k^{\frac{1}{k}} \rho^{c-\frac{1}{k}-n}, & \text{if } \alpha_k < \rho < \beta_{k-1} \\ \alpha_{k-1}^{-\frac{1}{k}} \rho^{c+\frac{1}{k}-n}, & \text{if } \beta_{k-1} \leq \rho \leq \alpha_{k-1}, \\ \rho^{c-n}, & \text{if } \rho > \alpha_{k_0}, \end{cases} \quad \text{for } k > k_0. \tag{5.1}$$

Svensson [7, Chapter 2] also showed that the weight has the following exponent sets

$$\underline{Q}_0(w) = (0, c), \quad \underline{S}_0(w) = (0, c], \quad \overline{S}_0(w) = (c, \infty), \quad \overline{Q}_0(w) = (c, \infty).$$

Theorem 5.1. *The weight w defined on $\mathbf{R}^n, n \geq 2$, by (5.1) is 1-admissible.*

Proof. Fist we can see that w is continuous since

$$w(\beta_{k-1}) = \alpha_{k-1}^{-\frac{1}{k}} \beta_{k-1}^{c+\frac{1}{k}-n} = \alpha_{k-1}^{\frac{2}{k}} \alpha_{k-1}^{-\frac{3}{2}\frac{2}{k}} \beta_{k-1}^{c+\frac{1}{k}-n} = \alpha_k^{\frac{1}{k}} \beta_{k-1}^{c-\frac{1}{k}-n}$$

and

$$w(\alpha_k) = \alpha_k^{-\frac{1}{k}} \alpha_k^{c+\frac{1}{k}-n} = \alpha_k^{c-n} = \alpha_k^{\frac{1}{k}} \alpha_k^{c-\frac{1}{k}-n}.$$

Additionally we also get that

$$\frac{\rho w'(\rho)}{w(\rho)} = \begin{cases} c - \frac{1}{k} - n, & \text{if } \alpha_k < \rho < \beta_{k-1}, \\ c + \frac{1}{k} - n, & \text{if } \beta_{k-1} < \rho < \alpha_{k-1}, \\ c - n, & \text{if } \rho > \alpha_{k_0}, \end{cases} \quad \text{for } k > k_0.$$

so it is clear that the quotient $\frac{\rho w'(\rho)}{w(\rho)}$ is bounded from above and below. In order to use Theorem 4.1 it remains to show that $\underline{Q}(\mu)$, where $d\mu = w\,dx$, is non-empty.

We use that $\mu(B_R) = \mu(B_{\alpha_{k_0}}) + \mu(B_R \setminus B_{\alpha_{k_0}})$, when $R \geq \alpha_{k_0}$, where we can calculate the second measure using polar coordinates as follows,

$$\mu(B_R \setminus B_{\alpha_{k_0}}) \simeq \int_{\alpha_{k_0}}^{R} \rho^{n-1} \rho^{c-n} d\rho = \int_{\alpha_{k_0}}^{R} \rho^{c-1} d\rho = R^c - \alpha_{k_0}^c$$

to see that $\mu(B_R) \simeq 1 + R^c - \alpha_{k_0}^c$. For $R > 1 > \alpha_{k_0}$ we now have the following

$$1 + R^c - \alpha_{k_0}^c \leq 2R^c$$

and also

$$1 + R^c - \alpha_{k_0}^c \geq R^c$$

so $\mu(B_R) \simeq R^c$.

For $0 < r < 1 < R$ we now get the following

$$\frac{\mu(B_r)}{\mu(B_R)} \simeq \frac{\mu(B_r)}{R^c} \lesssim \left(\frac{r}{R}\right)^c,$$

since $c \in \underline{S}_0(\mu)$. For $0 < r < R \leq 1$ we get that

$$\frac{\mu(B_r)}{\mu(B_R)} \lesssim \left(\frac{r}{R}\right)^q \quad \text{for all } q < c,$$

since $\underline{Q}_0(w) = (0, c)$. Furthermore we get for $R > r \geq 1$ that

$$\frac{\mu(B_r)}{\mu(B_R)} \simeq \left(\frac{r}{R}\right)^c.$$

The conclusion we can draw is that $(0, c) \subset \underline{Q}(\mu)$ and since we know that $\underline{Q}(\mu) \subset \underline{Q}_0(\mu) = (0, c)$ we can say that $\underline{Q}(\mu) = (0, c)$, which in particular means that $\underline{Q}(\mu)$ is non-empty, so Theorem 4.1 implies that w is 1-admissible. □

We have now shown that the first two weights in Svensson [7] are 1-admissible.

Chapter 6

A_p versus 1-admissibility

It is known that weights that are of class A_p are also p-admissible. The converse however is not always true, that is p-admissible weights are not necessarily of class A_p. In this chapter we will study some weights which are 1-admissible and see when they are and when they are not of class A_1 or A_p.

First we study a simple weight defined as follows

$$w(\rho) = \rho^{c-n} \tag{6.1}$$

for $c > 0$. In [3, Example 3.1] with $c \geq 1$ (and $\beta = 0$) it was shown that the weight has the following exponent sets

$$\underline{S}_0(w) = \underline{Q}_0(w) = \underline{Q}(w) = (0, c], \quad \bar{S}_0(w) = \overline{Q}_0(w) = \overline{Q}(w) = [c, \infty)$$

but we also get the same exponent sets for all $c > 0$.

Theorem 6.1. *The weight w in* (6.1) *is* 1*-admissible for all $c > 0$.*

Proof. We get that

$$\frac{\rho w'(\rho)}{w(\rho)} = c - n$$

so the result follows directly form Corollary 4.4. □

Now we shall also see that even though the weight ρ^{c-n} is always 1-admissible whether or not it is of class A_1 depends on the value of c.

Theorem 6.2. *The weight w in* (6.1) *is of class A_1 if and only if $c \leq n$.*

The sufficient part of Theorem 6.2 follows directly from Theorem 2.14 but for the reader's convenience we provide a complete proof.

Proof. For $c > n$ it is clear that $\inf_{B_r} w = 0$ for any ball B_r which contains the origin. Thus it is clear that for every $C > 0$ and all such balls,

$$\int_{B_r} w\,dx > Cr^n \inf_{B_r} w = 0,$$

showing that w is not of class A_1.

If $c = n$ then $w \equiv 1$ and is clearly of class A_1. For $c < n$ and for any ball $B(x_0, r)$ with $|x_0| \geq 2r$, we have the following estimates. For all $x \in B(x_0, r)$,

$$|x - x_0| < r \leq \tfrac{1}{2}|x_0|.$$

Now by the reverse triangle inequality, we get

$$|x| \geq |x_0| - |x - x_0| > |x_0| - r \geq \tfrac{1}{2}|x_0|,$$

from which it follows that

$$w(|x|) = |x|^{c-n} \leq (\tfrac{1}{2}|x_0|)^{c-n}.$$

With this we can estimate the integral as follows

$$\int_{B(x_0,r)} w\,dx \leq \int_{B(x_0,r)} (\tfrac{1}{2}|x_0|)^{c-n}\,dx = 2^{n-c}|x_0|^{c-n} \int_{B(x_0,r)} dx.$$

Since $r \leq \frac{1}{2}|x_0|$, we also have that

$$\inf_{B(x_0,r)} w = (r + |x_0|)^{c-n} \geq (\frac{1}{2}|x_0| + |x_0|)^{c-n} = \left(\frac{3}{2}\right)^{c-n} |x_0|^{c-n}.$$

With $C = 3^{n-c}$ we then get

$$\int_{B(x_0,r)} w\,dx \leq C \inf_{B(x_0,r)} w \int_{B(x_0,r)} dx.$$

For a ball $B(x_0, r)$ with $r \geq \frac{1}{2}|x_0|$, we notice that $B(x_0, r) \subset B(0, 3r)$. Because $w \geq 0$, we can estimate the integral by

$$\int_{B(x_0,r)} w\,dx \leq \int_{B(0,3r)} w\,dx.$$

Now we calculate the integral over the larger ball $B(0, 3r)$ using polar coordinates as follows,

$$\int_{B(0,3r)} w\,dx = \omega_{n-1} \int_0^{3r} \rho^{c-n}\rho^{n-1}\,d\rho = \frac{3^c\omega_{n-1}}{c} r^c,$$

where ω_{n-1} is the surface measure of the unit sphere in $\mathbf{R}^n$. We also get that

$$\inf_{B(x_0,r)} w = (r + |x_0|)^{c-n} \geq (r + 2r)^{c-n} = 3^{c-n} r^{c-n}.$$

Furthermore, we have that

$$\int_{B(x_0,r)} dx = \frac{\omega_{n-1}}{n} r^n,$$

so we can take $C = \frac{3^n n}{c}$ and get

$$\int_{B(x_0,r)} w\,dx \leq C \inf_{B(x_0,r)} w \int_{B(x_0,r)} dx.$$

This covers all cases and proves that $w \in A_1$ when $c < n$. □

Now we will see that a similar result holds for the weight defined in (5.1).

Theorem 6.3. *The weight w defined in* (5.1) *is not of class A_1 when $c \geq n$.*

Proof. Svensson [7, Chapter 2] showed that

$$\int_{B_{\alpha_k}} w\,dx \simeq \alpha_k^c$$

for any $k \geq k_0$. We thus get that

$$\frac{\displaystyle\int_{B_{\alpha_k}} w\,dx}{\alpha_k^n \inf\limits_{B_{\alpha_k}} w} \simeq \frac{\alpha_k^{c-n}}{\inf\limits_{B_{\alpha_k}} w}.$$

Now since

$$\inf_{B_{\alpha_k}} w \leq w(\alpha_{k+1}) = \alpha_{k+1}^{-\frac{1}{k+1}} \alpha_{k+1}^{c+\frac{1}{k+1}-n} = \alpha_{k+1}^{c-n},$$

we get

$$\frac{\displaystyle\int_{B_{\alpha_k}} w\,dx}{\alpha_k^n \inf\limits_{B_{\alpha_k}} w} \gtrsim \frac{\alpha_k^{c-n}}{\alpha_{k+1}^{c-n}} = \left(\frac{\alpha_k}{\alpha_{k+1}}\right)^{c-n} = \left(\frac{\alpha_k}{\alpha_k^2}\right)^{c-n} = \alpha_k^{n-c}.$$

If we now let $k \to \infty$ and thus $\alpha_k \to 0$, we see that

$$\lim_{k\to\infty} \frac{\displaystyle\int_{B_{\alpha_k}} w\,dx}{\alpha_k^n \inf\limits_{B_{\alpha_k}} w} = \infty \quad \text{if } n < c.$$

This shows that w is not of class A_1 if $c > n$. If $c = n$ we get, since $\beta_k \leq \alpha_k$, that

$$\inf_{B_{\alpha_k}} w \leq w(\beta_k) = \alpha_k^{-\frac{1}{k+1}} \beta_k^{\frac{1}{k+1}} = \alpha_k^{-\frac{1}{k+1}} \alpha_k^{\frac{3}{2}\frac{1}{k+1}} = \alpha_k^{\frac{1}{2(k+1)}}$$

hence we get that

$$\frac{\int_{B_{\alpha_k}} w\,dx}{\alpha_k^n \inf_{B_{\alpha_k}} w} \simeq \frac{\alpha_k^{n-n}}{\inf_{B_{\alpha_k}} w} \geq \alpha_k^{-\frac{1}{2(k+1)}} = 2^{(-2^k)(-\frac{1}{2(k+1)})} = 2^{\frac{2^{k-1}}{k+1}} \to \infty, \quad \text{as } k \to \infty,$$

which shows that w is not of class A_1 when $c = n$. □

Remark 6.4. The case where $c > n$ can also be shown by noticing that

$$w(\alpha_k) = \alpha_k^{c-n} \to 0, \quad \text{as } k \to \infty$$

so

$$\inf_{B_r} w = 0, \quad \text{for all } r > 0.$$

Which is not possible for an A_1-weight.

Next we show that the weight is not A_p for $p \leq \frac{c}{n}$.

Theorem 6.5. *The weight w defined in* (5.1) *is not of class A_p when $1 < p \leq \frac{c}{n}$.*

Proof. Take $p = \frac{c}{n}$. We will prove that w is not in A_p for this p by showing that for $k_0 > \frac{1}{c}$,

$$\int_{B_{\alpha_{k_0}}} w^{\frac{1}{1-p}}\,dx = \infty.$$

With polar coordinates we get that

$$\int_{B_{\alpha_{k_0}}} w^{\frac{1}{1-p}}\,dx \simeq \int_0^{\alpha_{k_0}} w(\rho)^{\frac{1}{1-p}} \rho^{n-1}\,d\rho.$$

Now since $w \geq 0$ we can get the following estimate

$$\int_0^{\alpha_{k_0}} w(\rho)^{\frac{1}{1-p}} \rho^{n-1}\,d\rho \geq \int_{\alpha_k}^{\beta_{k-1}} w(\rho)^{\frac{1}{1-p}} \rho^{n-1}\,d\rho$$

for all $k > k_0$. Now we can calculate the integral as follows

$$\int_{\alpha_k}^{\beta_{k-1}} w(\rho)^{\frac{1}{1-p}} \rho^{n-1}\, d\rho = \int_{\alpha_k}^{\beta_{k-1}} (\alpha_k^{\frac{1}{k}} \rho^{c-\frac{1}{k}-n})^{\frac{1}{1-p}} \rho^{n-1}\, d\rho$$

$$= \alpha_k^{\frac{1}{k(1-p)}} \left[\frac{\rho^{(c-\frac{1}{k}-n)\frac{1}{1-p}+n}}{(c-\frac{1}{k}-n)\frac{1}{1-p}+n} \right]_{\alpha_k}^{\beta_{k-1}=\alpha_k^{\frac{3}{4}}}$$

$$= \frac{\alpha_k^{\frac{1}{k(1-p)}}}{(c-\frac{1}{k}-n)\frac{1}{1-p}+n} \left(\alpha_k^{\frac{3}{4}((c-\frac{1}{k}-n)\frac{1}{1-p}+n)} - \alpha_k^{(c-\frac{1}{k}-n)\frac{1}{1-p}+n} \right).$$

Now we note that for $p = \frac{c}{n} > 1$,

$$(c - \frac{1}{k} - n)\frac{1}{1-p} + n = -\frac{1}{k(1-p)} = \frac{n}{k(c-n)}.$$

Hence, since $\frac{c}{n} > 1$ also implies that $c > n$, we get that

$$\int_{\alpha_k}^{\beta_{k-1}} w(\rho)^{\frac{1}{1-p}} \rho^{n-1}\, d\rho = \frac{\alpha_k^{-\frac{n}{k(c-n)}} \alpha_k^{\frac{n}{k(c-n)}}}{\frac{n}{k(c-n)}} \left(\alpha_k^{\frac{-n}{4k(c-n)}} - 1 \right)$$

$$= \frac{k(c-n)}{n} \left(2^{\frac{2^{k-2}n}{k(c-n)}} - 1 \right) \to \infty, \quad \text{as } k \to \infty.$$

Thus

$$\int_{B_{\alpha_{k_0}}} w^{\frac{1}{1-p}}\, dx = \infty$$

and w is not of class A_p for $p = \frac{c}{n}$. Theorem 2.15 thus implies that w is not of class A_p for any $p \leq \frac{c}{n}$. □

Now we will see that the weight is actually of class A_p for some larger p.

Theorem 6.6. *The weight w defined in* (5.1) *is of class A_p when $p > c$.*

Proof. By Theorem 5.1 w is 1-admissible, and hence by Theorem 2.9 it is also p-admissible for all $p \geq 1$. This means that the measure $d\mu = w\, dx$ is doubling and supports a p-Poincaré inequality. Now we want to use Theorems 2.17 and 2.18, so we need to show that $\inf \overline{Q}(\mu) = c$. Recall that Svensson [7, Chapter 2] showed that $\overline{Q}_0(\mu) = (c, \infty)$. In the proof of Theorem 5.1 we showed that $\mu(B_R) \simeq R^c$ for $R > 1$. Let $q > c$ be arbitrary. For $0 < r < 1 < R$ we get that

$$\frac{\mu(B_r)}{\mu(B_R)} = \frac{\mu(B_r)}{\mu(B_1)} \frac{\mu(B_1)}{\mu(B_R)} \gtrsim \left(\frac{r}{1}\right)^q \left(\frac{1}{R}\right)^c \geq \left(\frac{r}{R}\right)^q$$

and for $0 < r < R \le 1$

$$\frac{\mu(B_r)}{\mu(B_R)} \gtrsim \left(\frac{r}{R}\right)^q$$

because $\overline{Q}_0(\mu) = (c, \infty)$. And finally for $1 < r < R$

$$\frac{\mu(B_r)}{\mu(B_R)} \simeq \left(\frac{r}{R}\right)^c \ge \left(\frac{r}{R}\right)^q.$$

Thus $\overline{Q}(\mu) = (c, \infty)$ and we can apply Theorem 2.18 to get that

$$\operatorname{cap}_{p,\mu}^{\mathbf{R}^n}(\{0\}, B_r) \simeq r^{-p}\mu(B_r) \quad \text{for all } r > 0.$$

With this we can apply Theorem 2.17. Since the measure $d\mu = w\,dx$ is doubling and supports a p-Poincaré inequality Theorem 2.17 implies that w must be of class A_p. □

Theorem 6.7. *The weight w defined in* (3.1) *is not of class A_1 when $n = 2$.*

Proof. Svensson [7, Chapter 3] showed that

$$\int_{B_{\alpha_k}} w\,dx \simeq \alpha_k.$$

Since $\beta_k \le \alpha_k$ it now follows that

$$\inf_{B_{\alpha_k}} w \le w(\beta_k) = \alpha_k^{-1-\frac{1}{k+1}} \beta_k^{\frac{1}{k+1}}.$$

Hence

$$\frac{\int_{B_{\alpha_k}} w\,dx}{\alpha_k^2 \inf_{B_{\alpha_k}} w} \gtrsim \frac{\alpha_k}{\alpha_k^2 \alpha_k^{-1-\frac{1}{k+1}} \beta_k^{\frac{1}{k+1}}} = \frac{\alpha_k^{\frac{1}{k+1}}}{\beta_k^{\frac{1}{k+1}}} = \alpha_k^{-\frac{1}{k+1}\left(\frac{4+k}{3+k}-1\right)} = \alpha_k^{-\frac{1}{(k+1)(3+k)}},$$

where we used that $\beta_k = \alpha_k^{\frac{4+k}{3+k}}$. Thus we get that

$$\frac{\int_{B_{\alpha_k}} w\,dx}{\alpha_k^2 \inf_{B_{\alpha_k}} w} \gtrsim \alpha_k^{-\frac{1}{(k+1)(3+k)}} = 2^{(-2^k)(-\frac{1}{(k+1)(3+k)})} = 2^{\frac{2^k}{(k+1)(3+k)}} \to \infty, \quad \text{as } k \to \infty,$$

which shows that w is not of class A_1. □

Chapter 7

Discussion

In Chapter 3 we showed that one of the weights in Svensson [7] is 1-admissible. Furthermore in Theorem 3.4 we managed to generalize the weight and show that the generalized weight was 1-admissible. Additionally in Chapter 5 we showed that another weight from [7] is 1-admissible. Svensson [7] also defined a third weight, one could probably use the results from Chapter 4 to show that this weight is also 1-admissible, and if that is true a generalization such as in Theorem 3.4 should also be possible.

Finally in Chapter 6 we showed that the simple weight $w(\rho) = \rho^{c-n}$ is of class A_1 if and only if $c \leq n$. Whereas we for the more complicated weights from Svensson [7] only showed that they were not of class A_1 when $c \geq n$ and for $n = 2$. The obvious question that remains is if either of these weights are A_1 for any values of c and n. Additionally in Theorems 6.5 and 6.6 we showed that one of the weights is not of class A_p when $1 < p \leq \frac{c}{n}$ and is of class A_p when $p > c$. Whether or not the weight is A_p for the cases in between, that is when $\frac{c}{n} < p \leq c$, still remains an open question.

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