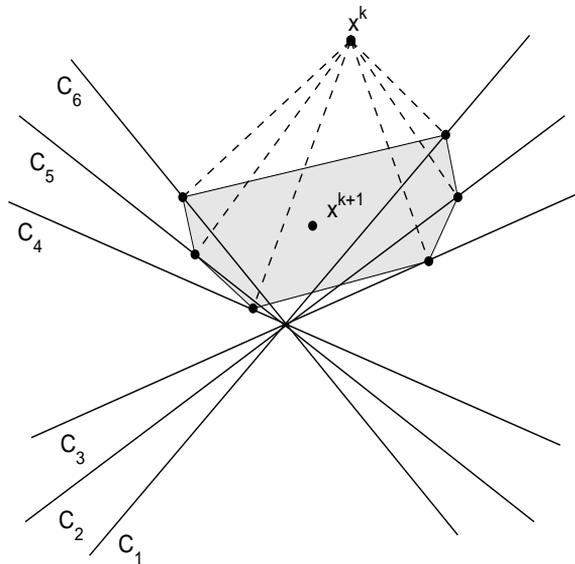


# Algebraic Reconstruction Methods

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## Algebraic Reconstruction Methods

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*Dedicated to  
Aronch, Ayleen  
and  
My parents*



# Abstract

Ill-posed sets of linear equations typically arise when discretizing certain types of integral transforms. A well known example is image reconstruction, which can be modeled using the Radon transform. After expanding the solution into a finite series of basis functions a large, sparse and ill-conditioned linear system occurs. We consider the solution of such systems. In particular we study a new class of iteration methods named DROP (for Diagonal Relaxed Orthogonal Projections) constructed for solving both linear equations and linear inequalities. This class can also be viewed, when applied to linear equations, as a generalized Landweber iteration. The method is compared with other iteration methods using test data from a medical application and from electron microscopy. Our theoretical analysis include convergence proofs of the fully-simultaneous DROP algorithm for linear equations without consistency assumptions, and of block-iterative algorithms both for linear equations and linear inequalities, for the consistent case.

When applying an iterative solver to an ill-posed set of linear equations the error usually initially decreases but after some iterations, depending on the amount of noise in the data, and the degree of ill-posedness, it starts to increase. This phenomenon is called semi-convergence. We study the semi-convergence performance of Landweber-type iteration, and propose new ways to specify the relaxation parameters. These are computed so as to control the propagated error.

We also describe a class of stopping rules for Landweber-type iteration for solving linear inverse problems. The class includes the well known discrepancy principle, and the monotone error rule. We unify the error analysis of these two methods. The stopping rules depend critically on a certain parameter whose value needs to be specified. A training procedure is therefore introduced for securing robustness. The advantages of using trained rules are demonstrated on examples taken from image reconstruction from projections.

Kaczmarz's method, also called ART (Algebraic Reconstruction Technique) is often used for solving the linear system which appears in image reconstruction. This is a fully sequential method. We examine and compare ART and its symmetric version. It is shown that the cycles of symmetric ART, unlike ART, converge to a weighted least squares solution if and only if the relaxation parameter lies between zero and two. Further we show that ART has faster asymptotic rate of convergence than symmetric ART. Also a stopping criterion is proposed and evaluated for symmetric ART.

We further investigate a class of block-iterative methods used in image recon-

struction. The cycles of the iterative sequences are characterized in terms of the original linear system. We define symmetric block-iteration and compare the behavior of symmetric and non-symmetric block-iteration. The results are illustrated using some well-known methods. A stopping criterion is offered and assessed for symmetric block-iteration.

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# Papers

The following papers are appended, and will be referred to by their Roman numerals. The papers [I,II] were published and the manuscript [III] was accepted for publication. The manuscript [IV] was submitted.

- [I] YAIR CENSOR, TOMMY ELFVING, GABOR T. HERMAN AND TOURAJ NIKAZAD, *On diagonally relaxed orthogonal projection methods*, SIAM J. Sci. Comput., 30 (2007/08), pp. 473–504.
- [II] TOMMY ELFVING AND TOURAJ NIKAZAD, *Stopping rules for Landweber-type iteration*, Inverse Problems, 23 (2007), pp. 1417–1432.
- [III] TOMMY ELFVING AND TOURAJ NIKAZAD, *Some Properties of ART-type Reconstruction Algorithms*, Mathematical Methods in Biomedical Imaging and Intensity-Modulated Radiation Therapy (IMRT), Y. Censor, M. Jiang and A.K. Louis, Editors, Edizioni della Normale, Pisa, Italy, 2008.
- [IV] TOMMY ELFVING AND TOURAJ NIKAZAD, *Some block-iterative methods used in image reconstruction*, (submitted March 2008).
- [V] TOMMY ELFVING AND TOURAJ NIKAZAD, *Semi-convergence and choice of relaxation parameters in Landweber-type algorithms*, (manuscript, April 2008).



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# 1

## Introduction

A mark-point in the history of medical imaging, was the discovery by Wilhelm Röntgen in 1895 of x-rays [22, 40]. The problem of generating medical images from measurements of the radiation around the body of a patient was considered much later. Hounsfield patented the first CT-scanner in 1972 (and was awarded, together with Cormack, in 1979 the Nobel Prize for this invention). This reconstruction problem belongs to the class of inverse problem, which are characterized by the fact that the information of interest is not directly available for measurements. The imaging device (the camera) provides measurements of a transformation of this information. In practice, these measurements are both imperfect (sampling) and inexact (noise).

The mathematical basis for tomographic imaging was laid down by Johann Radon already in 1917 [37]. The word tomography means 'reconstruction from slices'. It is applied in Computerized (Computed) Tomography (CT) to obtain cross-sectional images of patients. Fundamentally, tomographic imaging deals with reconstructing an image from its projections. The relationship between the unknown distribution (or object) and the physical quantity which can be measured (the projections) is referred to as the forward problem. For several imaging techniques, such as CT, the simplest model for the forward problem involves using the Radon transform  $R$ , see [2, 31, 35]. If  $\chi$  denotes the unknown distribution and  $\beta$  the quantity measured by the imaging device, we have

$$R\chi = \beta.$$

The discrete problem, which is based on expanding  $\chi$  in a finite series of basis-functions, can be written as

$$Ax = b, \tag{0.1}$$

where the vector  $b$  is a sampled version of  $\beta$  and the vector  $x$ , in the case of pixel-(2D) or voxel-(3D) basis, is a finite representation of the unknown object. The matrix  $A$ , typically large and sparse, is a discretization of the Radon transform. An approximative solution to this linear system could be computed by iterative methods, which only require matrix-vector products and hence do not alter the structure of  $A$ .

## 1 Semi-convergence behavior of Landweber-type iteration

When solving a set of linear ill-posed equations by an iterative method typically the iterates first improve, while at later stages the influence of the noise becomes more and more noticeable. This phenomenon is called semi-convergence by Natterer [35, p. 89]. In order to better understand the mechanism of semi-convergence, we take a closer look at the errors in the regularized solution using the following Landweber-type method

$$x^{k+1} = x^k + \lambda_k A^T M(b - Ax^k), \quad (1.1)$$

where  $\lambda_k$  is a relaxation parameter and  $M$  is a given symmetric positive definite matrix. Convergence result and recent extensions, including block versions of (1.1), can be found in [9, 29]. We now consider the following additive noise model

$$b = \bar{b} + \delta b.$$

Here  $\bar{b}$  is the noise free right-hand side and  $\delta b$  in the noise-component. The noise may come from both discretization errors and measurement errors. We also assume, without loss of generality, that  $x^0 = 0$ . Let

$$B = A^T M A, \text{ and } c = A^T M b.$$

Then using (1.1) with  $\lambda_k = \lambda$

$$\begin{aligned} x^k &= (I - \lambda B)x^{k-1} + \lambda c \\ &= \lambda \sum_{j=0}^{k-1} (I - \lambda B)^j c. \end{aligned}$$

Suppose

$$M^{\frac{1}{2}} A = U \Sigma V^T$$

is the singular value decomposition (SVD) of  $M^{\frac{1}{2}} A$ , where  $M^{\frac{1}{2}}$  is the square root of  $M$  (a good presentation of SVD can be found in, e.g., [4]). Then

$$B = (M^{\frac{1}{2}} A)^T (M^{\frac{1}{2}} A) = V \Sigma^T \Sigma V^T = F V V^T, \quad (1.2)$$

where

$$F = \text{diag}(\sigma_1^2, \sigma_2^2, \dots, \sigma_p^2, 0, \dots, 0), \text{ and } \sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_p > 0,$$

and assuming that  $\text{rank}(A) = p$ .

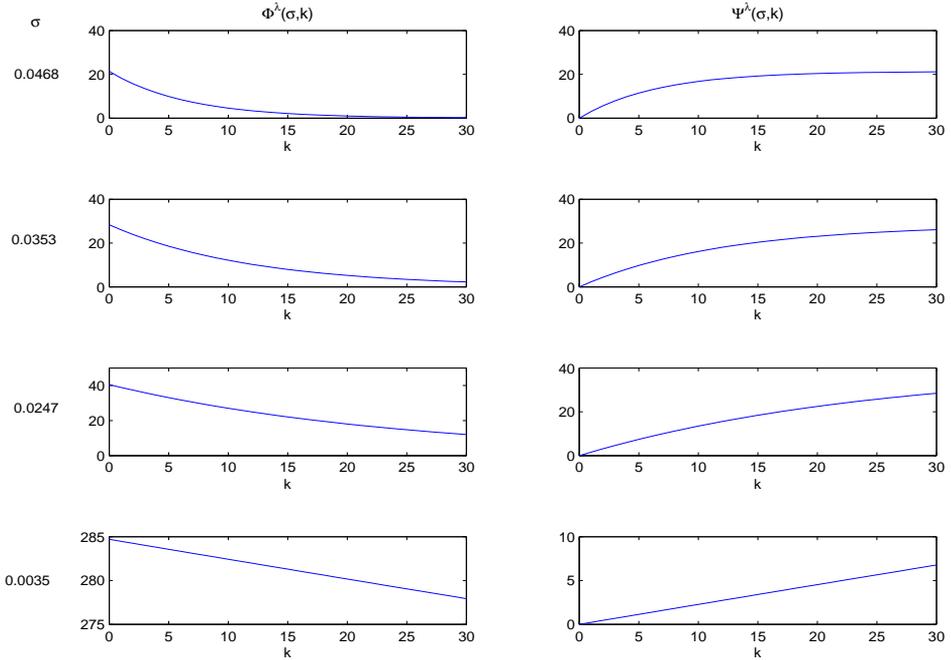
After some calculations it follows

$$x^k = \sum_{i=1}^p \{1 - (1 - \lambda \sigma_i^2)^k\} \frac{u_i^T M^{\frac{1}{2}} (\bar{b} + \delta b)}{\sigma_i} v_i. \quad (1.3)$$

The functions

$$\phi_i = 1 - (1 - \lambda \sigma_i^2)^k, \quad i = 1, 2, \dots, p$$

are called filter factors, see, e.g., [5] and [24, p. 138].



**Figure 1.1:** The behavior of  $\Phi$  (left) and  $\Psi$  (right) using different  $\sigma$ - values, with  $\lambda = 1.8/\sigma_1^2$ .

Let  $x^* = \operatorname{argmin} \|Ax - \bar{b}\|_M$  be the unique weighted least squares solution of minimal 2-norm. It is easy to show, see, e.g., paper [V] for details, that

$$e^{V,k} \equiv V^T(x^k - x^*) = D_1 \hat{b} + D_2 \delta b.$$

Here

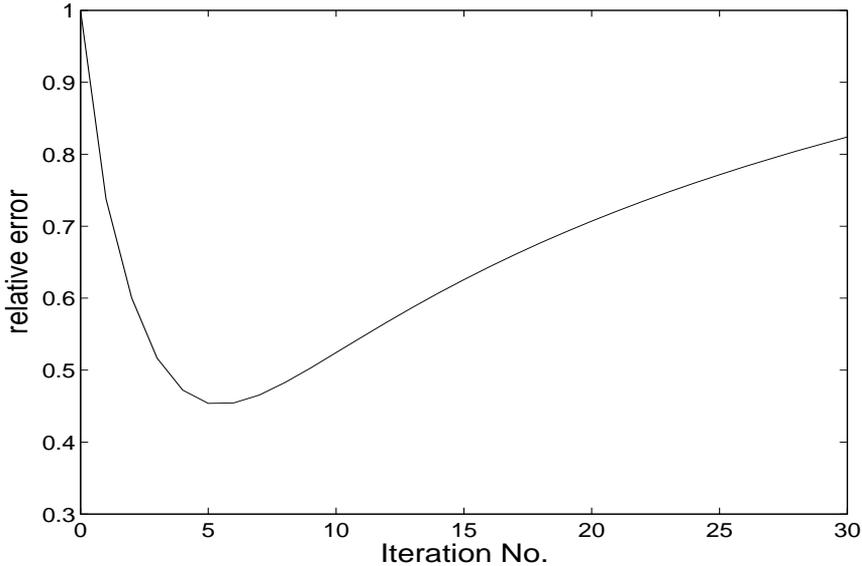
$$\begin{aligned} D_1 &\equiv -\operatorname{diag}\left(\frac{(1 - \lambda\sigma_1^2)^k}{\sigma_1}, \dots, \frac{(1 - \lambda\sigma_p^2)^k}{\sigma_p}, 0, \dots, 0\right), \\ D_2 &\equiv \operatorname{diag}\left(\frac{1 - (1 - \lambda\sigma_1^2)^k}{\sigma_1}, \dots, \frac{1 - (1 - \lambda\sigma_p^2)^k}{\sigma_p}, 0, \dots, 0\right), \end{aligned} \tag{1.4}$$

and

$$\hat{b} = U^T M^{\frac{1}{2}} \bar{b}, \quad \delta b = U^T M^{\frac{1}{2}} \delta b.$$

Let

$$\Phi^\lambda(\sigma, k) = \frac{(1 - \lambda\sigma^2)^k}{\sigma},$$



**Figure 1.2:** *Semi-convergence behavior.*

and

$$\Psi^\lambda(\sigma, k) = \frac{1 - (1 - \lambda\sigma^2)^k}{\sigma}.$$

Then the  $j$ th component of the projected error  $e^{V,k}$  is

$$e_j^{V,k} = -\Phi^\lambda(\sigma_j, k)\hat{b}_j + \Psi^\lambda(\sigma_j, k)\delta\hat{b}_j.$$

Hence the projected total error  $e_j^{V,k}$  has two components, an iteration-error (first term) and a noise-error. Figure 1.1 displays  $\Phi^\lambda(\sigma, k)$  and  $\Psi^\lambda(\sigma, k)$ , for a fixed  $\lambda$  and various  $\sigma$ , as a function of iteration index  $k$ . We establish some properties of these functions in paper [V]. It is seen that, for small values of  $k$  the noise-error is negligible and the iteration seems to converge to the exact solution. When the noise-error reaches the order of magnitude of the iteration-error, the propagated noise-error is no longer hidden in the regularized solution, and the total error starts to increase. Indeed the projected error goes to a constant value as the number of iteration goes to infinity. The typical overall error behavior is shown in figure 1.2.

In paper [V] we first study the semi-convergence behavior of algorithm (1.1). Based on this analysis new ways to specify the relaxation parameters are suggested. The parameters are computed so as to control the propagated noise component of the error. We also show that the resulting parameters are such that convergence of the iteration (1.1) is guaranteed. The advantages of using this choice of relaxation parameters are demonstrated on examples taken from image reconstruction from projections.

## 2 Projection algorithms

A common problem in different areas of mathematics and physical sciences consists of finding a point in the intersection of convex sets. This problem is referred to as the convex feasibility problem. Its mathematical formulation is as follows.

Suppose  $X$  is a Hilbert space and  $C_1, \dots, C_N$  are closed convex subsets with nonempty intersection  $C$ :

$$C = C_1 \cap \dots \cap C_N \neq \emptyset.$$

The convex feasibility problem is to find some point  $x$  in  $C$ . In image reconstruction using the fully discretized model each set  $C_i$  is a hyperplane or pairs of halfspaces, so called hyperslabs, see [11, p. 269-270]. A common solution approach to such problems is to use projection algorithms, see, e.g., [1], which employ orthogonal projections (i.e., nearest point mappings) onto the individual sets  $C_i$ . These methods can have different algorithmic structures (e.g., [6, 8] and [11, section 1.3]) some of which are particularly suitable for parallel computing, and they demonstrate nice convergence properties and/or good initial behavior patterns.

This class of algorithms has witnessed much progress in recent years and its member algorithms have been applied with success to fully-discretized models of problems in image reconstruction from projections (e.g., [25]), in image processing (e.g., [39]), and in intensity-modulated radiation therapy (IMRT) (e.g., [10]). Apart from theoretical interest, the main advantage of projection methods that makes them successful in real-world applications is computational. They commonly have the ability to handle huge-size problems of dimensions beyond which other, more sophisticated currently available, methods cease to be efficient. This is so because the building bricks of a projection algorithm are the projections onto the individual sets (that are assumed easy to perform) and the algorithmic structure is either sequential or simultaneous (or in-between). In paper [I] we study a new class of projection methods. This class when applied to linear equations, can also be seen as a generalized Landweber iteration. Another established class of iterations for solving linear equations is Krylov subspace methods, with CGLS (conjugate gradient applied to the normal equations) as a well known member. For low-noise and moderately ill-conditioned problems CGLS is usually very efficient. However for noisy and ill-conditioned problems (where the number of iterations is rather small before the noise component in the iterates starts to increase) projection methods become competitive, see also [34].

**2.1 Iterative algorithms** We are interested in iterative algorithms for solving the linear system (0.1). The algorithms can be generally classified as being either *sequential* or *simultaneous* or *block-iterative* - see, e.g., Censor and Zenios [11], and the review paper of Bauschke and Borwein [1] for a variety of specific algorithms of these kinds. In this section we will introduce them shortly and the related works that were done by us.

The Algebraic Reconstruction Technique (ART) is a fully sequential method, and has a long history and rich literature. Originally it was proposed by Kaczmarz [30], and independently, for use in image reconstruction by Gordon, Bender

and Herman, see [25]. It is of row-action type [11]. The vector of unknowns is updated at each equation of the system, after which the next equation is addressed. We define one *cycle* as one pass through all data. If the system of equations (0.1) is consistent, ART converges to a solution of this system. If the system is inconsistent, every sub-sequence of cycles through the system, converges, but not necessarily to a least squares solution [16]. Several ways to cope with inconsistency in ART have been suggested. Herman et al. [16, section 2.2] propose iterating both in  $x$  and a dual variable  $r$ . The resulting method converges towards the solution of a regularized least squares problem. Popa in a series of papers see, e.g., [36] also proposes iterating in both  $x$  and  $r$ . This method converges towards the minimum norm least squares solution of the linear system. As shown by Censor et al. [12], if relaxation parameter goes to zero then ART converges to a weighted least squares solution.

The symmetric ART (symART) [3] is another fully sequential method. To derive this method we first perform one cycle of ART followed by another cycle but now taking the equations in reverse order. Figure (2.1) illustrates how these two methods work. It is shown in paper [III] that the cycles of symmetric ART, unlike ART, converge to a weighted least squares solution if and only if the relaxation parameter lies between zero and two. Further we show that ART has a faster asymptotic rate of convergence than symmetric ART. Finally a stopping criterion is proposed and evaluated for symmetric ART. Although sequential methods are notoriously hard to parallelize, it should be noted that for sparse problems, the sequential row-action ART method can be parallelized by simultaneously projecting the current iterate onto a set of mutually orthogonal hyperplanes (obtained by considering equations whose sets of nonzero components are pairwise disjoint). For the case of image reconstruction from projections, such sets of equations can be obtained by considering parallel rays that are sufficiently far apart so as to pass through disjoint sets of pixels, see [7, section 5.1] and [17, section 5]. However, for image reconstruction, such a procedure produces very small granularity, and the amount of communications required between processors could make such a parallel implementation unattractive.

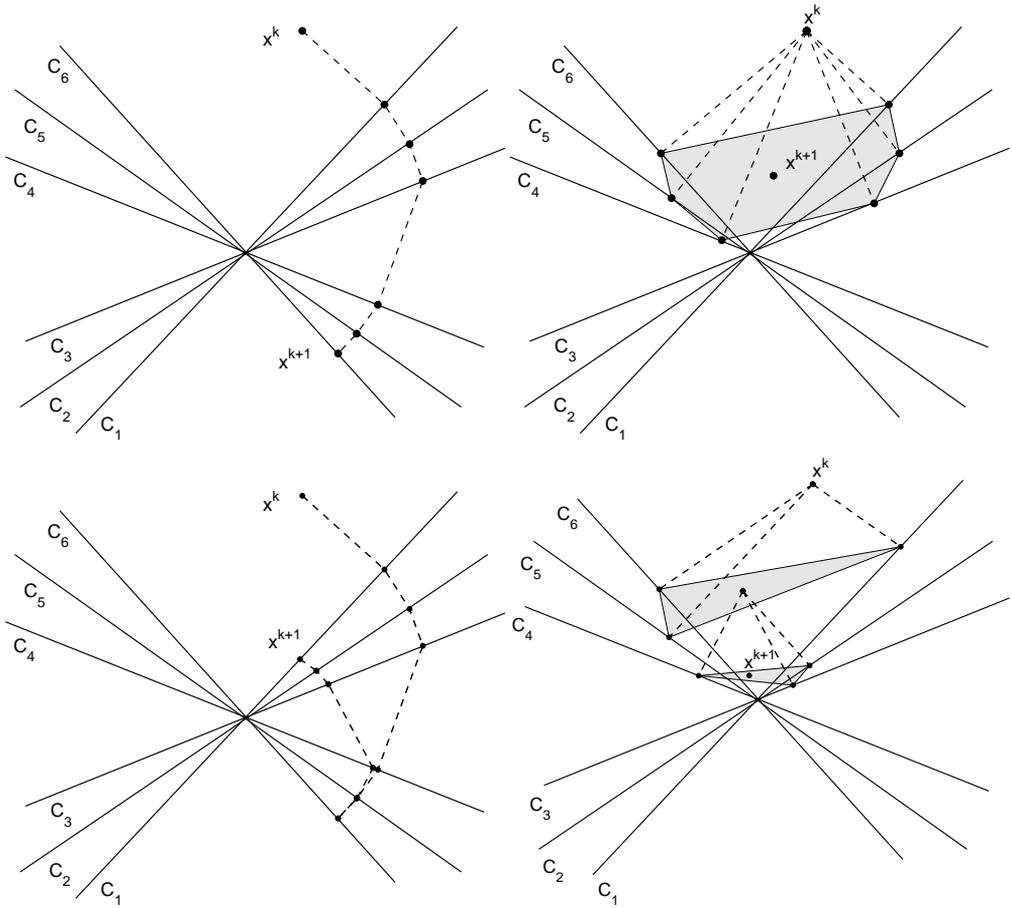
The prototype of simultaneous algorithms is the well-known Cimmino method [14]. In this method the current iterate  $x^k$  is first projected on all sets to obtain intermediate points

$$x^{k+1,i} = P_i(x^k), \quad i = 1, 2, \dots, N,$$

where  $P_i$  is the orthogonal (least Euclidean distance) projection onto  $C_i$ , and then the next iterate is

$$x^{k+1} = x^k + \lambda_k \left( \sum_{i=1}^N w_i x^{k+1,i} - x^k \right),$$

where  $w_i$  are fixed weights such that  $w_i > 0$  for all  $i$ , and  $\sum_{i=1}^N w_i = 1$ , and  $\{\lambda_k\}_{k \geq 0}$  are relaxation parameters, commonly confined to the interval  $\epsilon \leq \lambda_k \leq 2 - \epsilon$ , for some fixed but arbitrarily small  $\epsilon > 0$ , see figure (2.1). Besides its inherent parallelism, one advantage of the fully simultaneous algorithm over sequential



**Figure 2.1:** Left: ART method (top), symART method (bottom). Right: Cimmino method (top), block-iterative method (bottom).

methods, is its behavior in the inconsistent case. It is guaranteed to converge to a weighted least-squares solution which minimizes the weighted sum of the squares of the distances to the sets  $C_i$ ,  $i = 1, 2, \dots, N$ . This has been shown by Iusem and De Pierro [28], via a local convergence result, and globally by Combettes [15]. However, the slow rate of convergence of the method hampers its use, particularly for large and sparse unstructured systems. The term “rate of convergence” is here used in an informal manner to refer to the practical behavior of the algorithm during a finite number of iterations starting from the first iterate  $x^0$  and onwards.

We now move from the fully-simultaneous iteration to its “block-iterative” generalization. This algorithmic model of block iterations is a special case of *asynchronous iterations*, see, e.g., Frommer and Szyld [20] and Elsner, Koltracht and

Neumann [19]. Those were called in early days *chaotic relaxation* by Chazan and Miranker [13]. In recent literature in image reconstruction from projections the term “ordered subsets” is sometimes used for “block-iterative”, see, e.g., Hudson and Larkin [27]. The basic idea of a block-iterative algorithm is to partition the data  $A$  and  $b$  of the system  $Ax = b$  into “blocks” of equations (rows), it can also be done column wise, see, e.g., Elfving [17, 18], and treat each block according to the rule used in the simultaneous algorithm for the whole system, passing, e.g., cyclically over all the blocks. As shown in figure (2.1) block-iterative methods combine, during one cycle, simultaneous and sequential iteration. In paper [IV] the cycles of the iterative sequences are characterized in terms of the original linear system. We further define symmetric block-iteration and compare the behavior of symmetric and non-symmetric block-iteration. The results are illustrated using some well-known methods. Finally a stopping criterion is offered and assessed for symmetric block-iteration.

### 3 Stopping rules

All regularization methods make use of a certain regularization parameter that controls the amount of stabilization imposed on the solution. In iterative methods one can use the stopping index as regularization parameter. When an iterative method is employed, the user can also study on-line adequate visualizations of the iterates as soon as they are computed, and simply halt the iteration when the approximations reach the desired quality. This may actually be the most appropriate stopping rule in many practical applications, but it requires a good intuitive imagination of what to expect. In other situations the user will need the computer’s help to determine the optimal approximation, and this is the case we consider here. The stopping rule strategies naturally divide into two categories: rules which are based on knowledge of the norm of the errors, and rules which do not require such information.

If the error norm is known within reasonable accuracy, the perhaps most well known stopping rule is the discrepancy principle Morozov [33]. Another related rule is the monotone error rule by Hämarik and Tautenhahn [23]. Examples of the second category of methods are the L-curve criterion [24], and the generalized cross-validation criterion [21]. The performance of these parameter choice methods depends in a complex way on both regularization method and the inverse problem at hand. E.g., the results of using the discrepancy principle for the classical Landweber method [32] are quite good. However using the discrepancy principle for Cimmino’s method [14] requires special care as is demonstrated in papers [II, IV].

# 2

## Summary of papers

### **Paper I**

#### **On Diagonally Relaxed Orthogonal Projection Methods**

In the literature on reconstruction from projections, e.g., [26] and [38, Eq. (3)], researchers introduced diagonally-relaxed orthogonal projections (DROP) for heuristic reasons. However, there has been until now no mathematical study of the convergence behavior of such algorithms. Our paper makes a contribution to the convergence analysis.

We first consider a fully-simultaneous DROP algorithm for linear equations and prove its convergence without consistency assumptions. We also introduce general (block-iterative) algorithms both for linear equations and for linear inequalities and study their convergence, but only for the consistent case. Then we describe a number of iterative algorithms that we have implemented for the purpose of an experimental study. For the experiments a phantom based on a medical problem and another based on a problem from electron microscopy have been used to generate both noiseless and noisy projection data, and various algorithms have been applied to such data for the purpose of comparison. The results show that the use of DROP as an image reconstruction algorithm is not inferior to previously used methods. Those practitioners who used it without the mathematical justification offered here were indeed creating very good reconstructions. All our experiments are performed in a single processor environment. Further computational gains can be achieved by using DROP in a parallel computing environment with appropriate block choices but doing so and comparing it to other algorithms that were used in the comparisons made here calls for a separate study.

### **Paper II**

#### **Stopping Rules for Landweber-type Iteration**

A class of stopping rules for Landweber-type iterations for solving linear inverse problems is considered. The discrepancy principle (DP rule) and the monotone error rule (ME rule) are included in this class. Our analysis therefore unifies the DP and ME rule by showing that they both are special cases of a more general rule. We also unify the analysis of their error reduction properties and clarify the role

of the relaxation parameter. Also new results concerning the number of iterations needed in the DP and ME rule respectively are presented. We also shortly discuss possible errors in the matrix  $A$ , and show how the stopping rules can be modified to handle this case.

The DP rule is, stop when for the first time  $\|Ax^k - b\| \leq \tau\delta$  where  $\delta$ , the norm of the noise, is assumed known. We show that  $\tau \in (0, 2]$  for insuring error reduction. However the actual value of  $\tau$  is critical for the performance of the stopping rules. It was found, during our experiments, that generalized Landweber methods were quite sensitive to the choice of  $\tau$ . We therefore introduce a training procedure for securing a robust rule. The training is based on knowing the index where the error is minimal for certain training samples. The information gathered during the training phase is then used in the evaluation phase where unseen data is treated. We have found (experimentally) a scaling procedure, that allows using samples from a medium sized problem for predicting the stopping index for a large sized problem. The data samples all come from the field of image reconstruction from projections but differ in size and noise level.

The advantages of using a trained rule, cf. to using fixed values like  $\tau = 1, 2$  as suggested previously, are demonstrated on some examples taken from image reconstruction. In fact after training the stopping rules became quite robust and only small differences were observed between, e.g., the DP rule and ME rule.

### **Paper III**

#### **Some Properties of ART-type Reconstruction Algorithms**

We have analyzed the cyclic convergence of ART and symART for inconsistent data. Both methods converge to the same weighted least squares solution when the relaxation parameter  $\lambda$  goes to zero. Unlike ART, symART also converges to a weighted (with the weight matrix depending on  $\lambda$ ) least squares solution for any fixed  $\lambda$  between zero and two. It is also shown that ART has faster asymptotic rate of convergence than symART. As a by-product of our analysis we can apply the class of stopping rules discussed in paper [II] to symART. We discuss in particular the implementation of the DP rule combined with training, and report on numerical results obtained from some reconstruction problems. Also we study further the block-iterative version of DROP. It is shown by a counter example that a non-stationary version (called DROP2 in paper [I]) fails to converge.

### **Paper IV**

#### **Some Block-Iterative Methods used in Image Reconstruction**

We have studied cyclic convergence of a class of block-iterative methods used in image reconstruction from projections. These include the block Cimmino method and the block Kaczmarz's method. It is shown that the limit-points satisfy a certain system of linear equations. We also define symmetric block-iteration. Here the limit-points satisfy a weighted least squares problem. It is also shown that block-iteration always converges asymptotically faster than the corresponding symmetric block-iteration. We finally propose a stopping rule for symmetric block-iteration, and demonstrate its usefulness on some problems from image reconstruction.

## **Paper V**

### **Semi-Convergence and Choice of Relaxation Parameters in Landweber-type Algorithms**

We study the semi-convergence behavior of Landweber-type algorithms. The total error can be decomposed into two parts, the iteration-error and the noise-error. These two errors can be represented by two functions both depending on the iteration index, the relaxation parameter and the singular values of the operator. We derive some results on the behavior of these two functions. Based on this analysis we propose new ways to choose relaxation parameters. The parameters are computed so as to control the propagated noise-error. We also prove convergence of algorithm (1.1) using the new relaxation strategies. Finally we compare these strategies with two other choices using examples taken from image reconstruction from projections.

## **Notification**

The alphabetic order of authors in the five papers reflects approximatively equal inputs to the papers. It is of course natural that the adviser mostly inputs ideas and the student works out the details. Many improvements have emerged after the results of numerical experiments. All experimental work was done by the student.



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