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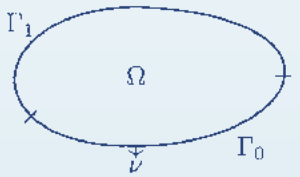
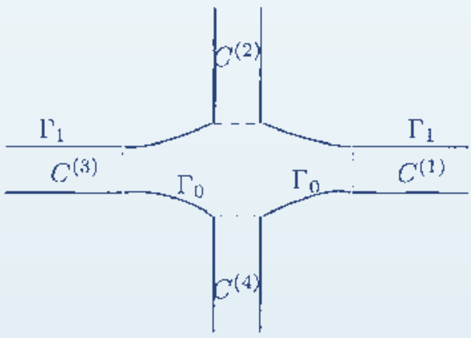
Reconstruction of solutions of Cauchy problems for elliptic equations in bounded and unbounded domains using iterative regularization methods

Pauline Achieng





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To my family

Abstract

Cauchy problems for elliptic equations arise in applications in science and engineering. These problems often involve finding important information about an elliptical system from indirect or incomplete measurements. Cauchy problems for elliptic equations are known to be disadvantaged in the sense that a small pertubation in the input can result in a large error in the output. Regularization methods are usually required in order to be able to find stable solutions.

In this thesis we study the Cauchy problem for elliptic equations in both bounded and unbounded domains using iterative regularization methods. In Paper I and II, we focus on an iterative regularization technique which involves solving a sequence of mixed boundary value well–posed problems for the same elliptic equation. The original version of the alternating iterative technique is based on iterations alternating between Dirichlet–Neumann and Neumann–Dirichlet boundary value problems. This iterative method is known to possibly work for Helmholtz equation. Instead we study a modified version based on alternating between Dirichlet–Robin and Robin-Dirichlet boundary value problems. First, we study the Cauchy problem for general elliptic equations of second order with variable coefficients in a limited domain. Then we extend to the case of unbounded domains for the Cauchy problem for Helmholtz equation. For the Cauchy problem, in the case of general elliptic equations, we show that the iterative method, based on Dirichlet–Robin, is convergent provided that parameters in the Robin condition are chosen appropriately. In the case of an unbounded domain, we derive necessary, and sufficient, conditions for convergence of the Robin–Dirichlet iterations based on an analysis of the spectrum of the Laplacian operator, with boundary conditions of Dirichlet and Robin types.

In the numerical tests, we investigate the precise behaviour of the Dirichlet-Robin iterations, for different values of the wave number in the Helmholtz equation, and the results show that the convergence rate depends on the choice of the Robin parameter in the Robin condition. In the case of unbounded domain, the numerical experiments show that an appropriate truncation of the domain and an appropriate choice of Robin parameter in the Robin condition lead to convergence of the Robin–Dirichlet iterations.

In the presence of noise, additional regularization techniques have to implemented for the alternating iterative procedure to converge. Therefore, in Paper III and IV we focus on iterative regularization methods for solving the Cauchy problem for the Helmholtz equation in a semi–infinite strip, assuming that the data contains measurement noise. In addition, we also reconstruct a radiation condition at infinity from the given Cauchy data. For the reconstruction of the radiation condition, we solve a well–posed problem for the Helmholtz equation in a semi–infinite strip. The remaining solution is obtained by solving an ill–posed problem. In Paper III, we consider the ordinary Helmholtz equation and use seperation of variables to analyze the problem. We show that the radiation condition is described by a non–linear well–posed problem that provides a stable oscillatory solution to the Cauchy problem. Furthermore, we show that the ill–posed problem can be regularized using the Landweber's iterative method and the discrepancy principle.

Numerical tests shows that the approach works well.
Paper IV is an extension of the theory from Paper III to the case of variable coefficients. Theoretical analysis of this Cauchy problem shows that, with suitable bounds on the coefficients, can iterative regularization methods be used to stabilize the ill–posed Cauchy problem.

Sammanfattning

Cauchyproblem för elliptiska ekvationer uppstår i tillämpningar inom vetenskap och ingenjörskonst. Dessa problem handlar ofta om att hitta viktig information om ett elliptiskt system från indirekta eller ofullständiga mätningar. Cauchyproblem för elliptiska ekvationer är kända för att vara illa-ställda i den meningen att en liten störning i indata kan resultera i ett stort fel i utdata. Det krävs vanligtvis regleringsmetoder för att det skall gå att hitta stabila lösningar.

I denna avhandling studerar vi Cauchyproblemet för elliptiska ekvationer i både begränsade och obegränsade områden med hjälp av iterativa regulariseringsmetoder. I Artikel I och Artikel II fokuserar vi på en iterativ regulariseringsteknik som innebär att en sekvens av välställda problem med blandade randvärden, för samma elliptiska ekvation, löses. Den ursprungliga versionen av den alternerande iterativa metoden är baserad på iterationer som alternerar mellan Dirichlet-Nuemann och Neumann-Dirichlet randvärdesproblem. Det är känt att denna iterativa metod eventuellt fungerar för Helmholtz-ekvation. Istället studerar vi en modifierad metod baserad på att vi alternerar mellan Dirichlet–Robin och Robin–Dirichlet randvärdeproblem. Först studerar vi Cauchy-problemet för allmänna elliptiska ekvationer av andra ordningen med variabla koefficienter i ett begränsat område. Sedan utökar vi teorin till fallet med obegränsade områden för Cauchy-problemet. För Cauchyproblemet, i fallet allmänna elliptiska ekvationer, visar vi att den iterativa metoden, baserad på Dirichlet–Robin villkor, är konvergent förutsatt att parametrarna i Robin-villkoret väljs på lämpligt sätt. I fallet med ett obegränsat område härleder vi nödvändiga, och tillräckliga, villkor för konvergens av Robin–Dirichlet-iterationerna baserat på en analys av spektrum av Laplace operatorn, med randvillkor av Dirichlet och Robin typ.

I de numeriska testerna undersöker vi det exakta beteendet hos Dirichlet–Robin iterationer, för olika värden på vågtalet i Helmholtz-ekvationen, och resultaten visar att konvergenshastigheten beror på valet av parameter i Robin-villkoret. I fallet med ett obegränsat område visar de numeriska experimenten att en lämplig trunkering av området och ett lämpligt val av parameter i Robin–villkoret, ledere till konvergens för Robin–Dirichlet-iterationerna.

Vid närvaro av brus måste ytterligare regleringstekniker implementeras för att den alternerande iterativa proceduren ska konvergera. Därför fokuserar vi i Artikel III och Artikle IV på iterativa regulariseringsmetoder för att lösa Cauchy-problemet för Helmholtz-ekvation i ett halv-oändligt band, under antagandet att data innehåller mätbrus. Dessutom rekonstruerar vi ett strålningsvillkor vid oändligheten från givna Cauchy-data. För rekonstruktionen av strålningsvillkoret löser vi ett välställt problem för Helmholtz-ekvationen i det halv-oändliga bandet. Den återstående lösningen erhålls genom att lösa ett illaställt problem. I Artikel III betraktar vi den vanliga Helmholtz-ekvationen och använder variabelseparation för att analysera problemet. Vi visar att strålningsvillkoret kan beskrivas av ett icke-linjärt välställt problem som ger en stabilt oscillerande lösning till Cauchy-problemet. Vidare, visar vi att det illa ställda problemet kan regulariseras med hjälp av Landwebers iterativa metod och diskrepansprincipen. Numeriska tester visar att tillvägagångssättet fungerar bra.

Artikel IV är en utvidgning av teorin från Artikel III till fallet med variabla koefficienter. Teoretisk analys av detta Cauchy-problem visar att, med lämpliga gränser för koefficienterna, kan iterativa regulariseringsmetoder användas för att stabilisera det illa ställda Cauchy-problemet.

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Contents

1 – Introduction

Inverse problems are encountered in various fields of science and engineering. These problems usually involve finding vital information about a system or a process from indirect or inadequate information. For instance, in acoustic and electromagnetic fields, scattering phenomena leads to inverse scattering problems. That is, properties of a medium, or an object, such as density or temperature distribution, size, shape, an inhomogenity in a material, a potential etc. are determined from acoustic and electromagnetic measurements of the medium, or the object. For example an inverse problem of determining the source of acoustical noise inside the cabin of a mid-size aircraft from accoustical noise field measurements inside the cabin [14, 15]. See also [13, 23, 34] which are devoted to the study of acoustic and electromagnetic waves and their various applications. Another example are inverse problems in medical imaging and non–destructive testing. In these cases, non–destructive testing techniques and tomographic techniques such as computerized tomography (CT), magnetic resonance imaging (MRI), etc., use signal measurements taken on the surface of a body, or a material, to detect internal flaws and defects. They also image internal organs, or tissues, without causing damage [11, 32, 36]. Other widely explored application areas are inverse problems in geophysics [37], astrophysics [4], corrosion detection [21], parameter identification [20, 30], just to mention a few.

One of the problems encountered when solving inverse problems is that they generally give rise to mathematical models that are ill–posed. A mathematical problem is said to be well–posed if all of the following characteristics holds [17]:

(i) A solution exists for all admissible data.

(ii) The solution is unique for all admissible data.

(iii) The solution depend continuously on the data.

For ill–posed problems, one or more of the above characteristics does not hold. For most inverse problems, the first two properties are usually fulfilled and in the case where they are not, the problem can somehow be adjusted or reformulated to meet these conditions. The last condition is difficult to repair whenever it is not satisfied. Therefore for problems where the third condition is not met, special treatments are considered as we shall see in this thesis. Furthermore, when solving a concerete problem, the above concepts of existence, uniqueness and continuity of a solution for admissible data must be properly defined. That is, one must clearly specify what kind of solution is sought, which norm continuity is measured and which data are considered admissible.

Cauchy problems for elliptic equations, like the Helmholtz equation, the Laplace equation, etc., are examples of inverse problems that do not satisfy the continuous dependence condition. We illustrate this with the Cauchy problem for Laplace equation in a unit–square:

Example 1.0.1

$$\begin{cases} \Delta u(x,y) = 0, & (x,y) \in (0,1)\times(0,1), \\ u(x,0) = f(x) & x \in (0,1), \\ u_y(x,0) = g(x) & x \in (0,1) \\ u(0,y) = u(1,y) = 0 & y \in (0,1) \end{cases} \tag{1.1}$$

Suppose that $f(x) = 0$ *and* $g(x) = \frac{\sin n\pi x}{n\pi}$, *for a specific* n. *Then by seperation of variables problem (1.1) has a unique solution given by*

$$u(x,y) = \frac{\sin(n\pi x)\sinh(n\pi y)}{(n\pi)^2}.$$

We observe that $\max_{x\in[0,1]} |g(x)| \to 0$ *as* $n \to \infty$ *but for a fixed* $y > 0$, $\max_{x\in[0,1]} |u(x,y)| \to \infty$, $n \to \infty$. *Therefore the solution does not depend continuously on the data and we conclude that it is ill–posed.*

1.1 Regularization of ill–posed problems

Ill–posed problems can be formulated as operator equations and the operator equation solved using various stablizing methods, depending on the nature of the problem, degree of ill–posedness, etc. In this section, we briefly discuss ill–posed operator equations and the theory of regularization in their abstract forms. The material is quite standard and can be found in [16, 24], among others.

1.1.1 Operator equations

Linear inverse problems can naturally be formulated as operator equations of the form

$$Kx = y \tag{1.2}$$

where K is a bounded linear operator mapping a linear space X into a linear space Y, y is the observed measurement and x is the unknown quantity to be determined. In this thesis we will consider the spaces X and Y as Hilbert spaces with inner products $\langle\cdot,\cdot\rangle_X$ and $\langle\cdot,\cdot\rangle_Y$ respectively. We assume that the inverse K^{-1} exists when considered as a mapping of $R(K)$ onto X. However, we do not assume that $R(K)$ is a closed subset of Y, otherwise that would mean that K^{-1} is bounded which is a too restrictive assumption to make.

We also suppose that the bounded linear operator K is compact. Compact operators possess properties useful in the analysis of inverse problems. Besides, most inverse problems can be formulated in terms of integral operators which are usually compact under suitable assumptions [16, 24]. The singular value decomposition for a compact operator K is defined as follows:

Definition 1.1.1 *Let* K *be a compact operator,* u_n *and* v_n *be complete orthonormal system for* X *and* Y *respectively. Then* $\{\sigma_n; u_n, v_n\}$ *is a singular system for*

K if $\sigma_1 \geq \sigma_2 \geq \cdots \geq 0$, $\sigma_n \to 0$ as $n \to \infty$ and

$$Ku_n = \sigma_n v_n, \quad K^* v_n = \sigma_n u_n \tag{1.3}$$

for all $n \in \mathbb{N}$. Here K^* is the adjoint of K.

We note that due to the assumption above that K^{-1} exists, all the singular values σ_n are strictly positive. The singular value decomposition always exists for compact operator [16]. With the expansion (1.3), the operator K can be diagonalized as

$$Kx = \sum_{n=1}^{\infty} \sigma_n \langle x, u_n \rangle_X v_n, \quad x \in X,$$

and such that the inifinite series converges in the norm of Y. The solution x to the operator equation (1.2) is given by

$$x = \sum_{n=1}^{\infty} \frac{\langle y, v_n \rangle_Y}{\sigma_n} u_n. \tag{1.4}$$

If $y \in R(K)$ then

$$\sum_{n=1}^{\infty} \frac{|\langle y, v_n \rangle_Y|^2}{\sigma_n^2} < \infty.$$

This is called the Picard criterion and gives a condition for existence of an exact solution. Basically, it says that the convergence of x given by (1.4) holds, hence a solution exist, if the coefficients $\langle y, v_n \rangle_Y$ decay sufficiently fast compared to the decay of the singular values σ_n as $n \to \infty$.

It follows from (1.4) that the operator equation (1.2) is ill–posed. The errors in y do not always stay bounded since the error components corresponding to small singular values are multipled by $1/\sigma_n$ and grows as n tends to infinity.

The ill–posedness of problem (1.2) becomes more severe the faster the singular values decay. This leads us to a criterion for quantification of the degree of ill-posedness.

Definition 1.1.2 *A problem is said to be mildly ill–posed if $\sigma_n = \mathcal{O}(n^{-\alpha})$ for some $\alpha \in R^+$ and severely ill–posed if $\sigma_n = \mathcal{O}(e^{-\alpha n})$.*

The Cauchy problem for the Laplace equation is an example of a severely ill–posed problem [7]. We also see this in Example 1.0.1. See also [16] for more examples. Next we give an example of a mildly ill–posed problem.

Example 1.1.3 Differentiation as an ill–posed problem
Let $K : L^2(0,1) \to L^2(0,1)$ be an integral operator defined as

$$(Ku)(y) = \int_0^y u(x)dx = \int_0^1 k(x,y)u(x)dx \tag{1.5}$$

with a kernel $k : (0,1) \times (0,1) \to \mathbb{R}$ given by

$$k(x,y) = \begin{cases} 1, & x \leq y \\ 0, & elsewhere. \end{cases}$$

Using the relation $\langle Ku, v\rangle = \langle u, K^*v\rangle$ *we compute* K^* *as*

$$\int_0^1 \int_0^1 k(x,y)u(x)v(y)dxdy = \int_0^1 u(x) \int_0^1 k(x,y)v(y)dydx$$

which implies that

$$(K^*v)(x) = \int_0^1 k(x,y)v(y)dy = \int_x^1 v(y)dy. \tag{1.6}$$

From (1.5) and (1.6) we obtain

$$\lambda u(x) = (K^*Ku)(x) = \int_x^1 \int_0^y u(z)dzdy. \tag{1.7}$$

From (1.7), we get the following boundary value problem for $u(x)$*:*

$$\begin{cases} u'' + \frac{1}{\lambda}u(x) = 0, \\ u'(0) = 0, \\ u(1) = 0. \end{cases} \tag{1.8}$$

Putting $\lambda = \sigma_n^2$ *and solving (1.8), we obtain the singular values and the singular functions of* K^*K *as*

$$\lambda_n = \sigma_n^2 = \frac{4}{(2n-1)^2\pi^2}, \quad u_n(x) = \sqrt{2}\cos\left(\frac{(2n-1)\pi}{2}x\right), \quad n = 1, 2, \ldots,$$

with $u_n(x)$ *as the orthonormal basis of* $L^2(0,1)$*. Applying the relation* $\sigma_n v_n = Ku_n$*, we compute* $v_n(y)$ *to get*

$$v_n(y) = \sqrt{2}\sin\left(\frac{(2n-1)\pi}{2}y\right).$$

Therefore given a function $f(y) \in L^2(0,1)$*, the Picard criterion gives*

$$2\sum_{n=1}^{\infty} \sigma_n^{-2}\left(\int_0^1 f(y)\sin\left(\frac{(2n-1)\pi}{2}y\right)dy\right)^2 < \infty,$$

which holds only if f *is differentiable and* $f' \in L^2(0,1)$*. Since the singular values decay as* $\mathcal{O}(1/n)$*, differentiation is a mildly ill–posed problem.*

1.1.2 Regularization

In most application problems the data $y \in R(K)$ is not known exactly. Instead we have noisy data $y^\delta \in Y$, with a certain noise level $\delta > 0$, such that

$$\|y - y^\delta\|_Y \le \delta. \tag{1.9}$$

Therefore, given y^δ, instead of solving the unpertubed operator equation (1.2), we solve a pertubed operator equation

$$Kx^\delta = y^\delta. \tag{1.10}$$

The challenge with solving (1.10) is that we cannot assume in general that the noisy data y^δ belong to the range of K. Besides, even if it does but the range of K is not closed then $K^{-1}y^\delta$ may not be a good approximation of $K^{-1}y$. The traditional numerical methods are not sufficient for solving ill–posed problems since they are highly sensitive to noise.

Regularization is the approximation of ill–posed problems by a family of neighbouring well–posed problems parameterized by a regularization parameter, say $\alpha > 0$, that controls the accuracy of the approximation [16]. That is, the unbounded inverse operator $K^{-1} : R(K) \to X$ is approximated by a family of bounded operators $R_\alpha : Y \to X$ such that for exact data y, $R_\alpha y$ converges to x as $\alpha \to 0$.

The regularization parameter α should be selected in such a way that if the noise level δ tends to zero then regularized solution x_α^δ tends to the exact solution x. We compute the error between the regularized solution $x_\alpha^\delta = R_\alpha y^\delta$, of the pertubed problem (1.10), and the solution $x = K^{-1}y$, of the unpertubed problem (1.2), to obtain

$$\begin{aligned}\|x_\alpha^\delta - x\|_X &\le \|R_\alpha y^\delta - R_\alpha y\| + \|R_\alpha y - x\| \le \|R_\alpha\|\|y^\delta - y\| + \|R_\alpha y - K^{-1}y\| \\ &\le \underbrace{\delta\|R_\alpha\|}_{(I)} + \underbrace{\|R_\alpha y - K^{-1}y\|}_{(II)}.\end{aligned} \tag{1.11}$$

Here, we see that the total error consists of the data error denoted by (I) and the approximation error denoted by (II). The data error does not always stay bounded for $\alpha \to 0$, since K^{-1} is unbounded and $R_\alpha \to K^{-1}$ as $\alpha \to 0$. On the other hand, the approximation error tends to zero as $\alpha \to 0$ because of the pointwise convergence of R_α to K^{-1} for exact data y. Thus, to ensure that the total error is minimal, an appropriate choice of α, should be dependent on δ, see [16, 24] for more details.

Consequently, we present the following definition for a convergent regularization method, as defined in [16].

Definition 1.1.4 *Let $K : X \to Y$ be a bounded linear operator between the Hilbert spaces X and Y , $\alpha_0 \in (0, \infty]$. For every $\alpha \in (0, \alpha_0)$, let $R_\alpha : Y \to X$ be a continuous operator. The family $\{R_\alpha\}$ is a regularization of K^{-1} if for all $y \in R(K)$, there exists a parameter choice rule $\alpha = (\delta, y^\delta)$ such that*

$$\limsup_{\delta \to 0}\{\|R_{\alpha(\delta,y^\delta)}y^\delta - K^{-1}y\| : y^\delta \in Y, \|y - y^\delta\| \le \delta\} = 0 \tag{1.12}$$

holds. Here

$$\alpha : \mathbb{R}^+ \times Y \to (0, \alpha)$$

is such that

$$\limsup_{\delta \to 0}\{\alpha(\delta, y^\delta) : y^\delta \in Y, \|y - y^\delta\| \le \delta\} = 0. \tag{1.13}$$

For a specific $y \in R(K)$, the pair (α, R_α) is called a convergent regularization method for solving (1.2) if (1.12) and (1.13) hold.

In the above definition of a convergent regularization method, we see that the regularization parameter α depends on the noise level. In what follows, we mention some of the well known parameter choice strategies.

The *discrepancy principle*, originally by Morozov [31] is a strategy based on the knowledge of the noise level. That is, if the estimate for the noise level is known, i.e., $\|y - y^\delta\| \leq \delta$, then the parameter $\alpha(\delta, y^\delta)$ is computed via a comparison between the residual norm $\|Kx_\alpha^\delta - y^\delta\|_Y$ and the bound δ as

$$\alpha(\delta, y^\delta) = \sup\{\alpha > 0 : \|Kx_\alpha^\delta - y^\delta\|_Y \leq \tau\delta\} \tag{1.14}$$

for $\tau > 1$. We obseve that a discrepancy in the order of δ is the best choice otherwise a smaller regularization parameter would imply less stability. We note that this is the strategy explored in our study in this thesis.

L-curve method, proposed by Lawson and Hanson [28] is an error-free strategy whereby the optimal value of α is chosen via a minimization of the residual norm and the solution norm. Another error-free method is the *Generalized cross-validation method* [40]. All the parameter choice rules mentioned above have been tackled in details in [16, 18], among others.

An example of a regularization method is the Tikhonov method. Phillips and Tikhonov [33, 38, 39] laid the foundation and made substantial development towards the theory of incorrectly posed problems and the Tikhonov regularization method. More details about Tikhonov method can be found in [16, 24].

1.2 Iterative regularization methods

In this thesis we focus on iterative regularization methods. These methods have been intensively developed and studied in literature. An iterative method finds an approximate solution by iteratively improving the solution in each iteration. Some of the advantages of iterative regularization methods are: they are easy to implement since most do not require modification of the operator equation as opposed to Tikhonov–type methods, they can be applied to general geometries and they can be used to regularize both constant and variable coefficients partial differential equations. Besides, majority of iterative methods have a self regularizing property in that early termination of the iterations have a regularization effect which implies that the iteration index plays the role of the regularization parameter.

Some of the commonly used iterative methods for solving ill–posed problems include: Landweber iterations and the Conjugate gradient method.

1.2.1 Landweber iteration

Landweber iteration is an iterative regularization method, originally proposed by Landweber[27]. Consider the operator equation (1.2). The Landweber iterations $x_p, p = 1, 2, \ldots,$ are given by

$$x_p = x_{p-1} + \omega K^*(y - Kx_{p-1}), \tag{1.15}$$

where $0 < \omega < 2/\|K\|^{-2}$ is a fixed constant and x_0 is an initial guess for the iterates. We note that the parameter ω, also known as the relaxation parameter, determines the convergence rate and the stability of the iterations. The optimal value usually depends on the problem at hand.

Theorem 1.2.1 *Consider the Landweber iteration (1.15). If $y \in R(K)$ then $x_p \to x$ as $p \to \infty$, otherwise $x_p \to \infty$.*

Definition (1.1.4), of a convergent regularization method, involves the regularization parameter α that tends to zero as δ tends to zero. Here, the iteration index p acts as the regularization parameter. This can be made to make sense by considering $\alpha = 1/p$ as the regularization parameter in the previous definition.

The Landweber method is a convergent regularization method in terms of Definition (1.1.4). That is, if the noise level is known then a stopping index can be chosen via the discrepancy principle as stated in the following definition, which is found in [16].

Definition 1.2.2 *Let x_p^δ be the pth iterate for the Landweber iteration (1.15) for solving the operator equation (1.2) with noisy data y^δ such that $\|y - y^\delta\|_Y \leq \delta$, $\delta > 0$. The stopping index $p_* = p_*(\delta, y^\delta)$, chosen according to the discrepancy principle, is the smallest index p such that*

$$\|y^\delta - Kx^\delta_{p(\delta,y^\delta)}\|_Y \leq \tau\delta \tag{1.16}$$

with $\tau > 1$. Morever, $p_*(\delta, y^\delta) = \mathcal{O}(\delta^{-2})$.

One of the disadvantages of using the Landweber method is the slow convergence, i.e. too many iterations are required before the stopping criterion (1.16) is reached. In that case, other fast converging iteration methods like the Conjugate gradient method (CG) exists. However, since the iteration number p acts as the regularization parameter and if the convergence is rapid then p should be carefully picked.

1.2.2 Conjugate gradient method

The conjugate gradient method is another iterative regularization method for solving linear equations [19]. Consider the operator equation (1.2). The Conjugate method is applied to the normal equation $K^*Kx = K^*y$. The method consists of finding iterates x_p, p=0,1, ..., that minimizes the residual

$$\|y - Kx_p\|_Y. \tag{1.17}$$

The standard conjugate gradient algorithm is as follows: select an initial guess x_0 and compute the residual and the search direction

$$r_0 = y - Kx_0 \quad \text{and} \quad z_1 = s_0 = K^*r_0$$

respectively. Then for $p = 1, 2, \ldots$, unless $s_{p-1} = 0$, perform the following step:

(i) Compute the step size $\alpha_p = \|s_{p-1}\|^2/\|Kz_p\|^2$.

(ii) Find the solution $x_p = x_{p-1} + \alpha_p z_p$.

(iii) Compute the residual $r_p = r_{p-1} - \alpha_p K z_p$, the conjugate direction $\beta_p = \|K^* r_p\|^2 / \|s_{p-1}\|^2$ and the search direction $z_{p+1} = K^* r_p + \beta_p z_p$.

The conjugate gradient iterations converge rapidly. In the case of noisy data the iterations are terminated according to the discrepancy principle.

We note that the conjugate gradient method converges for self-adjoint and positive definite operators. This method has been implemented to solve Cauchy problems for elliptic equations, see [6, 8, 35], among others.

1.3 Alternating iterative method

The alternating iterative method, proposed by Kozlov and Mazya [25], is an iterative method for solving ill–posed partial differential equations. Contrary to other regularization method used to solve ill–posed partial differential equations based on modification of the operator, e.g., the quasi–reversibility method [29], the alternating iterative method preserves the differential operator and therefore it is easy to implement. The algorithm involves alternatively solving a sequence of well–posed boundary value problem for the same equation. Regularizing property is achieved by a suitable change of boundary conditions.

Let us consider the following Cauchy problem for the Laplace equation on a bounded domain Ω with smooth boundary Γ divided into two disjoint boundaries Γ_0 and Γ_1, with smooth common boundary. We assume u is the exact solution to the following Cauchy problem:

$$\Delta u = 0 \quad \text{in} \quad \Omega, \quad u = f \quad \text{on} \quad \Gamma_0, \quad \partial_\nu u = g \quad \text{on} \quad \Gamma_0 \tag{1.18}$$

where f and g are the prescribed Cauchy data and ν is the outward unit normal to Γ.

To solve the Cauchy problem (1.18) using the alternating iterative method, we consider the following two mixed boundary value problems:

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ u = f & \text{on} \quad \Gamma_0, \\ \partial_\nu u = \psi & \text{on} \quad \Gamma_1 \end{cases} \tag{1.19}$$

and

$$\begin{cases} \Delta u = 0 & \text{in} \quad \Omega, \\ \partial_\nu u = g & \text{on} \quad \Gamma_0, \\ u = \eta, & \text{on} \quad \Gamma_1 \end{cases} \tag{1.20}$$

where (f, g) are the prescribed Cauchy data in (1.18) while η and ψ are functions which must be updated at each iteration. The problems (1.19) and (1.20) are well–posed. Then the alternating iterative algorithm for solving (1.18) is as follows:

(i) Pick an initial approximation for ψ on Γ_1 and solve problem (1.19) to obtain the first approximaximation u_0.

(ii) For odd numbered steps, u_{2n+1} is computed by solving problem (1.20) where η on Γ_1 is replaced by u_{2n}, previously constructed.

(iii) For even numbered steps, u_{2n+2} is computed by solving problem (1.19) with ψ replaced by u_{2n+1}.

This is the original alternative iterative algorithm proposed by Kozlov and Mazya: the Dirichlet–Neumann alternating iterative method. This algorithm converges for elliptic operators whose quadratic form is positive definite on $H^1(\Omega)$. Typical example is the Cauchy problem for the Laplace equation above, see [26]. Some extension of this algorithm to different elliptic operators can be found in [5, 12].

For problems where the quadratic form corresponding to the elliptic operator is not positive–definite, for instance Cauchy problems for Helmholtz–type equations, the algorithm does not necessarily converge, see [9]. In [9] a modified alternating algorithm for solving the Cauchy problem for the Helmholtz equation, dependent on an artificial interior boundary, is presented and its convergence demonstrated. In [1, 2, 10], modifications based on repalacing the Dirichlet–Neumann iterations on Γ_1 by Dirichlet–Robin iterations are presented. It is demonstrated in these papers that these algorithms converge under certain explicit conditions on the Robin parameter and the wave number for the Helmholtz equation.

Strategies for improving the convergence rate of the alternating iterative algorithms have also been developed and investigated, see [6, 8], etc. Besides, in the presense of noisy data, a stopping rule must be added to achieve convergence of the alternating iterative method. Finally, we note that this thesis is geared towards the study of alternating algorithms in unbounded domains.

2 – Summary of Papers

In this section we give a summary of the four appended papers.

2.1 Paper I: Analysis of Dirichlet–Robin iterations for solving the Cauchy Problem for Elliptic Equations

In this paper we prove convergence of the Dirichlet–Robin algorithm for Cauchy problem for general elliptic equations of second order with variable coefficients. In an earlier paper[10], it was demonstrated that the Robin–Dirichlet algorithm for Cauchy problem for the Helmholtz equation is convergent, even for large wave numbers.

We consider a general elliptic equation in divergence form in a bounded domain $\Omega \in \mathbb{R}^d$ with a Lipschitz boundary Γ divided into two disjoint parts Γ_0 and Γ_1, with a common boundary in Γ. We assume that the second order general elliptic operator considered is uniformly elliptic. Cauchy data are prescribed on the boundary Γ_0 and the goal is therefore to stably reconstruct the solution on the boundary Γ_1 where information is not provided.

We make two equivalent assumptions that quarantees convergence of the Dirichlet–Robin iterations. First, we assume that the elliptic operator with Dirichlet boundary condition is positive and second, that the elliptic operator with Robin boundary condition is also positive for appropriately chosen parameters in the Robin conditions. We prove equivalence of these two assumptions. With the assumptions in place, the Dirichlet–Robin algorithm is used to solve the Cauchy problem for the general elliptic equation considered. Further, we prove well–posedness result, in the Sobolev space $H^1(\Omega)$, of the mixed boundary value problems involved in the Dirichlet–Robin iterations. Convergence result of the Dirichlet–Robin iterations is also presented and proved.

In the numerical experiments, the Cauchy problem for the Helmholtz equation is considered. Precise behaviour of the Dirichlet–Robin iterations for different values of the wave number, k^2, is investigated. Also investigated is the convergence speed in relation to different values of Robin parameters.

2.2 Paper II: Robin–Dirichlet alternating iterative procedure for solving the Cauchy Problem for Helmholtz equation in an unbounded domain

In this paper we derive necessary and sufficient conditions for the convergence of the Robin–Dirichlet iterations for Cauchy problem for Helmholtz equation in unbounded domains. Sufficient conditions for convergence in bounded domains have been provided in [10] and [1] for the Robin–Dirichlet iterations for Cauchy problem for Helmholtz equation and for general elliptic equations of second order with variable coefficients respectively.

We consider the Helmholtz equation in a unbounded domain $\Omega \in \mathbb{R}^d$ $d \geq 2$ with a smooth boundary and with N cylindrical outlets to infinity with bounded inclusions in $\mathbb{R}^{d-1}$ i.e. for sufficiently large $|x|$ the domain Ω coincides with the union of N disjoint cylinders $\mathcal{C}^{(j)}$, $j = 1, \ldots, N$, which can be described in a certain cartesian coordinates $x^{(j)} = (y^{(j)}, z^{(j)})$, as

$$\mathcal{C}^{(j)} = \{x^{(j)} : y^{(j)} \in \omega^{(j)}, \, z^{(j)} \in \mathbb{R}\},$$

where the cross-sections $\omega^{(j)}$ are bounded domains in $\mathbb{R}^{d-1}$ with smooth boundaries. The boundary of Ω is denoted by Γ. We assume that a certain bounded[1] open set Γ_0 is chosen on the boundary Γ and the boundary of this set is also smooth. Let also Γ_1 be the interior of $\Gamma \setminus \Gamma_0$. Cauchy data is prescribed on Γ_0 and the goal is to reconstruct solution on the unbounded part of the boundary Γ_1.

As in Paper 1, we make two equivalent assumptions that quarantees convergence of the Robin–Dirichlet iterations. The first and main assumption concerns the real number k in the Helmholtz operator. We assume that $k^2 < \lambda_0^{(j)}$, where $\lambda_0^{(j)}$ is the first eigenvalue of the Dirichlet–Laplacian in the bounded cross-section $\omega^{(j)}$ or alternatively that $k^2 < \Lambda_0$, where Λ_0 is the smallest eigenvalue of the Dirichlet problem in the unbounded domain Ω. This restriction on k implies that the Helmholtz operator with Dirichlet boundary condition is positive–definite. The second assumption is that for appropriately chosen Robin parameters, the Helmholtz operator with Robin boundary condition is positive–definite. We proved that these two assumptions are equivalent. With these assumptions in place, the Robin–Dirichlet iterations are applied to solve the Cauchy problem for Helmholtz equation in Ω. We further explored the precise relationship between the first eigenvalue of the Robin–Laplacian and first eigenvalue of Dirichlet–Laplacian.

In the numerical experiments, we demonstrated that through appropriate truncation of the domain and appropriate choice of the Robin parameters, the Robin–Dirichlet iterations converge.

[1]This is a set where measurements are taken and it is reasonable to assume it bounded

2.3 Paper III: Reconstruction of the Radiation Condition and Solution for the Helmholtz Equation in a Semi–infinite Strip from Cauchy Data on an Interior Segment

In this paper we reconstruct a solution to the Helmholtz equation in a semi–infinite strip from measurements prescribed on a segment inside the semi–infinite strip. Moreover, we also reconsruct a unknown radiation condition at infinity. In an earlier paper [2], we demonstrated that a solution to the Helmholtz equation in a unbounded domain can be stably reconstructed using the Robin–Dirichlet iterations under suitable assumptions on the wavenumber k and the Robin parameters.

We consider the Helmholtz equation in a semi–infinite strip $\Omega = \{(x, y) : x \in (0, L_x), y \in (0, \infty)\}$, $L_x > 0$, with a boundary Γ consisting of two distinct parts, Γ_0 and Γ_2. The subsets Γ_0 and Γ_2 are defined as

$$\Gamma_0 = \{(x, y) : x \in (0, L_x),\ y = 0\} \quad \text{and} \quad \Gamma_2 = \{(x, y) : x = 0, L_x,\ y \in [0, \infty)\}.$$

In the interior of Ω, at $y = L_y > 0$, a segment Γ_1 is defined as

$$\Gamma_1 = \{(x, y) : x \in (0, L_x),\ y = L_y\}.$$

Homogeneous Neumann boundary conditions are prescribed on Γ_2 and Cauchy data on Γ_1. The aim is to reconstruct an unknown Dirichlet condition on Γ_0 and an unknown radiation condition at infinity.

The Cauchy problem is split into a well–posed problem, for reconstructing the radiation condition, and an ill–posed problem, for reconstruting the Dirichlet condition. Both problems are investigated using the method of separation of variables. We reconstruct the radiation condition by solving a non–linear problem which corresponds to the oscillating part of the solution to the Helmhotz equation. Further, we demonstrate that the parameter in the radiation condition depend continuously on the Cauchy data.

The problem of reconstructing the Dirichlet data is severely ill–posed in the sense that the solution does not depend continuously on the Cauchy data. The Landweber method together with the discrepancy principle is proposed to regularize it. Numerical experiments shows that the approach works well. In conclusion, we note that the numerical tests for the well–posed problem are not presented in this paper because our main focus was on the ill–posed problem. We however recognize that in applications the well–posed part is often the most important.

2.4 Paper IV: Reconstruction of the Radiation Condition and Solution for a variable coefficient Helmholtz Equation in a Semi–infinite Strip from Cauchy Data on an Interior Segment

In this paper we extend the results of [3] to the variable coefficient case. That is, we reconstruct a radiation condition at infinity and a solution to a variable coefficient Helmholtz equation in a semi–infinite strip from measurements prescribed on a segment inside the semi–infinite strip.

We consider a similar domain as in [3], i.e., let $\Omega = \{(x, y) : x \in (0, 1), y \in (0, \infty)\}$, with two distinct boundaries Γ_0 and Γ_∞ defined as

$$\Gamma_0 = \{(x, y) : x \in (0, 1),\ y = 0\} \quad \text{and} \quad \Gamma_\infty = \{(x, y) : x = 0, 1,\ y \in [0, \infty)\}.$$

In the interior of Ω, at $y = L > 0$, a segment Γ_L is defined as

$$\Gamma_L = \{(x, y) : x \in (0, 1),\ y = L\}.$$

We consider the Helmholtz equation, with wave number that depends on the space variables, i.e.

$$\Delta u + (k^2 + \gamma)u = 0 \quad \text{in} \quad \Omega$$

where k^2 is a positive constant and γ is a small pertubation, assumed to be bounded with compact support. Homogeneous Neumann boundary conditions are prescribed on Γ_∞ and Cauchy data on Γ_L. The aim is to reconstruct the unknown radiation condition at infinity and the unknown Dirichlet condition on Γ_0 using the given Cauchy data.

The problem is split into two sub–problems. The first consists of reconstructing the radiation condition and it is well–posed. We derive the equation for finding the parameter of the radiation condition, that holds at infinity, by solving two well–posed boundary value problems that describe the same solution: a Dirichlet problem and a Neumann problem for the variable coefficient Helmholtz equation in a unbounded sub–domain of Ω .

The second problem is the Cauchy problem of determining the unknown Dirichlet condition on Γ_0 and it is ill–posed. We reformulate the Cauchy problem into an operator equation, with compact support, defined by a solution of a well–posed boundary value problem. The challenge met is identifying the right function spaces, the natural inner products for the spaces as well as finding the adjoint operator. After that the operator equation can be solved using various suitable iterative regularization methods that exists.

Finally, we note that numerical experiments for this paper are missing since we unfortunately did not have enough time to do that. We will consider it as part of the future work.

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Papers

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