Master’s Thesis

Numerical experiments with FEMLAB® to support mathematical research.

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LiTH - MAT - EX - - 05 / 14 - - SE
Numerical experiments with FEMLAB® to support mathematical research.

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Using the finite element software FEMLAB® solutions are computed to Dirichlet problems for the Infinity-Laplace equation $\Delta_\infty(u) \equiv u_{xx}^2 + \Delta_\infty(u) = 0$. For numerical reasons $\Delta_q(u) = \text{div}(|\nabla u|^q \nabla u) = 0$, which (formally) approaches $\Delta_\infty(u) = 0$ as $q \to \infty$, is used in computation. A parametric nonlinear solver is used, which employs a variant of the damped Newton-Gauss method. The analysis of the experiments is done using the known theory of solutions to Dirichlet problems for $\Delta_\infty(u) = 0$, which includes AMLEs (Absolutely Minimizing Lipschitz Extensions), sets of uniqueness, critical segments and lines of singularity. From the experiments one main conjecture is formulated: For Dirichlet problems, which have a non-constant boundary function, it is possible to predict the structure of the lines of singularity in solutions in the Infinity-Laplace case by examining the corresponding Laplace case.

**Keyword**

FEMLAB, numerical experiments, Dirichlet problems for the Infinity-Laplace equation, AMLE, Absolutely Minimizing Lipschitz Extension, sets of uniqueness, critical segments, lines of singularity
Abstract

Using the finite element software FEMLAB® solutions are computed to Dirichlet problems for the Infinity-Laplace equation $\Delta_\infty(u) \equiv u^2_{xx} + 2u_x u_y u_{xy} + u^2_{yy} = 0$. For numerical reasons $\Delta_\infty(u) = \text{div}(|\nabla u|^q \nabla u) = 0$, which (formally) approaches $\Delta_\infty(u) = 0$ as $q \to \infty$, is used in computation. A parametric nonlinear solver is used, which employs a variant of the damped Newton-Gauss method. The analysis of the experiments is done using the known theory of solutions to Dirichlet problems for $\Delta_\infty(u) = 0$, which includes AMLEs (Absolutely Minimizing Lipschitz Extensions), sets of uniqueness, critical segments and lines of singularity. From the experiments one main conjecture is formulated: For Dirichlet problems, which have a non-constant boundary function, it is possible to predict the structure of the lines of singularity in solutions in the Infinity-Laplace case by examining the corresponding Laplace case.

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Nomenclature

Symbols

$L_u(V)$ denotes the least constant $L \in [0, \infty]$ for which $|u(x) - u(y)| \leq L|x - y|$, $x, y \in V$; $V \subset \mathbb{R}^n$, $u : V \to \mathbb{R}$.

$q(m) = n$ The parametric solver has made $m$ iterations to reach the value $n$ of the parameter $q$.

$PQ$ denotes the Euclidean distance $\left(\sum_{i=1}^{n} (x_i(P) - x_i(Q))^2\right)^{1/2}$ between the points $P, Q \in \mathbb{R}^n$.

Abbreviations

q-Laplace denotes the equation $\Delta_q(u) = 0$.

AMLE Absolutely Minimizing Lipschitz Extension.
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Chapter 1

Introduction

The aim of this thesis is to attempt to investigate the properties of solutions $u$ to Dirichlet problems for the Infinity-Laplace equation \[ \Delta_\infty(u) \equiv u^2_{xx} + 2u_x u_y u_{xy} + u^2_{yy} = 0 \] on convex bounded domains. This investigation is done by numerical experiments for variety of Dirichlet problems, using the finite element software FEMLAB®\(^{1}\), where an approximation to $\Delta_\infty(u) = 0$ is used in computation. The results of the computation are then analyzed using the known theory concerning solutions to Dirichlet problems for $\Delta_\infty(u) = 0$. The hope is to find indications of the behaviour of the solution $u$ on the interior of the domain.

Chapter 2 details the theory needed for our analysis.
Chapter 3 details the nonlinear solver used by FEMLAB®.
Chapter 4 details the experiments performed and results.
Chapter 5 contains a discussion of results.
Chapter 6 details numerical and mathematical conclusions drawn from the experiments and some suggestions of future work.
The Appendix contains an (non-exhaustive) investigation of the convergence of solutions obtained from FEMLAB®. The settings used in FEMLAB® are also described.

\(^{1}\)FEMLAB is a registered trademark of COMSOL AB.
Chapter 2

Theoretical background

At the center of this thesis is the relation between the Lipschitz extension of a function \( \phi \), defined on the boundary \( \partial D \) of a convex bounded domain \( D \) in \( \mathbb{R}^2 \), into the domain and solutions to a Dirichlet problem, with boundary function \( \phi \), for the Infinity Laplace equation.

### 2.1 Dirichlet problems for the Infinity-Laplace equation

We study solutions \( u \) to Dirichlet problems for the Infinity-Laplace equation \( \Delta_\infty(u) = 0 \). These PDEs are formulated as follows: Find \( u \) such that

\[
\begin{cases}
\Delta_\infty(u) &\equiv u_x^2 u_{xx} + 2u_x u_y u_{xy} + u_y^2 u_{yy} = 0 &\text{in } D \\
u & = \phi &\text{on } \partial D
\end{cases}
\]  

(2.1)

Throughout this thesis we assume that the domain \( D \) is convex and bounded, and that the boundary function \( \phi \) satisfies (at least) a Lipschitz condition. The exact properties of the solution space for \( u \) are unknown. We know by Savin [12] that \( u \in C^1(D) \). Our experiments will concern \( u \in C^2(D) \cap C(\bar{D}) \) (classical solutions), where the solution \( u \) is not entirely in this space i.e. on a subdomain \( V \subset D \) it holds that \( u \not\in C^2(V) \).

### 2.2 AMLE

A function \( f \) is absolutely minimal in an arbitrary domain \( \Omega \) in \( \mathbb{R}^n \), if \( L_f(E) = L_f(\partial E) < \infty \) for every bounded domain \( E \) such that \( E \subseteq \Omega \).

Suppose that \( g \in \partial E \) satisfies a Lipschitz condition with optimal constant \( L_g(\partial E) = \alpha < \infty \). It is known that \( g \) can be extended into \( E \) by \( h \), without violating the Lipschitz condition, in such a way that for every subdomain \( F \subseteq E \) we have that \( L_h(F) = L_h(\partial F) < \infty \). We then say that the function \( h \) is an AMLE of \( g \) into \( E \).

Solutions \( u \) to (2.1) such that \( u \in C^2(D) \) are of special interest to us, as seen by the following:

**Theorem 2.1.** Let \( \Omega \) be an arbitrary region in \( \mathbb{R}^n \). If \( u \in C^2(\Omega) \), then \( u \) is absolutely minimal in \( \Omega \) iff \( \Delta_\infty(u) = 0 \) in \( \Omega \).

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Chapter 2. Theoretical background

Proof:
p. 559 (Aronsson [1]).

Theorem 2.2. Assume that \( f \in \partial E \) satisfies a Lipschitz condition. For a given extension problem of \( f \) into the bounded domain \( E \), there is a solution which is absolutely minimizing in \( E \).

Proof:
p. 560 (Aronsson [1]).

From Theorem 2.1 and Theorem 2.2 we have (specially) that: For functions \( u \in C^2(D) \) it holds that \( u \) is an AMLE of \( \phi \) into \( D \) iff \( u \) is a solution to (2.1).

It is possible to make a more general statement: Jensen [9] has proven that (2.1) has a unique viscosity\(^1\) solution \( u \in C(\bar{D}) \) satisfying \( u = \phi \) on \( \partial D \). In fact it is sufficient to require that \( \phi \in C(\partial D) \). Furthermore Jensen has shown that the unique viscosity solution \( u \) is an AMLE of \( \phi \) into \( D \). Since we in our experiments only deal with continuous boundary conditions it follows there will always be an unique AMLE solution to the PDE in question.

2.3 Formulation of the PDE using an \( L_\infty \)-norm

It is also instructive to formulate (2.1) in terms of an \( L_\infty \)-norm. Before we do this we review a few concepts and results needed for the formulation: It is well known (Rademacher’s theorem (Heinonen [10])) that any function \( u \), which satisfies a Lipschitz condition in a domain \( G \in \mathbb{R}^n \), is differentiable a.e. in \( G \). Thus \( |\nabla u| \) is bounded and by continuity \( |\nabla u| \) is measurable, and so \( |\nabla u| \in L_\infty(G) \).

Furthermore
\[
\| \nabla u \|_{L_\infty(G)} = \sup_{x \in G} \| \nabla u(x) \|.
\]
Recall that if \( m(G)^2 < \infty \) and \( f \in L_\infty(G) \), then
\[
\| f \|_{L_\infty(G)} = \lim_{p \to \infty} \left( \int_G |f|^p \, dx \right)^{1/p} \quad \text{(Kufner, Oldrich, Fučík [11]).}
\]

Thus, if \( m(G) < \infty \), we have
\[
\| \nabla u \|_{L_\infty(G)} = \lim_{p \to \infty} \left( \int_G |\nabla u|^p \, dx \right)^{1/p}.
\]

For all functions \( u \) satisfying a Lipschitz condition
\[
\| \nabla u \|_{L_\infty(D)} = \sup_{x, y \in D, x \neq y} \frac{|u(x) - u(y)|}{|x - y|}
\]
which is the optimal Lipschitz condition \( u \) on \( D \). This is proven by a standard (but non-trivial) argument using the theory of weak derivatives.

---
\(^1\)Viscosity solutions are defined from a comparison principle. Viscosity solutions are treated further in section 2.7. See (Aronsson, Crandall, Juutinen [4]) for a complete treatment of viscosity solutions.
\(^2\)\( m(\ldots) \) is the Lebesgue measure.
2.4. Equations approximating $\Delta_{\infty}(u) = 0$

The formal Euler equation for minimizing $\| \nabla u \|_{L_{\infty}(D)}$ is $\Delta_{\infty}(u) = 0$. We now wish to formulate (2.1) in terms of the $L_{\infty}$-norm. The $L_{\infty}$-norm is non-additive, which can be expressed through the following relation: If $\omega \subseteq E$ then $\| \nabla v \|_{L_{\infty}(E)} = \max(\| \nabla v \|_{L_{\infty}(\omega)}, \| \nabla v \|_{L_{\infty}(E-\omega)})$. Because of the non-additivity of the $L_{\infty}$-norm the formulation of (2.1) using the $L_{\infty}$-norm becomes: Find $u|_{\partial D} = \phi$ such that for every subdomain $V \subseteq D$, $\| \nabla u \|_{L_{\infty}(V)} \leq \| \nabla v \|_{L_{\infty}(V)}$ for all extensions $v$ of $u$ into $V$ i.e. $u$ is an AMLE of $\phi$ into $D$.

Compare this with the case of

\[
\int_{D} |\nabla u|^{2} \text{dx}
\]

(2.2)

and

\[
\int_{D} |\nabla u|^{p} \text{dx}
\]

(2.3)

If we want to solve the problem of minimizing (2.2), given that $u = \phi$ on $\partial D$, then it is sufficient to establish that $\int_{D} |\nabla u|^{2} \text{dx} \leq \int_{D} |\nabla v|^{2} \text{dx}$ for all extensions $v$ of $\phi$ into $D$. Since (2.2) is additive, it then follows that for every subdomain $V \subseteq D$ we have $\int_{V} |\nabla u|^{2} \text{dx} \leq \int_{V} |\nabla v|^{2} \text{dx}$, for all extensions $v$ of $u$ into $V$. (2.3) is also additive, so an analogous argument can be made for (2.3).

Note that the formal Euler equation for minimizing (2.2) is $\Delta(u) = 0$ and for (2.3) is $\text{div}(|\nabla(u)|^{p-2}\nabla u) = 0$.

2.4 Equations approximating $\Delta_{\infty}(u) = 0$

For numerical purposes a useful observation is that $\Delta_{\infty}(u) = 0$ is the formal limit of the Euler equation for minimizing a functional of $u$, where $u$ is subject to given boundary conditions on $\partial D$ \footnote{We make this observation due to the fact that $\Delta_{\infty}(u) = 0$ cannot be formulated in weak form (Aronsson, personal communication), and so any analysis with finite elements will be precluded.}. There are two functionals for which the above holds: $\int_{D} |\nabla u|^{p}$ (Aronsson, Crandall, Juutinen [4]) and $\int_{D} e^{K|\nabla u|^{2}}$ (Evans, Yifeong [8]). The formal Euler equations for minimizing these are, respectively,

$$\Delta_{p}(u) = \text{div}(|\nabla u|^{p-2}\nabla u) = 0$$

and

$$\Delta_{K}(u) = \text{div}(e^{K|\nabla u|^{2}}\nabla u) = 0.$$  

By setting $q = p - 2$ we get $\Delta_{q}(u) = \text{div}(|\nabla u|^{q}\nabla u)$. Simple calculation yields

$$\Delta_{q}(u) = 0 \quad \text{if} \quad q \geq 0 \quad \Rightarrow \quad \frac{|\nabla u|^{2}\Delta u}{q} + \Delta_{\infty}(u) = 0. \quad (2.4)$$

Furthermore

$$\Delta_{K}(u) = 0 \quad K \geq 0 \quad \Rightarrow \quad \frac{\Delta u}{2K} + \Delta_{\infty}(u) = 0. \quad (2.5)$$

Then it is clear that $\Delta_{q}(u) = 0$ and $\Delta_{K}(u) = 0$ approach $\Delta_{\infty}(u) = 0$ as $q \to \infty$ and $K \to \infty$ respectively. Exactly how large $q$ or $K$ have to be for $\Delta_{K}(u) = 0$ or $\Delta_{q}(u) = 0$ to approximate $\Delta_{\infty}(u) = 0$ well, will in the case of $q$ will depend on

\[\frac{|\nabla u|^{2}\Delta u}{q} + \Delta_{\infty}(u) = 0.\]
the behavior of $|\nabla u|$ and $\Delta u$ over the domain of $u$ (see (2.4)), while in the case of $K$
 depend on the behavior of $\Delta u$ over the domain of $u$ (see (2.5)). It is difficult to foresee
the exact behavior of these quantities, but certainly a Dirichlet condition which satisfies
a Lipschitz condition for $\alpha$, where $\alpha$ is large, will in most cases require larger $q$.

In our computations the $q$-formulation will be used and this is the one we most
often will refer to (see Appendix).

2.5 Sets of uniqueness and critical segments

An important part of the analysis of our experiments is the concept of sets of uniqueness,
with respect to functions satisfying a Lipschitz condition $\alpha$ on $D^4$. Let $g$ be a
function on $\partial D$ which satisfies the Lipschitz condition with optimal constant $\alpha$ and
let $f$ be an extension of $g$ into $D$, such that $f$ satisfies the same Lipschitz condition.
Furthermore let $P$ and $Q$ be points such that $P \in D$ and $Q \in \partial D$.

Then it follows that:

$$g(Q) - \alpha PQ \leq f(P) \leq g(Q) + \alpha PQ, \quad \forall P, Q.$$ 

Trivially we then have:

$$\sup_{Q \in \partial D} (g(Q) - \alpha PQ) \leq f(P) \leq \inf_{Q \in \partial D} (g(Q) + \alpha PQ), \quad \forall P.$$ 

These two bounds define two functions:

$$l(P) = \sup_{Q \in \partial D} (g(Q) - \alpha PQ) \quad \text{(lower bound)}$$

and

$$u(P) = \inf_{Q \in \partial D} (g(Q) + \alpha PQ) \quad \text{(upper bound)}.$$ 

Both of these functions are valid extensions of $g$, i.e. they satisfy the optimal Lipschitz
condition $\alpha$. Consequently all extensions of $g$ agree at a point $P \notin \partial D$ iff
$l(P) = u(P)$. The set of uniqueness is now defined by $H = \{P|P \notin \partial D, l(P) = u(P)\}$. From the above it can be proved that the set $D$ is traversed by straight lines
(wholly, partly or not at all (i.e. when $H$ is empty)), starting and ending on the boundary,
on which all extensions (of $g$) agree. These lines are called critical segments. The critical
segments satisfy $|g(Q_1) - g(Q_2)| = \alpha Q_1Q_2$, where $Q_1, Q_2 \in \partial D$. Furthermore,
if $f$ is an extension of $g$ into $D$ (such that $f$ satisfies the same Lipschitz condition as $g$)
and $P$ a point on a critical segment $\nu$ then $f$ is a linear function on $\nu$, $f$ is differentiable
at $P$ and $\text{grad} f(P) = \alpha \vec{e}$, where $\vec{e}$ is unit vector along $\nu$ in the direction of increasing
$f$ (Aronsson [1]).

\footnote{$D$ need not be assumed to be convex in this section.}
2.6 Known solutions to $\Delta_\infty (u) = 0$

The explicitly known solutions to $\Delta_\infty (u) = 0$ are:

1. $u(x, y) = |x|^{4/3} - |y|^{4/3}$ (Aronsson [1]).
2. $u(r, \theta) = \sqrt{r} e^{\theta/2}$, where $(r, \theta)$ are polar coordinates (Aronsson [1]).
3. $u(x, y) = M \arctan \left( \frac{y-y_0}{x-x_0} \right) + N$, where $(x_0, y_0) \not\in D$ (Aronsson [2]).
4. Every function $u(x, y) \in C^2(\Omega)$, where $\Omega$ is an arbitrary domain, such that $|\nabla u| = \text{constant}$ e.g. $u(x, y) = Ax + By + C$ and $u(x, y) = \sqrt{(x-x_0)^2 + (y-y_0)^2}$, where $(x_0, y_0) \not\in D$ (Aronsson [2]).

2.7 Lines of singularity

By analyzing (1) we can illustrate some further points regarding solutions to the Dirichlet problem for $\Delta_\infty (u) = 0$. In this case we have $u_{xx} = \frac{4}{9} |x|^{-2/3}$, $u_{yy} = -\frac{4}{9} |y|^{-2/3}$ and $u_{xy} = 0$. It is clear that $u_{xx}$ and $u_{yy}$ are unbounded on the coordinate axes. The coordinate axes are in this instance so called lines of singularity (since the solution has (some) noncontinuous second partial derivatives on the axes). The streamlines in the solution join the axes (see figure (a) below) instead of running in parallel to the axes, which would be the case if the solution had continuous second partial derivatives on the axes. To see why this is the case see figure (c) on the next page. Here we can view the streamlines as solutions to a nonlinear PDE $y' = F(x, y)$. If we assume that $u \in C^2$ then, by the basic theory of ODE, all streamlines must be unique. But since they are not we conclude that $u \not\in C^2$ on the line of singularity (the axis in this case). This characteristic streamline structure is useful when trying to identify lines of singularity in the streamline plots of solutions to arbitrary Dirichlet problems for $\Delta_\infty (u) = 0$.

On the open quadrants $O_i$ $(i=1,2,3,4)$, we have $u \in C^\infty (O_i)$ but $u$ is not a classical solution ($u \not\in C^2(\mathbb{R}^2)$), because of the lines of singularity. But $u \in C^{1,1/3}(\mathbb{R}^2)$ i.e. the first partial derivatives of $u$ satisfy a Hölder condition for $\mu = 1/3$. Clearly the class of classical solutions is too small to always contain the sought solutions. Instead

5see Aronsson [5] for an implicit solution.
6This is not a universally established concept.
we use the larger class of viscosity solutions. A viscosity solution can have (at least) a line of singularity and so \( u(x, y) = x^{4/3} - y^{4/3} \in C^{1,1/3}(\mathbb{R}^2) \) is such a solution. **Theorem 2.1** does not apply here, but Aronsson \([3]\) has proved that this solution is still absolutely minimal in \( \mathbb{R}^2 \).

As was alluded to in the beginning of this chapter Savin \([12]\) has shown that for a solution \( v \), to an arbitrary Dirichlet problem for \( \Delta_{\infty}(v) = 0 \) in an arbitrary domain \( \Omega \) in the plane, it holds that \( v \in C^1(\Omega) \). Consequently it is the case that on the lines of singularity, of a function \( v \) in the plane, it must hold that \( v \in C^1 \) but \( v \notin C^2 \).

\( C^2 \)-solutions \( u \) to \( \Delta_{\infty}(u) = 0 \) have the following streamline properties:

(a) streamlines are either convex curves or straight lines (given that \( \text{grad}(u) \not\equiv 0 \) in the domain \( D \)).

(b) \( \Delta_{\infty}(u) = 0 \) can be written as

\[
\Delta_{\infty}(u) = \frac{1}{2} \text{grad}\{\text{grad}(u)^2\} \cdot \text{grad}(u) = 0.
\]

Then it is clear that \( |\text{grad}(u)| \) is constant on the streamlines of the solution \( u \) (Aronsson [2]). We can use this to identify lines of singularity. If we suspect that a streamline might be (or at least be part of) a line of singularity we plot \( u \) on the line in question. If \( u \) is not a linear function of arc length it is clear that \( |\text{grad}(u)| \not\equiv \) constant on the line, which implies that the \( u \) is not in \( C^2 \) on the line and so we can draw the conclusion that the examined line is a line of singularity. Using this method we can only find lines of singularity that are streamlines (either entirely or with the exception of a finite number of points, as in the case of \( u(x, y) = |x|^{4/3} - |y|^{4/3} \), where the singular lines pass through the origin, but no streamlines). To verify that a plotted line is indeed a streamline we use a symmetry argument. In section 2.2 we saw that Jensen \([9]\) has proved that there is a unique viscosity solution \( u \in D \) to a given Dirichlet problem for \( \Delta_{\infty}(u) = 0 \) such that the boundary function \( \phi \in C(\partial D) \). If the domain and the boundary condition are symmetric we have, by the uniqueness of \( u \) and the invariance of the gradient under reflection, that there is a straight streamline along the line of symmetry.

We differentiate between **verifiable** and **non-verifiable** lines of singularity. We refer to lines of singularity that *are* streamlines, established through the above symmetry argument, as verifiable lines of singularity and those that *are not* as non-verifiable lines of singularity.
Chapter 3

Numerical approach

3.1 FEMLAB®

All experiments described in this thesis were conducted using FEMLAB®. FEMLAB® is a software application developed by COMSOL AB for solving a variety of engineering and mathematical problems by utilizing the finite element method. Our analysis will concern Dirichlet problems for \( \Delta_q(u) = 0 \)\(^1\) for finite values of \( q \).

Computation is done using a stationary nonlinear solver algorithm, which is a variant of the damped Newton method. The nonlinear solver works on nonlinear stationary PDE problems, such as the Dirichlet problem

\[
\begin{cases}
\nabla \cdot (-c\nabla u - \alpha u + \gamma) + \beta \cdot \nabla u + \alpha u = f & \text{in } \Omega \\
hu = r & \text{on } \partial\Omega
\end{cases}
\]

(3.1)

where one or more of \( \alpha, \beta, \gamma, \alpha, c, f, h \) or \( r \) can be functions of the solution \( u \) or its spatial derivatives. FEMLAB® denotes (3.1) as the coefficient formulation of a PDE.

The general formulation of (3.1) is

\[
\begin{cases}
\nabla \cdot \Gamma = F & \text{in } \Omega \\
hu = r & \text{on } \partial\Omega
\end{cases}
\]

where one or more of \( \Gamma, F, h \) or \( r \) can be functions of the solution \( u \) or its spatial derivatives. \( \Gamma \) is a vector, while \( F, h \) and \( r \) are scalars. It follows from this that the solver can be used to solve Dirichlet problems for \( \Delta_q(u) = 0 \) (or \( \Delta_K(u) = 0 \)). The PDE

\[
\begin{cases}
\Delta_q(u) = 0 & \text{in } D \\
u = \phi & \text{on } \partial D
\end{cases}
\]

written in general form becomes

\[
\begin{cases}
\nabla \cdot ((10^{-3} + u_x^2 + u_y^2)^q, (10^{-3} + u_x^2 + u_y^2)^q) = 0 & \text{in } D \\
u = \phi & \text{on } \partial D
\end{cases}
\]

For computational reasons (not treated here) FEMLAB®’s User Manual recommends the use of the general form when solving nonlinear PDEs.

\(^1\)For numerical purposes we use \( \Delta_q(u) = \nabla \cdot ((10^{-3} + |\nabla u|^2)^q \cdot \nabla u) \). The addition of the constant in the expression, speeds up calculation in most situations. The reason for this is beyond the scope of this text.
The following is an outline of the nonlinear method used by FEMLAB®:

First the PDE is subjected to finite element discretization: the domain \( D \) is triangulated (meshed), the PDE is multiplied by an arbitrary test function \( v_i \), integrated over the domain \( D \), Green’s formula is applied along with boundary conditions and the Ansatz

\[
    u = \sum_j U_j v_j
\]

is made. The resulting equation is written on residual form \( r(U) = 0 \), where \( r(U) \) is the residual vector and \( U \) is the solution vector e.g. \( U = (U_1, U_2, ..., U_{34000})^T \), if there are 34000 degrees of freedom. An initial guess \( U^{(0)} \) is made and then the solution process proceeds as follows: Suppose \( U^{(n)} \) is the \( n+1 \):th solution vector. The \( n+1 \):th Newton step vector \( p_n \) is determined by solving

\[
    \frac{\delta r(U^{(n)})}{\delta U} p_n = -r(U^{(n)})
\]

with a user-selected linear solver, where \( \frac{\delta r(U^{(n)})}{\delta U} \) is the Jacobian of \( r \) at \( U^{(n)} \) with respect to \( U \). Then the next solution vector is given by \( U^{(n+1)} = U^{(n)} + \lambda p_n \), where \( \lambda \in (0, 1] \) is the damping factor.

The relative error for the solution vector \( U^{(n+1)} \) is defined as

\[
    \text{r.e.} = \left( \frac{1}{N} \sum_i \left( \frac{|p_{n_i}|}{W_i} \right)^2 \right)^{1/2}
\]

where \( N \) is the number of degrees of freedom, \( W_i \) is a weight factor related to scaling of the variables in the solution vector \( U^{(n+1)} \) and \( p_{n_i} \) signifies the \( i \):th element in the newton step vector \( p_n \).

If \( \text{[r.e. for } U^{(n+1)} \text{]} > \text{[r.e. for } U^{(n)} \text{]} \), then \( \lambda \) is reduced and \( U^{(n+1)} \) recomputed.

The damping factor is reduced until \( \text{[r.e. for } U^{(n+1)} \text{]} \leq \text{[r.e. for } U^{(n)} \text{]} \) and then the next Newton step vector \( p_{n+1} \) is calculated with \( U^{(n+1)} \) in (3.3). The iteration continues until the relative error underflows a user-defined limit, at which point the algorithm terminates and the solution vector \( U \) is evaluated in (3.2). If the damping factor underflows the user-defined minimum damping factor, during the solution process, then the algorithm terminates and the most recently computed solution vector \( U \) is evaluated in (3.2).

There is no guarantee that we will find the solution, i.e. there is no way to ensure convergence. In most cases, though, Newton solvers perform well, but it is to be expected that difficulties arise when treating nonlinear equations such as \( \Delta_\infty(u) = 0 \), where the solver will have difficulties finding the domain of attraction. The damping factor enlarges the domain of attraction, but it is the initial guess \( U^{(0)} \) which is of the most importance for the solver. The nonlinearity of the equations we are studying makes it almost impossible to find good guesses. To overcome this difficulty we use the parametric nonlinear solver (with initial guess \( U^{(0)} = 0 \)) which computes a solution with \( q=0 \) (Laplace equation, a linear problem) and next this solution is used as the initial solution vector for the calculation with \( q=0.1 \), etc. up to e.g. \( q=40 \).

\(^2\)Note however that the description is of a general nature, for the exact workings of the solver the reader is referred to the FEMLAB® User Manual (COMSOL AB [7]).

\(^3\)We used FEMLAB®’s default linear solver: UMFPACK(Direct), which is a direct solver. For a complete description see the FEMLAB® User Manual (COMSOL AB [7]).

\(^4\)For \( n = 0 \) this relation is assumed to be true, in order to start the iterative process.
It is important to verify that the solver can find convergent solutions when starting with the initial guess $U^{(0)} = 0$. We use the explicitly known solutions (section 2.6) and the fact that some theoretical requirements must be fulfilled by (some) solutions to test FEMLAB®’s solver. This is done in Appendix A, where different aspects of convergence of FEMLAB®’s solutions are examined. We concentrate on the worst case scenario where we do not know the solution in advance, so we do not use the known solutions as initial value in this analysis, even though this would speed up the calculation considerably and almost certainly guarantee convergence, and instead we use $U^{(0)} = 0$.

Computation was performed on a laptop (AMD Celeron 1.6 GHz, 512 MB DDR RAM).
Chapter 4

Numerical experiments

4.1 General notes concerning experiments

Experiments 1-10 mainly concern identifying lines of singularity. Experiments 1-3 also illustrate critical segments and sets of uniqueness. In experiments 11-13 we investigate the connection between the stochastic process 'Tug of war' and the solution $u$ to a Dirichlet problem for $\Delta_\infty(u) = 0$.

In many experiments we use the symmetry of boundary condition and domain for a given Dirichlet problem to find potential lines of singularity. The symmetry enforces a straight streamline along the line of symmetry (see section 2.7), which can be (at least) part of a line of singularity. It is easy to verify that a straight streamline in a solution $u$ is a line of singularity, but this requires that the streamline runs along a line of symmetry. In many experiments we have indication of straight streamlines, which do not run along lines of symmetry. In this case one obviously cannot conclude with authority that the line in question is a line of singularity.

In several experiments curved lines of singularity seem to occur. To verify that these lines are indeed lines of singularity we first need to establish that they are streamlines and then curves need to be fitted to the 'lines'. If the evaluation of the solution $u$ along such a curve, yields the result that the solution $u$ is not a linear function of the arc length along the curve, then we have found a line of singularity. This type of analysis is beyond the scope of this text. Since we have no way of verifying the existence of a curved line of singularity, we always refer to such a line as a potential curved line of singularity.

The stochastic game 'Tug of war' is conducted as follows: A marker is placed inside a domain in the plane. There are two players: A and B. On the boundary of the domain a continuous payoff function is defined, which may change sign and vanish on part of the boundary. Inside the domain a grid of equidistant points is assigned. The game starts with a marker being placed on an arbitrary point in the grid. At each turn in the game each player has the same probability of being allowed to move the marker. The objective of player A is to maximize his payoff, while on the other hand B wants to minimize A's payoff. The game ends when any part of the boundary is reached. A wins the value of the boundary function at the point of contact with the boundary. If we let the distance between the grid points go to zero, then we pass from a discrete to a continuous game, and it is this game we refer to in our experiments concerning 'Tug of war'. The hypothesis, formulated Sheffield et al. (Berkeley), is that

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if the game starts in an arbitrary point $x_0$ then the expected value of player A’s profit ($E[\text{player A’s profit at a point } x_0]$) is the same as the solution of the Dirichlet problem, with boundary conditions identical to those in the ‘Tug of war’, evaluated at point $x_0$: $\Delta_\infty(x_0)$ or in the case of our experiments: $\Delta_q(x_0)$ for sufficiently large $q$.

### 4.2 Observations and problems encountered when conducting experiments

In some experiments a disturbing phenomenon was observed. The computation resulted in what seemed to indicate a linear model, which of course is an absurdity. After some investigation the problem was isolated. The non-zero part of the Dirichlet data were relatively small compared to the domain, e.g. on a square of radius 2 it not a good idea to use a non-zero Dirichlet condition of order $10^{-2}$ or less. My conjecture is that the effect of having radically different scales between Dirichlet condition and the size of the domain is that the effect of the Dirichlet condition will disappear gradually in the many matrix operations involved in the solution process.

Using a step size larger than 0.1, when using the parametric solver, will in some instances result in non-convergence, especially when boundary conditions give rise to several “domains of influence” (see Appendix). Therefore we have in all cases used step size 0.1.

In most cases the use of the adaptive version of the solver was rejected. In our experiments it proved to be too time-consuming and did not improve the solutions, when compared to the tactic of identifying the problem areas, i.e. areas in vicinity of a rapid variation in the boundary function, and manually refining the mesh in these areas.
4.3 Numerical experiments and results

4.3.1 Experiment 1 - Deformed cone I

Verifiable straight lines of singularity: none.
Non-verifiable straight lines of singularity: none.
Potential curved lines of singularity: none.

The Dirichlet boundary condition satisfies a Lipschitz condition of $\alpha = 1$. Most of the solution $u$ belongs to the set of uniqueness, which is the part of the domain covered by straight streamlines i.e. the critical segments. The bubble on the outer boundary does not affect the solution in a major way. This is completely consistent with the theory covered in section 2.5, since the bubble does not change the Lipschitz condition on the boundary.

The line between points (1,0) and (2.2,0) seems to be a line of singularity at first glance. But (1f) shows that $u$ is a linear function on this line. Since the line is a streamline (enforced by the symmetry of the Dirichlet condition and the domain) it cannot be a line of singularity.
Plot of solution $u$ on the line between $(1,0)$ and $(2.2,0)$
4.3.2 Experiment 2 - Deformed cone II

Verifiable straight lines of singularity: none.
Non-verifiable straight lines of singularity: none.
Potential curved lines of singularity: none.

This is a modification of 'Deformed cone I'. Here the inward bubble breaks the Lipschitz condition $\alpha = 1$ and so now the set of uniqueness only contains a single critical segment: the straight line between (-1.8,0) and (-1,0). Figure (2f) shows that the solution $u$ is linear function of arc length on the line, which is consistent with it being a critical segment.

Even though most of the solution lies outside the set of uniqueness, the solution is not disturbed in a major way by breaking the Lipschitz condition. The structure of the solution is indeed almost exactly the same as in the previous case 'Deformed cone I', so it seems that breaking the Lipschitz condition only results in a localized disturbance. However, note that the disturbance from the inward bubble is relatively much greater than the disturbance resulting from the outward bubble. This is expected since the solution is non-unique, except for one critical segment.

Note that the line between points (1,0) and (2.2,0) has the same properties as in the previous experiment 'Deformed cone I': figure (2g) shows that $u$ is a linear function on this line. Since the line is a streamline (enforced by the symmetry of the Dirichlet condition and the domain) it cannot be a line of singularity.
Plot of solution $u$ on the line between (-1.8,0) and (-1,0)

Plot of solution $u$ on the line between (1,0) and (2,2,0)
4.3.3 Experiment 3 - Clothes line

Verifiable straight lines of singularity: none.
Non-verifiable straight lines of singularity: four.
Potential curved lines of singularity: none.

The Dirichlet boundary condition satisfies a Lipschitz condition of $\alpha = 1$. If the solution is an AMLE then the solution should be linear function of arc length on the line between $(0,-1)$ and $(0,1)$ i.e. a critical segment should exist between these two points. We see that this is the case by examining figure (3f). Naturally this does not imply that the solution is an AMLE, rather it is an indication of this.

We have four likely candidates for lines of singularity. Figures (3g),(3h),(3i) and (3j) indicate that these are all non-verifiable lines of singularity, since the solution $u$ is not a linear function on these lines.
Chapter 4. Numerical experiments

(3f) Plot of solution \( u \) on the line between (0, -1) and (0, 1)

(3g) Plot of solution \( u \) on the line between (0, -1) and (-1, 0)

(3h) Plot of solution \( u \) on the line between (0, -1) and (1, 0)

(3i) Plot of solution \( u \) on the line between (-1, 0) and (0, 1)

(3j) Plot of solution \( u \) on the line between (1, 0) and (0, 1)
4.3.4 Experiment 4 - One peak

Verifiable straight lines of singularity: one.
Non-verifiable straight lines of singularity: two.
Potential curved lines of singularity: two.
The symmetry of the Dirichlet boundary condition and domain implies that the solution $u$ has a straight streamline between $(0,-1)$ and $(0,1)$. Figure (4f) indicates that the line segment between $(0,0.2)$ and $(0,1)$ is a verifiable line of singularity. It looks like this line is part of two curved lines of singularity, which join up at $(0,0.2)$: one starting at (-1,-1), the other in (1,-1) and both ending in (0,1). Figures (4h) and (4i), which plot straight segments, indicate that these lines indeed start out as non-verifiably singular.
22 Chapter 4. Numerical experiments

(4f) Plot of solution $u$ on the line between $(0, 0.2)$ and $(0, 1)$

(4g) Plot of solution $u$ on the line between $(0, -1)$ and $(0, 0.2)$

(4h) Plot of solution $u$ on the line between $(-1, -1)$ and $(-0.4, -0.4)$

(4i) Plot of solution $u$ on the line between $(1, -1)$ and $(0.4, -0.4)$
4.3. Numerical experiments and results

4.3.5 Experiment 5 - Variant one peak

Verifiable straight lines of singularity: one.
Non-verifiable straight lines of singularity: two.
Potential curved lines of singularity: two.

The symmetry of the Dirichlet boundary condition and domain implies that the solution \( u \) has a straight streamline between (0,-1) and (0,1). Figure (5g) indicates that the line segment between (0,0.2) and (0,1) is a verifiable line of singularity, although one can infer that it certainly is closer to being in \( C^2 \) than its counterpart in 'One peak'. The contour plot (figure (5c)) indicates the the ridge between (0,0.2) and (0,1) is smoother than the corresponding ridge in 'One peak' seen in figure (4c). The lines described in figures (5h) and (5i) are non-verifiable lines of singularity, as in 'One peak', that each seem to be part of a curved line of singularity.
Chapter 4. Numerical experiments

Plot of solution $u$ on the line between $(0, -1)$ and $(0, 0.2)$

Plot of solution $u$ on the line between $(0, 0.2)$ and $(0, 1)$

Plot of the solution $u$ on line between $(-1, -1)$ and $(-0.4, -0.4)$

Plot of the solution $u$ on the line between $(1, -1)$ and $(0.4, -0.4)$
4.3.6 Experiment 6 - Twin peaks

Verifiable straight lines of singularity: one.
Non-verifiable straight lines of singularity: none.
Potential curved lines of singularity: two.

The symmetry in the Dirichlet boundary condition and domain implies a straight streamline between (0,-1) and (0,1). This line, which is plotted figure (6f), is clearly a verifiable line of singularity between (approximately) points (0,0) and (0,1). On the segment between (0,-1) and (0,0) the solution \( u \) looks like it is linear, so on this line the solution may have continuous partial second derivatives.

There seems to be two curved streamlines, one between (-1,-1) and (-0.5,1) and the other between (1,-1) and (0.5,1). The curvature of these lines has prevented the plotting of a straight segment on the curve, although it does look like they start out as approximatively straight from the points (-1,-1) and (1,-1).
4.3.7 Experiment 7 - Asymmetric twin peaks

Verifiable straight lines of singularity: none.
Non-verifiable straight lines of singularity: two.
Potential curved lines of singularity: # unclear.

Non-verifiable lines of singularity start in (-1,-1) and (1,-1). This is verified by the solution $u$ being a nonlinear function of arc length on the lines in figures (7f) and (7g). The two lines quickly become curved, and so it is hard to further verify their potential singularity. Lines of singularity or not, they join up at approximately the point (-0.17,0.3) and continue to the boundary at (-0.5,1) as seen in figure (7e). Two other curved lines of singularity seem to start at points (0,1) and (0.5,1), but they quickly disappear.
Plot of solution $u$ on the line between $(1,-1)$ and $(0.4,-0.4)$

(7f)

Plot of solution $u$ on the line between $(-1,-1)$ and $(-0.6,-0.6)$

(7g)
4.3.8 Experiment 8 - Unit corner

Verifiable straight lines of singularity: one.
Non-verifiable straight lines of singularity: none.
Potential curved lines of singularity: none.

By symmetry, in Dirichlet boundary condition and domain, we have that the diagonal is a streamline and then in figure (8f) we see that it is clearly a verifiable line of singularity. This can also be proved rigorously (Aronsson, personal communication).
4.3.9 Experiment 9 - Unit saddle

Verifiable straight lines of singularity: two.
Non-verifiable straight lines of singularity: none.
Potential curved lines of singularity: none.

By symmetry, in Dirichlet boundary condition and domain, we have that the two diagonals are streamlines (with the exception of the stationary point of the solution at the origin) and so figure (9f) and (9g), clearly show that they are verifiable lines of singularity, since $u$ is not a linear function of arc length on these lines.
4.3. Numerical experiments and results

Plot of solution $u$ on line between $(-1,-1)$ and $(1,1)$

Plot of solution $u$ on the line between $(1,-1)$ and $(-1,1)$
4.3.10 Experiment 10 - Four side disturbed square

Verifiable straight lines of singularity: two.
Non-verifiable straight lines of singularity: none.
Potential curved lines of singularity: eight.

By symmetry we have two straight streamlines (with the exception of the stationary point of the solution at the origin) running between (-5,0) and (5,0), and between (0,-5) and (0,5). The figures (10f) and (10g) verify that they are verifiable lines of singularity.

Each "bubble" has two potential curved lines of singularity its vicinity. At points (0,-3.75), (0,3.75), (-4.3,0) and (4.3,0) the singular lines split into three: one straight verifiable and two potentially curved. The curved lines are best seen in figures (10d) and (10e).
4.3. Numerical experiments and results

q(251) = 25  Surface: u

Plot of solution u on the line between (0,-5) and (0,5)

Plot of solution u on the line between (-5,0) and (5,0)
4.3.11 Experiment 11 - Tug of war rectangle

This is a tug of war scenario. Note that $\int_{\partial D} f \, ds \approx 2 \int_{\partial D} g \, ds$. In figure (11f) we plot the solutions $u_1$ and $u_2$ along the line a, described in figure (11e). We see that the lines intersect at $x = 10$. For $x < 10$ the Dirichlet condition $\alpha$ is most favorable for player A, while for $x > 10$ the Dirichlet condition $\beta$ is better.

Here it is hard to formulate a hypothesis regarding the strategies of player A and B. The results of the experiment are hard to interpret, but using simulation one could possibly formulate a hypothesis regarding the strategies. However this is beyond the scope of this text.

Note that this experiment was somewhat problematic for FEMLAB® (see Table A.2), something that affects the reliability of results.
4.3. Numerical experiments and results

(11e) 

-15 -10 -5 0 5 10 15 

-10 -5 0 5 10 

<table>
<thead>
<tr>
<th>-0.05</th>
<th>0</th>
<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
<th>0.3</th>
<th>0.35</th>
</tr>
</thead>
</table>

solution $u_1$ evaluated along line a

(11f) 

-20 -15 -10 -5 0 5 10 15 20 

-0.05 0 0.05 0.1 0.15 0.2 0.25 0.3 

<table>
<thead>
<tr>
<th>dotted</th>
<th>solution $u_1$ evaluated along line a</th>
</tr>
</thead>
<tbody>
<tr>
<td>solid</td>
<td>solution $u_2$ evaluated along line a</td>
</tr>
</tbody>
</table>
4.3.12 Experiment 12 - Tug of war square

This is a tug of war scenario on a square. Note that \( \int_{\partial D} f \, ds \approx 2 \int_{\partial D} g \, ds \). The two cases we want to compare are:

1. \( u_1(0, 0), \) where \( u_1 \) is the solution of the Dirichlet problem \( \alpha \) for \( \Delta q = 20 \) and \( (u_1) = 0 \)

and

2. \( u_2(0, 0), \) where \( u_2 \) is the solution of the Dirichlet problem \( \beta \) for \( \Delta q = 20 \) and \( (u_2) = 0 \).

In case (1) all of A’s possible strategies for moving the marker are the same, that is A wants to move his marker toward the point (0,5). In case (2) A’s strategy is not as evident. He can move either in the direction of the point (0,5) or point (0,-5), and in either case he will move toward lesser profit than in (1) and thus A’s expected profit should be greater in case (1). This is confirmed by the results of the experiment: \( u_1(0, 0) = 0.3759 \) and \( u_2(0, 0) = 0.3202 \).

Note that this experiment was somewhat problematic for FEMLAB® (see Table A.2), something that affects the reliability of results.
4.3. Numerical experiments and results

4.3.13 Experiment 13 - Tug of war decline of expected value

This is a tug of war scenario, conducted on two rectangles with Dirichlet boundary conditions as seen in figure (13a) and (13c). Note that \( \int_{\partial D} f ds \approx \int_{\partial D} g ds \). Our hypothesis is that player A’s expected profit will decline as we move away from the non-zero part of the boundary. Furthermore we expect that A’s expected profit will decline faster when the rectangle is shrunk y-wise. These hypotheses are confirmed by figures (13b), (13d) and (13g).

By computing the logarithm of the solutions \( u_1 \) and \( u_2 \) we see in figures (13e) and (13f) that the solutions decline exponentially. In (13g) we see that the solutions decline exponentially at different rates in two intervals. For \( \log(u_1) \) the (approximate) intervals are \((-40 < x < -10)\) and \((-10 < x < 38)\), and for \( \log(u_2) \) the (approximate) intervals are \((-40 < x < -20)\) and \((-20 < x < 38)\). Of course we disregard the very tail end \((38 < x < 40)\) in this analysis, since the the solutions \( u_1 \) and \( u_2 \) are forced to zero by the zero boundary condition.

Note that this experiment was somewhat problematic for FEMLAB® (see Table A.2), something that affects the reliability of results.
log(u1) and log(u2) evaluated along the line between (-40,0) and (40,0)
4.3.14 Experiment 14 - Harmonic circle

This experiment is conducted to investigate if \( u \) behaves as a harmonic function in the sense that

\[
\text{area}(D) = \frac{1}{2\pi} \int_{\partial D} u \, ds.
\]

The experiment resulted in \( \frac{1}{2\pi} \int_{\partial D} u \, ds = 0.1628 \), while \( u(0, 0) = 0.3994 \). So we have \( u(0, 0) > \frac{1}{2\pi} \int_{\partial D} u \, ds \).
4.4 Structure of singular solutions - exemplified in two cases

4.4.1 Case 1 - Path to singular solution

Figure 4.1: Formation of singular lines in the solution to the 'Unit saddle' Dirichlet problem for $\Delta_q(u) = 0$ as $q$ increases.
4.4.2 Case 2 - Laplace vs. q-Laplace

When examining the plots below, Laplace solution to the left and q-Laplace to the right, it is evident that the two solutions are akin. If the streamlines gather around a line in the Laplace solution then we see below that this line will become a line of singularity in the q-Laplace solution. If it holds generally remains to be proved, but these plots certainly provide a strong indication. Since $\Delta_q(u) = 0$ approaches $\Delta_\infty(u) = 0$ as $q \to \infty$, the above conjecture is also applicable to the Infinity-Laplace equation, that is we can predict the structure of lines of singularity in the solution to a Dirichlet problem, with non-constant boundary function, for $\Delta_\infty(u) = 0$ by studying the solution to the corresponding Dirichlet problem for $\Delta u = 0$ equivalently $\Delta_q(u)|_{q=0} = 0$.

Figure 4.2: Clothes line

Figure 4.3: One peak

Figure 4.4: Variant one peak
Figure 4.5: Twin peaks

Figure 4.6: Asymmetric twin peaks

Figure 4.7: Unit corner

Figure 4.8: Unit saddle
4.4. Structure of singular solutions - exemplified in two cases

Figure 4.9: Four side disturbed square

Figure 4.10: Tug of War - rectangle (one "bubble")

Figure 4.11: Tug of War - rectangle (two "bubbles")

Figure 4.12: Tug of War - square (one "bubble")
Figure 4.13: Tug of War - square (two "bubbles")

Figure 4.14: Tug of War - decline of expected value (broad rectangle)

Figure 4.15: Tug of War - decline of expected value (narrow rectangle)

Figure 4.16: Harmonic circle
4.5 Illustration of the concept of AMLE

Here the concept of AMLE is illustrated. The Dirichlet boundary condition satisfies a Lipschitz condition of $\alpha = 1$. In the $q$-Laplace ($q=40$) solution we see that the contour lines are equidistant and this is consistent with the solution $u$ satisfying $|\nabla u| \equiv \text{constant}$. Compare this to the Laplace ($q=0$) solution, where the cone has an inward bend. Functions $u \in C^1(\Omega)$, such that $|\nabla u| \equiv \text{constant}$, form an important subclass of AMLE.
Chapter 5

Discussion of results

We have not been able to draw many useful conclusions from the experiments, with some exceptions. We have formulated a conjecture regarding lines of singularity and the connection between the Laplace solution to a given Dirichlet problem and the corresponding Infinity-Laplace solution.

Our experiments show that it seems a likely conjecture that lines of singularity appear in solutions for Dirichlet problems, with non-constant boundary functions\(^1\), to \(\Delta_q(u) = 0\). Furthermore there is anecdotal evidence that breaking a Lipschitz condition, by modifying the Dirichlet condition belonging to \(\Delta_q(u) = 0\), only results in a localized disturbance around the area of modification, in the resulting solution. An attempt has been made to modify the Dirichlet condition in the 'Clothes line' experiment, so as to break the Lipschitz condition. This attempt did not yield any conclusive result. There is need of a large experiment set, if we are to formulate any worthwhile conjecture.

There is a need for numerical and analytical methods that verify the existence of lines of singularity, especially those which seem to run along curves. This would advance the understanding of lines of singularity in solutions to the Dirichlet problems for the Infinity-Laplace equation.

The tug of war experiments follow the intuition in the square and decline of expected value case, but in the rectangle case (experiment 11) it is difficult to formulate any hypothesis regarding the strategies of the two players.

The lion's share of the work on this thesis has concerned realizing the experiments. The quest for true AMLE has led down many paths, not all of them constructive. At one point FEMLAB\(\text{®}\) was rejected for a promising strategy, which later proved unusable. Although we cannot know absolutely that FEMLAB\(\text{®}\)’s solutions are in all ways correct, it seems likely that the solutions at least resemble the true solutions (the AMLEs) structurally. Knowing the approximative structure of an AMLE can at the very least give hints of which strategies should be employed when formulating proofs. Note however the results of our experiments do not in themselves constitute proofs. The experiments can indicate which strategies are the most rewarding, and which are too difficult to be effectively pursued.

\(^1\)The 'Deformed cone I' experiment provides a potential exception to this. Here the piece-wise constant boundary function does not seem to give rise to a line of singularity.

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Chapter 6
Conclusions and future work

6.1 Conclusions

6.1.1 Numerical

A boundary function, for a Dirichlet problem for $\Delta_\infty(u) = 0$, can have clearly visible "domains of influence". A simple example of this is if the boundary function is zero, except for a limited part of the boundary. The non-zero part of the boundary function will then have a "domain of influence", which will increase with increasing absolute value of the integral of the boundary function. When the boundary function has several distinct "domains of influence" that are in the vicinity of each other, our experiments indicate that the solution process is impeded. When the boundary condition does not create problems with "domains of influence" then our investigations have shown that it is likely to assume that FEMLAB® produces solutions that are valid approximations of the Infinity-Laplace solution to the Dirichlet problem being examined.

6.1.2 Mathematical

After analyzing the results of our experiments the following hypotheses are proposed:

(1) Lines of singularity are a characteristic feature of solutions to Dirichlet problems for $\Delta_q(u) = 0$ (sufficiently large $q$), where the boundary condition is a non-constant function\(^1\). Since $\Delta_q(u) = 0$ approaches $\Delta_\infty(u) = 0$ as $q \to \infty$, this statement applies to $\Delta_\infty(u) = 0$.

(2) For Dirichlet problems, with non-constant boundary functions, we propose that one can predict the structure of the lines of singularity in the Infinity-Laplace case by examining the Laplace case. This is our main hypothesis.

\(^1\)The 'Deformed cone I' experiment provides a potential exception to this. Here the piece-wise constant boundary function does not seem to give rise to a line of singularity.

Hansson, 2005.
6.2 Future work

By expanding the experiment set, where domain and boundary condition are symmetric, more properties of solutions $u$ to Dirichlet problems for $\Delta_\infty(u) = 0$, containing straight lines of singularity, could be found through systematic investigation.

A method needs to be developed to determine the existence of curved lines of singularity. Such a method would greatly advance our knowledge of lines of singularity in solutions $u$ to Dirichlet problems for $\Delta_\infty(u) = 0$.

Expand the experiment set to include 3d models. Most of the theory which applies to 2d case can also be used in the 3d case. For experimentation to be possible a computer with at least 2 GB RAM would need to be used, since 3d modelling is extremely memory intensive, due to the large number of degrees of freedom needed for finite element discretization of the PDE on the domain in question.

The study of the relation between solutions to Dirichlet problems for $\Delta_\infty(u) = 0$ and the "Tug of War" game is at this point in its early stages. Knowledge of the strategies of the players is central if we are to fully understand this relation. Hopefully the upcoming papers by Sheffield et al. (Berkeley) concerning "Tug of War" will provide some new insights, which will inspire the creation of a more exhaustive experiment set (than the one in this thesis).
Bibliography


Appendix A

On the convergence of solutions obtained from FEMLAB®

The results of this section are anecdotal, but still provide some idea of the convergence of solutions obtained from FEMLAB®.

A.0.1 Verification of FEMLAB®’s solver using known solutions

The plots below describe the difference between known solutions and FEMLAB®’s solution of $\Delta_q(u) = 0$ and $\Delta_K(u) = 0$, with the known solutions (section 2.6) as Dirichlet boundary conditions. Note that initial solution used is $U_0 = 0$.

In (4.3.1), (4.3.2), (4.3.3) and (4.5) the solver produces solutions with properties which the known theory of AMLE functions predicts, i.e. FEMLAB®’s solution is linear with correct slope (the optimal Lipschitz constant satisfied by the boundary function) in the places where it should be.

In Table A.1 the computation times for $\Delta_q(u) = 0$ and $\Delta_K(u) = 0$ are displayed. In our experiments we have chosen to use the q-formulation, which is more of a happenstance than anything else. The entries in the table seem to indicate that the two formulations are approximatively equivalent time wise, but note that this only a weak indication. It could be the case that one of the formulations reaches a stable state in the solution process faster than the other one. This will not be examined here, but this would certainly be of major importance when performing large scale experiments. The important point, for us, is that the two formulations seem to deliver approximatively equivalent solutions, that is the solutions deviate from the true solutions in a comparable manner.

<table>
<thead>
<tr>
<th>Boundary cond.</th>
<th>K</th>
<th>q</th>
</tr>
</thead>
<tbody>
<tr>
<td>$2x+2y+2$</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$\arctan(y/(x+2))$</td>
<td>12</td>
<td>12</td>
</tr>
<tr>
<td>$</td>
<td>x</td>
<td>^{4/3} -</td>
</tr>
<tr>
<td>$\sqrt{r^{4/2}}$</td>
<td>17</td>
<td>17</td>
</tr>
<tr>
<td>$\sqrt{(x-2)^2 + y^2}$</td>
<td>20</td>
<td>20</td>
</tr>
</tbody>
</table>

Table A.1: Time of calculation (minutes) for verification experiments (parameter step length: 0.1) performed on a laptop (1.6 GHz, 512 MB DDR RAM).
A.0.2 Comparison of consecutive parametric solution steps

<table>
<thead>
<tr>
<th>q</th>
<th>0.1 - 0</th>
<th>5 - 4.9</th>
<th>10 - 9.9</th>
<th>15 - 14.9</th>
<th>19 - 18.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deformed cone I</td>
<td>0.016552</td>
<td>0.0032778</td>
<td>0.0001069</td>
<td>5.171 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>Deformed cone II</td>
<td>0.016552</td>
<td>0.0032807</td>
<td>0.0001072</td>
<td>5.2537 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>Clothes line</td>
<td>0.024284</td>
<td>0.00059682</td>
<td>0.000186</td>
<td>8.901 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>One peak</td>
<td>0.031963</td>
<td>0.00079919</td>
<td>0.00023605</td>
<td>0.00012004</td>
<td>-</td>
</tr>
<tr>
<td>Variant one peak</td>
<td>0.015261</td>
<td>0.00026548</td>
<td>7.3365 · 10^{-9}</td>
<td>3.8125 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>Twin peaks</td>
<td>0.015531</td>
<td>0.00028924</td>
<td>8.2124 · 10^{-9}</td>
<td>4.2851 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>Asym. Twin peaks</td>
<td>0.016642</td>
<td>0.00032559</td>
<td>9.0593 · 10^{-9}</td>
<td>4.311 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>Unit corner</td>
<td>0.011271</td>
<td>0.00050452</td>
<td>0.00016774</td>
<td>8.3069 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>Unit saddle</td>
<td>0.0056535</td>
<td>0.00025225</td>
<td>8.3986 · 10^{-9}</td>
<td>4.1833 · 10^{-9}</td>
<td>-</td>
</tr>
<tr>
<td>Four side d. sq.</td>
<td>0.12824</td>
<td>0.0028233</td>
<td>0.0008332</td>
<td>0.00047304</td>
<td>-</td>
</tr>
<tr>
<td>ToW rect. (\alpha)</td>
<td>0.040262</td>
<td>0.0007656</td>
<td>0.00022998</td>
<td>0.00020685</td>
<td>0.00020877</td>
</tr>
<tr>
<td>ToW rect. (\beta)</td>
<td>0.04664</td>
<td>0.00054114</td>
<td>0.0018976</td>
<td>0.0096118</td>
<td>0.01572</td>
</tr>
<tr>
<td>ToW sq. (\alpha)</td>
<td>0.04234</td>
<td>0.0007154</td>
<td>0.00019184</td>
<td>0.00030773</td>
<td>-</td>
</tr>
<tr>
<td>ToW sq. (\beta)</td>
<td>0.046424</td>
<td>0.00045656</td>
<td>0.0005757</td>
<td>0.0015664</td>
<td>-</td>
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<tr>
<td>ToW dep sm. rect.</td>
<td>0.063176</td>
<td>0.00112</td>
<td>0.0003466</td>
<td>0.00038068</td>
<td>-</td>
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<tr>
<td>ToW dep lar. rect.</td>
<td>0.02667</td>
<td>0.00076314</td>
<td>0.00023509</td>
<td>0.00011942</td>
<td>-</td>
</tr>
<tr>
<td>Harmonic circle</td>
<td>0.030341</td>
<td>0.00060268</td>
<td>0.00016887</td>
<td>7.6685 · 10^{-9}</td>
<td>-</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>q</th>
<th>20 - 19.9</th>
<th>25 - 24.9</th>
<th>30 - 29.9</th>
<th>35 - 34.9</th>
<th>40 - 39.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deformed cone I</td>
<td>4.2941 · 10^{-9}</td>
<td>4.7383 · 10^{-9}</td>
<td>4.7707 · 10^{-9}</td>
<td>4.6032 · 10^{-9}</td>
<td>4.4843 · 10^{-9}</td>
</tr>
<tr>
<td>Deformed cone II</td>
<td>3.0947 · 10^{-9}</td>
<td>2.9285 · 10^{-9}</td>
<td>3.255 · 10^{-9}</td>
<td>3.2769 · 10^{-9}</td>
<td>3.2754 · 10^{-9}</td>
</tr>
<tr>
<td>Clothes line</td>
<td>5.2086 · 10^{-9}</td>
<td>3.4343 · 10^{-9}</td>
<td>2.4796 · 10^{-9}</td>
<td>2.0229 · 10^{-9}</td>
<td>1.7807 · 10^{-9}</td>
</tr>
<tr>
<td>One peak</td>
<td>8.9441 · 10^{-9}</td>
<td>7.9366 · 10^{-9}</td>
<td>7.5166 · 10^{-9}</td>
<td>7.5918 · 10^{-9}</td>
<td>7.571 · 10^{-9}</td>
</tr>
<tr>
<td>Variant one peak</td>
<td>3.574 · 10^{-9}</td>
<td>3.3276 · 10^{-9}</td>
<td>3.1682 · 10^{-9}</td>
<td>9.9753 · 10^{-9}</td>
<td>6.373 · 10^{-9}</td>
</tr>
<tr>
<td>Twin peaks</td>
<td>3.89 · 10^{-9}</td>
<td>3.833 · 10^{-9}</td>
<td>9.2792 · 10^{-9}</td>
<td>0.00013776</td>
<td>0.0014187</td>
</tr>
<tr>
<td>Asym. Twin peaks</td>
<td>3.8719 · 10^{-9}</td>
<td>3.3618 · 10^{-9}</td>
<td>5.6896 · 10^{-9}</td>
<td>0.00019098</td>
<td>0.00028261</td>
</tr>
<tr>
<td>Unit corner</td>
<td>4.9344 · 10^{-9}</td>
<td>3.2753 · 10^{-9}</td>
<td>3.2301 · 10^{-9}</td>
<td>1.7448 · 10^{-9}</td>
<td>1.359 · 10^{-9}</td>
</tr>
<tr>
<td>Unit saddle</td>
<td>2.5147 · 10^{-9}</td>
<td>1.6945 · 10^{-9}</td>
<td>1.2554 · 10^{-9}</td>
<td>1.1308 · 10^{-9}</td>
<td>1.2243 · 10^{-9}</td>
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<tr>
<td>Four side d. sq.</td>
<td>0.0023777</td>
<td>0.0009474</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ToW rect. (\alpha)</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
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<tr>
<td>ToW rect. (\beta)</td>
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<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ToW sq. (\alpha)</td>
<td>0.00031333</td>
<td>0.0013572</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ToW sq. (\beta)</td>
<td>0.00092904</td>
<td>0.005045</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ToW dep sm. rect.</td>
<td>0.00057113</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>ToW dep lar. rect.</td>
<td>7.5691 · 10^{-9}</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Harmonic circle</td>
<td>5.4023 · 10^{-9}</td>
<td>4.9435 · 10^{-9}</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

Table A.2: Absolute value of maximal difference between consecutive parametric solution steps. Note: dep = decline of expected value.

Equidistant points were assigned to the domain in question, and then the absolute value of the maximal difference, over all points, of two consecutive solutions in the parametric solution process, e.g. q=5 and q=4.9, were computed and stored in Table A.2. This was achieved by using a MATLAB® script, which computes the entries after extracting data from the FEMLAB® solution. In Table A.3 the number of points assigned to

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1MATLAB is a registered trademark of The Mathworks, Inc.
the domain of each experiment are given (the difference in numbers is due to differing size of domains). If the values decrease as we move from left to right in Table A.2, then this is an indication of convergence. Note that FEMLAB®’s solver does not take the entries of the table into account during computation; if it did it would take action when convergence is deteriorating in the sense described in this table. If the values do not decrease we take this as an indication that the solver has problems handling the Dirichlet boundary condition of the experiment at hand.

A boundary function, for a Dirichlet problem for $\Delta_{\infty}(u) = 0$, can have clearly visible "domains of influence". When the boundary function has several distinct "domains of influence" that are in the vicinity of each other, our experiments indicate that the solution process is impeded. See Table A.2 and figures (6b), (10e), (11d) and (12d). The shape of the domain also seems to play a role. In figures (11b) and (13b) the non-zero part of the boundary function has a very local "domain of influence" and Table A.2 indicates that this is problematic for the solver. In contrast ’simple’ Dirichlet conditions, e.g. that of ‘One peak’, seem to indicate monotone convergence to the final solution.

In cases where Dirichlet conditions do not heavily affect the solution studied, the solution has reached stable state at approximately $q = 15$. Note that the absence of entries for higher values of $q$ signify lack of convergence in the solution process for these parameter values.

We must be careful when drawing conclusions from Table A.2. See figure (12a), which at first glance does not seem problematic, since there is only one "domain of influence" which is non-local, i.e. the Dirichlet condition is ’simple’. It is not the case that ’Tug war square $\alpha$’ is exceptionally difficult to handle for the solver, but it is harder to handle than other ‘simple’ Dirichlet conditions (for example those of ’Clothes line’, ’Twin peaks’ etc.). The explanation for this is at this point unknown to the author.

<table>
<thead>
<tr>
<th>Experiment</th>
<th># points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Def. cone I</td>
<td>102304</td>
</tr>
<tr>
<td>Def. cone II</td>
<td>100836</td>
</tr>
<tr>
<td>Cl. line</td>
<td>40000</td>
</tr>
<tr>
<td>One peak</td>
<td>40000</td>
</tr>
<tr>
<td>V. one peak</td>
<td>40000</td>
</tr>
<tr>
<td>Twin peaks</td>
<td>40000</td>
</tr>
<tr>
<td>Asym. Twin p.</td>
<td>40000</td>
</tr>
<tr>
<td>Unit corner</td>
<td>40000</td>
</tr>
<tr>
<td>Unit saddle</td>
<td>40000</td>
</tr>
<tr>
<td>Four s. d. sq.</td>
<td>250000</td>
</tr>
<tr>
<td>ToW rect. $\alpha$</td>
<td>80000</td>
</tr>
<tr>
<td>ToW rect. $\beta$</td>
<td>80000</td>
</tr>
<tr>
<td>ToW sq. $\alpha$</td>
<td>250000</td>
</tr>
<tr>
<td>ToW sq. $\beta$</td>
<td>250000</td>
</tr>
<tr>
<td>ToW dep sm.</td>
<td>100000</td>
</tr>
<tr>
<td>ToW dep lar.</td>
<td>200000</td>
</tr>
<tr>
<td>Harm. circ.</td>
<td>32040</td>
</tr>
</tbody>
</table>

Table A.3: # equidistant points assigned to domain of experiments.
A.0.3 Settings used in FEMLAB®

The figures (A.a-A.i) describe the most important of the settings used in calculation. The settings here detailed have worked well in our experiments, but we do not claim that they are optimal.

In the Model Navigator (A.a) we selected FEMLAB → PDE modes → PDE, General Form → Stationary analysis, as this was best suited for handling $\Delta q(u) = 0$.

No adaption was used, instead we manually refined problem areas, i.e. areas in the vicinity of a rapidly varying boundary function. On average 16000 triangles were used in the mesh of any given model, which resulted in about twice as many test functions (degrees of freedom).

The selection in figure (A.h) (Parametric → predictor → Constant) causes the initial solution guess $U^{(0)}$ for the current parameter value to be the solution vector of the previous parameter value. In figure (A.e) the box ‘Current solution’ is checked and ‘Parameter value: automatic’ is selected to make this work.

The settings in the ‘Nonlinear’ box (figure (A.i)) have been selected with the objective of obtaining a solution, where $q$ is relatively large while the relative error is as small as possible (too small relative error will hinder the solution process). ‘Maximum number of iterations’ have been set to the maximum number (=10000) in order to avoid the solution process being stopped due to a large number of iterations; instead this is done manually if it seems that the solution process has stalled.

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Appendix A. On the convergence of solutions obtained from FEMLAB®

A.d

A.e
Appendix A. On the convergence of solutions obtained from FEMLAB®

A.

A.h

A.i
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