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N.B.: When citing this work, cite the original article.

Original Publication:
http://dx.doi.org/10.1016/j.aam.2006.07.007
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http://www.elsevier.com/

Postprint available at: Linköping University Electronic Press
http://urn.kb.se/resolve?urn=urn:nbn:se:liu:diva-18010
A regular decomposition of the edge-product space of phylogenetic trees

Jonna Gill∗ Svante Linusson† Vincent Moulton‡ Mike Steel§

Abstract

We investigate the topology and combinatorics of a topological space called the edge-product space that is generated by the set of edge-weighted finite labelled trees. This space arises by multiplying the weights of edges on paths in trees, and is closely connected to tree-indexed Markov processes in molecular evolutionary biology. In particular, by considering combinatorial properties of the Tuflley poset of labelled forests, we show that the edge-product space has a regular cell decomposition with face poset equal to the Tuflley poset.

Keywords: Trees, forests, partitions, poset, regular cell complex, recursive coatom ordering

MSC: 05C05, 92D15
1 Introduction

For a tree $T$, we let $V(T)$ and $E(T)$ denote the sets of vertices and edges of $T$ respectively. For a fixed finite set $X$ we are interested in the (finite) set of binary (i.e. trivalent) trees $T$ that have $X$ as their set of leaves (degree one vertices). They correspond to all the possible “phylogenetic trees” for $X$. See [15]. We study the space of all such trees with edge weights between 0 and 1. To each binary tree with edge weights we associate a point in a high-dimensional space in the following way. Given a map $\lambda: E(T) \to [0,1]$ define

$$p = p(T,\lambda): \binom{X}{2} \to [0,1]$$

by setting, for all $x, y \in X$,

$$p(x,y) = \prod_{e \in P(T;x,y)} \lambda(e),$$

where $P(T;x,y)$ is the set of edges in the path in $T$ from $x$ to $y$.

Let $E(X,T) \subset [0,1]^{\binom{X}{2}}$ denote the image of the map

$$\Lambda_T : [0,1]^{E(T)} \to [0,1]^{\binom{X}{2}}, \ \lambda \mapsto p(T,\lambda)$$

and let $E(X)$ be the union of the subspaces $E(X,T)$ of $[0,1]^{\binom{X}{2}}$ over all binary trees $T$ with $X$ as its set of leaves. We call $E(X)$ the edge-product space for trees on $X$. Note that a different notion of the space of trees was suggested by Billera, Holmes and Vogtmann in [2]. The main difference is that they add edge weights along paths to give a distance between leaves, rather than multiplying them. The edge-product space can be seen as a compactification of the space studied in [2], since we allow zero as an edge weight.

Apart from their intrinsic interest, a central motivation for investigating edge-product spaces is that they are intimately connected with tree-indexed Markov process in molecular evolutionary biology [7, 15], as we now briefly outline. In these models there is a fixed matrix $Q$ of transition rates between states of some set (e.g. nucleotide bases, amino acids), which forms a stationary and time-reversible Markov process. The process operates for some duration $d(e)$ on each edge $e$ of $T$. Let $\lambda: E(T) \to [0,1]$ be defined by $\lambda(e) = e^{-d(e)}$, and allow $\lambda(e)$ to equal 0 in order to model ‘site saturation’ (i.e. the limiting value as $d(e) \to \infty$). The Markov process, parameterised by the pair $(T,\lambda)$, induces a (marginal) joint probability distribution on the set of state assignments to $X$. Furthermore it can be shown that two pairs $(T,\lambda)$ and $(T',\lambda')$ induce the same joint probability distribution precisely if $p(T,\lambda) = p(T',\lambda')$ (by extending the approach of [16] which established this result when $Q$ is a symmetric $2 \times 2$ matrix). Consequently, the edge-product space defined above is homeomorphic to the quotient space where trees with $\lambda$–valued edge weights are identified if they induce the same Markov process at the leaves for a fixed rate matrix $Q$.

In [12] it was shown that $E(X)$ has a natural CW–complex structure for any finite set $X$, and a combinatorial description of the associated face poset, called the Tuffley poset was given. It was also conjectured that $E(X)$ is a regular cell complex. Here we prove that this conjecture holds.
Theorem 1.1. The edge-product space $E(X)$ has a regular cell decomposition with face poset given by the Tuffley poset.

In particular, $E(X)$ is homeomorphic to the geometric realization of the Tuffley poset (see [3, 12.4(ii)]). Note that as a consequence of this theorem we obtain an affirmative answer to the main conjecture in [1].

Our main technical tool is a proof that any interval of the Tuffley poset (with an artificial $\hat{0}$ added) has a recursive coatom ordering. This implies amongst other things that the order complex of every open interval is a sphere. Note that we do not describe an explicit recursive coatom ordering that is valid for any given interval. Instead, we have developed what we believe to be a new method for establishing the existence of recursive coatom orderings. In particular, we define a class of coatom orderings and show that for any interval we can always choose some ordering from that class that satisfies the conditions for a recursive coatom ordering.

We now describe the contents of the paper. In Section 2 we review some properties of “$X$–forests” and of the Tuffley poset, which can be regarded as an order relation on the collection of all $X$–forests. In Section 3 we prove that there exists a shelling order for the chains in every interval of the Tuffley poset which we require in order to prove Theorem 1.1 in Section 4. Finally, in Section 5 we present the proofs of some technicalities that we used in Section 3. As many of the cases are straightforward checks, where appropriate we will refer the reader to [8] where the full details are presented.

Acknowledgement: We thank Anders Björner for valuable comments on an earlier version of this manuscript. We also thank Richard Hain and especially Robert Daverman for their help with the topological results stated in the Appendix. Steel thanks the Marsden Fund.

2 Trees, Forests and the Tuffley poset

In this section we review some material concerning trees and related structures. Throughout this paper $X$ will be a finite set.

An $X$–tree $T$ is a pair $(T; \phi)$ where $T$ is a tree, and $\phi : X \to V(T)$ is a map such that all vertices in $V(T) - \phi(X)$ have degree greater than two. Note that we do not require $\phi$ to be injective. We call $X$ labels and the vertices in $V(T) - \phi(X)$ unlabelled. Two $X$–trees $(T_1; \phi_1)$ and $(T_2; \phi_2)$ are isomorphic if there is a graph isomorphism $\psi : V(T_1) \to V(T_2)$ such that $\phi_2 = \psi \circ \phi_1$. For an $X$–tree $T = (T; \phi)$ we let $E(T)$ denote $E(T)$, the set of edges of $T$. By a contraction of an edge $e = \{v, u\}$ in an $X$–tree $T = (T; \phi)$, we mean contraction of the edge in $T$, with identification of $u$ and $v$ and label $\phi^{-1}(u) \cup \phi^{-1}(v)$ for the new vertex.

An $X$–forest is a collection $\alpha = \{(A, T_A) : A \in \pi\}$ where

(i) $\pi$ forms a set partition of $X$, and
We let $\mathcal{S}(X)$ denote the set of $X$--forests. We order the elements of $\mathcal{S}(X)$ by letting $\beta \leq \alpha$ if the trees in $\beta$ can be obtained from the trees in $\alpha$ by contracting certain edges, and deleting certain other edges, with any resulting unlabelled vertices of degree 2 being suppressed.

The poset $\mathcal{S}(X)$ was first defined (slightly differently) by Christopher Tuffley [16], and it is thus called the Tuffley poset on $X$. In Figure 1 we show the Hasse diagram of $\mathcal{S}(\{1, 2, 3\})$.

Given an $X$--forest $\alpha$ and an edge $e \in E(\alpha)$, we denote the $X$--forest obtained by contracting $e$ with $e^c(\alpha)$, and the $X$--forest obtained by deleting $e$ and suppressing any resulting vertices of degree 2 with $e^d(\alpha)$. From now on we will with an edge deletion always include the suppression of any degree two vertices. When $\alpha$ is clear from the context we will simply write $e^c$ and $e^d$. Furthermore,\[|E(e^c(\alpha))| = |E(\alpha)| - 1,\]and\[|E(\alpha)| - 3 \leq |E(e^d(\alpha))| \leq |E(\alpha)| - 1.\]

We will say that the edge deletion $\alpha \mapsto e^d(\alpha)$ is safe if $|E(e^d(\alpha))| = |E(\alpha)| - 1$. We say that a vertex in an $X$--tree is unsupported if it is unlabelled and of degree 3. We can easily conclude that for an $X$--forest $\alpha$, an edge deletion $\alpha \mapsto e^d(\alpha)$ is safe if and only if neither endpoint of the edge $e$ in $\alpha$ is unsupported.

We define an elementary operation on an element of $\mathcal{S}(X)$ to be either an edge contraction, or a safe edge deletion. The covering relation in the poset will be denoted by the symbol $\preceq$. 

Figure 1: The Tuffley poset $\mathcal{S}(X)$ for $X = \{1, 2, 3\}$.
The following result which is a restatement of Theorem 4.2 of [12] describes $S(X)$ in terms of these operations, and establishes some further structural properties of the Tuffley poset.

**Theorem 2.1.** Suppose that $X$ is a finite set and $\alpha, \beta \in S(X)$. Then the following statements hold.

(i) $\beta \leq \alpha$ if and only if $\beta$ can be obtained from $\alpha$ by any sequence of contraction and deletion operations, in which case we can insist that all contractions occur first, and that all the subsequent deletions are safe.

(ii) $\beta \lessdot \alpha$ if and only if $\beta$ can be obtained from $\alpha$ by one elementary operation.

(iii) $S(X)$ is a pure poset, and for an element $\alpha = \{(A, T_A) : A \in \pi\}$ of $S(X)$ its rank, denoted $\rho(\alpha)$, is given by

$$\rho(\alpha) = |E(\alpha)|.$$ 

(iv) $S(X) \cup \{\hat{0}\}$ is thin (that is, all intervals of length 2 contain exactly four elements).

(v) The maximal elements of $S(X)$ are precisely the binary $X$–trees with $|X|$ leaves. They have rank $2|X| - 3$.

(vi) The minimal elements of $S(X)$ are precisely the $X$–forests with no edges. Hence, there is a minimal element for each set partition of $X$.

(vii) Suppose $\alpha$ is an $X$–forest, and that $\alpha$ has an interior vertex $v$ labelled by $m$. Construct an $X'$–forest $\beta$ by removing $v$ from $\alpha$ and giving edge number $i$ incident with $v$, $1 \leq i \leq \deg(v)$, a new vertex $v_i$ that is labelled by $m_i$, $m_i \notin X$ and $m_i \neq m_j$ if $i \neq j$. Then $[\hat{0}, \alpha]$ is isomorphic to $[0, \beta]$.

### 3 Recursive coatom orderings

In this section we establish the following theorem.

**Theorem 3.1.** There is a recursive coatom ordering for each interval $[\hat{0}, \Gamma] \subset S(X) \cup \{\hat{0}\}$. In particular, every such interval is shellable.

The proof of Theorem 3.1 is provided in Section 3.3

**Corollary 3.2.** The order complex of any interval $(\alpha, \beta)$ in $S(X) \cup \{\hat{0}\}$ is homeomorphic to a sphere of dimension $\rho(\beta) - \rho(\alpha)$. In particular, the M"obius function $\mu$ of the Tuffley poset is given by $\mu(\alpha, \beta) = (-1)^{\rho(\beta) - \rho(\alpha)}$.

**Proof:** We know that intervals in $S(X) \cup \{\hat{0}\}$ are thin (see Theorem 2.1(iv)) and admit a recursive coatom ordering. The result now follows from [4, Theorem 4.7.24(ii)] (see page 13). The M"obius function is equal to the reduced Euler characteristic of the order complex of the open interval $(\alpha, \beta)$ (see [3, 9.14]). \qed
3.1 Preliminaries and definitions

Definition 3.3. A recursive coatom ordering for an interval $[\hat{0}, \Gamma]$ is an ordering $\alpha_1, \ldots, \alpha_t$ of its coatoms that satisfies the following two conditions:

(V1) For all $i < j$ and $\gamma < \alpha_i, \alpha_j$ there is a $k < j$ and an element $\beta$ such that $\beta < \alpha_k, \alpha_j$ and $\gamma \leq \beta$.

(V2) For all $j = 1, \ldots, t$, $[\hat{0}, \alpha_j]$ admits a recursive coatom ordering in which the coatoms that come first in the ordering are those that are covered by some $\alpha_k$ where $k < j$.

A recursive coatom ordering of an interval has several implications, see e.g. [3].

The following notation is used. The edges incident with a vertex $v$ in an $X$–forest will be denoted $e_1, \ldots, e_n$ (or $f_1, \ldots, f_n$). If the other vertex incident with an edge $e_i$ has degree 3, the two other edges incident with that vertex will be denoted $e_{i1}$ and $e_{i2}$, else $e_{i1}$ and $e_{i2}$ are not defined. See Figure 2. Remember that the coatom obtained from an $X$–forest $\Gamma$ by contraction of the edge $e_i$ is $e_i^c(\Gamma)$, and the coatom obtained by safe deletion of the edge $e_i$ is $e_i^d(\Gamma)$.

[equation]

Convention 3.4. $e_1, \ldots, e_n$ (or $f_1, \ldots, f_n$) are not in general fixed labels for the edges incident with a vertex $v$. This convention is made to avoid having to write $e_{i1}, \ldots, e_{in}$ where $i_1, \ldots, i_n$ is a permutation of $\{1, \ldots, n\}$. If for example some condition uses $e_1$ and $e_2$, it will be true for all $e_{i1}$ and $e_{i2}$ with the same properties. The same applies to numbered components of an $X$–forest. This concerns all sections in this text.

Convention 3.5. By Theorem 2.1 (vii) we may replace any $X$–forest $\alpha$ with labels on some internal vertices by a forest $\beta$ of trees without labels on internal vertices and lower interval $[\hat{0}, \beta]$ isomorphic to $[\hat{0}, \alpha]$. We may therefore throughout the proof of the existence of a recursive coatom ordering assume that no internal vertices are labelled. We may also during the proof without loss of generality assume that each leaf has exactly one label.

To prove that there exists a recursive coatom ordering $\alpha_1, \ldots, \alpha_t$ for $[\hat{0}, \Gamma]$, a new condition for coatom orderings called (V3) is used. The idea is to replace the conditions for a recursive coatom ordering with a stronger but easier condition. In Theorem 3.16 it will be proven that all coatom orderings satisfying the condition (V3) are recursive coatom orderings. The following two definitions are made to simplify condition (V3) that will be given in Definition 3.8, and to facilitate dealing with (V3) later.
Definition 3.6. Let $\Gamma$ be an $X$–forest, and $v$ an interior vertex of $\Gamma$ incident with the edges $e_1, \ldots, e_n$. Let $v_i$ be the other vertex incident with $e_i$, $1 \leq i \leq n$. To simplify notation, we use the following symbols for coatoms of $[0,\Gamma]$:

\[
\begin{align*}
\hat{c}^d_i(\Gamma) &= \begin{cases} 
 e^d_i(\Gamma) & \text{if } \deg(v_i) \neq 3, \deg(v) \geq 4 \\
 e^d_i(\Gamma) \text{ or } e^d_2(\Gamma) & \text{if } \deg(v_i) = 3
\end{cases} \\
\ddot{c}^d_i(\Gamma) &= \begin{cases} 
 e^d_i(\Gamma) & \text{if } \deg(v_i) \neq 3, \deg(v) \geq 4 \\
 \{e^c_i(\Gamma), e^c_2(\Gamma)\} & \text{if } \deg(v_i) = 3
\end{cases}
\end{align*}
\]

Note that $\hat{c}^d_i(\Gamma)$ and $\ddot{c}^d_i(\Gamma)$ are not defined if $\deg(v_i) \neq 3$, $\deg(v) = 3$. When no confusion arises, $\Gamma$ is often omitted.

Definition 3.7. Let $\mathcal{C}$ and $\mathcal{D}$ be disjoint sets of coatoms, and suppose there is a given coatom ordering. If all elements in $\mathcal{C}$ come before every element in $\mathcal{D}$ in the ordering, then $\mathcal{C} \preceq \mathcal{D}$. We will use $\mathcal{C} \not\preceq \mathcal{D}$ to denote the condition that at least one element in $\mathcal{C}$ comes after some element in $\mathcal{D}$.

Definition 3.8. Take an $X$–forest $\Gamma$, consisting of the non-trivial components $K_1, \ldots, K_m$ (i.e. with at least one edge each). If a coatom ordering of $[0,\Gamma]$ satisfies the following conditions, then it is said to satisfy the condition (V3) (Recall Convention 3.4):

\begin{enumerate}
\item[(V3)]
\begin{enumerate}
\item[a)] $\{\gamma | \gamma < \Gamma, \gamma = e^c$ or $\gamma = e^d \text{ where } e \in \bigcup_{i=1}^{\ell} K_i \} \not\preceq \{\gamma | \gamma < \Gamma, \gamma = e^c$ or $\gamma = e^d \text{ where } e \in \bigcup_{i=1}^{m-\ell} K_i \}$ for all $1 \leq \ell \leq m - 1$.
\item[b)] If $v$ is an interior vertex in $\Gamma$, $n = \deg(v) = 3$ and $\ddot{e}^d_3$ is defined then $\{e^c_1, e^c_2\} \not\preceq \{\ddot{e}^d_3\}$ and $\{\ddot{e}^d_3\} \not\preceq \{e^c_1, e^c_2\}$.
\item[c)] If $v$ is an interior vertex in $\Gamma$ and $n = \deg(v) \geq 4$ then $\{e^c_1, e^c_2\} \not\preceq \{\ddot{e}^d_3, \ddot{e}^d_4, \ldots, \ddot{e}^d_n\}$ and $\{\ddot{e}^d_3, \ddot{e}^d_4, \ldots, \ddot{e}^d_n\} \not\preceq \{e^c_1, e^c_2\}$.
\item[d)] If $v$ is an interior vertex in $\Gamma$ and $n = \deg(v) \geq 4$ then $\{e^c_1, e^c_2, \ldots, e^c_k, \ddot{e}^d_k\} \not\preceq \{e^c_{k+1}, e^c_{k+2}, \ldots, e^c_n, \ddot{e}^d_n\}$, where $1 \leq k \leq n - 1$.
\end{enumerate}
\end{enumerate}

The sub-conditions of (V3) are symmetric, hence the reversal of a coatom ordering satisfying (V3) also satisfies (V3).

Definition 3.9. Let $\Gamma$ be an $X$–forest and $v$ a vertex in $\Gamma$. The coatom $\alpha$ is said to be near $v$ if $\alpha$ is obtained by (safe) deletion or contraction of an edge $e_i$ incident with $v$ or contraction of $e_{i1}$ or $e_{i2}$.

The above definition is made since the sub-conditions (V3b), (V3c), and (V3d) only deal with the coatoms near an interior vertex $v$ in an X–forest.

### 3.2 Outline of proof of Theorem 3.1

To prove Theorem 3.1 the following method is used. A class of recursive coatom orderings is created. This class has the property that if $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_t$ is a coatom ordering in the class, then for each $1 \leq j \leq t$ the class contains a
coatom ordering for $[\hat{0}, \alpha_j]$ in which the coatoms that come first in the ordering are those that are covered by some $\alpha_k$ where $k < j$. In particular, the class is defined to be all coatom orderings satisfying the condition $(V3)$.

That property $(V1)$ of Definition 3.3 follows from $(V3)$ is shown in Lemma 3.10. The property which implies $(V2)$ of Definition 3.3 is shown in Lemma 3.12 and Theorem 3.15 with the help of Lemma 3.13 and Lemma 3.14. This is done by first finding certain orderings of the coatoms near each interior vertex in $\alpha_j$, and then combining them to a coatom ordering for $[\hat{0}, \alpha_j]$.

The results are put together in Theorem 3.16 to show that a coatom ordering satisfying $(V3)$ is a recursive coatom ordering. Finally Theorem 3.1 follows from Theorem 3.15 which implies that there is always a coatom ordering of $[\hat{0}, \Gamma]$ satisfying $(V3)$, and Theorem 3.16.

Since the proofs of Lemmas 3.10 – 3.14 are very technical, they will be presented separately in Section 5.

### 3.3 There is a recursive coatom ordering for $[\hat{0}, \Gamma]$}

The following lemma is obviously necessary, and will be proven in Section 5.1.

**Lemma 3.10.** Let $\Gamma$ be an $X$–forest, and let $\alpha_1, \ldots, \alpha_t$ be the coatoms of $[\hat{0}, \Gamma]$. If $\alpha_1, \ldots, \alpha_t$ satisfies $(V3)$, then it also satisfies part $(V1)$ of Definition 3.3.

To prove that a coatom ordering satisfying $(V3)$ also satisfies part $(V2)$ of Definition 3.3, Lemma 3.12 and Theorem 3.15 are needed.

The following definition is useful since $A \triangle B$ is a necessary condition for the possibility of ordering $A \cup B$ so that the ordering satisfies $A \bowtie B$ and $(V3)$.

**Definition 3.11.** Let $\Gamma$ be an $X$–forest, and let $A$ and $B$ be disjoint sets of coatoms of $[\hat{0}, \Gamma]$. Then $A$ and $B$ are said to be compatible with the condition $(V3)$ if $A \bowtie B$ is not forbidden by any single sub-condition of $(V3)$. If $A$ and $B$ are compatible with $(V3)$, we write $A \triangle B$. If $A$ and $B$ are not compatible with $(V3)$, we write $A \not\triangle B$. Since the condition $(V3)$ is symmetric, $A \triangle B \iff B \triangle A$.

**Lemma 3.12.** Let $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_t$ be a coatom ordering of $[\hat{0}, \Gamma]$ satisfying $(V3)$. Fix $j$, and consider the interval $[\hat{0}, \alpha_j]$. Let $A = \{ \gamma \mid \gamma < \alpha_k, \alpha_j \text{ and } k < j \}$ and let $B = \{ \gamma \mid \gamma < \alpha_k, \alpha_j \text{ and } k > j \}$ (The sets $A$ and $B$ are disjoint since $S(X) \cup \{ \hat{0} \}$ is thin). Then $A \triangle B$.

The above lemma is shown in Section 5.3. To prove Theorem 3.15 the following method is used. For each interior vertex a coatom ordering for the coatoms near that vertex is found, an ordering that satisfies $A \bowtie B$ and $(V3)$. Then these coatom orderings are combined to a coatom ordering of all coatoms. Hence these orderings must agree on the order of coatoms that are near more than one vertex, which we will now make sure by the following investigation and Lemmas 3.13 – 3.14.

Let $v_1$ and $v_2$ be adjacent interior vertices, and denote the edge between them $e_1$. Let the edges incident with $v_1$ be denoted $e_1, \ldots, e_n$. The coatoms near $v_1$
which are also near \( v_2 \) are the following.

- \( e_1^1, e_2^1, \) and \( e_3^1 \) if \( \text{deg}(v_1) = 3, \ \text{deg}(v_2) \geq 4 \)
- \( e_1^1, e_2^1, e_3^1, e_1^2, \) and \( e_1^3 \) if \( \text{deg}(v_1) = 3, \ \text{deg}(v_2) = 3 \)
- \( e_1^1, e_2^2, \) and \( e_3^1 \) if \( \text{deg}(v_1) \geq 4, \ \text{deg}(v_2) = 3 \)
- \( e_1^1, e_2^1, \) and \( e_1^2 \) if \( \text{deg}(v_1) \geq 4, \ \text{deg}(v_2) \geq 4 \)

If \( v_2 \) is adjacent to an interior vertex \( v_3 \) then some coatoms can be near both \( v_1 \) and \( v_3 \), but in that case these coatoms are near both \( v_1 \) and \( v_2 \) too.

**Lemma 3.13.** Let \( \alpha \) be an \( X \)-forest, and let \( v \) be an interior vertex of \( \alpha \) with degree 3. Suppose the coatoms near \( v \) are partitioned into two sets \( C \) and \( D \) where \( C \triangleleft D \). Then there is always an ordering of \( C \cup D \) that satisfies (V3) and \( C \triangleright D \), and has a given order of \( e_1^1, e_2^2, \) and \( e_1^3, e_1^2 \) (if they are defined) that is not forbidden by (V3b) or \( C \triangleleft D \).

**Lemma 3.14.** Let \( \alpha \) be an \( X \)-forest, and let \( v \) be an interior vertex of \( \alpha \) with \( \text{deg}(v) \geq 4 \). Suppose the coatoms near \( v \) are partitioned into two sets \( C \) and \( D \) where \( C \triangleleft D \). Then there is always an ordering of \( C \cup D \) that satisfies (V3) and \( C \triangleright D \), and has a given order of those of \( e_1^1 \) and \( e_2^2 \) that are in \( C \), and the same for \( D \).

Lemma 3.13 and Lemma 3.14 are proven in Section 5.4.

**Theorem 3.15.** Let \( \alpha \) be an \( X \)-forest, and let \( A \) and \( B \) be a disjoint bipartition of the coatoms of \( [0, \alpha] \). If \( A \triangleleft B \) then there is a coatom ordering for \( [0, \alpha] \) satisfying (V3) and \( A \triangleright B \).

**Proof:** Let \( A_v = \{ \gamma \in A \mid \gamma \text{ is near } v \} \) and let \( B_v = \{ \gamma \in B \mid \gamma \text{ is near } v \} \) for an interior vertex \( v \) in \( \alpha \). Then \( A \triangleleft B \) implies that \( A_v \triangleleft B_v \).

It is now possible to find a coatom ordering that satisfies (V3) in the following way. Choose a component of \( \alpha \), and then choose an interior vertex \( v_0 \) in that component. Take an ordering of the coatoms near that vertex that satisfies (V3) and \( A_{v_0} \triangleright B_{v_0} \). This is possible by Lemmas 3.13 and 3.14 with \( C = A_{v_0} \) and \( D = B_{v_0} \). Then for each of the vertices \( v_i, i \in I \), adjacent to the first vertex, choose an ordering of the coatoms near \( v_i \) that does not contradict the earlier chosen ordering and satisfies (V3) and \( A_{v_i} \triangleright B_{v_i} \). This is possible since the coatoms near two adjacent vertices are exactly those possible to choose order of in Lemma 3.13 and Lemma 3.14.

Then continue recursively in the same way with the vertices adjacent to them, until all interior vertices in the component are dealt with. It is now easy to combine the orderings for all interior vertices in the component to produce a coatom ordering that satisfies (V3) and does not contradict \( A \triangleright B \).

Do the same for each component. In the end, combine the orderings of the coatoms in the components by choosing one component that has at least two coatoms in \( A \) (if such a component exists) and put the first coatom of its ordering first in the ordering of \( A \) and the last coatom in \( A \) of its ordering last in the ordering of \( A \). Do the same with \( B \). Since \( A \triangleleft B \) this implies that the result is a coatom ordering of \( [0, \alpha] \) satisfying (V3) and \( A \triangleright B \). \( \square \)
Theorem 3.16. Let \( \Gamma \) be an \( X \)-forest, and let \( \alpha_1, \ldots, \alpha_t \) be a coatom ordering for \([\hat{0}, \Gamma]\). If \( \alpha_1, \ldots, \alpha_t \) satisfies \((V3)\), then it is a recursive coatom ordering.

Proof: Assume \( \alpha_1, \ldots, \alpha_t \) is a coatom ordering for \([\hat{0}, \Gamma]\) satisfying \((V3)\). It will be shown by induction over \( |E(\Gamma)| \) that \( \alpha_1, \ldots, \alpha_t \) is then a recursive coatom ordering. Remember that if \( \alpha \) is a coatom of \([\hat{0}, \Gamma]\) then \( |E(\alpha)| = |E(\Gamma)| - 1 \).

I) If \( |E(\Gamma)| \leq 1 \) there is only one or two coatoms. In both these cases all possible coatom orderings satisfy \((V3)\) and are recursive coatom orderings.

II) Assume that if \( 0 \leq |E(\Gamma)| \leq q \) \((q \geq 1)\), then every coatom ordering of \([\hat{0}, \Gamma]\) satisfying \((V3)\) is a recursive coatom ordering.

III) Take an \( X \)-forest \( \Gamma \) such that \( |E(\Gamma)| = q + 1 \). Let \( \alpha_1, \ldots, \alpha_t \) be a coatom ordering for \([\hat{0}, \Gamma]\) satisfying \((V3)\).

Lemma 3.10 implies that \((V3) \Rightarrow (V1)\).

Fix \( j, 1 \leq j \leq t \). Let \( A = \{ \gamma \mid \gamma \succ \alpha_k, \alpha_j \text{ and } k < j \} \). Since \( S(X) \cup \{\hat{0}\} \) is thin, the remaining coatoms in \([\hat{0}, \alpha_j]\) are \( B = \{ \gamma \mid \gamma \succ \alpha_k, \alpha_j \text{ and } k > j \} \).

From Lemma 3.12 and Theorem 3.15 it now follows that there is a coatom ordering for \([\hat{0}, \alpha_j]\) satisfying \((V3)\) and \( A \triangleleft B \). By the induction assumption it is a recursive coatom ordering. Hence \((V2)\) is also satisfied. Thus \( \alpha_1, \ldots, \alpha_t \) is a recursive coatom ordering for \([\hat{0}, \Gamma]\).

\(\square\)

Proof of Theorem 3.1: Let \( \Gamma \) be an \( X \)-forest. The existence of a coatom ordering satisfying \((V3)\) for \([0, \Gamma] \subseteq S(X) \cup \{\hat{0}\}\) is implied by Theorem 3.15 with \( \alpha = \Gamma, A = \emptyset, \text{ and } B = \{ \gamma \mid \gamma \prec \Gamma \} \). That this coatom ordering is a recursive coatom ordering for \([0, \Gamma]\) follows from Theorem 3.16. \(\square\)

3.4 An example of a coatom ordering satisfying \((V3)\)

Figure 3 shows an \( X \)-forest \( \Gamma \). The coatom ordering of \([0, \Gamma] \) given by \( g_1^c e_1^d e_2^d e_3^d g_4^d g_5^d f_1^d g_6^d e_4^d e_5^d f_2^d g_7^d f_3^d f_4^d f_5^d \) satisfies \((V3)\), and thus it is a recursive coatom ordering.

![Figure 3: The X-forest Γ](image)
4 The edge-product space is a regular cell complex

In this section we will assume that the reader is familiar with basic concepts of point-set topology. We will also make use of some purely topological results that for convenience we state in the Appendix.

To an \( X \)-tree \( T \), we associate the closed ball \( B(T) = [0, 1]^{E(T)} \) and open ball \( \text{Int}(B(T)) = (0, 1)^{E(T)} \). More generally, for an \( X \)-forest \( \alpha = \{ (A, T_A) : A \in \pi \} \), we let \( B(\alpha) = \prod_{A \in \pi} B(T_A) \) and let \( \text{Int}(B(\alpha)) = \prod_{A \in \pi} \text{Int}(B(T_A)) \). Note that \( B(\alpha) \) (respectively \( \text{Int}(B(\alpha)) \)) is homeomorphic to a closed (respectively open) ball of dimension \( \sum_{A \in \pi} |E(T_A)| \) and accordingly we will refer to this quantity as the dimension of \( \alpha \), denoted \( \dim(\alpha) \).

Given an \( X \)-tree \( T = (T; \phi) \) and map \( \lambda : E(T) \to [0, 1] \) define \( p_{(T, \lambda)} : \binom{X}{2} \to [0, 1] \) by setting
\[
p_{(T, \lambda)}(x, y) = \prod_{e \in p(T; \phi(x), \phi(y))} \lambda(e),
\]
where the empty product is taken as 1.

We can extend the correspondence \( \lambda \mapsto p_{(T, \lambda)} \) to \( X \)-forests as follows. Given an \( X \)-forest \( \alpha = \{ (A, T_A) : A \in \pi \} \) let \( \psi_\alpha : B(\alpha) \to [0, 1][\binom{X}{2}] \) be defined by setting, for \( \lambda = (\lambda_A : A \in \pi) \),
\[
\psi_\alpha(\lambda)(x, y) = \begin{cases} p_{(T_A, \lambda_A)}(x, y), & \text{if } \exists A \in \pi \text{ with } x, y \in A; \\ 0, & \text{otherwise.} \end{cases}
\]

We begin by proving two useful lemmas. Let \( \delta(B(\alpha)) \) denote the boundary of the ball \( B(\alpha) \). The following lemma describes a useful property of the map \( \psi_\alpha \).

**Lemma 4.1.** Let \( \alpha = \{ (A, T_A) : A \in \pi \} \) be an \( X \)-forest. Then,
\[
\psi_\alpha(\text{Int}(B(\alpha))) \cap \psi_\alpha(\delta(B(\alpha))) = \emptyset.
\]

**Proof:** Suppose \( \psi_\alpha(\text{Int}(B(\alpha))) \cap \psi_\alpha(\delta(B(\alpha))) \neq \emptyset \) - we will show that this leads to contradictions. This assumption implies that for some \( \lambda_1 \in \text{Int}(B(\alpha)) \), and \( \lambda_2 \in \delta(B(\alpha)) \) we have \( \psi_\alpha(\lambda_1)(x, y) = \psi_\alpha(\lambda_2)(x, y) \) for all \( x, y \in X \). Now, if there exists an edge \( e \) of \( \alpha \) with \( \lambda_2(e) = 0 \) then select a pair \( x, y \in X \) that are separated by \( e \) but contained in the same component of \( \alpha \). Then, \( \psi_\alpha(\lambda_1)(x, y) = \psi_\alpha(\lambda_2)(x, y) = 0 \), and this implies that \( \lambda_1 \in \delta(B(\alpha)) \), a contradiction. Thus we may suppose that for every edge \( e \) of \( \alpha \) we have \( \lambda_2(e) > 0 \) and so therefore also \( \psi_\alpha(\lambda_1)(x, y) = \psi_\alpha(\lambda_2)(x, y) > 0 \) for all \( x, y \) that belong to any component tree \( T_A \) of \( \alpha \). Now if we let \( d_i(x, y) := -\log(\psi_\alpha(\lambda_i))(x, y) \) for all \( x, y \) in \( T_A \), then \( d_i \) describes, for each pair \( x, y \in X \), the sum of the real-valued weights \( -\log(\lambda_i)(e) \) over all edges \( e \) of \( T_A \) that separate \( x \) and \( y \) (i.e. \( d_i \) is a distance function on \( X \) induced by this edge weighting). Now, it is a well-known and easily established result that two edge weightings of an \( X \)-tree induce the same distance function on \( X \) if and only if the two edge weightings are identical (see e.g. Lemma 2.2(i))
of [12]). Consequently, since \(d_1 = d_2\) it follows that \(\lambda_1\) and \(\lambda_2\) agree on each edge of \(T_A\). Since this applies for each component \(A \in \pi\) it follows that \(\lambda_1 = \lambda_2\). But this is impossible since \(\lambda_1 \in \text{Int}(\mathcal{B}(\alpha))\) and \(\lambda_2 \in \delta(\mathcal{B}(\alpha))\). \(\Box\)

Using the following lemma we will later be able to restrict our attention to trees as opposed to forests.

**Lemma 4.2.** If \(\alpha = \{(A, T_A) : A \in \pi\}\) then

\[
\psi_\alpha(\mathcal{B}(\alpha)) \cong \prod_{A \in \pi} (\psi(\mathcal{B}(T_A))).
\]

**Proof:** This follows from Lemma A.3 of the Appendix, taking \(I = \pi\), and for \(A \in I\), \(Z_A = \mathcal{B}(T_A)\), and for \(\lambda, \lambda' \in \mathcal{B}(T_A)\), writing \(\lambda R_A \lambda'\) if and only if \(p(T, \lambda|A) = p(T_A, \lambda'|A)\). \(\Box\)

We now recall the definition of a regular cell complex. In [3, Section 12.4] it states that a family of balls (homeomorphs of \(B^d, d \geq 0\)) in a Hausdorff space \(Y\) is a set of closed balls of a *regular cell complex* if and only if the interiors of the balls partition \(Y\) and the boundary of each ball is a union of other balls.

Consider the set

\[
\mathcal{C} := \{\psi_\alpha(\mathcal{B}(\alpha)) : \alpha \in \mathcal{S}(X)\}.
\]

We claim that this forms a set of closed balls of a regular cell complex (decomposition of \(\mathcal{E}(X)\)) where the boundary of each ball \(\psi_\alpha(\mathcal{B}(\alpha))\), denoted \(\delta(\psi_\alpha(\mathcal{B}(\alpha)))\), is defined by \(\delta(\psi_\alpha(\mathcal{B}(\alpha))) := \psi_\alpha(\delta(\mathcal{B}(\alpha)))\) (so that, by Lemma 4.1, the interior of each ball \(\psi_\alpha(\mathcal{B}(\alpha))\) is given by \(\text{Int}(\psi_\alpha(\mathcal{B}(\alpha))) = \psi_\alpha(\mathcal{B}(\alpha)) - \delta(\psi_\alpha(\mathcal{B}(\alpha))) = \psi_\alpha(\text{Int}(\mathcal{B}(\alpha)))\)).

To help prove our claim we first present a proposition that is a reformulation of some results appearing in [12]. Let

\[
\mathcal{S}(X)_{<\alpha} := \{\beta \in \mathcal{S}(X) : \beta < \alpha\}.
\]

**Proposition 4.3.** The following statements hold:

(i) \(\mathcal{E}(X)\) is the disjoint union of the elements of

\[
\{\psi_\alpha(\text{Int}(\mathcal{B}(\alpha))) : \alpha \in \mathcal{S}(X)\}.
\]

(ii) For \(\alpha \in \mathcal{S}(X)\), \(\delta(\psi_\alpha(\mathcal{B}(\alpha)))\) is the union of the elements of

\[
\{\psi_\beta(\mathcal{B}(\beta)) : \beta \in \mathcal{S}(X)_{<\alpha}\},
\]

and the disjoint union of the elements of

\[
\{\psi_\beta(\text{Int}(\mathcal{B}(\beta))) : \beta \in \mathcal{S}(X)_{<\alpha}\}.
\]

(iii) If \(\alpha\) is an \(X\)-tree, then for each \(y \in \psi_\alpha(\mathcal{B}(\alpha))\), \(\psi_\alpha^{-1}(y)\) is a contractible regular cell complex.
Proof: Part (i) and (ii) follow from [12, Theorem 3.3] and the definition of \( \delta(\psi_\alpha(B(\alpha))) \). Part (iii) is [12, Proposition 6.5].

By Proposition 4.3(i), the interiors of the elements of \( \mathcal{C} \) partition \( \mathcal{E}(X) \), and by Proposition 4.3(ii) the boundary of each element of \( \mathcal{C} \) is equal to the union of other elements in \( \mathcal{C} \). Hence to show that \( \mathcal{C} \) is the set of closed balls of a regular cell complex it suffices to prove the following.

**Theorem 4.4.** For all \( \alpha \in \mathcal{S}(X) \), the set \( \psi_\alpha(B(\alpha)) \) is homeomorphic to \( [0,1]^{\dim(\alpha)} \).

Proof: By Lemma 4.2 it suffices to prove the theorem for \( \alpha \in \mathcal{S}(X) \) an \( X \)-tree.

To prove the theorem we use induction on \( \dim(\alpha) \). It can easily be checked that the result holds for \( \dim(\alpha) = 0, 1, 2, 3 \).

Now suppose that \( d := \dim(\alpha) > 3 \), and that \( \psi_\beta(B(\beta)) \) is homeomorphic to \( [0,1]^{\dim(\beta)} \) for all \( \beta \in \mathcal{S}(X) \) such that \( \dim(\beta) < d \).

By Proposition 4.3(ii) and the inductive hypothesis, \( \delta(\psi_\alpha(B(\alpha))) \) is a regular cell complex, with set of closed balls equal to \( \{ \psi_\beta(B(\beta)) : \beta \in \mathcal{S}(X)_{<\alpha} \} \).

Moreover, this complex has face poset isomorphic to \( (\mathcal{S}(X)_{<\alpha}, \leq) \) (cf. [12, Theorem 3.3]). By Theorem 2.1 the poset \( [\hat{0}, \alpha] \) obtained by adding a minimal and a maximal element to \( (\mathcal{S}(X)_{<\alpha}, \leq) \) is thin and graded (graded means pure with a unique minimal and maximal element) with length \( d + 1 \), and by Theorem 3.1 \( [\hat{0}, \alpha] \) has a recursive coatom ordering. It follows by [4, Theorem 4.7.24(i)] \(^1\) that \( \psi_\alpha(\delta(B(\alpha))) \) is homeomorphic to \( \delta([0,1]^d) \), the \((d-1)\)-dimensional sphere.

It now follows that the set \( \psi_\alpha(B(\alpha)) \) is homeomorphic to \( [0,1]^d \) by applying Proposition 4.3(iii) together with Corollary A.2 of the Appendix with \( g = \psi_\alpha \), \( B = B(\alpha) \), and \( Z = \psi_\alpha(B(\alpha)) \).

5 Proof of some combinatorial lemmas

In this section we give the proofs of Lemmas 3.10 – 3.14. Since many of the proofs require cases that are straight-forward to check (but quite detailed to write out), when appropriate we will refer the reader to [8] where the full details are presented.

5.1 Reformulation of \((V1)\) with implications

In this section we prove Lemma 3.10.

\(^1\)Denoting the face poset of a cell complex \( \Delta \) by \( \mathcal{F}(\Delta) \), this theorem states that if \( P \) is a graded poset of length \( d + 2 \), then \( P \cong \mathcal{F}(\Delta) \cup \{ \hat{0}, \hat{1} \} \) for some shellable regular cell decomposition \( \Delta \) of the \( d \)-sphere if and only if \( P \) is thin and admits a recursive coatom ordering.
Recall part (V1) of Definition 3.3: For all i < j and γ < α_i, α_j there is a k < j and an element β such that γ ≤ β < α_k, α_j.

An equivalent formulation is:

For each interior vertex v of Γ, where deg(v) = n, the following conditions apply (recall Convention 3.4):

\[ V1a) \{ e_1^e, e_2^e \} \neq \{ e_3^e, e_4^e \} \quad \text{when } n = 3, e_{31}, e_{32} \text{ are defined}; \]
\[ V1b) \{ e_1^e, e_2^e \} \neq \{ e_3^e, \ldots, e_n^e \} \quad \text{when } n \geq 4; \]
\[ V1c) \{ e_1^e, e_2^e \} \neq \{ e_3^e, \ldots, e_n^e \} \quad \text{when } n = 4, e_1^d, e_2^d \not\in \Gamma; \]
\[ V1d) e_1^d e_2^d \notin \{ e_3^e, \ldots, e_n^e \} \quad \text{when } n \geq 4, e_1^d \leq \Gamma; \]
\[ V1e) e_1^d e_2^d \notin \{ e_3^d, \ldots, e_n^d \} \quad \text{when } n \geq 4, e_1^d \leq \Gamma; \]
\[ V1f) \quad \text{Furthermore, if there is a component } K \text{ in } \Gamma \text{ with only one edge } e, \text{ then the coatoms } e^e \text{ and } e^d \text{ are not the two first coatoms in the ordering.} \]

That the new formulation of (V1) is equivalent to the original one, follows from the investigation of common elements of \([0, \alpha_1]\) and \([0, \alpha_2]\) which is made in Section 5.2.

**Proof of Lemma 3.10:** It is easy to show that (V3b) ⇒ (V1a), (V3c) ⇒ (V1b), (V3c) ⇒ (V1c), (V3d) ⇒ (V1c), and (V3a) ⇒ (V1f). Thus it only remains to show that (V1d) holds. Suppose the coatom ordering satisfies (V3) but not (V1d). Then some \(e_i^e\), \(2 \leq i \leq n\), has to come before \(e_i^e\) in the ordering because of (V3d). But then \(\{ e_1^e, e_i^e \} \cap \{ e_2^d, \ldots, e_n^d \}\), which contradicts (V3c). Hence (V3c) and (V3d) imply (V1d). \[\square\]

## 5.2 Common elements of \([0, \alpha_1]\) and \([0, \alpha_2]\)

Let \(\Gamma\) be an X–forest, and let \(\alpha_1\) and \(\alpha_2\) be different coatoms of \([0, \Gamma]\). To reformulate the condition (V1) it is important to find the common elements of \([0, \alpha_1]\) and \([0, \alpha_2]\). For every pair \(\alpha_1, \alpha_2\) there is either some \(\delta\) such that \([0, \alpha_1] \cap [0, \alpha_2] = [0, \delta_1]\) or \(\delta_1\) and \(\delta_2\) such that \([0, \alpha_1] \cap [0, \alpha_2] = [0, \delta_1] \cup [0, \delta_2]\) (see Lemma 5.1).

To reformulate (V1) we also need to find all \(\beta \in S(X)\) such that \(\delta_i < \beta < \alpha_j\) when \(\delta_i\) is not covered by \(\alpha_j\), \(i = 1, 2, j = 1, 2\).

From Lemma 5.1 it now follows that the first and second formulation of (V1) are equivalent. This lemma is also needed in the proofs of Lemma 5.3 and Lemma 5.12.

**Lemma 5.1.** Let \(\Gamma\) be an X–forest, and let \(\alpha_1\) and \(\alpha_2\) be distinct coatoms of \([0, \Gamma]\). For every pair of \(\alpha_1\) and \(\alpha_2\), Table 1 gives \(\delta\) such that \([0, \alpha_1] \cap [0, \alpha_2] = [0, \delta]\) or \(\delta_1\) and \(\delta_2\) such that \([0, \alpha_1] \cap [0, \alpha_2] = [0, \delta_1] \cup [0, \delta_2]\). Furthermore, Table 1 (column 3) states whether or not \(\delta_i\) is covered by \(\alpha_1\) and \(\alpha_2\), and if not, column 4 gives \(\beta\) such that \(\delta_i < \beta < \alpha_j\) for \(j = 1, 2\).

If nothing else is specified, all operations are made on \(\Gamma\). This means that for example \(f_1 f_2^d = f_1 (f_2^d(\Gamma))\). Let \(e_1, \ldots, e_n\) be the edges incident with a
vertex \( v_0 \) in \( \Gamma \). Let the other vertex incident with \( e_i \) be \( v_i \), and let \( m_i = \deg(v_i) \). If \( m_1 \geq 3 \), let \( f_2, \ldots, f_m \) be the other edges incident with \( v_1 \). Let \( g \) be an edge of \( \Gamma \) that is not adjacent to \( e_1 \).

Let \( \alpha = \{ e_1, e_2, \ldots, e_m \} \) be a coatom ordering of \([0, \Gamma]\) satisfying (V3). Fix \( j \), and consider the interval \([0, \alpha_j]\). Let \( A = \{ \gamma | \gamma < \alpha_k, \alpha_j \text{ and } k < j \} \) and let \( B = \{ \gamma | \gamma < \alpha_k, \alpha_j \text{ and } k > j \} \). The sets \( A \) and \( B \) are disjoint since \( S(X) \cup \{0\} \) is thin.

**Lemma 5.3.** If \( v \) is an interior vertex of \( \alpha_j \), \( A_v = \{ \gamma \in A | \gamma \text{ is near } v \} \), and \( B_v = \{ \gamma \in B | \gamma \text{ is near } v \} \), then \( A_v \cap B_v \).

**Proof:** Let \( A' = \{ \beta \in \Gamma | \gamma < \beta \text{ and } \gamma \in A_v \} \) and \( B' = \{ \beta \in \Gamma | \gamma < \beta \text{ and } \gamma \in B_v \} \). Then \( A' \subseteq \{ \alpha_1, \ldots, \alpha_{j-1} \} \) and \( B' \subseteq \{ \alpha_{j+1}, \ldots, \alpha_t \} \). This lemma will be proven by assuming \( A_v \cap B_v \) for some interior vertex \( v \) in \( \alpha_j \), and then deducing that \( \alpha_1, \ldots, \alpha_t \) does not satisfy (V3) (which is a contradiction) by showing that

<table>
<thead>
<tr>
<th>( \alpha_1, \alpha_2 )</th>
<th>( \delta ) or ( \delta_1, \delta_2 )</th>
<th>( \text{Is } \delta &lt; \alpha_1, \alpha_2 ? )</th>
<th>( \delta &lt; \beta &lt; \alpha_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_1^d, e_2^d )</td>
<td>( \delta = g^d e_1^d )</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( e_1^d, e_2^d )</td>
<td>( \delta = g^d e_2^d )</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( e_1^d, e_2^d )</td>
<td>( \delta = g^d e_3^d )</td>
<td>yes</td>
<td>yes</td>
</tr>
<tr>
<td>( e_1^d, e_2^d )</td>
<td>( \delta = 0 \text{ if } n = m_1 = 1 )</td>
<td>if (</td>
<td>E(\Gamma)</td>
</tr>
<tr>
<td>( e_1^d, e_2^d )</td>
<td>( \delta = e_2^d \ldots e_n^d \text{ if } n \geq 4, m_1 = 1 )</td>
<td>no</td>
<td>no</td>
</tr>
<tr>
<td>( e_1^d, e_2^d )</td>
<td>( \delta = e_2^d \ldots e_n^d \text{ if } n \geq 4, m_1 = 1 )</td>
<td>no</td>
<td>no</td>
</tr>
</tbody>
</table>

Table 1: Description of elements less than both \( \alpha_1 \) and \( \alpha_2 \)

**Proof:** With the help of the definition of \( \leq \) it is rather straight-forward to obtain the results in Table 1.

### 5.3 \( A \) and \( B \) are compatible with (V3)

The following definition is used in Sections 5.4 and 5.3.

**Definition 5.2.** Let \( \Gamma \) be an \( X \)-forest and \( e \) an edge in \( \Gamma \), and let \( \alpha \) be \( e^c(\Gamma) \), \( e^d(\Gamma) \), \( e^d(\Gamma) \) or \( \hat{e}^d(\Gamma) \). The symbol \( \langle \alpha \rangle \) denotes \( \alpha \) if \( \alpha \) is defined and \( \alpha < \Gamma \).

In this section we prove Lemma 3.12. To do this, we require some preliminary results.

Let \( \alpha_1, \ldots, \alpha_j, \ldots, \alpha_t \) be a coatom ordering of \([0, \Gamma]\) satisfying (V3). Fix \( j \), and consider the interval \([0, \alpha_j]\). Let \( A = \{ \gamma | \gamma < \alpha_k, \alpha_j \text{ and } k < j \} \) and let \( B = \{ \gamma | \gamma < \alpha_k, \alpha_j \text{ and } k > j \} \). The sets \( A \) and \( B \) are disjoint since \( S(X) \cup \{0\} \) is thin.

**Lemma 5.3.** If \( v \) is an interior vertex of \( \alpha_j \), \( A_v = \{ \gamma \in A | \gamma \text{ is near } v \} \), and \( B_v = \{ \gamma \in B | \gamma \text{ is near } v \} \), then \( A_v \cap B_v \).

**Proof:** Let \( A' = \{ \beta \in \Gamma | \gamma < \beta \text{ and } \gamma \in A_v \} \) and \( B' = \{ \beta \in \Gamma | \gamma < \beta \text{ and } \gamma \in B_v \} \). Then \( A' \subseteq \{ \alpha_1, \ldots, \alpha_{j-1} \} \) and \( B' \subseteq \{ \alpha_{j+1}, \ldots, \alpha_t \} \). This lemma will be proven by assuming \( A_v \cap B_v \) for some interior vertex \( v \) in \( \alpha_j \), and then deducing that \( \alpha_1, \ldots, \alpha_t \) does not satisfy (V3) (which is a contradiction) by showing that
\( A' \triangleleft \{ \alpha_j \} \triangleleft B' \) is forbidden by one of the sub-conditions (V3b), (V3c), and (V3d).

First, note that since all coatoms near \( v \) are obtained by contracting or deleting edges in the same component of \( \alpha_j \) and (V3) is symmetric, \( A_0 \triangleleft B_0 \) implies that \( A_0 \triangleleft B_0 \) is forbidden by one of the sub-conditions (V3b), (V3c), and (V3d).

To simplify the proof, we use the following notation. Let \( C \subseteq A_0 \) and \( D \subseteq B_0 \). Then \( C \triangleleft D \iff C' \triangleleft \{ \alpha_j \} \triangleleft D' \) means that \( C' = \{ \beta \triangleleft \gamma \triangleleft \alpha_j, \beta \text{ where } \gamma \in C \} \) and \( D' = \{ \beta \triangleleft \gamma \triangleleft \alpha_j, \beta \text{ where } \gamma \in D \} \). Since \( S(X) \cup \{ 0 \} \) is thin, \( C' \) and \( D' \) are well defined and are always disjoint. Observe that if \( C \triangleleft D \iff C' \triangleleft \{ \alpha \} \triangleleft D' \), then \( D \triangleleft C \iff D' \triangleleft \{ \alpha \} \triangleleft C' \). Since (V3) is symmetric it is not necessary to check the reverse of any condition.

Based on the result in Lemma 5.1, we can divide the rest of the proof into cases as below. For each case, (V3b), (V3c), and (V3d) are checked.

The interior vertex \( v \) in \( \alpha_j \) corresponds to one interior vertex \( v' \) or two interior vertices \( v'_1 \) and \( v'_2 \) in \( \Gamma \). Let the edges incident with \( v' \) or \( v'_1 \) be denoted \( e_0, ..., e_n \), and let the edges incident with \( v'_2 \) be denoted \( f_0 = e_0, f_1, ..., f_m \).

**Case 1:** The coatom \( \alpha_j \) is obtained from \( \Gamma \) by contracting the edge \( e_0 \) incident with \( v'_1 \) and \( v'_2 \), where \( \text{deg}(v'_1) \geq 3 \) and \( \text{deg}(v'_2) \geq 3 \). This case is illustrated in Figure 4, where \( n, m \geq 2 \).

**Case 2:** The coatom \( \alpha_j \) is obtained from \( \Gamma \) by deleting the edge \( e_0 \) incident with \( v' \), where \( \text{deg}(v') \geq 4 \).

We will deduce the desired contradiction for the following case. Suppose \( m \geq 3 \) and that \( A_0 \triangleleft B_0 \) is forbidden by (V3c). If \( e_1, e_2, ..., e_n, f_1, ..., f_m \in B_0 \), then by Lemma 5.1 \( \{ e_1, e_2 \} \triangleleft \{ e_1, e_2, ..., e_n, f_1, ..., f_m \} \iff \{ e_1, e_2 \} \triangleleft \{ e_1, e_2, ..., e_n, f_1, ..., f_m \} \iff \{ e_1, e_2, f_1, ..., f_m \} \triangleleft \{ e_1, e_2, f_1, ..., f_m \} \). If \( f_0 \triangleleft \Gamma \), then (V3c) implies that \( \{ e_1, e_2, f_0 \} \triangleleft \{ e_1, e_2, f_0 \} \). If \( f_0 \triangleleft \Gamma \), then \( n = 2 \) and \( f_0 = \{ e_1, e_2 \} \). This means that \( \{ f_0, f_0 \} \triangleleft \{ f_1, f_1, ..., f_m, f_m \} \) which is forbidden by (V3c). This is a contradiction since \( \alpha_1, ..., \alpha_t \) satisfies (V3).

The other cases are dealt with similarly (see [8, Section 5.2] for details).

**Case 2:** The coatom \( \alpha_j \) is obtained from \( \Gamma \) by deleting the edge \( e_0 \) incident with \( v' \), where \( \text{deg}(v') \geq 4 \).

We deduce the desired contradiction for the following case. Suppose \( n \geq 4 \) and that \( A_0 \triangleleft B_0 \) is forbidden by (V3d). Then for some \( 1 \leq k \leq n - 1 \) \( e_1, e_2, ..., e_k, e_{k+1}, e_{k+2}, ..., e_n, f_{k+1}, ..., f_m \in B_0 \). By Lemma 5.1 we have \( \{ e_1, e_2, ..., e_k, e_{k+1}, e_{k+2}, ..., e_n, f_{k+1}, ..., f_m \} \iff \{ e_1, e_2, ..., e_k, e_{k+1}, e_{k+2}, ..., e_n, f_{k+1}, ..., f_m \} \). Since \( e_0 \triangleleft \Gamma \), \( e_0 \) has to come before or after \( e_{k+1} \) in the coatom ordering. This implies that \( \{ e_0, e_1, ..., e_k, f_{k+1}, ..., f_m \} \triangleleft \{ e_0, e_0, e_{k+1}, e_{k+2}, ..., e_n, e_{k+1}, ..., e_n, f_{k+1}, ..., f_m \} \) which are both forbidden.
by \((V3d)\). This is a contradiction since \(\alpha_1, \ldots, \alpha_t\) satisfies \((V3)\).

The other cases are dealt with similarly (see [8, Section 5.2] for details).

The remaining cases concern coatoms \(\alpha_j\) obtained from \(\Gamma\) by contracting or deleting the edge \(g\) not incident with \(v'\). In this situation \(\deg(v) = \deg(v')\).

Let \(e_0\) be incident with \(v'\) and \(v'_0\) in \(\Gamma\). If \(g\) is not adjacent to any edge \(e_i\), then the coatoms near \(v\) are obtained from \(\alpha_j\) by the same operations as the coatoms near \(v'\) in \(\Gamma\). Furthermore, if a coatom \(\gamma\) near \(v\) is obtained from \(\alpha_j\) by a certain operation, then \(\gamma\) is covered by the coatom near \(v'\) which is obtained from \(\Gamma\) by the same operation. Hence, if \(A_e \triangleleft B_e\) is forbidden by one of \((V3b)\), \((V3c)\), and \((V3d)\), then \(A \triangleleft B\) obviously is forbidden by the same condition. This also applies when \(g\) is adjacent to \(e_0\), if \(\alpha_j\) is obtained from \(\Gamma\) by contracting \(g\) and \(\deg(v'_0) \geq 4\), or if \(\alpha_j\) is obtained from \(\Gamma\) by deleting \(g\) and \(\deg(v'_0) \geq 5\). There are now two cases left.

\textbf{Case 3:} The coatom \(\alpha_j\) is obtained from \(\Gamma\) by contracting the edge \(g\) adjacent to \(e_0\), where \(\deg(v'_0) = 3\).

The conditions \((V3b)\), \((V3c)\) and \((V3d)\) can be checked in a similar way to Cases 1 and 2. Details can be found in [8, Section 5.2].

\textbf{Case 4:} The coatom \(\alpha_j\) is obtained from \(\Gamma\) by deleting the edge \(g\) adjacent to \(e_0\), where \(\deg(v'_0) = 4\).

The conditions \((V3b)\), \((V3c)\) and \((V3d)\) can be checked in a similar fashion to Cases 1 and 2. Details can be found in [8, Section 5.2].

By checking the conditions for all cases it is found that \(A_e \triangleleft B_e\) is not forbidden by any of the conditions \((V3b)\), \((V3c)\), and \((V3d)\). Thus \(A_e \triangleright B_e\) for every interior vertex \(v\) in \(\alpha_j\).

\(\square\)

We will now prove that \(A \triangleleft B\) is not forbidden by the condition \((V3a)\) in Lemma 5.5 and Lemma 5.6. The following notation will be helpful.

\textbf{Definition 5.4.} If \(\gamma\) is a coatom of \([\hat{0}, \beta]\) which is obtained by contraction or deletion of an edge of the component \(K\) in \(\beta\), then we write \(\gamma \in K\).

Recall Convention 3.4. The \(X\)-forest \(\alpha_j\) is obtained from \(\Gamma\) by contracting or deleting an edge \(e_0\). Sometimes contraction or deletion of an edge results in splitting of a component in two or more parts (recall Convention 3.5). Let \(K_1, \ldots, K_{t+1}\) be the non-trivial components in \(\Gamma\), and suppose \(e_0 \in K_{t+1}\). Thus \(\alpha_j\) has the non-trivial components \(K_1, \ldots, K_t, K'_1, \ldots, K'_s\) where \(s \geq 0\). Let \(v_1\) and \(v_2\) be the vertices incident with \(e_0\), let \(e_0, \ldots, e_m\) be the edges incident with \(v_1\), and let \(e_0, f_1, \ldots, f_m\) be the edges incident with \(v_2\). Then \(\deg(v_1) = n + 1\) and \(\deg(v_2) = m + 1\).

\textbf{Lemma 5.5.} If \(s \geq 2\), then \(\{\gamma \leq \alpha_j \mid \gamma \in \bigcup_{i=1}^{t} K'_i\} \not\triangleleft \{\gamma \leq \alpha_j \mid \gamma \in \bigcup_{i=k+1}^{s} K'_i\}\) for all \(1 \leq k \leq s - 1\).

\textbf{Proof:} Suppose \(\{\gamma \mid \gamma \in \bigcup_{i=1}^{t} K'_i\} \not\triangleleft \{\gamma \mid \gamma \in \bigcup_{i=k+1}^{s} K'_i\}\) for some \(1 \leq k \leq s - 1\). We show that \(\alpha_1, \ldots, \alpha_t\) does not satisfy \((V3)\), a contradiction from which the lemma follows. The following two cases arise. The details are straight-forward
and similar to those in Lemma 5.3. They will be omitted here, but can be found in [8, Section 5.3].

**Case 1:** $v_1$ is unlabelled, $v_2$ is labelled, and $\alpha_j = e_0^j(\Gamma)$. By Convention 3.5 it follows that $s = n$, and in $\alpha_j$ that $e_i \in K'_i$ and $e_i$ is incident with a leaf for $1 \leq i \leq n$.

It is easy to show that $\{e_1^c, e_1^d, \ldots, e_k^c, e_k^d\} \subset \{e_{k+1}^c, e_{k+1}^d, \ldots, e_n^c, e_n^d\}$ implies that $\alpha_1, \ldots, \alpha_t$ does not satisfy (V3).

**Case 2:** The vertices $v_1$ and $v_2$ are unlabelled and $\alpha_j = e_0^j(\Gamma)$. Hence $s = 2$.

Using $\{e_1^c, (e_1^d), \ldots, e_n^c, (e_n^d)\} \subset \{f_1^c, \langle \tilde{f}_1^d \rangle, \ldots, f_m^c, \langle \tilde{f}_m^d \rangle\}$, it is easy to show that $\alpha_1, \ldots, \alpha_t$ does not satisfy (V3). □

**Lemma 5.6.** \{\gamma \prec \alpha_j \mid \gamma \in \bigcup_{i=1}^{k} K_i\} \not\subset \{\gamma \prec \alpha_j \mid \gamma \in \bigcup_{i=1}^{\ell} K_i\} \text{ for all } 1 \leq k \leq \ell \text{ if } s > 0, \text{ and for all } 1 \leq k \leq \ell - 1 \text{ if } s = 0.

**Proof:** Let $C_1 = \{\gamma \prec \alpha_j \mid \gamma \in \bigcup_{i=1}^{k} K_i\}$, $C_2 = \{\gamma \prec \alpha_j \mid \gamma \in \bigcup_{i=1}^{\ell} K_i\}$, and $C_3 = \{\gamma \prec \alpha_j \mid \gamma \in \bigcup_{i=1}^{\ell-1} K_i\}$. Also, let $D_1 = \{\beta \prec \Gamma \mid \beta \in \bigcup_{i=1}^{k} K_i\}$, $D_2 = \{\beta \prec \Gamma \mid \beta \in \bigcup_{i=1}^{\ell} K_i\}$, and $D_3 = \{\beta \prec \Gamma \mid \beta \in K_{\ell+1}\}$. The lemma now states that $C_1 \not\subset (C_2 \cup C_3)$.

Note that $D_i = \{\beta \prec \Gamma \mid \gamma \prec \alpha_j, \beta \gamma \in C_i\}$ for $i = 1, 2$. The set $D_3$ is the disjoint union of $\{\beta \prec \Gamma \gamma \prec \alpha_j, \beta \gamma \in C_3\}$ and the set $D'$ defined by

$$D' = \begin{cases} 
\{e_0^d, (e_0^d)\} & \text{if } \alpha_j = e_0^d, n \neq 3, m \neq 3 \text{ or if } \alpha_j = e_0^d; \\
\{e_0^d, e_0^c, (e_1^d), (e_2^d), (e_0^d)\} & \text{if } \alpha_j = e_0^d, n = 3, m \neq 3; \\
\{e_0^d, e_0^c, (e_1^d), (e_2^d), (f_1^d), (f_2^d), (f_3^d)\} & \text{if } \alpha_j = e_0^d, n = m = 3.
\end{cases}$$

Now, suppose $C_1 \not\subset (C_2 \cup C_3)$. Then it is easy to deduce that $D_1 \not\subset (D_2 \cup D_3)$ if $s \geq 1$ and that $(D_1 \cup D_3) \not\subset D_2$ or $D_1 \not\subset (D_2 \cup D_3)$ if $s = 0$, since $C_1 \not\subset (C_2 \cup C_3)$. □

**Proof of Lemma 3.12:**

From Lemma 5.3 follows that $A \triangle B$ is not forbidden by (V3b), (V3c), or (V3d). Lemmas 5.5 and 5.6 imply that $A \triangle B$ is not forbidden by (V3a). Thus $A \triangle B$.

### 5.4 The order of the coatoms near two vertices

In this section the following notation will be used. Let $\alpha$ be an $X$–forest and $\mathcal{C}$ a set of coatoms of $\alpha$. In an ordering of $\mathcal{C}$, $[\beta]$ denotes $\beta$ if $\beta \in \mathcal{C}$.

**Proof of Lemma 3.13:** It is easy to show that there always exists an order of $e_1^c, e_2^c, e_3^c$, $(e_1^c)$, and $(e_1^c)$ that satisfies (V3b) and $\mathcal{C} \triangle D$.

Since $\deg(v) = 3$, it follows that (V3) is satisfied if (V3b) is satisfied. Order the elements of $\mathcal{C} \cup D$ so that $\mathcal{C} \triangle D$ and $e_1^c, e_2^c, e_3^c$, $(e_1^c)$ and $(e_1^c)$ have the
given order. Then put \( e_1^2, e_3^3, (e_1^1) \) as far from each other as possible in the ordering without violating \( C \sqsubset D \), and do the same with \( e_1^3 \) and \( e_2^2 \). It is now easy to see that this ordering satisfies (V3) and \( C \sqsubset D \), and has the given order of \( e_1^1, e_2^2, e_3^3, (e_1^1) \) and \( (e_1^2) \).

**Proof of Lemma 3.14:** If \( \gamma_1, \ldots, \gamma_k \) is an ordering of \( C \) such that \( \{\gamma_1, \ldots, \gamma_i\} \cap \{\gamma_{i+1}, \ldots, \gamma_k\} \cup D \) for all \( 1 \leq i \leq k-1 \), and \( \delta_1, \ldots, \delta_\ell \) is an ordering of \( D \) such that \( C \cup \{\delta_1, \ldots, \delta_\ell\} \cap \{\delta_{\ell+1}, \ldots, \delta_\ell\} \) for all \( 1 \leq i \leq \ell - 1 \), then the ordering \( \gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_\ell \) of \( C \cup D \) satisfies (V3) and \( C \sqsubset D \).

The symmetry of (V3) implies that if it is possible to find orderings as above for \( C \), then it is also possible to find the desired orderings for \( D \). Hence it suffices to find such orderings of \( C \). In this case the sub-conditions of (V3) that need to be checked are (V3c) and (V3d).

If \( D = \emptyset \), we may give \( C \) the ordering \( e_2^2, e_3^3, \ldots, e_2^2, e_3^3 \). If \(|C| \leq 2\), then give \( C \) any ordering.

Now suppose \( D \neq \emptyset \) and \(|C| \geq 3\). If there exists an edge \( f_1 \) such that \( f_1^1 \in C \) and \( f_1^2 \in D \), then \(|C| \geq 3\) and (V3c) imply that there is some \( f_2^3 \in C \). If also \( f_2^3 \in C \), then (V3c) implies that there is some \( f_3^3 \in C \). Hence either \( f_2^3 \in C \), \( f_2^3 \in D \), or there is some \( f_3^3 \in C \) such that \( f_2 \neq e_1 \). Now it is easy to show that the orderings \( f_1^1 f_2^3 \ldots f_2^3 \) or \( f_1^1 f_2^3 f_2^3 \ldots f_2^3 \) of \( C \) will do.

If \( f_1^1 \in C \Rightarrow f_1^3 \in C \) for all edges \( f_1 \) incident with \( v \), then (V3d) implies that there exists an edge \( f_1 \) such that \( f_1^3 \in C \) and \( f_1^1 \in D \). If \( f_2^3 \) and/or \( f_2^3 \) are in \( C \), then \( f_2^3 \) and/or \( f_2^3 \) are also in \( C \). Then the orderings \( f_1^3 f_2^3 \ldots f_2^3 \) of \( C \) will do.

Thus in all cases above it is possible to choose an ordering that has the given order of those of \( e_1^1 \) and \( e_2^2 \) that are in \( C \).

**Appendix**

In this appendix we present some topological results that we use in Section 4. For basic terminology and results concerning point-set topology see, for example, [13].

**Proposition A.1.** Suppose that \( B \) is a ball, with boundary \( S \), a sphere. If \( f : S \to S \) is a continuous map such that \( f^{-1}(p) \) is contractible for each \( p \in S \), then \( f \) extends to a continuous map \( F : B \to B \) such that \( F \) restricted to \( \text{Int}(B) \) is injective and \( F(\text{Int}(B)) = \text{Int}(B) \).

**Proof.** The conditions on \( f \) ensure that \( f \) is a cellular map, and therefore it can be approximated by homeomorphisms – for the definition of these terms, and the justification of the last statement, see [9] and (for the dimension 4 case) [14]. The existence of an extension \( F \) with the properties promised by Proposition A.1 now follows by Lemma 1 of [9].

Now, given topological spaces \( Y, Z \), and a map \( f : Y \to Z \), let \( R_f \) be the equivalence relation on \( Y \) induced by \( f \), i.e. the equivalence relation that identifies
elements of $Y$ that are mapped by $f$ to the same element of $Z$. Recall that if $Y$ is compact, and $f$ is a continuous surjection with $Z$ Hausdorff, then $Z$ is homeomorphic to the quotient space $Y/R_f$. With this fact and the previous proposition in hand, we now present a result that is used in the proof of Theorem 4.4.

**Corollary A.2.** Let $Z$ be a Hausdorff topological space, and let $B$ be a $d$-dimensional ball with boundary $S$, a sphere. Suppose that $g : B \to Z$ is a continuous surjection whose restriction to $\text{Int}(B)$ is bicontinuous and injective. Suppose furthermore that $g(S)$ is homeomorphic to $S$, $g(S) \cap g(\text{Int}(B)) = \emptyset$, and that for each $q \in g(S)$, $g^{-1}(q)$ is contractible. Then $Z$ is homeomorphic to $B$.

**Proof:** By assumption the space $Z$ is homeomorphic to the quotient space $B/R_g$. Now, by Proposition A.1, the map $g|_S$ extends to a continuous map $F : B \to B$ that maps $\text{Int}(B)$ injectively onto $\text{Int}(B)$. In particular, it follows that $B$ is homeomorphic to $B/R_F$. Now, $R_F = R_g$ since $F$ and $g$ are both injective on $\text{Int}(B)$, $F(\text{Int}(B)) \cap F(S) = g(\text{Int}(B)) \cap g(S) = \emptyset$, and $F|_S = g$. Consequently, $B/R_F = B/R_g$, from which it immediately follows that $Z$ is homeomorphic to $B$.

We conclude this appendix with a result that is used in the proof of Lemma 4.2.

**Lemma A.3.** Let $I$ be a finite index set. For each $i \in I$, let $Z_i$ be compact topological space, and $R_i$ be an equivalence relation on $Z_i$ such that $Y_i := Z_i/R_i$ is Hausdorff. Let $R$ be the (product) equivalence relation on $Z := \prod_{i \in I} Z_i$ defined by $(z_i)_{i \in I} R (z'_i)_{i \in I}$ if and only if $z_i R_i z'_i$ for each $i \in I$. Then $Z/R$ is homeomorphic to $\prod_{i \in I} Y_i$.

**Proof:** Set $Y := \prod_{i \in I} Y_i$, and let $f : Z \to Y$ be the map that takes $(z_i)_{i \in I}$ to $([z_i])_{i \in I}$, where, for $z_i \in Z_i$, $[z_i]$ denotes the equivalence class of $z_i$ under relation $R_i$.

Set $W = Z/R$, and let $g : Z \to W$ be the associated quotient mapping. By the universal property of quotient mappings, there is a continuous mapping $h : W \to Y$ with $h \circ g = f$. By standard properties of quotient mappings $h$ is a bijection. But since $Z$ is compact, so is $W$, and since each $Y_i$ is Hausdorff, so is $Y$. Hence $h$ is a continuous bijection from a compact space to a Hausdorff space, and is therefore a homeomorphism.

**References**


