The $k$-assignment Polytope and the Space of Evolutionary Trees

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Linköping 2004
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LiU-TEK-LIC-2004:46
ISBN 91-85295-45-0
ISSN 0280-7971

Printed by UniTryck, Linköping 2004
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Abstract

This thesis consists of two papers.

The first paper is a study of the structure of the $k$-assignment polytope, whose vertices are the $m \times n$ $(0, 1)$-matrices with exactly $k$ $1$'s and at most one $1$ in each row and each column. This is a natural generalisation of the Birkhoff polytope and many of the known properties of the Birkhoff polytope are generalised. Two equivalent representations of the faces are given, one as $(0, 1)$-matrices and one as ear decompositions of bipartite graphs. These tools are used to describe properties of the polytope, especially a complete description of the cover relation in the face lattice of the polytope and an exact expression for the diameter.

The second paper studies the edge-product space $\mathcal{E}(X)$ for trees on $X$. This space is generated by the set of edge-weighted finite trees on $X$, and arises by multiplying the weights of edges on paths in trees. These spaces are closely connected to tree-indexed Markov processes in molecular evolutionary biology. It is known that $\mathcal{E}(X)$ has a natural $CW$-complex structure, and a combinatorial description of the associated face poset exists which is a poset $S(X)$ of $X$-forests. In this paper it is shown that the edge-product space is a regular cell complex. One important part in showing that is to conclude that all intervals $[0, \Gamma]$, $\Gamma \in S(X)$, have recursive coatom orderings.

Acknowledgements

I would like to thank my supervisor professor Svante Linusson for all invaluable support and help during this work. Thanks also to the Swedish Research Council which has been supporting my graduate studies. I would also like to thank Peter Rand, Johan Lundvall and David Byers for help with $\LaTeX$ and other things. Finally I would like to thank God and my family for all patience, help and encouragement during my studies.
Introduction

This thesis consists of two papers.

Paper 1

The first paper is a study of a polytope called the $k$-assignment polytope. This polytope is a generalisation of the well-known Birkhoff polytope $B_n$ which has the $n \times n$ permutation matrices as its vertices, and has been studied from a lot of different viewpoints, see for example [1, 2, 3, 5]. The Birkhoff polytope has many names; two other usual names are 'The polytope of doubly stochastic matrices' and 'The assignment polytope'. There are still problems left about the Birkhoff polytope, for example the volume is a challenging and interesting problem which is still not solved.

A natural generalisation of permutation matrices occurring both in optimisation and in theoretical combinatorics is $k$-assignments. A $k$-assignment is $k$ entries in a matrix that are required to be in different rows and columns. This can also be described as placing $k$ non-attacking rooks on a chessboard. The $k$-assignment polytope $M(m, n, k)$ is the polytope whose vertices are the $m \times n \,(0, 1)$-matrices with exactly $k$ 1:s and at most one 1 in each row and each column.

The origin of our interest in the $k$-assignment polytope is the conjecture by G. Parisi on the so called Random Assignment Problem [9], which was immediately generalised by D. Coppersmith and G. Sorkin to $k$-assignments [4]. Both of these conjectures were proved in 2003 [6, 8]. In the proof [6] the generalisation to $k$-assignments was a necessary ingredient for the proof. Since there exist general methods [7] to compute the expected values of an
optimisation problem with random variables using the face lattice of the polytope we became interested in studying the faces of the $k$-assignment polytope.

This paper gives two equivalent representations of the faces of $M(m, n, k)$, one as $(0, 1)$-matrices and one as ear decompositions of bipartite graphs. Then these representations are used to describe the cover relation in the face lattice of the polytope, and to give an exact expression for the diameter.

Paper 2

The second paper studies the edge-product space $E(X)$ for trees on $X$, where $X$ is a fixed finite set. The edge-product space for trees on $X$ is generated by the set of edge-weighted finite trees on $X$, and arises by multiplying the weights of edges on paths in trees.

One reason for investigating these spaces is that they are closely connected to tree-indexed Markov processes in molecular evolutionary biology, see [10]. In [10] it was shown that $E(X)$ has a natural $CW$–complex structure for any finite set $X$, and a combinatorial description of the associated face poset was given. This combinatorial description is a poset $S(X)$ of $X$-forests.

In this paper it is shown that the edge-product space is a regular cell complex. One important part in showing that is to conclude that all intervals $[0, \Gamma]$, $\Gamma \in S(X)$, have recursive coatom orderings. The paper is a joint work with Linusson, Moulton and Steel, and my contribution is to prove that each interval $[0, \Gamma]$ has a recursive coatom ordering.

References


The $k$-assignment polytope

Jonna Gill$^*$  Svante Linusson$^\dagger$

Abstract

In this paper we study the structure of the $k$-assignment polytope, whose vertices are the $m \times n$ (0,1)-matrices with exactly $k$ 1:s and at most one 1 in each row and each column. This is a natural generalisation of the Birkhoff polytope and many of the known properties of the Birkhoff polytope are generalised. Two representations of the faces are given, one as certain (0,1)-matrices and one as ear decompositions of bipartite graphs. These tools are used to describe properties of the polytope, especially a complete description of the cover relation in the face lattice of the polytope and an exact expression for the diameter.

1 Introduction

The Birkhoff polytope and its properties have been studied from different viewpoints, see e.g. [1, 2, 3, 5]. The Birkhoff polytope $B_n$ has the $n \times n$ permutation matrices as vertices and is known under many names, like `The polytope of doubly stochastic matrices' or `The assignment polytope', for example. A natural generalisation of permutation matrices occurring both in optimisation and in theoretical combinatorics is $k$-assignments. A $k$-assignment is $k$ entries in a matrix that are required to be in different rows and columns. This can also be described as placing $k$ non-attacking rooks on a chess-board.

Let $M(m, n, k)$ denote the polytope in $\mathbb{R}^{m \times n}$ whose vertices are the $m \times n$ (0,1)-matrices with exactly $k$ 1:s and at most one 1 in each row and each column. It will be called ‘The $k$-assignment polytope’ and this paper is devoted to determine some of its combinatorial properties. The origin of our interest in the $k$-assignment polytope is the conjecture by G. Parisi on the so called Random Assignment Problem [12], which was immediately generalised by D. Coppersmith and G. Sorkin to $k$-assignments [4].

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Both of these conjectures were proved in 2003 [7, 11]. In the proof [7] the generalisation to $k$-assignments was a necessary ingredient for the proof. Since there exist general methods [8] to compute the expected values of an optimisation problem with random variables using the face lattice of the polytope we became interested in studying the faces of the $k$-assignment polytope.

In section 2 a description of the points in $M(m, n, k)$ in terms of inequalities and equalities is given, and the dimension and the facets of $M(m, n, k)$ are described.

In section 3 a representation by matrices of the faces of $M(m, n, k)$ is given, and some properties following from this representation. For example, exactly when one face is a facet of another face.

In section 4 a representation by bipartite graphs of the faces of $M(m, n, k)$ is given. It is equivalent to the representation of the faces by matrices in the former section, but it is still useful to have both descriptions. Some properties following from the bipartite graph representation will be shown. In section 4.1 an ear decomposition of these bipartite graphs is constructed.

In section 5 the diameter of $M(m, n, k)$ is studied, and an explicit formula for the diameter is given for all values on $m, n$ and $k$ in Theorem 5.6 and Theorem 5.7.

2 Some basic properties of the $k$-assignment polytope

Lemma 2.1. The polytope $M(m, n, k)$ has $\binom{m}{k} \cdot \binom{n}{k} \cdot k!$ vertices.

The proof will be omitted since it is trivial.

The points in $M(m, n, k)$ are described by real $m \times n$ matrices $X = [x_{ij}]$. If $V^1, \ldots, V^T$, where $V^r = [v^r_{ij}]$ for all $r$, are the vertices of $M(m, n, k)$, then

$$M(m, n, k) = Conv\{V^1, \ldots, V^T\} = \{\sum_{t=1}^{T} \lambda_t V^t; \sum_{t=1}^{T} \lambda_t = 1, \lambda_t \geq 0 \text{ for all } t\}.$$

But it also is possible to describe the points in $M(m, n, k)$ with equalities and inequalities, as was first proved by Mendelsohn and Dulmage for $m = n$ in [10].
Theorem 2.2. The points of $M(m,n,k)$ are precisely \{ $X \in \mathbb{R}^{m \times n}; x_{ij} \geq 0$ for all $i,j$, $\sum_{i,j} x_{ij} = k$, $\sum_{i} x_{ij} \leq 1$ for all $j$, $\sum_{j} x_{ij} \leq 1$ for all $i$. \}

Proof: Let $P = \{ X \in \mathbb{R}^{m \times n}; x_{ij} \geq 0$ for all $i,j$, $\sum_{i,j} x_{ij} = k$, $\sum_{i} x_{ij} \leq 1$ for all $j$, $\sum_{j} x_{ij} \leq 1$ for all $i$. \}.

It is to be shown that $M(m,n,k) = P$. First, take $X \in M(m,n,k)$. Then $X = \sum_{t=1}^{T} \lambda_t V_t$ for some $\lambda_1, \ldots, \lambda_T$ where $\sum_{t=1}^{T} \lambda_t = 1, \lambda_t \geq 0$. This also means that $x_{ij} = \sum_{t=1}^{T} \lambda_t v_{ij}^t$.

$v_{ij}^t \in \{0,1\}$ for all $i,j,t$ $\Rightarrow$ $x_{ij} = \sum_{t=1}^{T} \lambda_t v_{ij}^t \geq 0$ for all $i,j$

$$\sum_{i,j} v_{ij}^t = k \quad \Rightarrow \quad \sum_{i,j} x_{ij} = \sum_{t=1}^{T} \sum_{i,j} \lambda_t v_{ij}^t = \sum_{t=1}^{T} \lambda_t k = k$$

$$\sum_{i=1}^{m} v_{ij}^t \leq 1 \text{ for all } j \quad \Rightarrow \quad \sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \sum_{t=1}^{T} \lambda_t v_{ij}^t \leq \sum_{t=1}^{T} \lambda t \sum_{i=1}^{m} v_{ij}^t \leq \sum_{t=1}^{T} \lambda_t = 1 \text{ for all } j$$

Analogously $\sum_{j=1}^{n} x_{ij} \leq 1$ for all $i$.

This shows that $M(m,n,k) \subseteq P$.

To show that $P \subseteq M(m,n,k)$, take an arbitrary element $X \in P$. Let $R_i$ be the sum of all elements in row $i$ in $X$, $1 \leq i \leq m$, and let $C_j$ be the sum of all elements in column $j$ in $X$, $1 \leq j \leq n$. Then create a new $(m+n-k) \times (m+n-k)$ matrix $X'$ in the following way:

- $x'_{ij} = x_{ij}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$.
- $x'_{i,j} = 0$ if $n+1 \leq j \leq m+n-k$ and $1 \leq i \leq m+n-k$.
- $x'_{i,n+1} = \ldots = x'_{i,n+(m-k)} = \frac{1-R_i}{m-k}$ for $1 \leq i \leq m$.
- $x'_{m+1,j} = \ldots = x'_{m+(n-k),j} = \frac{1-C_j}{n-k}$ for $1 \leq j \leq n$. 

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Now it is easy to see that $X_0 \in B_{m+n-k}$, why it is possible to write $X'$ as a convex combination of $(m + n - k) \times (m + n - k)$ permutation matrices: 

$$X' = \begin{bmatrix} n & m-k \\
0 & \vdots \\
\vdots & \vdots \\
\frac{1-C_1}{n-k} & \frac{1-C_2}{n-k} & \frac{1-C_3}{n-k} & \cdots & \frac{1-C_{m-1}}{n-k} \end{bmatrix} \in M(m+n-k).$$

Every $V_s$ now has zeros in its lower right corner, why it has to have $m-k$ 1:s in its upper right corner, and $n-k$ 1:s in its lower left corner. This means that there is $k$ 1:s in $V_s$, and there is at most one 1 in each row and each column of $V_s$. Then $V_1, \ldots, V_S$ are vertices in $M(m,n,k)$, why $X \in M(m,n,k)$. It is now shown that $P \subseteq M(m,n,k)$, why $P = M(m,n,k)$.

Now Theorem 2.2 can be used to determine the dimension of $M(m,n,k)$ and the equations of the facets.

**Definition 2.3.** Let $M$ be a polytope described by equalities and inequalities. Remove as many inequalities and equalities describing $M$ as possible without changing $M$. Then the remaining inequalities and equalities are said to be **independent**, since none of them can be removed without changing $M$.

The dimension is given by subtracting the number of independent equalities from the dimension of the space (which is $mn$), and the facets are given by replacing one of the independent inequalities with an equality. To show that an inequality is independent of all other conditions it is removed, and then a point satisfying the other conditions but not lying in the polytope is found. Since some of the conditions above are implied by the other conditions, and some inequalities are forced to be equalities for some values on $k$, $n$ and $m$, different cases have to be treated differently. By symmetry we can assume that $n \leq m$. 

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The cases now are:

- \( k < n \leq m \); \( k = 1 \) or \( k > 1 \)
- \( k = n < m \); \( k = 1 \) or \( k > 1 \)
- \( k = n = m \)

**\( k < n \leq m \)**: The dimension of \( M(m, n, k) \) is \( (mn - 1) \) here, since there is only one equality in this case.

If \( k = 1 \), then \( M(m, n, k) = \{ X \in \mathbb{R}^{m \times n}; x_{ij} \geq 0 \text{ for all } i, j, \sum_{i} x_{ij} = 1 \} \), the \( mn \)-simplex. Thus each facet is obtained by replacing one inequality \( x_{rs} \geq 0 \) with the equality \( x_{rs} = 0 \), why there are \( mn \) different facets.

If \( k > 1 \), it is easy to see that all conditions are independent. Thus there are \( (mn + m + n) \) facets, each of them obtained by replacing one of the \( (mn + m + n) \) inequalities with an equality.

**\( k = n < m \)**: In this case it is easy to see that \( M(m, n, k) = \{ X \in \mathbb{R}^{m \times n}; x_{ij} \geq 0 \text{ for all } i, j, \sum_{i} x_{ij} = 1 \text{ for all } j, \sum_{j} x_{ij} \leq 1 \text{ for all } i \} \). The equalities \( \sum_{i} x_{ij} = 1 \) for all \( j \) are independent, since if \( \sum_{i} x_{is} \) is not forced to be 1, then the point \( X = [x_{ij}] \) where \( x_{jj} = 1 \) for all \( j \) except \( j = s \), and \( x_{ij} = 0 \) elsewhere is allowed, but \( X \) is not in \( M(m, n, k) \). So there are \( n \) independent equalities, why the dimension is \( (m - 1)n \).

If \( k = 1 \) \((n = 1)\), then \( M(m, n, k) \) is the \( m \)-simplex with \( m \) facets.

If \( k > 1 \), then all conditions are independent. Thus there are \( (mn + m) \) inequalities, and each of the \((mn + m)\) facets is obtained by replacing one of the inequalities with an equality.

**\( k = n = m \)**: This is the Birkhoff polytope, with \( mn = n^2 \) facets, each of them obtained by replacing one of the inequalities \( x_{ij} \geq 0 \) with \( x_{ij} = 0 \). The dimension is \((m - 1)(n - 1) = (n - 1)^2\).

### 3 Representation of the faces by matrices

There is a one-to-one correspondence between faces of the Birkhoff polytope \( B_n \) and \( n \times n \) \((0,1)\)-matrices with a special property called total support. This correspondence is described in [2, Section 2]

**Definition 3.1.** An \( n \times n \) \((0,1)\) matrix \( A = [a_{ij}] \) is said to have **total support** if \( a_{rs} = 1 \) implies that there is an \( n \times n \) permutation matrix \( P = [p_{ij}] \) with \( p_{rs} = 1 \) and \( P \leq A \) (i.e. \( p_{ij} \leq a_{ij} \) for all \( i, j \)).

The face corresponding to the matrix \( A \) with total support is denoted \( F_B(A) \), and the vertices of \( F_B(A) \) are exactly all permutation matrices \( P \) such that \( P \leq A \).
There is a similar correspondence between the faces of $M(m, n, k)$ and $(m + n - k) \times (m + n - k)$ $(0, 1)$-matrices with another special property here called extended support. This correspondence will now be described.

The faces of $M(m, n, k)$ are obtained by replacing some of the inequalities describing $M(m, n, k)$ with equalities.

**Definition 3.2.** An $(m + n) \times (m + n)$ $(m + n - k)$ matrix $C = [c_{ij}]$ which satisfies the following conditions is said to have **extended support**:

- $C$ has total support.
- $c_{ij} = 0$ if $n + 1 \leq j \leq m + n - k$ and $m + 1 \leq i \leq m + n - k$.
- $c_{i,n+1} = c_{i,n+2} = \ldots = c_{i,m+n-k}$ if $1 \leq i \leq m$.
- $c_{m+1,j} = c_{m+2,j} = \ldots = c_{m+n-k,j}$ if $1 \leq j \leq n$.

Matrices of order $(m + n - k)$ will from now on be divided into four parts in the following way:

$$
C = \begin{bmatrix}
  C_{\alpha} & C_{\beta} & m \\
  C_{\gamma} & C_{\delta} & n - k
\end{bmatrix}
$$

The following applies to matrices with extended support: $C_{\delta}$ contains only zeros. The number of zero rows in $C_{\beta}$ will be denoted $r_0$ or $r_0(C)$, and the number of zero columns in $C_{\gamma}$ will be denoted $k_0$ or $k_0(C)$. If $C \neq 0$, it follows that $r_0 \leq k$ and $k_0 \leq k$.

Suppose we have a matrix $C$ with extended support. Then $C$ also has total support. Take a permutation matrix $P$ such that $P \leq C$. Now divide $P$ in the same way as $C$. Since $P_{\beta}$ contains only zeros, there must be $m - k$ 1:s in $P_{\beta}$ and $n - k$ 1:s in $P_{\gamma}$. Now the other $k$ 1:s in $P$ have to be in $P_{\alpha}$, which means that $P_{\alpha}$ is a vertex of $M(m, n, k)$. $P_{\alpha}$ does not change if the 1:s in $P_{\beta}$ are permuted according to columns, or if the 1:s in $P_{\gamma}$ are permuted according to rows. That means that there are $\Psi := (m - k)! \cdot (n - k)!$ different permutation matrices $P \leq C$ which have $P_{\alpha}$ in common.
All possible (0, 1)-matrices $P_\alpha$ must now satisfy the following conditions:

- $p_{ij} = 0$ if $c_{ij} = 0$, for $1 \leq i \leq m, 1 \leq j \leq n$.
- $\sum_{j=1}^{n} p_{ij} = 1$ if $c_{i,n+1} = 0$, for $1 \leq i \leq m$.
- $\sum_{j=1}^{m} p_{ij} \leq 1$ if $c_{i,n+1} = 1$, for $1 \leq i \leq m$.
- $\sum_{i=1}^{m} p_{ij} = 1$ if $c_{m+1,j} = 0$, for $1 \leq j \leq n$.
- $\sum_{i=1}^{m} p_{ij} \leq 1$ if $c_{m+1,j} = 1$, for $1 \leq j \leq n$.
- $\sum_{i=1}^{m} \sum_{j=1}^{n} p_{ij} = k$.

Any matrix $P_\alpha$ satisfying these conditions can be extended to a permutation matrix $P \leq C$. Note that $P \leq C$ is valid for any permutation matrix $P$ made from $P_\alpha$, which means there are exactly $\Psi$ permutation matrices that have $P_\alpha$ in common. All the (0, 1)-matrices satisfying the conditions for $P_\alpha$ are exactly the vertices of a face of $M(m,n,k)$, since the conditions describe a face of $M(m,n,k)$. Let the face of $M(m,n,k)$ corresponding to $C$ be denoted $F^{M}(C)$.

The function that takes a matrix $X$ and gives the matrix $X_\alpha$ is a projection of $R^{(m+n-k)\times(m+n-k)}$ on $R^{m\times n}$. Call this function $\pi$. Then $\pi$ will project all vertices of $F^{B}(C)$ on the vertices of $F^{M}(C)$, why $F^{B}(C)$ will be projected by $\pi$ on $F^{M}(C)$. The number of vertices of $F^{B}(C)$ projected by $\pi$ on one arbitrary vertex of $F^{M}(C)$ is $\Psi$.

**Theorem 3.3.** There is a one-to-one correspondence between matrices $C$ with extended support and faces $F^{M}$ of $M(m,n,k)$.

All permutation matrices of order $(m+n-k)$ that are projected by $\pi$ on vertices of $F^{M}(C)$ are exactly all permutation matrices $P$ such that $P \leq C$.

This theorem implies that if $Q_1, \ldots, Q_\tau$ are the vertices of a face of $M(m,n,k)$ corresponding to a matrix $C$, then $P_1, \ldots, P_\tau\Psi$ (which are all the permutation matrices of order $(m+n-k)$ that are projected by $\pi$ on $Q_1, \ldots, Q_\tau$) are the vertices of $F^{B}(C)$.

**Proof:** As shown above a matrix $C$ with extended support corresponds to a face of $M(m,n,k)$. Now take a face $F^{M}$ of $M(m,n,k)$. This face can be described by the conditions describing $M(m,n,k)$ with some of the inequalities replaced by equalities such as

$x_{rs} = 0, \sum_{i=1}^{m} x_{is} = 1$ and $\sum_{j=1}^{n} x_{rj} = 1$ \hspace{1cm} (X = [x_{ij}] \in F^{M})$.
Let $C = [c_{ij}]$ be the $(0, 1)$-matrix of order $(m + n - k)$ such that

- $c_{ij} = 0$ if $x_{ij} = 0$ in $F^M$.
- $c_{n+1,s} = \ldots = c_{m+n-k,s} = 0$ if $\sum_{i=1}^{m} x_{is} = 1$.
- $c_{r,m+1} = \ldots = c_{r,m+n-k} = 0$ if $\sum_{j=1}^{m} x_{rj} = 1$.
- $C_Q = 0$.
- $c_{ij} = 1$ elsewhere.

It is easy to see that any vertex of $F^M$ can be extended to a permutation matrix $P$ of order $(m + n - k)$ such that $P \leq C$ in the same way as earlier. Now let $P$ be a permutation matrix such that $P \leq C$. Then $P_s$ satisfies the same conditions as the points in $F^M$, which means that $P_s$ is a vertex of $F^M$.

Finally it is to be shown that $C$ has total support. If $c_{ij} = 1$ in $C_s$, then there is at least one vertex of $F^M$ having a 1 in that position, and if $c_{ij} = 1$ in $C_Q$ ($C_r$), then there is at least one vertex of $F^M$ having no 1 in row $i$ (column $j$), why at least one extension of that vertex to a permutation matrix has a 1 in that position. Thus $C$ has total support, why $C$ also has extended support and corresponds to the face $F^M$.

**Corollary 3.4.** If $C$ and $D$ have extended support, then $C \lesssim D \iff F^B(C) \subset F^B(D) \iff F^M(C) \subset F^M(D)$.

**Proof:** The first equivalence follows easily from Definition 3.1, and the second follows from Theorem 3.3.

Let $Q_1, \ldots, Q_l$ be different vertices of $M(m, n, k)$. Let $C = [c_{ij}]$ be the $(0, 1)$-matrix of order $(m + n - k)$ such that $c_{ij} = 1$ if at least one vertex $Q_s$, has a 1 in position $(i,j)$, $c_{n+1,s} = \ldots = c_{m+n-k,s} = 1$ if at least one vertex $Q_s$ has a 1 in column $s$, $c_{r,m+1} = \ldots = c_{r,m+n-k} = 1$ if at least one vertex $Q_s$ has a 1 in row $r$, and $c_{ij} = 0$ elsewhere. Then $F^M(C)$ is the smallest face of $M(m, n, k)$ containing the vertices $Q_1, \ldots, Q_l$. The vertices $Q_1, \ldots, Q_l$ are exactly the vertices of a face of $M(m, n, k)$ if and only if per $C = t\Psi$.

**Proof:** The construction of $C$ implies that $F^M(C)$ contains the vertices $Q_1, \ldots, Q_l$ and that $C$ has extended support. A $(0, 1)$-matrix $D$ of order $(m + n - k)$ with extended support containing the vertices $Q_1, \ldots, Q_l$ must at least have 1:s in all positions where $C$ has 1:s, because else the matrix does not contain all of the vertices $Q_1, \ldots, Q_l$. This means that $C \leq D$, why $C$ is the smallest matrix with extended support containing $Q_1, \ldots, Q_l$. Hence $F^M(C)$ is the smallest face of $M(m, n, k)$ containing $Q_1, \ldots, Q_l$.

Theorem 3.3 implies that

$Q_1, \ldots, Q_l$ are the vertices of a face of $M(m, n, k)$

$\Downarrow$

$P_1, \ldots, P_{t\Psi}$ are the vertices of a face of the Birkhoff polytope
and by [2, Theorem 2.1]

\[ P_1, \ldots, P_t \] are the vertices of a face of the Birkhoff polytope

\[ \per C = t \Psi \]

\( \Box \)

**Definition 3.6.** (As the terms are used in [2, Section 2].) Let \( A \) be a \((0,1)\)-matrix of order \( n \) with total support. \( A \) is said to be **decomposable** if there are permutation matrices \( P \) and \( Q \) such that

\[
PAQ = \begin{bmatrix}
A_1 & 0 & \cdots & 0 \\
A_3 & A_2 & & \\
& & \ddots & \\
& & & \ddots & 0 \\
0 & \cdots & 0 & A_t
\end{bmatrix}
\]

where \( A_1 \) and \( A_2 \) are quadratic. If \( n = 1 \), then \( A \) is decomposable if and only if \( A = 0 \). Since \( A \) has total support also \( A_3 = 0 \) when \( A \) is decomposable. If \( A \) (which has total support) is not decomposable, then \( A \) is said to be **indecomposable**.

**Theorem 3.7.** [2, Section 2] A quadratic \((0,1)\)-matrix \( A \) has total support if and only if there are permutation matrices \( P \) and \( Q \) such that \( PAQ \) is a direct sum of indecomposable matrices, i.e. \( PAQ = A_1 \oplus \cdots \oplus A_t \) where \( A_i \) is indecomposable for \( i = 1, \ldots, t \).

Then \( \dim F(A_1 \oplus \cdots \oplus A_t) = \dim F(A_1) + \cdots + \dim F(A_t) \).

If \( B \) is a matrix, let \( \|B\| \) denote the sum of all elements in \( B \). If \( B \) is a set, let \( |B| \) denote the number of elements in \( B \).

**Theorem 3.8.** [2, Corollary 2.6] Let \( A \) be a \((0,1)\)-matrix of order \( n \) with total support. Let \( P \) and \( Q \) be permutation matrices such that \( PAQ = A_1 \oplus \cdots \oplus A_t \), where \( A_i \) is indecomposable for \( i = 1, \ldots, t \). Then \( \dim F^B(A) = \|A\| - 2n + t \).

**Theorem 3.9.** Let \( C \) be a matrix with extended support. Take permutation matrices \( P \) and \( Q \) such that \( PCQ = C_1 \oplus \cdots \oplus C_t \), where \( C_i \) is indecomposable for \( i = 1, \ldots, t \). Then \( \dim F^M(C) = \|C\| - r_0 - k_0 + t - 2 \) in the case \( k < n \leq m \), and \( \dim F^M(C) = \|C\| - r_0 - n + t - 1 \) in the case \( k = n < m \).

**Proof:** Suppose there is a polytope defined on the variables \( x_{ij} \), \((i,j) \in I \) \((I \) is a set of indices). The image of the polytope under the projection
that removes the variable $x_{rs}$ ($r, s \in I$) has the same dimension as the polytope if and only if there are $\lambda_{ij}$ such that $\sum_{(i,j) \in I} \lambda_{ij} \cdot x_{ij} = \text{constant}$, $\lambda_{rs} \neq 0$. Otherwise the image of the polytope has dimension one less than the polytope. This is easy to see since the dimension of a polytope is the number of variables it is defined on minus the number of independent equalities these variables satisfy. So if $x_{rs}$ is removed, also one of the equalities is removed (the independent equalities can be rewritten so that they are removed by projections the resulting image will still have the same dimension as $F^B(C)$). When these variables in the last column (there are $(m-r_0)$ such variables) are included in the conditions $\sum_{j=1}^{m+n-k} x_{ij} = 1$, so when they are removed by projections the resulting image will still have the same dimension as $F^B(C)$. The remaining variables in the last row (there are $(n-k_0)$ such variables) are treated analogously. If $k < m$, then there is at least one row in $C_\beta$ with only 1:s and the corresponding variables are included in the conditions $\sum_{i=1}^{m} x_{ij} = 1$ for $n + 1 \leq j \leq m + n - k - 1$, so when these $(m - k - 1)$ variables are removed by a projection the resulting image will still have the same dimension as $F^B(C)$. If $k = m$, then $C_\beta$ does not exist. The variables in $C_\gamma$ are treated in analogue.

Now the remaining variables in $C_\gamma$ and $C_\beta$ are not included in any remaining equality, so the image of the projection that removes them has dimension $\dim F^B(C) - \#(\text{variables left})$. In the case $k < n \leq m$ there remain $(m - k - 1)(m - r_0 - 1) + (n - k - 1)(n - k_0 - 1)$ variables in $C_\gamma$ and $C_\beta$ when all variables possible are removed without changing the dimension. In the case $k = n < m$ there remain $(m - k - 1)(m - r_0 - 1)$ variables in $C_\gamma$.

Thus the dimension of $F^B(C)$ is $\|A\| - 2(m + n - k) + t = \|C_\alpha\| + (m - k)(m - r_0) + (n - k)(n - k_0) - 2(m + n - k) + t = \|C_\alpha\| + (m - k - 1)(m - r_0 - 1) + (n - k - 1)(n - k_0 - 1) - r_0 - k_0 + t - 2$. This means that $\dim F^M(C) = \|C_\alpha\| - r_0 - k_0 + t - 2$ in the case $k < n \leq m$, and $\dim F^M(C) = \|C_\alpha\| - r_0 - n + t - 1$ in the case $k = n < m$.

In [2] it is described exactly when $F^B(D)$ is a facet of $F^B(C)$, given that $C$ is indecomposable and that $D$ has total support.

**Theorem 3.10.** [2, Corollary 2.11] Let $C = [c_{ij}]$ and $D = [d_{ij}]$ be $(0,1)$-matrices, where $C$ is indecomposable and $D$ has total support. Then $F^B(D)$ is a facet of $F^B(C)$ if and only if one of the following holds.
1. $D$ is indecomposable and there exist $r$ and $s$ such that $c_{rs} = 1$ and $D$ is obtained from $C$ by replacing $c_{rs}$ with 0.

2. $D$ is decomposable and there exist permutation matrices $P$ and $Q$ such that

$$PCQ = \begin{bmatrix}
C_1 & 0 & \cdots & 0 & E_k \\
E_1 & C_2 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & C_{k-1} & 0 \\
0 & 0 & \cdots & E_{k-1} & C_k
\end{bmatrix},$$

$$PDQ = C_1 \oplus \cdots \oplus C_k,$$

where for $i = 1, \ldots, k$, $C_i$ is indecomposable and $\|E_i\| = 1$.

The following theorem describes exactly when $F^M(D)$ is a facet of $F^M(C)$ given that $C$ is indecomposable and both $C$ and $D$ have extended support.

**Theorem 3.11.** Let $C$ be an indecomposable matrix with extended support and let $D$ be a matrix with extended support. Then $F^M(D)$ is a facet of $F^M(C)$ if and only if one of the following holds.

1. $F^B(D)$ is a facet of $F^B(C)$ and there are $r, s$ such that $c_{rs} = 1$, $c_{rs}$ is in $C_\alpha$, and $D$ is obtained from $C$ by replacing $c_{rs}$ with 0.

2. $F^B(D)$ is a facet of $F^B(C)$ and there are $r, s$ such that $c_{rs} = 1$, $c_{rs}$ is in $C_\beta$ and $m - k = 1$ or $c_{rs}$ is in $C_\gamma$ and $n - k = 1$, and $D$ is obtained from $C$ by replacing $c_{rs}$ with 0.

3. $D$ is indecomposable and there is an $i$ or a $j$ such that one of the following holds.
   - $c_{i, n+1} = 1$ and $D$ is obtained from $C$ by replacing $c_{i, n+1}, \ldots, c_{i, n+m-k}$ with zeros.
   - $c_{m+1, j} = 1$ and $D$ is obtained from $C$ by replacing $c_{m+1, j}, \ldots, c_{m+n-k, j}$ with zeros.

4. Nothing above applies and there are $r$, $s$ and $t_D$, $r \geq 0$, $s \geq 0$, $t_D - r - s \geq 0$, such that $D$ has $t_D$ indecomposable components and is obtained from $C$ by replacing $r$ 1:s from $C_\alpha$, $s$ rows of 1:s from $C_\beta$, and $t_D - r - s$ columns of 1:s from $C_\gamma$ with zeros.

**Proof:** The proof of theorem 3.9 says that

$$\dim F^M(A) = \dim F^B(A) - f(m, n, k, r_0(A), k_0(A)).$$
$k < n \leq m$: $f(m,n,k,r_0(A),k_0(A)) = (m-k-1)(m-r_0(A)-1) + (n-k-1)(n-k_0(A)-1) \\
dim F^M(A) = \|A\alpha\| - r_0(A) - k_0(A) + t_A - 2$

$k = n < m$: $f(m,n,k,r_0(A),k_0(A)) = (m-k-1)(m-r_0(A)-1) \\
dim F^M(A) = \|A\alpha\| - r_0(A) + t_A - 1$

$C$ is indecomposable (i.e. $t_C = 1$) and describes the face $F^M(C)$.

1. If a 1 is removed from $C$ in $C_\alpha$ to obtain $D$, then the value of $f(m,n,k,r_0,k_0)$ does not change, and hence $F^M(D)$ is a facet of $F^M(C)$ if and only if $F^B(D)$ is a facet of $F^B(C)$.

2. If $m-k = 1$ and a 1 is removed from $C$ in $C_\beta$ or $n-k = 1$ and a 1 is removed from $C$ in $C_\gamma$, the above also applies, since in the first case the value of $f$ is independent of $r_0$ and in the second case the value of $f$ is independent of $k_0$.

3. Else if a row of 1:s in $C_\beta$ or a column of 1:s in $C_\gamma$ is removed from $C$ to obtain $D$ and $D$ is indecomposable, then $\dim F^M(C) - \dim F^M(D) = 1$ why $F^M(D)$ is a facet of $F^M(C)$.

4. The only possibility to create a facet of $F^M(C)$ in any other way than above is to remove $r$ 1:s from $C_\alpha$, $s$ rows of 1:s from $C_\beta$, and $t_D - r - s$ columns of 1:s from $C_\gamma$ (where $r \geq 0$, $s \geq 0$, and $t_D - r - s \geq 0$) to obtain a matrix $D$, in such a way that $D$ has total support and has $t_D$ indecomposable components. Then $\dim F^M(C) - \dim F^M(D) = \|C_\alpha\| - \|D_\alpha\| + r_0(D) - r_0(C) + [k_0(D) - k_0(C)] + 1 - t_D = 1$. Else $\dim F^M(C) - \dim F^M(D) \neq 1$. 

\[\square\]

4 Representation of the faces by bipartite graphs

Definition 4.1. [9, Chapter 4.1] A bipartite graph $G$ is said to be elementary if each edge of $G$ lies in some complete matching of $G$.

The definition in [9] also requires $G$ to be connected, which are not done here, nor in [1]. But each component of an elementary graph $G$ will be elementary according to the original definition.
In [1, Section 2] it is shown that there is a one-to-one correspondence between faces of $B_n$ and elementary graphs with $2n$ nodes. Every vertex $P$ of $B_n$ corresponds to a complete matching where the edge $(i, j)$ is in the matching if and only if $p_{ij} = 1$. A face of $B_n$ corresponds to the elementary graph that is the union of the complete matchings corresponding to the vertices of the face. It is easy to see that this representation is equivalent to the representation with $n \times n$ $(0, 1)$-matrices, and that a matrix $A$ with total support is equivalent to an elementary graph $G (a_{ij} = 1 \iff (i, j) \in G)$. If $G$ is an elementary graph, then the corresponding face of $B_n$ is denoted $F^B(G)$.

There is a similar correspondence between the faces of $M(m, n, k)$ and extended elementary graphs with $2(m + n - k)$ nodes which will now be described.

**Definition 4.2.** Let $G = (V, E)$ be a bipartite graph where $V = V_1 \cup V_2$ and $|V_1| = |V_2| = m - n + k$. Let $V_1 = L \cup XR$ where $L$ is the first $m$ nodes in $V_1$ and $XR$ is the last $n - k$ nodes, and let $V_2 = R \cup XL$ where $R$ is the first $n$ nodes in $V_2$ and $XL$ is the last $m - k$ nodes. Then $G$ is called **extended elementary** if it satisfies all of the following.

- $G$ is elementary.
- There are no edges between nodes in $XR$ and nodes in $XL$.
- Every node in $L$ is adjacent to all or none of the nodes in $XL$.
- Every node in $R$ is adjacent to all or none of the nodes in $XR$.

An example of an extended elementary graph can be seen in figure 1.

![Extended elementary graph](image)

**Figure 1:** Extended elementary graph and doped elementary graph

This extended elementary graph $G$ is equivalent to a matrix $A$ with extended support $((i, j) \in G \iff a_{ij} = 1)$, why a face $F^M$ corresponding to a matrix $A$ also corresponds to the graph $G$. The representations of
*M*(m, n, k) by extended elementary graphs and by matrices with extended support are therefore equivalent. No information is lost if *XR* is replaced with one node *NR* and *XL* is replaced with one node *NL*.

**Definition 4.3.** Let *H* be a graph constructed from an extended elementary graph *G* by replacing *XR* with one node *NR* and replacing *XL* with one node *NL*. If *G* has edges from some node to *XL* or *XR*, then let *H* have an edge from that node to *NL* or *NR*, respectively. Then *H* is called **doped elementary**. An example is seen in figure 1.

**Definition 4.4.** A vertex in the polytope does not correspond to an ordinary complete matching in a doped elementary graph, but to a doped matching which is defined as follows: The doped matching consists of a *k*-matching of *L* and *R*, together with edges from *NL* and *NR* to all unmatched nodes in *L* and *R*, respectively (there is *m* − *k* unmatched nodes in *L*, and *n* − *k* unmatched nodes in *R*). Figure 2 shows a doped matching.

Since no information is lost there is a one-to-one correspondence between doped elementary graphs and extended elementary graphs, and therefore a one-to-one correspondence between doped elementary graphs and faces of *M*(m, n, k).

**Lemma 4.5.** Take two doped elementary graphs *H*₁ and *H*₂. Then *H*₁ ⊆ *H*₂ ⇔ *F*^M*(H₁) ⊊ *F*^M*(H₂).

**Proof:** Follows from Corollary 3.4. □

**Theorem 4.6.** The face lattice of *M*(m, n, k) is isomorphic to the lattice of all doped elementary subgraphs of *K*ₘ₊¹,ₙ₊¹ \ (m + 1, n + 1) ordered by inclusion.

**Proof:** It is easy to see that *M*(m, n, k) is represented by the graph *K*ₘ₊¹,ₙ₊¹ \ (m + 1, n + 1). The empty graph with (m + 1) + (n + 1) nodes
is doped elementary and corresponds to $\emptyset$. There is a one-to-one correspondence between doped elementary graphs and faces of $M(m,n,k)$, and by Lemma 4.5 the order is preserved, why the two lattices are isomorphic.

The indecomposable components of a matrix with extended support correspond to the components of the corresponding extended elementary graph. An extended elementary graph has the same number of components as the corresponding doped elementary graph, because in an extended elementary graph, all nodes in $XR$ belong to one component, and all nodes in $XL$ belong to one component.

**Theorem 4.7.** If $F^M$ is a face of $M(m,n,k)$ and the doped elementary graph $H$ representing $F^M$ has $e$ edges and $t$ components, then $\dim F^M = e - m - n + t - a$ where $a = 2$ in the case $k < n \leq m$, $a = 1$ in the case $k = n < m$ and $a = 0$ in the case $k = n = m$.

**Proof:** Since $e = \|C_\alpha\| + (m - r_0) + (n - k_0)$, the result follows from Theorem 3.9.

**Theorem 4.8.** Let $H$ be a doped elementary graph. Then $F^M(H)$ is a one-dimensional face of $M(m,n,k) \iff H$ contains exactly one cycle.

**Proof:** With notation as in the previous theorem, $H$ has $m + n + a$ nodes and $t$ components. If each component in $H$ were a tree, then $H$ would have $m + n + a - t$ edges. Now $\dim H = 1 \iff e = m + n + a - t + 1$, so there is one more edge than if each component in $H$ were a tree. This is equivalent to that $H$ contains exactly one cycle.

### 4.1 Ear decomposition

Ear decompositions of bipartite graphs are described in [9]. They were introduced in [6].

**Definition 4.9.** [9, Chapter 4.1] Let $x$ be an edge. Join its endpoints by a path $E_1$ of odd length (the first ear). Then a sequence of bipartite graphs can be constructed as follows: If $G_{s-1} = x + E_1 + \cdots + E_{s-1}$ has already been constructed, add a new ear $E_s$ by joining any two nodes in different colour classes of $G_{s-1}$ by an odd path ($= E_s$) having no other node in common with $G_{s-1}$. The decomposition $G_s = x + E_1 + \cdots + E_s$ will be called an ear decomposition of $G_s$, and $E_i$ will be called an ear ($i = 1, \ldots, s$).

**Theorem 4.10.** [9, Theorem 4.1.6] A bipartite graph $G$ is elementary if and only if each component of $G$ has an ear decomposition.
**Theorem 4.11.** [1, page 6] If $G$ is an elementary bipartite graph, then the number of ears in an ear decomposition of $G$ is equal to the dimension of $F_B(G)$.

Since doped elementary graphs are not elementary graphs, a slightly different kind of ear decomposition is more convenient to use here.

**Definition 4.12.** In each step of an ear decomposition a new ear is added. When new nodes are added because of the new ear they are said to be activated. This means that an ear begins and ends in activated nodes, and has no other activated nodes.

**Definition 4.13.** Let $G$ be a connected extended elementary graph. Suppose there is an ear decomposition. An ordinary ear that has $2(m-k)$ nodes in $XL$ and $L$ or $2(n-k)$ nodes in $R$ and $XR$, and no other nodes, is called an extended ear. See figure 3.

**Definition 4.14.** Let $H$ be a connected doped elementary graph. A doped ear is a modified ear whose only endpoint is $NL$ or $NR$ and that has $m-k-1$ edges between $NL$ and not activated nodes in $L$ or $n-k-1$ edges between $R$ and $NR$, respectively. See figure 3.

A doped ear decomposition is a modified ear decomposition that except normal ears with nodes in $L$ and $R$, has one (if $k = m$ or $k = n$) or two doped ears.

![Figure 3: Extended ear and doped ear](image)

**Theorem 4.15.** A bipartite graph $H = (V_1 \cup V_2, E)$ where $V_1 = L \cup NR$, $V_2 = NL \cup R$, $|NL| = |NR| = 1$, and where there is no edge between $NL$ and $NR$, is doped elementary if and only if every component not containing $NR$ or $NL$ has an ear decomposition and every component containing at least one of $NR$ and $NL$ has a doped ear decomposition.

**Proof:** Suppose there is such a graph $H$ where every component has an ear decomposition or a doped ear decomposition. Construct a graph $G$ by replacing the node $NL$ with $m-k$ nodes in a set $XL$ and letting every
node in $XL$ be adjacent to the same nodes in $L$ as $NL$, and by replacing
the node $NR$ with $n - k$ nodes in a set $XR$ in the same manner.

Consider the components $K_H$ and $K_G$ in $H$ and $G$ containing $NL$ and $XL$,
respectively. Then $K_H = x + E_1 + \ldots + E_s$, where $E_j$ is the doped ear.
Then $K_G = x + E_1 + \ldots + E_j' + \ldots + E_s + E_{s+1} + \ldots E_{s+t}$, where
the first node in $XL$ replaces $NL$, $E_j'$ is an extended ear which begins in
the first node in $XL$ and ends in a previously activated node in $L$ (there is such a node
since the first node in $XL$ is activated) and has the same other nodes in $L$ as $E_j$, and $E_{s+1}, \ldots, E_{s+t}$ are ears consisting of one edge each between $XL$
and $L$. An example is seen in figure 4, where $E_{s+1}, \ldots, E_{s+t}$ are omitted.

![Diagram showing the change from a doped ear decomposition to an extended ear decomposition](image)

Figure 4: Extension of ear decomposition

The same can be done with the components in $H$ and $G$ containing $NR$
and $XR$, respectively. Theorem 4.10 now implies that $G$ is elementary,
and the construction of $G$ and its ear decomposition implies that $G$ is an
extended elementary graph, and that $H$ is a doped elementary graph.

Suppose $H$ is doped elementary. Let $G$ be the corresponding extended
elementary graph. Then the components of $G$ have ear decompositions.
The ear decomposition of the component $K_G$ containing $XL$ can be re-
arranged into an ear decomposition with an extended ear according to Lemma
4.16 below (the same applies to the component containing $XR$). Now
$K_G = x + E_1 + \ldots + E_j + \ldots + E_s + E_{s+1} + \ldots E_{s+t}$, where $E_j'$ is an extended
ear and $E_{s+1}, \ldots, E_{s+t}$ are all ears consisting of one edge between $XL$
and $L$ not incident with the first node in $XL$. Then $K_H = x + E_1 + \ldots + E_j + \ldots + E_s$, where
$E_j$ is a doped ear corresponding to the extended ear $E_j'$. The same
can be done with the components containing $XR$ and $NR$. The compo-
nants of $H$ containing $NR$ and $NL$ now have doped ear decompositions,
and the other components can keep the same ear decompositions as in $G$.

Lemma 4.16. Let $G$ be an extended elementary graph. An ear decom-
position for a component in $G$ containing $XL$ or $XR$ can always be changed
into an ear decomposition with an extended ear containing $XL$ or $XR$, respectively.
Proof: Suppose there is an ear decomposition for the component containing \( XL \) (the component containing \( XR \) can be treated in analogue). Ears consisting of one edge each can be placed anywhere in the ear decomposition after the activation of its endpoints.

Ears can contain different patterns, and by changing the patterns in the ears, this ear decomposition can be changed to an ear decomposition with an extended ear containing \( XL \). Six patterns and two different types of ears will be characterised, and then changes between different patterns and ears will be described. In the figures a node in a circle is an earlier activated node, and ears have numbers sometimes to tell in which order they come in the ear decomposition.

**Figure 5: Pattern 1, 2 and 3**

**Pattern 1, 2 and 3:** Pattern 1 begins with an edge from an activated node in \( XL \) to a node in \( L \) followed by a number (\( \geq 1 \)) of edges between \( L \) and \( XL \), and then an edge from a node in \( L \) to a node in \( R \) follows. Pattern 2 begins with an edge from an activated node in \( L \) to a node in \( XL \) followed by an edge to a node in \( L \) and an edge to a node in \( R \). Pattern 3 is just like pattern 2 but it has an edge from a node in \( R \) to a node in \( L \) instead of beginning in \( L \).

**Pattern 4, 5 and 6** Pattern 4 begins with an edge from an activated node in \( L \) to a node in \( XL \) followed by a number (\( \geq 1 \)) of edges between \( XL \) and \( L \), and then an edge from a node in \( L \) to a node in \( R \). Pattern 5 is just like pattern 4 but it has an edge from a node in \( R \) to a node in \( L \) instead of beginning in \( L \). Pattern 6 is all patterns except patterns 1–5.

**Ears of type 7 and 8** Ears of type 7 are ears consisting of one edge each between \( XL \) and \( L \). Ears of type 8 are ears that have at least 3 edges, and
only has edges between $XL$ and $L$. There is also the starting edge $x$ for the ear decomposition.

Pattern 1, 2, 3, 4 and 5 will, with the help of convenient ears of type 7, be changed to pattern 6 and ears of type 7 and 8, so that in the end at most one ear contain pattern 2 or 3 (no ear will if $x$ is between $XL$ and $L$). If the last ear contains more than one part of pattern 2 or 3, all of the parts except one will be changed to pattern 6. At last ears of type 8 will, with the help of convenient ears of type 7, be changed to ears of type 7 and one ear of type 8. Here follows a table with the changes, which will also be described by figures.

<table>
<thead>
<tr>
<th>Patterns/Types of ears</th>
<th>Patterns/Types of ears</th>
<th>Example in</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 + 7</td>
<td>6 + 8</td>
<td>figure 7</td>
</tr>
<tr>
<td>4 + 7</td>
<td>2 + 8</td>
<td>figure 8</td>
</tr>
<tr>
<td>5 + 7</td>
<td>3 + 8</td>
<td>figure 8</td>
</tr>
<tr>
<td>2 + 7</td>
<td>6 + 8</td>
<td>figure 9</td>
</tr>
<tr>
<td>3 + 7 + 7</td>
<td>6 + 6 + 8</td>
<td>figure 10</td>
</tr>
<tr>
<td>2/3 + 2/3 + 7</td>
<td>6 + 2/3 + 8</td>
<td>figure 11</td>
</tr>
<tr>
<td>8 + 8(+7)(+7)</td>
<td>8 + 7(+7)(+7)</td>
<td>figure 12</td>
</tr>
</tbody>
</table>

**Figure 7:** Pattern 1 changes to pattern 6

**Figure 8:** Pattern 4 and 5 changes to pattern 2 and 3
Figure 9: Pattern 2 changes to pattern 6

Figure 10: Pattern 3 changes to pattern 6

Figure 11: Two instances of pattern 2/3 changes to one instance of pattern 2/3 and one of pattern 6

Figure 12: Two ears of type 8 changes to one ear of type 8
One ear of type 2, 3, or the edge $x$ is the first one to activate a node in $XL$.

An ear of type 8 can together with two ears of type 7 be replaced by two ears of type 7 and one ear of type 8 that begins in the first activated node in $XL$ and ends in a node in $L$ activated by the ear of type 2, 3, or the edge $x$. See figure 13.

![Figure 13: The last ear of type 8 changes to an extended ear of type 8](image)

Now the ear of type 8 is an extended ear, and can be placed directly after the ear of type 2, 3 or the edge $x$ in the ear decomposition. Then all other ears can come in their order, before the ear of type 2 or 3 if it exists and after the ear of type 8. See figure 14 where all important ears are shown.

![Figure 14: Extended ear decomposition](image)

Now this is an ear decomposition of the component containing $XL$ with an extended ear.

**Theorem 4.17.** Let $H$ be a doped elementary graph, corresponding to the face $F^M(H)$ of $M(m,n,k)$. Then the number of ears not being doped ears in a doped ear decomposition of $H$ is equal to the dimension of $F^M(H)$.

**Proof:** Let $G$ be the extended elementary graph corresponding to $H$. If $NL$ belongs to component $K_{2}^{H}$ in $H$, then the ear decomposition of $K_{2}^{H}$
can be extended to an ear decomposition for the corresponding component \( K^2_G \) in \( G \). This extension is described in the proof of Theorem 4.15. There are \( m - r_0 \) edges between \( NL \) and \( L \), and there are \((m - r_0)(m - k)\) edges between \( XL \) and \( L \). The doped ear in \( K^2_H \) is changed to an extended ear in \( K^2_G \), and the extended ear contains \( m - k \) more edges than the doped ears. All other ears remain as they are. Now there are \((m - k - 1)(m - r_0 - 1) - 1\) edges between \( XL \) and \( L \) that are not part of any ear, and since all vertices in \( XL \) and \( R \) are previously activated, each of these edges will be a new ear. If the doped ear is not counted, \( K^2_G \) will have \((m - k - 1)(m - r_0 - 1)\) more ears than \( K^2_H \), and if there is a component \( K^1_G \) containing \( XR \), it will have \((n - k - 1)(n - k_0 - 1)\) more ears than the corresponding component \( K^1_H \).

As seen in the end of the proof of Theorem 3.9, the difference between the number of ears is exactly the difference between the dimension of \( F^M(H) \) (\( = F^M(G) \)) and \( F^B(G) \). The result now follows from Theorem 4.11.

## 5 The diameter of \( M(m, n, k) \)

The graph of a polytope is the graph whose nodes are the vertices of the polytope and whose edges are the one-dimensional faces of the polytope. The diameter of the polytope is the diameter of its graph, which is the smallest number \( \delta \) such that between any two nodes in the graph there is a path with at most \( \delta \) edges.

In this section the diameter of \( M(m, n, k) \), which is denoted \( \delta(M(m, n, k)) \), will be computed. The algorithm given in the proofs of Theorem 5.4 and Theorem 5.7 can be used to find a path with at most \( \delta(M(m, n, k)) \) edges between two given vertices of \( M(m, n, k) \).

**Definition 5.1.** Let \( H^{(1)} \) and \( H^{(2)} \) be doped elementary graphs corresponding to the vertices \( v^{(1)} \) and \( v^{(2)} \) of \( M(m, n, k) \). Let \( b_L(H^{(1)}, H^{(2)}) \) be the number of nodes in \( L \) adjacent to \( NL \) in \( H^{(1)} \) but not in \( H^{(2)} \), and let \( b_R(H^{(1)}, H^{(2)}) \) be the number of nodes in \( R \) adjacent to \( NR \) in \( H^{(1)} \) but not in \( H^{(2)} \). If \( b = \max(b_L, b_R) \), then \( b \) is the **difference** of \( v^{(1)} \) and \( v^{(2)} \). Note that \( b_L \) and \( b_R \) are well defined.

**Theorem 5.2.** If two vertices of \( M(m, n, k) \) are the vertices of a one-dimensional face of \( M(m, n, k) \), then the difference of the vertices is at most 1.

**Proof:** Let \( H^{(1)} \) and \( H^{(2)} \) be the corresponding doped elementary graphs and \( H = H^{(1)} \cup H^{(2)} \). Suppose \( b_L(H^{(1)}, H^{(2)}) \geq 2 \) (\( b_R \) is treated in analogue). In \( H \) then at least four nodes in \( L \) have degree 2. It is easy to see that each of these four nodes have to be contained in a cycle or in a path.
from \(NL\) to \(NR\) (no vertex in \(L\) with degree 1 is adjacent to a vertex in \(R\) with degree 2 and vice versa).

Since there are edges from each of these four nodes to \(NL\), at most two of them can be contained in one single cycle, and two paths from \(NR\) to \(NL\) form a cycle. Hence there are at least two cycles in \(H\), why Theorem 4.8 implies that the vertices can not be the vertices of a one-dimensional face of \(M(m, n, k)\). Thus if the vertices are the vertices of a one-dimensional face, then the difference of the vertices is at most 1.

\[\square\]

**Corollary 5.3.** If the difference of two vertices of \(M(m, n, k)\) is \(b\), then the shortest path between the two vertices has at least \(b\) edges.

**Theorem 5.4.** If \(k \geq 2\), then \(\delta(M(k + 2, k + 2, k)) \leq 2\).

**Proof:** The idea of proof is taking the graph \(H\) of the doped matchings corresponding to two arbitrary vertices, and then finding an intermediate vertex which differ by at most 1 from the two vertices.

This will be proven by induction over \(k\). Suppose \(\delta(M(r + 2, r + 2, r)) \leq 2\) when \(r < k\). Then an intermediate vertex can be found for all pairs of vertices in \(M(r + 2, r + 2, r)\). Now look at \(M(k + 2, k + 2, k)\). The cases possible to reduce are when \(H\) has a path of length three in \(L\) and \(R\) that is not a subset of a cycle of length 4, and when \(H\) has two cycles of length 4 in \(L\) and \(R\) and \(k \geq 5\). These cases with reduction, intermediate vertices in the reduced case intermediate vertex in the original case are shown in figure 15 and figure 16. Of course it is possible to reduce the case when the two vertices share an edge between \(L\) and \(R\), since removing that edge and its incident nodes decreases \(m, n\) and \(k\) by 1, and the intermediate vertex then can be augmented by adding them again.

![Figure 15: The first case possible to reduce in the induction](image)

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The base cases, i.e. all cases that are not possible to reduce, together with intermediate vertices are shown in figure 17. When there is an alternative for the second vertex in the figure, the alternative vertex is obtained by altering the edges in the small cycle made of edges from the second vertex and its alternative edges. In one case where \( k = 2 \) only a part of each vertex is sketched since there are several possible positions for the remaining edges. Hence each base case have at most one cycle of length four in \( L \) and \( R \) when \( k \geq 5 \), and elsewhere paths of at most length two in \( L \) and \( R \). In addition, no edge between \( L \) and \( R \) are shared by the two vertices, why there must be an even number of paths of length one in \( L \) and \( R \). Let \( p_0 \) be the number of cycles of length four. Each paths endpoints must be adjacent to \( NL \) or \( NR \), and there are at most four edges incident to each of \( NL \) and \( NR \). Thus \( p_1 + p_2 \leq 4 \), \( p_0 \leq 1 \) when \( k \geq 5 \) and \( k = 2p_0 + p_1/2 + p_2 \). This implies that all cases where \( k \geq 7 \) is possible to reduce, and that the cases in figure 17 are all base cases.

Take two vertices of \( M(k + 2, k + 2, k) \). If \( H \) is one of the base cases, then there is a path of length two between the vertices. If \( H \) is not one of the base cases, it is possible to reduce the case according to figure 15. The two marked nodes and all edges incident with them are removed (if more than one edge from each vertex is removed, new vertices have to be added in the
Figure 17: Base cases for the induction proof
resulting graph). The resulting graph has \((m', n', k') = (m - 1, n - 1, k - 1)\), why there is an intermediate vertex because of the induction assumption. This intermediate vertex is then modified to be an intermediate vertex between the two original vertices by adding one edge and perhaps move some other edge (the modification depends on the intermediate vertex in the reduced case, why there are some different cases in the figures). Hence there is a path of at most length 2 between the vertices.

Thus \(\delta(M(k + 2, k + 2, k)) \leq 2\) for all \(k \geq 2\).

**Corollary 5.5.** If \(\max(m, n) \leq k + 2\), then \(\delta(M(m, n, k)) \leq 2\).

**Theorem 5.6.** The diameter of \(M(m, n, k)\) when \(\max(m, n) < k + 2\) or \(k = 1\):

\[
\begin{align*}
\star & \quad \delta(M(m, n, 1)) = 1 \\
\star & \quad \delta(M(k, k, k)) = \begin{cases} 1 & \text{if } k \leq 3 \\
2 & \text{if } k \geq 4 \\
\end{cases} \\
\star & \quad \delta(M(k + 1, k, k)) = \delta(M(k, k + 1, k)) = \begin{cases} 1 & \text{if } k \leq 2 \\
2 & \text{if } k \geq 3 \\
\end{cases} \\
\star & \quad \delta(M(k + 1, k + 1, k)) = \begin{cases} 1 & \text{if } k = 1 \\
2 & \text{if } k \geq 2 \\
\end{cases}
\end{align*}
\]

**Proof:** To prove that \(\delta(M(m, n, 1)) = 1\), take two different vertices and let \(H\) be as in the proof of Theorem 5.2. Then \(H\) has only two edges between \(L\) and \(R\), why \(H\) has only one cycle. So the vertices, which were arbitrary, are the vertices of a one-dimensional face of \(M(m, n, k)\).

Observation: The number of cycles in a doped elementary graph does not decrease if some new nodes in \(L\) and some new nodes in \(R\) are added, and edges are added between the new nodes and \(NL\) and \(NR\), respectively. This increases \(m\) and \(n\). Hence Theorem 4.8 implies that \(\delta(M(m_1, n_1, k)) \leq \delta(M(m_2, n_2, k))\) if \(m_1 \leq m_2\) and \(n_1 \leq n_2\).

The proof of the rest is now simple. Since the nodes in \(H\) have at most degree 2, \(H\) has to have at least 8 edges if there is two cycles in \(H\). There is at most \(2(m + n - k)\) edges in \(H\). By Theorem 4.8 now follows that \(M(2, 2, 2), M(3, 2, 2)\) and \(M(3, 3, 3)\) have diameter 1.

Figure 18 shows that \(M(4, 3, 3), M(4, 4, 4)\) and \(M(3, 3, 2)\) have diameter 2, why the above observation completes the proof.

**Theorem 5.7.** If \(\max(m, n) \geq k + 2\), then \(\delta(M(m, n, k)) = \min(\max(m, n) - k, k)\).

**Proof:** Let \(G\) be the doped elementary graph corresponding to \(M(m, n, k)\). Take two arbitrary vertices \(v_0\) and \(v_L\). They correspond to two doped
matchings $H_0$ and $H_L$. Note that a doped matching is determined by its $k$ edges between $L$ and $R$.

Suppose $\max(m,n) - k \leq k$. One can assume that $H_0$ has edges from node $j$ in $L$ to node $j$ in $R$, for $1 \leq j \leq k$. Now denote the first $k + 2$ nodes in $L$ and $R$ with $LU$ and $RU$, respectively, and denote the rest of the nodes in $L$ and $R$ with $LD$ and $RD$, respectively (see figure 19).

A new doped matching $H_2$ is defined as follows: Let $E_0'$ be the edges of $H_L$ between $LU$ and $RU$, where $s = |E_0'|$. Let $LU_{RD}$ be all nodes in $LU$ adjacent to nodes in $RD$ in $H_L$, and let $RU_{LD}$ be all nodes in $RU$ adjacent to nodes in $LD$ in $H_L$. One can without loss of generality assume that $|LU_{RD}| \leq |RU_{LD}|$. Put $t := |LU_{RD}|$. Let $E_t$ be a set of $t$ independent edges between $LU_{RD}$ and $RU_{LD}$. Let $H_2$ have the edges in $E_s'$ and $E_t$, and then add $k - s - t$ edges between $LU$ and $RU$ so that $H_2$ has $k$ independent edges between $LU$ and $RU$. Now $H_2$ is determined. There are examples of $v_0$, $v_L$ and $v_2$ in figure 19.

Let $G'$ be the subgraph of $G$ with nodes $LU$ and $RU$ and all edges between $LU$ and $RU$ in $G$. Note that $H_0$ and $H_2$ are identical outside $G'$. Then the restrictions $H_0'$ and $H_2'$ of $H_0$ and $H_2$ to $G'$ are matchings corresponding to vertices in $M(k + 2, k + 2, k)$, why Corollary 5.5 implies that there is a matching $H_1'$ in $G'$ such that $H_0' \cup H_1'$ and $H_2' \cup H_1'$ have at most one cycle each. This matching $H_1'$ can be extended to a doped matching $H_1$ in $G$, and all its new edges coincide with the edges of $H_0$ and $H_2$ outside $G'$, why $H_0 \cup H_1$ and $H_2 \cup H_1$ have at most one cycle each. In figure 20 an example is shown.

Now, for $i = 1, \ldots, t$, let the doped matching $H_{i+2}$ be constructed from $H_{i+1}$ in the following way: Remove edge number $i$ in $E_t$ and one edge between $LU$ and $RU$ not in $E_t$ or $H_L$ from $H_{i+1}$, and then add the two edges in $H_L$ adjacent to edge number $i$ in $E_t$.

Figure 18: Vertices in $M(4,3,3)$ or $M(4,4,4)$ and $M(3,3,2)$
Figure 19: Construction of $v_2$ from $v_0$ and $v_L$

Figure 20: Example of $v_1$, an intermediate vertex of $v_0$ and $v_2$

From the construction of $H_{i+2}$ it is easy to see that $H_{i+2} \cup H_{i+1}$ has at most one cycle. The two added edges together with edge number $i$ in $E_t$ are a part of a path of length 5 from $NL$ to $NR$, and the other removed edge is a part of a path of length 3 from $NL$ to $NR$. All other parts of $H_{i+2}$ and $H_{i+1}$ are identical, why there can only be one cycle (which is of length 8).

Now $H_{t+2}$ contains all edges of $H_L$ between $L$ and $R$ except some eventual edges incident with nodes in $LD$. Since $H_{t+2}$ contains $t$ edges from $H_L$ incident with nodes in $LD$, and $|LD| = m - k - 2$ there are $b \leq m - k - t - 2$
edges in $H_L$ between $L$ and $R$ that are not contained in $H_{t+2}$.

For $i = t + 2 + 1, \ldots, t + 2 + b$, let the doped matching $H_i$ be obtained from $H_{i-1}$ by adding one edge between $LD$ and $R$ belonging to $H_L$ but not $H_{i-1}$, and then removing one edge not belonging to $H_L$ between $L$ and $R$. If there is an edge between $L$ and $R$ in $H_{i-1}$ that is incident with the same node in $R$ as the added edge, then that edge is the one to be removed. Otherwise it does not matter which edge is removed.

Since $H_i$ and $H_{i-1}$ have the same edges except one between $L$ and $R$, $H_i \cup H_{i-1}$ has at most one cycle for all $i$, and $H_{t+2+b} = H_L$.

See figure 21 for examples of $H_i$ and $H_{i-1}$ in both cases.

Since $H_i \cup H_{i-1}$ has at most one cycle for $i = 1, \ldots, b + 2 + t$ Theorem 5.2 implies that there is a path between $v_0$ and $v_L$ of at most length $t + 2 + b \leq t + 2 + \max(m,n) - k - t - 2 = \max(m,n) - k$. The two vertices were arbitrary, hence $\delta(M(m,n,k)) \leq \max(m,n) - k \leq k$.

If $m - k > k$ and $n - k \leq k$ then there are (at least) $m - 2k$ nodes in $L$ adjacent to $NL$ in both $H_0$ and $H_L$. Let $G'$ be the subgraph of $G$ that lacks these $m - 2k$ nodes and all edges incident with them. Then $G'$ corresponds to the graph $M(2k,n,k)$.

If $m - k > k$ and $n - k > k$, then there are also $n - 2k$ nodes in $R$ adjacent to $NR$ in both $H_0$ and $H_L$. Let $G'$ be the subgraph of $G$ that lacks these $m - 2k$ nodes in $L$ and $n - 2k$ nodes in $R$ and all edges incident with them. Then $G'$ corresponds to the graph $M(2k,2k,k)$.
Let $H_0^0$ and $H_0^L$ be the restrictions of $H_0$ and $H_L$ to $G'$. Then there are (in both cases above) a sequence of doped matchings $H_0^0, H_1^0, \ldots, H_\ell^0$ where $H_\ell = H_L$ and $\ell \leq \max(m, n) - k = k$, such that $H_\ell^0 \cup H_{\ell-1}^0$ has at most one cycle for $i = 1, \ldots, \ell$. Let $H_i$ be the doped matching in $G$ with the same edges between $L$ and $R$ as $H_0^i$, for $i = 1, \ldots, \ell$. Then $H_i \cup H_{i-1}$ has at most one cycle for $i = 1, \ldots, \ell$, why Theorem 5.2 implies that there is a path between $v_0$ and $v_L$ of at most length $k$.

Thus $\delta(M(m, n, k)) \leq \min(\max(m, n) - k, k)$ when $\max(m, n) \geq k + 2$.

Let $H_0$ be the same doped matching as before, and let $H_L$ be the doped matching with edges between the last $k$ nodes in $L$ and $R$, respectively. Then the difference of $v_0$ and $v_L$ is $\min(\max(m, n) - k, k)$, why Corollary 5.3 implies that $\delta(M(m, n, k)) \geq \min(\max(m, n) - k, k)$.

Thus $\delta(M(m, n, k)) = \min(\max(m, n) - k, k)$. $\Box$

References


Paper 2
The edge-product space of evolutionary trees has a regular cell decomposition

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Abstract
We investigate the topology and combinatorics of a topological space that is generated by the set of edge-weighted finite trees. This space arises by multiplying the weights of edges on paths in trees and is closely connected to tree-indexed Markov processes in molecular evolutionary biology. We show that this space is a regular cell complex.

Keywords: Trees, forests, partitions, poset, contractibility

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1 Introduction

For a tree $T$, we let $V(T)$ and $E(T)$ denote the sets of vertices and edges of $T$ respectively. For a fixed finite set $X$ we let $T(X)$ denote the (finite) set of trees $T$ that have $X$ as their set of leaves (degree one vertices). Given a map $\lambda : E(T) \to [0,1]$ define

$$p = p_{(T,\lambda)} : \binom{X}{2} \to [0,1]$$

by setting, for all $x,y \in X$,

$$p(x,y) = \prod_{e \in P(T;x,y)} \lambda(e),$$

where $P(T;x,y)$ is the set of edges in the path in $T$ from $x$ to $y$.

Let $E(X,T) \subset [0,1]\binom{X}{2}$ denote the image of the map

$$\Lambda_{T} : [0,1]^{E(T)} \to [0,1]\binom{X}{2}, \quad \lambda \mapsto p_{(T,\lambda)}$$

and let $E(X)$ be the union of the subspaces $E(X,T)$ of $[0,1]\binom{X}{2}$ over all $T \in T(X)$. We call $E(X)$ the edge-product space for trees on $X$.

The main motivation for the study of this space is the connection to the tree reconstruction problem and Markov process on phylogenetic trees, see [8]. In [8] it was shown that $E(X)$ has a natural CW–complex structure for any finite set $X$, and a combinatorial description of the associated face poset was given. It was also conjectured that $E(X)$ is a regular cell complex. Here we prove that this conjecture holds.

**Theorem 1.1.** The edge-product space $E(X)$ has a regular cell decomposition with face poset given by the Tuffley poset.

In Section 2 we review some properties of $X$-trees and of the Tuffley poset which is an order relation on all $X$-trees. In Section 3 we prove that there exists a shelling order for the chains in every interval of the Tuffley poset which we need to finish the proof of Theorem 1.1 in Section 4. Finally we save the proofs of some technicalities of the recursive coatom ordering to Section 5.

2 Preliminaries

In this section we review some material concerning trees and related structures that is presented in [8]. Throughout this paper $X$ will be a finite set.
2.1 Trees

An $X$-tree $T$ is a pair $(T; \phi)$ where $T$ is a tree, and $\phi : X \rightarrow V(T)$ is a map such that all vertices in $V - \phi(X)$ have degree greater than two. We call the vertices in $V - \phi(X)$ unlabelled. Two $X$-trees $(T_1; \phi_1)$ and $(T_2; \phi_2)$ are isomorphic if there is a graph isomorphism $\alpha : V(T_1) \rightarrow V(T_2)$ such that $\phi_2 = \alpha \circ \phi_1$. For an $X$-tree $T = (T; \phi)$ we let $E(T)$ denote the set of edges of $T$.

A collection of bipartitions or splits of $X$ is called a split system on $X$. We will write $A \sqcup B$ to denote the split $\{A, B\}$. Given a split system $\Sigma$ on $X$ and a subset $Y$ of $X$, let

$$\Sigma|_Y = \{B \cap Y | C \cap Y : B|_C \in \Sigma, B \cap Y \neq \emptyset, C \cap Y \neq \emptyset\},$$

called the restriction of $\Sigma$ to $Y$. If $x = B|_C \in \Sigma$, and $B \cap Y \cap C \cap Y$ is contained in $\Sigma|_Y$ then we will denote $B \cap Y \cap C \cap Y$ by $x|_Y$. A split system $\Sigma$ is said to be pairwise compatible if, for any two splits $A|B$ and $C|D$ in $\Sigma$, we have

$$\emptyset \in \{A \cap C, A \cap D, B \cap C, B \cap D\}.$$

Given an $X$-tree, $T = (T; \phi)$, and an edge $e$ of $T$, delete $e$ from $T$ and denote the vertices of the two connected components of the resulting graph by $U$ and $V$. If we let $A = \phi^{-1}(U)$ and $B = \phi^{-1}(V)$ then it is easily checked that $A|B$ is a split of $X$, and that different edges of $T$ induce different splits of $X$. We say that the split $A|B$ corresponds to the edge $e$ (and vice versa). Let $\Sigma(T)$ denote the set of all splits of $X$ that are induced by this process of deleting one edge from $T$. The following fundamental result is due to Buneman [1].

**Proposition 2.1.** Let $\Sigma$ be a split system on $X$. Then, there exists an $X$-tree $T$ such that $\Sigma = \Sigma(T)$ if and only if $\Sigma$ is pairwise compatible. Furthermore, in this case, $T$ is unique up to isomorphism.

Thus we may regard pairwise compatible split systems and (isomorphism classes of) $X$-trees as essentially equivalent. This allows us to make the following definitions that will be useful later.

- Given an $X$-tree $T$ and a non-empty subset $Y$ of $X$ let $T|_Y$ be the $Y$-tree for which $\Sigma(T|_Y) = \Sigma(T)|_Y$.

- For an $X$-tree $T$ and a $Y$-tree $T'$, where $Y \subseteq X$, we say that $T$ displays $T'$ if $\Sigma(T') \subseteq \Sigma(T|_Y) (= \Sigma(T)|_Y)$. 

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A further concept that will be useful to us is the notion of a tree metric, which we will now describe. Suppose that \( T = (T;\phi) \) is an \( X \)-tree, and \( w : E(T) \rightarrow \mathbb{R}^{>0} \). Let \( d_{(T,w)} : (X^2) \rightarrow \mathbb{R}^{>0} \) be defined by

\[
d_{(T,w)}(x,y) = \sum_{e \in P(T;\phi(x),\phi(y))} w(e).
\]

Any function \( d : (X^2) \rightarrow \mathbb{R}^{>0} \) that can be written in this way is said to be a tree metric (with representation \((T,w)\)). Recall that a topological embedding is a map between two topological spaces that is one-to-one and bicontinuous (i.e. a map that is a homeomorphism onto its image). Part (i) of the following lemma is a classic result - see for example Buneman [1]. For part (ii) the map described is injective by part (i), and it is bicontinuous by Theorem 2.1 of [7].

**Lemma 2.2.**

(i) If \( d \) and \( d' \) are tree metrics on \( X \) with representations \((T,w)\) and \((T',w')\) respectively, then \( d = d' \) if and only if \( T \) is isomorphic to \( T' \) and \( w = w' \).

(ii) For each \( X \)-tree \( T \) the map from \((\mathbb{R}^{>0})^{E(T)}\) to \( \mathbb{R}^{(X^2)} \) defined by \( w \mapsto d_{(T,w)} \) is a topological embedding.

### 2.2 The Tuffley Poset

An \( X \)-forest is a collection \( \alpha = \{(A,T_A) : A \in \pi \} \) where

(i) \( \pi \) forms a partition of \( X \), and

(ii) \( T_A \) is an \( A \)-tree for each \( A \in \pi \).

We let \( S(X) \) denote the set of \( X \)-forests. A partial order can be placed on \( S(X) \) as follows [8]. Let \( \alpha = \{(A,T_A) : A \in \pi \} \) and \( \beta = \{(B,T'_B) : B \in \pi' \} \) be two \( X \)-forests. We write \( \beta \leq \alpha \) precisely if the following two conditions hold.

(O1) The partition \( \pi' \) is a refinement of the partition \( \pi \).

(O2) If \( A = \cup_{B \in J} B \) for some \( A \in \pi \) and \( J \subseteq \pi' \) then

(i) for all \( B \in J \), \( T_A \) displays \( T'_B \), and

(ii) for all \( B, C \in J \) with \( B \neq C \) there exists \( F \mid G \in \Sigma(T_A) \) with \( B \subseteq F \) and \( C \subseteq G \).
Informally this order relation translates as follows - \( \beta \leq \alpha \) if the trees in \( \beta \) can be obtained from the trees in \( \alpha \) by collapsing certain edges, and deleting certain other edges, with any resulting unlabelled vertices of degree 2 being suppressed.

The poset \( S(X) \) was first defined (slightly differently) by Christopher Tuffley [10], and it is thus called the Tuffley poset on \( X \).

We now review some important properties of the Tuffley poset.

Let \( \alpha = \{(A, T_A) : A \in \pi \} \in S(X) \). Select one of the elements of \( \alpha \) – say \((A, T_A)\) – together with a split \( x = B|C \in \Sigma(T_A) \). Delete \((A, T_A)\) from \( \alpha \) and replace it by either one of the following:

- \((A, T'_A)\) where \( \Sigma(T'_A) = \Sigma(T_A) - \{B|C\} \), an operation that we call edge contraction (on \( x \));
- the pair \((B, T_A|B)\) and \((C, T_A|C)\), an operation that we call edge deletion (on \( x \)).

Given an \( X \)-forest, \( \alpha = \{(A, T_A) : A \in \pi \} \), let

\[
\Sigma(\alpha) = \cup_{A \in \pi} \Sigma(T_A).
\]

In view of Proposition 2.1, we may view \( \Sigma(\alpha) \) as the set of edges in \( \alpha \). We will not make any distinction in notation between an edge and the corresponding split.

Clearly, for any \( x \in \Sigma(\alpha) \) the set obtained by contraction on \( x \), denoted \( x^c(\alpha) \), or by edge deletion on \( x \), denoted \( x^d(\alpha) \), results in an \( X \)-forest. When \( \alpha \) is clear from the context we will simply write \( x^c \) and \( x^d \). Furthermore,

\[
|\Sigma(x^c(\alpha))| = |\Sigma(\alpha)| - 1, \quad (1)
\]

and

\[
|\Sigma(\alpha)| - 3 \leq |\Sigma(x^d(\alpha))| \leq |\Sigma(\alpha)| - 1. \quad (2)
\]

We will say that the edge deletion \( \alpha \mapsto x^d(\alpha) \) is safe if \( |\Sigma(x^d(\alpha))| = |\Sigma(\alpha)| - 1 \).

The following easily checked lemma provides the graph theoretic interpretation of a safe edge deletion, where we say that a vertex in an \( X \)-tree is unsupported if it is unlabelled and of degree 3.

**Lemma 2.3.** For an \( X \)-forest \( \alpha \), an edge deletion \( \alpha \mapsto x^d(\alpha) \) is safe if and only if neither endpoint of the edge \( e \) that corresponds to \( x \) in \( \alpha \) is unsupported.
We define an elementary operation on an element of $S(X)$ to be either an edge contraction, or a safe edge deletion. The covering relation in a poset will be denoted by the symbol $\preceq$.

The following result which is a restatement of Theorem 4.2 of [8] describes $E(X)$ in terms of these operations, and establishes some further structural properties of the Tuffley poset.

**Theorem 2.4.** Suppose that $X$ is a finite set and $\alpha, \beta \in S(X)$. Then the following statements hold.

(i) $\beta \preceq \alpha$ if and only if $\beta$ can be obtained from $\alpha$ by any sequence of contraction and deletion operations, in which case we can insist that all contractions occur first, and that all the subsequent deletions are safe.

(ii) $\beta \triangleleft \alpha$ if and only if $\beta$ can be obtained from $\alpha$ by one elementary operation.

(iii) $S(X)$ is a pure poset, and for an element $\alpha = \{(A, T_A) : A \in \pi\}$ of $S(X)$ its rank, denoted $\rho(\alpha)$, is given by

$$\rho(\alpha) = |\Sigma(\alpha)|.$$ 

(iv) $S(X)$ is thin.

(v) The maximal elements of $S(X)$ are precisely the elements $\alpha$ for which $\alpha = \{(X, T)\}$ and with $|\Sigma(T)| = 2|X| - 3$.

(vi) The minimal elements of $S(X)$ are precisely the $X$-forests of the form $\alpha = \{(A, T_A) : A \in \pi\}$, with $\Sigma(T_A) = \emptyset$ for all $A \in \pi$.

(vii) Suppose $\alpha$ is an $X$-forest, and that $\alpha$ has an interior vertex $v$ labelled by $m$. Construct an $X'$-forest $\beta$ by removing $v$ from $\alpha$ and giving edge number $i, 1 \leq i \leq \deg(v)$, a new vertex $v_i$ that is labelled by $m_i$, $m_i \not\in X$ and $m_i \neq m_j$ if $i \neq j$. Then $[0, \alpha]$ is isomorphic to $[0, \beta]$.

### 3 Recursive coatom orderings

In this section the following theorem will be proven.

**Theorem 3.1.** For each interval $[0, \alpha] \subset S(X) \cup \{\emptyset\}$ there exist a recursive coatom ordering. (Hence every such interval is shellable)
3.1 Preliminaries and definitions

**Definition 3.2.** A *recursive coatom ordering* for an interval $[0, \Gamma]$ is an ordering $\alpha_1, \ldots, \alpha_t$ of its coatoms that satisfies:

(V1) For all $i < j$ and $\gamma < \alpha_i, \alpha_j$ there is a $k < j$ and an element $\beta$ such that $\beta < \alpha_k, \alpha_j$ and $\gamma \leq \beta$.

(V2) For all $j = 1, \ldots, t$, $[0, \alpha_j]$ admits a recursive coatom ordering in which the coatoms that come first in the ordering are those that are covered by some $\alpha_k$ where $k < j$.

A recursive coatom ordering of an interval has several implications, see e.g. [2].

The edges incident with a vertex $v$ will be denoted $x_1, \ldots, x_n$ (or corresponding for $y$). If the other vertex incident with an edge $x_i$ has degree 3, the two other edges incident with that vertex will be denoted $x_{i1}$ and $x_{i2}$. Else $x_{i1}$ and $x_{i2}$ are said not to exist.

**Definition 3.3.** Let $\Gamma$ be an $X$-forest, and $v$ an interior vertex of $\Gamma$ incident with the edges $x_1, \ldots, x_n$. Let $v_i$ be the other vertex incident with $x_i$, $1 \leq i \leq n$. The coatoms will be denoted as follows: The coatom obtained from $\Gamma$ by contraction of the edge $x_i$ is $x_{ci}(\Gamma)$. The coatom obtained by safe deletion of the edge $x_i$ is $x_{di}(\Gamma)$. If deletion of $x_i$ is not safe, $x_{di}(\Gamma)$ is said not to exist.

The following symbols are also used: The symbol $\_x_{di}(\Gamma)$ denotes $x_{di}(\Gamma)$ if $\deg(v) \geq 4$, $\deg(v_i) \neq 3$, and one of $x_{ci}^{d_1}(\Gamma)$ and $x_{ci}^{d_2}(\Gamma)$ if $\deg(v_i) = 3$. The symbol $\_x_{ci}(\Gamma)$ denotes $x_{ci}(\Gamma)$ if $\deg(v) \geq 4$, $\deg(v_i) \neq 3$, and both of $x_{ci}^{d_1}(\Gamma)$ and $x_{ci}^{d_2}(\Gamma)$ if $\deg(v_i) = 3$. (Note that $\_x_{ci}(\Gamma)$ and $\_x_{ci}(\Gamma)$ does not exist when $\deg(v) = 3$, $\deg(v_i) \neq 3$.)

When no confusion arises, $\Gamma$ is often omitted.

To express a coatom $\alpha$ (of some interval in $S(X)$) which may not exist, the notation $\langle \alpha \rangle$ will be used.

**Definition 3.4.** Let $\mathcal{C}$ and $\mathcal{D}$ be disjoint sets of coatoms, and suppose there is a given coatom ordering. If all elements in $\mathcal{C}$ come before every element in $\mathcal{D}$ in the ordering, then $\mathcal{C} \triangleleft \mathcal{D}$.

**Definition 3.5.** Let $\Gamma$ be an $X$-forest and $v$ a vertex in $\Gamma$. The coatom $\alpha$ is said to be *near* $v$ if $\alpha$ is obtained by (safe) deletion or contraction of an edge $x_i$ incident with $v$ or contraction of $x_{i1}$ or $x_{i2}$.

To prove that there exists a recursive coatom ordering $\alpha_1, \ldots, \alpha_t$ for $[0, \Gamma]$, a new condition called (V3) is used.
Convention 3.6. \( x_1, \ldots, x_n \) (or \( y_i \)) are not in general fixed labels for the edges incident with a vertex \( v \). If for example some condition uses \( x_1 \) and \( x_2 \), it will be true for all \( x_i \) and \( x_j \) with the same properties. The same applies to numbered components of an \( X \)-forest. This concerns all sections in this text.

Definition 3.7. Take an \( X \)-forest \( \Gamma \), containing the non-trivial (with at least one edge) components \( K_1, \ldots, K_m \). If a coatom ordering of \([0, \Gamma]\) satisfies the following conditions, then it is said to satisfy the condition (V3) (Recall Convention 3.6):

\[
\text{(V3)} \quad \begin{align*}
\text{a) (Mixing condition)} & \quad \{ \gamma \mid \gamma \prec \Gamma, \, \gamma = x^c \text{ or } \gamma = x^d \text{ where } x \in \bigcup_{i=1}^{\ell} K_i \} \not\subseteq \{ \gamma \mid \gamma \prec \Gamma, \gamma = x^c \text{ or } \gamma = x^d \text{ where } x \in \bigcup_{i=\ell+1}^{m} K_i \} \text{ for all } 1 \leq \ell \leq m-1. \\
\text{b) If } v \text{ is an interior vertex in } \Gamma, \, n = \deg(v) = 3 \text{ and } x_{31}, x_{32} \text{ exist then } \{ x_1, x_2 \} \not\subseteq \{ x_3 \} \text{ and } \{ x_3 \} \not\subseteq \{ x_1, x_2 \}. \\
\text{c) If } v \text{ is an interior vertex in } \Gamma \text{ and } n = \deg(v) \geq 4 \text{ then } \{ x_1, x_2 \} \not\subseteq \{ x_3, x_4, \ldots, x_n \} \text{ and } \{ x_3, x_4, \ldots, x_n \} \not\subseteq \{ x_1, x_2 \}. \\
\text{d) If } v \text{ is an interior vertex in } \Gamma \text{ and } n = \deg(v) \geq 4 \text{ then } \{ x_1, x_2, \ldots, x_k, x_{k+1}, \ldots, x_n \} \not\subseteq \{ x_k, x_{k+1}, \ldots, x_n \}, \text{ where } 1 \leq k \leq n-1.
\end{align*}
\]

The conditions are obviously possible to reverse, why the reversal of a coatom ordering satisfying (V3) also satisfies (V3).

Definition 3.8. Let \( \Gamma \) be an \( X \)-forest, and let \( \mathcal{A} \) and \( \mathcal{B} \) be disjoint sets of coatoms of \([0, \Gamma]\). Then \( \mathcal{A} \) and \( \mathcal{B} \) are said to be compatible with the condition (V3) if \( \mathcal{A} \not\subseteq \mathcal{B} \) is not forbidden by any single sub-condition of (V3).

If \( \mathcal{A} \) and \( \mathcal{B} \) are compatible with (V3), we write \( \mathcal{A} \cup \mathcal{B} \). If \( \mathcal{A} \) and \( \mathcal{B} \) are not compatible with (V3), we write \( \mathcal{A} \not\cup \mathcal{B} \).

Since the condition (V3) is symmetric, \( \mathcal{A} \cup \mathcal{B} \Leftrightarrow \mathcal{B} \cup \mathcal{A} \).

That \( \mathcal{A} \cup \mathcal{B} \) is obviously a necessary (but maybe not sufficient) condition for the possibility of ordering \( \mathcal{A} \cup \mathcal{B} \) so that the ordering satisfies \( \mathcal{A} \not\cup \mathcal{B} \) and (V3).

Definition 3.9. Two edges in a graph are said to be adjacent if they have a common endpoint.

3.2 There is a recursive coatom ordering for \([0, \Gamma]\)

Corollary 3.10 (of Theorem 3.1). The order complex of any interval \([\alpha, \beta]\) in \( S(X) \) is homotopy equivalent to a sphere.
Proof: Let $\Gamma$ be an $X$-forest. Take the interval $[0, \Gamma]$, turn it upside down and call the resulting poset $P$. Let $\alpha_1, \ldots, \alpha_t$ be a coatom ordering for $[0, \Gamma]$ satisfying (V3). By the proof of Theorem 3.1 there exists such a coatom ordering. Then $\alpha_1, \ldots, \alpha_1$ also satisfies (V3). Theorem 3.11 implies that these are recursive coatom orderings. This means that $\alpha_1, \ldots, \alpha_t$ and $\alpha_t, \ldots, \alpha_1$ are recursive atom orderings of $P$ satisfying (V3). Now construct CL-labellings $\lambda$ and $\mu$ of $P$ in the way described in [4, Theorem 3.2]:

Put $\lambda(0, \alpha_i) = i$ and $\mu(0, \alpha_i) = -i$, $1 \leq i \leq t$. Let $F(\alpha_j)$ be the set of all atoms of $\alpha_j$ that cover some $\alpha_i$ where $i < j$, and let $H(\alpha_j)$ be the set of all atoms of $\alpha_j$ that cover some $\alpha_k$ where $k > j$. Then choose an atom ordering for $\alpha_j$ satisfying (V3) and $F(\alpha_j) \triangleleft H(\alpha_j)$. By theorem 3.12 this is possible.

Choose $\lambda(\alpha_j, \beta) < \lambda(0, \alpha_j)$ if $\beta \in F(\alpha_j)$ and $\lambda(\alpha_j, \beta) > \lambda(0, \alpha_j)$ if $\beta \in H(\alpha_j)$ and let $\lambda(\alpha_j, \beta) < \lambda(\alpha_j, \gamma)$ if $\beta$ comes before $\gamma$ in the atom ordering.

The reverse of the atom ordering for $\alpha_j$ satisfies (V3) and $H(\alpha_j) \triangleleft F(\alpha_j)$. Put $\mu(\alpha_j, \beta) = -\mu(0, \alpha_j)$ for all atoms $\beta$ of $\alpha_j$. Then $\mu(\alpha_j, \beta) < \mu(0, \alpha_j)$ if $\beta \in H(\alpha_j)$, $\mu(\alpha_j, \beta) > \mu(0, \alpha_j)$ if $\beta \in F(\alpha_j)$ and $\mu(\alpha_j, \gamma) < \mu(\alpha_j, \beta)$ if $\gamma$ comes before $\beta$ in the reverse atom ordering.

Continue in this way, until two complete CL-labellings are obtained. Now $\lambda = -\mu$. Every interval in $P$ has exactly one rising maximal chain with respect to $\lambda$ since it is a CL-labelling. Since $\lambda = -\mu$, every interval now has exactly one decreasing maximal chain with respect to $\mu$. Thus the Möbius function for each interval is $-1$ to the power of the length of that interval. The Möbius function is the same when the poset $P$ is turned upside down again.

The Möbius function of $(x, y)$ have the same value as the reduced Euler characteristics of the order complex of the open interval $(x, y)$ (see [2]). The interval $(x, y)$ is pure and shellable, why this order complex is homotopy equivalent to a wedge of spheres of the same dimension, and the Euler characteristics then is $\pm 1$ times the number of spheres. This implies that there is only one sphere, why the order complex of any interval $[a, b]$ in $S(X)$ is homotopy equivalent to a sphere. 

Proof of Theorem 3.1: Let $\Gamma$ be an $X$-forest. The existence of a coatom ordering satisfying (V3) for $[0, \Gamma] \subset S(X)$ is implied by Lemmas 3.14 - 3.16 if the interval $[0, \Gamma]$ is considered with $A = \emptyset$ and $B = \{\gamma \mid \gamma \in \Gamma\}$. An example of a coatom ordering satisfying (V3) is shown in section 3.5. Then this coatom ordering is a recursive coatom ordering for $[0, \Gamma]$ according to Theorem 3.11.

Theorem 3.11. Let $\Gamma$ be an $X$-forest, and let $\alpha_1, \ldots, \alpha_t$ be a coatom ordering for $[0, \Gamma]$. If $\alpha_1, \ldots, \alpha_t$ satisfies (V3), then it is a recursive coatom ordering.
Proof: Assume $\alpha_1, \ldots, \alpha_t$ is a coatom ordering for $[0, \Gamma]$ satisfying (V3). It will be shown by induction over $|\Sigma(\Gamma)|$ that $\alpha_1, \ldots, \alpha_t$ is then a recursive coatom ordering. Remember that if $\alpha$ is a coatom of $[0, \Gamma]$ then $|\Sigma(\alpha)| = |\Sigma(\Gamma)| - 1$.

I) If $|\Sigma(\Gamma)| \leq 1$ there is only one or two coatoms. In these cases all possible coatom orderings satisfy (V3) and are recursive coatom orderings.

II) Assume that if $0 \leq |\Sigma(\Gamma)| \leq q (q \geq 1)$, then every coatom ordering of $[0, \Gamma]$ satisfying (V3) is a recursive coatom ordering.

III) Take an $X$-forest $\Gamma$ such that $|\Sigma(\Gamma)| = q + 1$. Let $\alpha_1, \ldots, \alpha_t$ be a coatom ordering for $[0, \Gamma]$ satisfying (V3). It has to be shown that parts (V1) and (V2) of Definition 3.2 are satisfied.

Lemma 3.17 says that (V3) $\Rightarrow$ (V1).

Take a $j$, $1 \leq j \leq t$. Let $A = \{ \gamma \mid \gamma \leq \alpha_k, \alpha_j \text{ and } k < j \}$ and $B = \{ \gamma \mid \gamma \leq \alpha_k, \alpha_j \text{ and } k > j \}$.

Theorem 3.12 says that there is a coatom ordering for $[0, \alpha_j]$ satisfying (V3) and $A \triangle B$, which by the induction assumption is a recursive coatom ordering. Hence (V2) is satisfied.

Thus $\alpha_1, \ldots, \alpha_t$ is a recursive coatom ordering for $[0, \Gamma]$.

Now it is proved by induction that if $\alpha_1, \ldots, \alpha_t$ is a coatom ordering for $[0, \Gamma]$ satisfying (V3), then it is a recursive coatom ordering $\square$

Theorem 3.12. Let $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_t$ be a coatom ordering of $[0, \Gamma]$ satisfying (V3). Fix $j$, and consider the interval $[0, \alpha_j]$.

Let $A = \{ \gamma \mid \gamma \leq \alpha_k, \alpha_j \text{ and } k < j \}$ and let $B = \{ \gamma \mid \gamma \leq \alpha_k, \alpha_j \text{ and } k > j \}$.

The sets $A$ and $B$ are disjoint since $S(X)$ is thin.

Then there is a coatom ordering for $[0, \alpha_j]$ satisfying (V3) and $A \triangle B$.

Proof: The theorem follows directly from Lemmas 3.13 – 3.16. To achieve coatom orderings of the coatoms near each interior vertex that do not contradict each other, begin with an ordering of the coatoms near one interior vertex and then recursively choose an ordering of the coatoms near each adjacent vertex that does not contradict it. That is possible since the coatoms near two vertices are exactly those possible to choose orders of by Lemmas 3.14 and 3.15. $\square$

Lemma 3.13. The same assumptions as in Theorem 3.12 is made here. If $v$ is an interior vertex of $\alpha_j$, let $A_v = \{ \gamma \in A \mid \gamma \text{ is near } v \}$ and let $B_v = \{ \gamma \in B \mid \gamma \text{ is near } v \}$. Then $A_v \triangle B_v$. This is shown in section 5.2.
Lemma 3.14. Let α be an $X$-forest, and let $v$ be an interior vertex of $\alpha$ with degree 3. Suppose the coatoms near $v$ are partitioned into two sets $C$ and $D$ where $C \cup D$. Then there is always an ordering of $C \cup D$ that satisfies (V3) and $C \subset D$, and has a given order (that does not violate (V3b)) of those of $x_1^v, x_2^v, x_3^v, x_1^{v1}$ and $x_2^{v2}$ that are in $C$, and the same for $D$. This is shown in section 5.1.

Lemma 3.15. Let $\alpha$ be an $X$-forest, and let $v$ be an interior vertex of $\alpha$ with $\deg(v) \geq 4$. Suppose the coatoms near $v$ are partitioned into two sets $C$ and $D$ where $C \cup D$. Then there is always an ordering of $C \cup D$ that satisfies (V3) and $C \subset D$, and has a given order of those of $x_1^v$ and $x_2^v$ that are in $C$, and the same for $D$. This is shown in section 5.1.

Lemma 3.16. Suppose there are orderings of the coatoms near each interior vertex $v$ in $\alpha_j$ that do not contradict each other in the cases when some coatoms are near more than one interior vertex.

Then it is possible to find a combination of the orderings of the coatoms near each interior vertex of $\alpha_j$ such that the result is a coatom ordering for $[0, \alpha_j]$ satisfying (V3) and $A \cup B$. This is shown in section 5.4.

3.3 Reformulation of (V1) with implications

Lemma 3.17. Let $\Gamma$ be an $X$-forest, and let $\alpha_1, \ldots, \alpha_t$ be the coatoms of $[0, \Gamma]$. If $\alpha_1, \ldots, \alpha_t$ satisfies (V3), then it also satisfies part (V1) of Definition 3.2.

Part (V1) of Definition 3.2 is: For all $i < j$ and $\gamma < \alpha_i, \alpha_j$ there is a $k < j$ and an element $\beta$ such that $\gamma \leq \beta < \alpha_k, \alpha_j$.

An equivalent formulation is:

For each interior vertex $v$ of $\Gamma$, where $\deg(v) = n$, the following must apply (Recall Convention 3.6):

1. $\{x_1^v, x_2^v\} \not\subset \{x_3^v, x_4^v\}$ when $n = 3$, $x_3^v, x_4^v$ exist
2. $\{x_1^v, x_2^v\} \not\subset \{x_3^v, \ldots, x_n^v\}$ when $n \geq 4$
3. $\{x_1^v, x_2^v\} \not\subset \{x_3^v, \ldots, x_n^v\}$ when $n \geq 4$, $x_3^v, x_4^v$ exist
4. $\{x_1^v, x_2^v\} \not\subset \{x_3^v, \ldots, x_n^v\}$ when $n \geq 4$, $x_1^v, x_2^v$ exist
5. $\{x_1^v, x_2^v\} \not\subset \{x_3^v, \ldots, x_n^v, x_5^v\}$ when $n \geq 4$, $x_5^v$ exists
6. Furthermore, if there is a component $K$ in $\Gamma$ with only one edge $x$, then the coatoms $x^e$ and $x^d$ are not the two first coatoms in the ordering.

That the new formulation of (V1) is equivalent to the original one, follows from the investigation of common elements of $[0, \alpha_i]$ and $[0, \alpha_j]$ which is made in section 3.4.
Proof of Lemma 3.17: It is easy to see that \((V3b) \Rightarrow (V1a), (V3c) \Rightarrow (V1b), (V3c) \Rightarrow (V1c), (V3d) \Rightarrow (V1e),\) and \((V3a) \Rightarrow (V1f).\) Now \((V1d)\) is left. Suppose the coatom ordering satisfies \((V3)\) but not \((V1d).\) Then some \(x_i^c, 2 \leq i \leq n,\) has to come before \(x_1^c\) in the ordering because of \((V3d).\) But then \(\{x_1^c, x_i^c\} \cup \{x_2^d, \ldots, x_n^d\}\) contradicts \((V3c).\) Hence \((V3c)\) and \((V3d)\) implies \((V1d).\)

Thus \((V3) \Rightarrow (V1).\) □

3.4 Common elements of \([\hat{0}, \alpha_i]\) and \([\hat{0}, \alpha_j]\)

Let \(\Gamma = \{(A, T_A) : A \in \pi_T\}.\) Let \(\alpha_1, \ldots, \alpha_t\) be the coatoms of \(\Gamma.\) To deal with the conditions \((V1), (V2)\) and \((V3)\) it is important to know the common elements of \([\hat{0}, \alpha_i]\) and \([\hat{0}, \alpha_j]\) (when \(\alpha_i \neq \alpha_j).\)

We know that \(\alpha_i\) and \(\alpha_j\) can be obtained from \(\Gamma\) by one elementary operation each, that is, by an edge contraction or by a safe edge deletion. These edges can be denoted \(z_i\) and \(z_j,\) where \(z_i \in \Sigma(T_{A_i})\) and \(z_j \in \Sigma(T_{A_j}),\) \(A_i, A_j \in \pi_T.\) If \(A_i = A_j\) then \(z_i = B_i | B_j \cup C, z_j = B_j | B_i \cup C.\)

There are different cases depending on how \(\alpha_i\) and \(\alpha_j\) are obtained from \(\Gamma.\) For each case below we find \(\delta\) or \(\delta_1\) and \(\delta_2\) such that

\[
\begin{align*}
\gamma \leq \alpha_i, \alpha_j & \iff \gamma \leq \delta \text{ or } \\
\gamma \leq \alpha_i, \alpha_j & \iff \gamma \leq \delta_1 \text{ or } \gamma \leq \delta_2
\end{align*}
\]

That this is really the case is easy to show by the definition of \(\leq\) and the investigation below.

The fact that \(S(X)\) is thin implies that if \(\gamma \prec \alpha_i\) then there is a unique \(1 \leq \ell \leq t\) such that \(\gamma \prec \alpha_{\ell}.\) From the above now follows that the first and second formulation of \((V1)\) are equivalent.

\(z_i, z_j\) not adjacent:

Let \(\delta\) be obtained by first applying one of the elementary operations on \(\Gamma,\) and then applying the other elementary operation on the result. Then \(\delta \prec \alpha_i, \alpha_j.\)

\(z_i, z_j\) adjacent:

Let \(v\) be the vertex incident to both \(z_i\) and \(z_j.\)

If \(\alpha_i = z_i^c(\Gamma), \alpha_j = z_j^c(\Gamma)\) (or vice versa) and \(z_i \neq z_j,\) let \(\delta = z_j^c(\alpha_i) = z_i^c(\alpha_j).\) Then \(\delta \prec \alpha_i, \alpha_j.\)

If \(\alpha_i = z_i^c(\Gamma)\) and \(\alpha_j = z_j^c(\Gamma),\) let \(\delta_1 = z_j^c(\alpha_i) = z_i^c(\alpha_j) \prec \alpha_i, \alpha_j.\)

Let \(\delta_2\) be obtained from \(\Gamma\) by deletion of all edges incident with \(v\) except \(z_i\) and \(z_j.\)
Then $\delta_2 < \alpha_i, \alpha_j$ if and only if $\deg(v) = 3$ and the vertex $v'$ not incident with $z_i$ or $z_j$ but adjacent to $v$ has not degree $3$. If the third edge incident with $v$ in this case is $x$, then $\delta_2 = x^d(\alpha_i) = x^d(\alpha_j)$.

Else if $\deg(v) = 3$ and $\deg(v') = 3$ then $\delta_2 < \alpha_i, \alpha_j$, and the X-forests $\beta$ satisfying $\delta_2 < \beta < \alpha_i$ are exactly all $\beta$ such that $\beta = x^c(\alpha_i), x$ incident with $v'$ but not with $v$. Then $\beta < x^c(\Gamma), \alpha_i$. The corresponding applies for $\alpha_j$. This gives condition $(V1a)$.

Else if $\deg(v) \geq 4$ then $\delta_2 < \alpha_i, \alpha_j$, and the X-forests $\beta$ satisfying $\delta_2 < \beta < \alpha_i$ are exactly all $\beta$ such that $\beta = x^d(\alpha_i), x \neq z_i, z_j$ incident with $v$. Then $\beta < x^d(\Gamma), \alpha_i$. The corresponding applies for $\alpha_j$. This gives condition $(V1b)$.

If $\alpha_i = z^d_i(\Gamma)$ and $\alpha_j = z^d_j(\Gamma)$, then let $\delta$ be obtained from $\Gamma$ by deletion of $z_i$ and $z_j$. Then $\delta < \alpha_i, \alpha_j$ if and only if $\deg(v) \geq 5$. In that case $\delta = z^d_i(\Gamma) = z^d_j(\Gamma)$.

Else $\delta < \alpha_i, \alpha_j$, and the X-forests $\beta$ satisfying $\delta < \beta < \alpha_i$ are exactly all $\beta$ such that $\beta = x^c(\alpha_i), x \neq z_i, z_j$ incident with $v$. Then $\beta < x^c(\Gamma), \alpha_i$. The corresponding applies for $\alpha_j$. This gives condition $(V1c)$.

$z_1 = z_2$:

Let $v_1$ and $v_2$ be the endpoints of $z_i$. Suppose $\alpha_i = z^c_i$ and $\alpha_j = z^d_i$. If $\deg(v_1) \geq 4$, let $\delta_1$ be obtained from $\alpha_i$ by deleting all edges incident with $v_1$. If $\deg(v_2) \geq 4$, let $\delta_2$ be obtained from $\alpha_j$ by deleting all edges incident with $v_2$. Then $\delta_1, \delta_2 < \alpha_i, \alpha_j$.

The X-forests $\beta_{v_1}$ satisfying $\delta_1 < \beta_{v_1} < \alpha_i$ are exactly all $\beta_{v_1}$ such that $\beta_{v_1} = x^d(\alpha_i), x \neq z_i$ incident with $v_1$. Then $\beta_{v_1} < x^d(\Gamma), \alpha_i$. The corresponding applies to the X-forests $\beta_{v_2}$ satisfying $\delta_2 < \beta_{v_2} < \alpha_i$. This gives condition $(V1d)$.

The X-forests $\beta_{v_1}$ satisfying $\delta_1 < \beta_{v_1} < \alpha_i$ are exactly all $\beta_{v_1}$ such that $\beta_{v_1} = x^c(\alpha_i) or \beta_{v_1} = x^d(\alpha_i), x \neq z_i$ incident with $v_1$. Then $\beta_{v_1} < x^c(\Gamma), \alpha_i$ or $\beta_{v_1} < x^d(\Gamma), \alpha_i$. The corresponding applies to the X-forests $\beta_{v_2}$ satisfying $\delta_2 < \beta_{v_2} < \alpha_i$. This gives condition $(V1e)$.

If $v_1$ and $v_2$ are both leaves, then let $\delta = 0$. The X-forests $\beta$ satisfying $\delta < \beta < \alpha_i$ are exactly all $\beta$ such that $\beta = x^c(\alpha_i) or \beta = x^d(\alpha_i)$ where $x \neq z_i$. Then $\beta < x^c(\Gamma), \alpha_i or \beta < x^d(\Gamma), \alpha_i$. This gives condition $(V1f)$.

### 3.5 A coatom ordering of $\hat{[0, \Gamma]}$ satisfying (V3)

Here is an example of a coatom ordering of $\hat{[0, \Gamma]}$ that satisfies (V3).

Do the following for each component in $\Gamma$. Begin with an interior vertex $v_1$ and order the coatoms near $v_1$ in the following way:
Denote the edge with the following way. For each vertex \( v \) then add the coatoms near the vertices adjacent to \( v \).

Then add the coatoms near the vertices adjacent to \( v \) to the order in the following way. For each vertex \( v_i \) adjacent to \( v \) do:

- If \( \deg(v_1) = 3 \):
  \[ \langle x_{11}, \langle x_{21} \rangle, \langle x_{31} \rangle, x_{12}, x_{13}, \langle x_{12} \rangle, \langle x_{32} \rangle \rangle. \]

- If \( \deg(v_1) = n \), \( n \geq 4 \) and \( n \) is even:
  \[ x_{1}^{c_1}, x_{2}^{c_1}, x_{3}^{c_1}, \ldots, x_{n-1}^{c_1}, x_{n}^{c_1}, x_{1}^{c_2}, x_{2}^{c_2}, x_{3}^{c_2}, \ldots, x_{n-1}^{c_2}, x_{n}^{c_2}. \]

- If \( \deg(v_1) = n \), \( n \geq 4 \) and \( n \) is odd:
  \[ x_{1}^{c_1}, x_{2}^{c_1}, x_{3}^{c_1}, \ldots, x_{n-1}^{c_1}, x_{n}^{c_1}, x_{1}^{c_2}, x_{2}^{c_2}, x_{3}^{c_2}, \ldots, x_{n-1}^{c_2}, x_{n}^{c_2}. \]

Denote the edge with \( v_1 \) and \( v_2 \) as endpoints \( x_1 \).

- If \( \deg(v_1) = 3 \): Let \( x_{11}^{c_1} \) and \( x_{12}^{c_1} \) keep their relative order to \( x_{11}^{c_1} \) and \( x_{12}^{c_1} \) (if they exist), and order the rest of the coatoms in the following way:
  \[ \langle x_{11}, \langle x_{21} \rangle, \langle x_{31} \rangle, x_{12}, x_{13}, \langle x_{12} \rangle, \langle x_{32} \rangle \rangle. \]

- If \( \deg(v_1) = n \), \( n \geq 4 \) and \( n \) is even: Let \( x_{1}^{c_1} \) and \( x_{2}^{c_1} \) have the same order as before, and chose the appropriate of the following orders:
  - \( x_{1}^{c_1}, x_{2}^{c_1}, x_{3}^{c_1}, \ldots, x_{n-1}^{c_1}, x_{n}^{c_1}, x_{1}^{c_2}, x_{2}^{c_2}, x_{3}^{c_2}, \ldots, x_{n-1}^{c_2}, x_{n}^{c_2}. \) (\( x_{1}^{c_1} \) before \( x_{2}^{c_1} \))
  - \( x_{1}^{c_1}, x_{2}^{c_1}, x_{3}^{c_1}, \ldots, x_{n-1}^{c_1}, x_{n}^{c_1}, x_{1}^{c_2}, x_{2}^{c_2}, x_{3}^{c_2}, \ldots, x_{n-1}^{c_2}, x_{n}^{c_2}. \) (\( x_{2}^{c_1} \) before \( x_{1}^{c_1} \))
  - \( x_{11}^{c_1}, x_{12}^{c_1}, x_{13}^{c_1}, \ldots, x_{n_1}^{c_1}, x_{n_2}^{c_1}, x_{12}^{c_2}, x_{13}^{c_2}, \ldots, x_{n_1}^{c_2}, x_{n_2}^{c_2}. \) (\( x_{11}^{c_1} \) between \( x_{11}^{c_1} \) and \( x_{12}^{c_1} \))

- If \( \deg(v_1) = n \), \( n \geq 4 \) and \( n \) is odd: Let \( x_{1}^{c_1} \) and \( x_{2}^{c_1} \) have the same order as before, and chose the appropriate of the following orders:
  - \( x_{1}^{c_1}, x_{2}^{c_1}, x_{3}^{c_1}, \ldots, x_{n-1}^{c_1}, x_{n}^{c_1}, x_{1}^{c_2}, x_{2}^{c_2}, x_{3}^{c_2}, \ldots, x_{n-1}^{c_2}, x_{n}^{c_2}. \) (\( x_{1}^{c_1} \) before \( x_{2}^{c_1} \))
  - \( x_{1}^{c_1}, x_{2}^{c_1}, x_{3}^{c_1}, \ldots, x_{n-1}^{c_1}, x_{n}^{c_1}, x_{1}^{c_2}, x_{2}^{c_2}, x_{3}^{c_2}, \ldots, x_{n-1}^{c_2}, x_{n}^{c_2}. \) (\( x_{2}^{c_1} \) before \( x_{1}^{c_1} \))
  - \( x_{11}^{c_1}, x_{12}^{c_1}, x_{13}^{c_1}, \ldots, x_{n_1}^{c_1}, x_{n_2}^{c_1}, x_{12}^{c_2}, x_{13}^{c_2}, \ldots, x_{n_1}^{c_2}, x_{n_2}^{c_2}. \) (\( x_{11}^{c_1} \) between \( x_{11}^{c_1} \) and \( x_{12}^{c_1} \))

Then treat the coatoms near the vertices adjacent to \( v_i \) in a similar way, and then deal with the vertices adjacent to \( v_i \), and so on, until all vertices in the component are dealt with.

Now for each component there is a coatom ordering for all coatoms near any vertex in the component. At last, mix the coatom orderings of the components by placing the first coatom in the coatom ordering of some component first in the total coatom ordering, and the last coatom in the coatom ordering of that component last in the total coatom ordering. Then the mixing condition is satisfied.

The result is a coatom ordering of \([0, 1] \) satisfying (V3), hence it is a recursive coatom ordering of \([0, 1] \).
4 The edge-product space is a regular cell complex

In this section we will assume that the reader is familiar with basic concepts of point-set topology. We will also make use of some purely topological results that for convenience we state in the Appendix.

To an $X$–tree $T$, we associate the closed ball $B(T) = [0, 1]^{E(T)}$ and open ball $\text{Int}(B(T)) = (0, 1)^{E(T)}$. More generally, for an $X$–forest $\alpha = \{(A, T_A) : A \in \pi\}$, we let $B(\alpha) = \prod_{A \in \pi} B(T_A)$ and let $\text{Int}(B(\alpha)) = \prod_{A \in \pi} \text{Int}(B(T_A))$. Note that $B(\alpha)$ (respectively $\text{Int}(B(\alpha))$) is homeomorphic to a closed (respectively open) ball of dimension \(\sum_{A \in \pi} |E(T_A)|\) and accordingly we will refer to this quantity as the dimension of $\alpha$, denoted $\dim(\alpha)$.

Given an $X$–tree $T = (T, \phi)$ and map $\lambda : E(T) \to [0, 1]$ define $p(\tau, \lambda) : \left(\frac{\lambda}{2}\right) \to [0, 1]$ by setting

$$p(\tau, \lambda)(x, y) = \prod_{e \in P(T, \phi(x), \phi(y))} \lambda(e).$$

We can extend the correspondence $\lambda \mapsto p(\tau, \lambda)$ to $X$–forests as follows. Given an $X$–forest $\alpha = \{(A, T_A) : A \in \pi\}$ let $\psi_{\alpha} : B(\alpha) \to [0, 1]^{\left(\frac{\lambda}{2}\right)}$ be defined by setting, for $\lambda = (\lambda_A : A \in \pi)$,

$$\psi_{\alpha}(\lambda)(x, y) = \begin{cases} p(\tau_A, \lambda_A)(x, y), & \text{if } \exists A \in \pi \text{ with } x, y \in A; \\ 0, & \text{otherwise.} \end{cases}$$

We begin by proving two useful lemmas. Let $\delta(B(\alpha))$ denote the boundary of the ball $B(\alpha)$. The following lemma describes a useful property of the map $\psi_{\alpha}$.

**Lemma 4.1.** Let $\alpha = \{(A, T_A) : A \in \pi\}$ be an $X$–forest. Then,

$$\psi_{\alpha}(\text{Int}(B(\alpha))) \cap \psi_{\alpha}(\delta(B(\alpha))) = \emptyset.$$

**Proof:** Suppose $\psi_{\alpha}(\text{Int}(B(\alpha))) \cap \psi_{\alpha}(\delta(B(\alpha))) \neq \emptyset$ - we will show that this leads to contradictions. This assumption implies that for some $\lambda_1 \in \text{Int}(B(\alpha))$, and $\lambda_2 \in \delta(B(\alpha))$ we have $\psi_{\alpha}(\lambda_1)(x, y) = \psi_{\alpha}(\lambda_2)(x, y)$ for all $x, y \in X$. Now, if there exists an edge $e$ of $\alpha$ with $\lambda_2(e) = 0$ then select a pair $x, y \in X$ that are separated by $e$ but contained in the same component of $\alpha$. Then, $\psi_{\alpha}(\lambda_1)(x, y) = \psi_{\alpha}(\lambda_2)(x, y) = 0$, and this implies that $\lambda_1 \in \delta(B(\alpha))$, a contradiction. Thus we may suppose that for every edge $e$ of $\alpha$ we have $\lambda_2(e) > 0$ and so therefore also $\psi_{\alpha}(\lambda_1)(x, y) = \psi_{\alpha}(\lambda_2)(x, y) > 0$ for all $x, y$ that belong to any component tree $T_A$ of $\alpha$. Now if we let
\[ d_t(x,y) := -\log(\psi_\alpha(\lambda_1))(x,y) \]
for all \( x,y \) in \( T_A \), then since \( d_1 = d_2 \) we have
that \( \lambda_1 \) agrees with \( \lambda_2 \) on the edges of \( T_A \) by Lemma 2.2. Since this applies
for each components \( A \in \pi \) it follows that \( \lambda_1 = \lambda_2 \). But this is impossible
since \( \lambda_1 \in Int(B(\alpha)) \) and \( \lambda_2 \in \delta(B(\alpha)). \)

Using the following lemma we will later be able to restrict our attention to
trees as opposed to forests.

**Lemma 4.2.** If \( \alpha = \{(A, T_A) : A \in \pi\} \) then
\[
\psi_\alpha(B(\alpha)) \cong \prod_{A \in \pi} (\psi(B(T_A))).
\]

**Proof:** This follows from Lemma 6.2 in the Appendix, taking \( I = \pi \),
and for \( A \in I, X_A = B(T_A) \), and writing \( \lambda R \lambda' \) if and only if \( p(T, \lambda | A) = p(T, \lambda' | A) \). \( \square \)

We now recall the definition of a regular cell complex. In [2, Section 12.4]
it states that a family of balls (homeomorphs of \( B^d, d \geq 0 \)) in a Hausdorff
space \( Y \) is a set of closed balls of a regular cell complex if and only if the
interiors of the balls partition \( Y \) and the boundary of each ball is a union
of other balls.

Consider the set
\[
C := \{\psi_\alpha(B(\alpha)) : \alpha \in S(X)\}.
\]
We claim that this forms a set of closed balls of a regular cell complex
(decomposition of \( E(X) \)) where the boundary of each ball \( \psi_\alpha(B(\alpha)) \),
denoted \( \delta(\psi_\alpha(B(\alpha))) \), is defined by \( \delta(\psi_\alpha(B(\alpha))):=\psi_\alpha(\delta(B(\alpha))) \)
(so that, by Lemma 4.1, the interior of each ball \( \psi_\alpha(B(\alpha)) \) is given by \( Int(\psi_\alpha(B(\alpha))) = \psi_\alpha(B(\alpha)) - \delta(\psi_\alpha(B(\alpha))) = \psi_\alpha(Int(B(\alpha))) \)).

To help prove our claim we first present a proposition that is a reformulation
of some results appearing in [8]. Let
\[
S(X)_{<\alpha} := \{\beta \in S(X) : \beta < \alpha\}.
\]

**Proposition 4.3.** The following statements hold:

(i) \( E(X) \) is the disjoint union of the elements of
\[
\{\psi_\alpha(Int(B(\alpha))) : \alpha \in S(X)\}.
\]

(ii) For \( \alpha \in S(X), \delta(\psi_\alpha(B(\alpha))) \) is the union of the elements of
\[
\{\psi_\beta(B(\beta)) : \beta \in S(X)_{<\alpha}\},
\]
and the disjoint union of the elements of
\[
\{\psi_\beta(Int(B(\beta))) : \beta \in S(X)_{<\alpha}\}.
\]
(iii) If $\alpha$ is an $X$-tree, then for each $y \in \psi_{\alpha}(B(\alpha))$, $\psi_{\alpha}^{-1}(y)$ is a contractible regular cell complex.

**Proof:** Part (i) and (ii) follow from [8, Theorem 3.3] and the definition of $\delta(\psi_{\alpha}(B(\alpha)))$. Part (iii) is [8, Proposition 6.5].

By Proposition 4.3(i), the interiors of the elements of $C$ partition $E(X)$, and by Proposition 4.3(ii) the boundary of each element of $C$ is equal to the union of other elements in $C$. Hence to show that $C$ is the set of closed balls of a regular cell complex it suffices to prove the following.

**Theorem 4.4.** For all $\alpha \in S(X)$, the set $\psi_{\alpha}(B(\alpha))$ is homeomorphic to $[0,1]^{\dim(\alpha)}$.

**Proof:** By Lemma 4.2 it suffices to prove the theorem for $\alpha \in S(X)$ an $X$-tree.

To prove the theorem we use induction on $\dim(\alpha)$. It can easily be checked that the result holds for $\dim(\alpha) = 0, 1, 2, 3$.

Now suppose that $d := \dim(\alpha) > 3$, and that $\psi_{\alpha}(B(\alpha))$ is homeomorphic to $[0,1]^m$ for all $m < d$.

By Proposition 4.3(ii) and the inductive hypothesis, $\delta(\psi_{\alpha}(B(\alpha)))$ is a regular cell complex, with set of closed balls equal to

$$\{\psi_{\beta}(B(\beta)) : \beta \in S(X)_{<\alpha}\}.$$

Moreover, this complex has face poset isomorphic to $(S(X)_{<\alpha}, \leq)$ (cf. [8, Theorem 3.3]). By Theorem 2.4 the poset $[0,\alpha]$ obtained by adding a minimal and a maximal element to $(S(X)_{<\alpha}, \leq)$ is thin and graded (graded means pure with a unique minimal and maximal element) with length $d+1$, and by Theorem 3.1 $[0,\alpha]$ has a recursive coatom ordering. It follows by part (i) of Theorem 4.5 below that $\psi_{\alpha}(\delta(B(\alpha)))$ is homeomorphic to $\delta([0,1]^d)$, the $(d-1)$-dimensional sphere.

It now follows that the set $\psi_{\alpha}(B(\alpha))$ is homeomorphic to $[0,1]^d$ by applying Proposition 4.3(iii) together with Lemma 6.1 of the appendix with $g = \psi_{\alpha}$, $B_{\alpha} = [0,1]^d$, and $Z = \psi_{\alpha}(B_{\alpha})$.

**Definition 4.1.** $F(\Delta)$ is the face poset of the cell complex $\Delta$.

**Theorem 4.5 (Björner).** [3, Theorem 4.7.24] Let $P$ be a graded poset of length $d + 2$. Then

(i) $P \cong F(\Delta) \cup \{\hat{0}, \hat{1}\}$ for some shellable regular cell decomposition $\Delta$ of the $d$-sphere $\iff P$ is thin and admits a recursive coatom ordering.
(ii) \( P \cong \mathcal{F}(\Delta) \cup \{0, 1\} \) for some shellable regular cell decomposition \( \Delta \) of the \( d \)-ball \iff \( P \) is subthin and admits a recursive coatom ordering.

A graded poset with \( \hat{1} \) is called subthin if every interval \([x, y]\) of length 2 and \( y \neq \hat{1} \) has cardinality 4 and every such interval with \( y = \hat{1} \) has cardinality 3 or 4, with at least one of cardinality 3.

5 Proof of some combinatorial lemmas

In this section the more intricate parts of the proof of Theorem 3.1 are shown.

5.1 The order of the coatoms near two vertices can be manipulated

**Proof of Lemma 3.14:** Due to the reversibility of \((V3)\) it is only necessary to investigate the case when at least two of \( x_1^c \), \( x_2^c \) and \( x_3^c \) are in \( C \). The proof method is to give the desired orderings satisfying \((V3)\) and \( C \cup D \) for every case. (Recall Convention 3.6.)

If \( x_1^c \) or \( x_3^c \) are in \( C \) then they will have the given order to those of \( x_1^c \), \( x_2^c \) and \( x_3^c \) that are in \( C \), where the given order does not violate \((V3b)\), and the same for \( D \). They will not be shown below among the other coatoms, but they are there anyway. The elements in \( C \) and \( D \) will be separated by the symbol \( | \). The following coatom orderings now satisfy \((V3)\):

- \( x_1^c, x_2^c, x_3^c \in C \):
  \[ (x_1^c, x_2^c), (x_1^c, x_3^c), (x_2^c, x_3^c), (x_2^c, x_1^c), (x_3^c, x_1^c), (x_3^c, x_2^c) \] where \( x_1^c, x_2^c \) and \( x_3^c \) have the given order.

- \( x_1^c, x_2^c \in C \), \( x_3^c \in D \):
  \[ (x_1^c, x_2^c), (x_1^c, x_3^c), (x_2^c, x_3^c), (x_2^c, x_1^c), (x_3^c, x_1^c), (x_3^c, x_2^c) \] where \( x_1^c \) and \( x_2^c \) have the given order.

- \( x_2^c, x_3^c \in C \), \( x_1^c \in D \):
  \[ (x_2^c, x_3^c), (x_2^c, x_1^c), (x_3^c, x_1^c), (x_3^c, x_2^c) \] where \( x_2^c \) and \( x_3^c \) have the given order.

It is easy to check that these orderings satisfy \((V3)\), since the only sub-condition of \((V3)\) to check is \((V3b)\). \( \Box \)
Proof of Lemma 3.15: If $\gamma_1, \ldots, \gamma_k$ is an ordering of $C$ such that 
${\gamma_1, \ldots, \gamma_i} \cup {\gamma_{i+1}, \ldots, \gamma_k} \cup D$ for all $1 \leq i \leq k - 1$, and $\delta_1, \ldots, \delta_\ell$ is an ordering of $D$ such that $C \cup {\delta_1, \ldots, \delta_i} \cup {\delta_{i+1}, \ldots, \delta_\ell}$ for all $1 \leq i \leq \ell - 1$, then the ordering $\gamma_1, \ldots, \gamma_k, \delta_1, \ldots, \delta_\ell$ of $C \cup D$ satisfies $(V3)$.

The reversibility of $(V3)$ implies that if it is possible to find such orderings of $C$, then it is also possible to find the desired orderings of $D$. Hence it suffices to find such orderings of $C$.

In this case the sub-conditions of $(V3)$ needed to be checked is $(V3c)$ and $(V3d)$. One way to guarantee that the second condition in $(V3c)$ is satisfied, is to put an $x^c$ early in the ordering of $C$ and then put $x^d$ last in the ordering of $C$ if it is in $C$.

$x^c_1, x^d_1 \in C$:

The condition $(V3d)$ implies that there is an $x_2$ such that $x^c_2 \in C$ and $x^d_2 \in D$ or vice versa. If $x^c_2 \in C$ then $(V3c)$ implies that there is an $x_3$ such that $x^d_3 \in C$. The following coatom orderings now satisfy $(V3)$.

$x^c_1$ before $x^d_1: \quad \{ x^c_1 \ldots x^d_1 \text{ if } x^c_2 \in C, x^d_2 \in D \}
\quad \{ x^c_1 \ldots x^c_3 \text{ else, since } x^c_2 \in C, x^d_2 \in D \}

x^d_1$ before $x^c_1: \quad \{ x^d_1 \ldots x^c_1 \text{ if } x^d_2 \in C, x^c_2 \in D \}
\quad \{ x^d_1 \ldots x^d_3 x^c_1 \text{ else if } x^c_2 \in C, \text{ since } x^c_2 \in C \text{, } x^c_2 \in D \}
\quad \{ x^d_1 \ldots x^d_3 x^d_1 \text{ else, since } x^d_2 \in C, x^c_2 \ldots, x^n_2 \in D \}

Thus it is always possible to choose an arbitrary order of $x^c_1$ and $x^d_1$. The remaining question is whether it is possible to choose the order $x^c_1, x^d_1, x^c_2$ or not. That is indeed possible. From above it follows that it is always possible to get the orders $x^c_1, x^c_2, x^d_1$ and $x^c_1, x^c_2, x^d_2$. If these three coatoms are at the positions $p, p + r$ and $p + r + s$ ($p, r, s \geq 1$) in the total order, then it is possible to do the following without violating $(V3)$:

In the first case the coatom in position $p + r$ can be removed and put directly after $x^c_1$ in position $p + r + s$. In the last case the coatom in position $p + r$ can be removed and put directly before $x^c_1$ in position $p$. The result is the desired orders.

So it is always possible to get an arbitrary order (not violating $(V3b)$) of $x^c_1, x^c_1, x^d_2$, or of $x^c_1$ and $x^d_1$, when they are in $C$.

$x^c_1, x^c_2 \in C, x^c_1 \in D$:

If there is an $x_2$ such that $x^c_2 \in C$ and $x^d_2 \in D$ or vice versa, then the above orderings with $x^d_1$ replaced with $x^c_1$ satisfy $(V3)$. Else $x^c_2 \in C$ implies $x^d_2 \in C$, and $x^d_2 \in C$ implies $x^c_2 \in C$. The condition $(V3c)$ then implies that $x^c_2 \in C$, why $x^c_3, x^d_2 \in C$. Then the following coatom orderings satisfy $(V3)$:
First suppose $x_1^e \in C$. Then $x_1^d \in D$. If $x_3^d \in D$ and $(V3c)$ implies $x_3^c \in C$. This means that either $x_3^c \in C$, or $x_1^e$ is the only element in $C$. If there is no edge $x_1$ such that $x_1^c \in C$, then $x_1^e, \ldots, x_n^e \in D$, and all orderings satisfy $(V3)$. Thus the following orderings satisfy $(V3)$:

$$\begin{cases} 
  x_1^c | x_2^d & \text{if } x_1^e \in C \\
  \ldots & \text{else, since } x_1^e, \ldots, x_n^e \in D 
\end{cases}$$

It is easy to check that the above orderings satisfy $(V3)$. $\Box$

5.2 $A_v$ and $B_v$ are compatible with $(V3)$

**Proof of Lemma 3.13:** This lemma will be shown by assuming $A_v \not\subseteq B_v$ for some interior vertex $v$ in $\alpha_j$ and then deducing that $\alpha_1, \ldots, \alpha_t$ does not satisfy $(V3)$. Since all coatoms near $v$ are obtained by contracting or deleting edges in the same component of $\alpha_j$, it is not necessary to check the condition $(V3a)$.

**Definition 5.1.** Let $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_t$ be a given coatom ordering for $[\emptyset, \Gamma]$. Let $C'$ and $D'$ be disjoint sets of coatoms of $[\emptyset, \alpha_j]$.

Then $C' \triangleleft D' \iff C \triangleleft \{ \alpha_j \} \triangleleft D$ means that $C' = \{ \gamma \mid \gamma \triangleleft \alpha_j, \beta \text{ where } \beta \in C \}$ and $D' = \{ \gamma \mid \gamma \triangleleft \alpha_j, \beta \text{ where } \beta \in D \}$. Since $S(X)$ is thin, $C$ and $D$ are well defined and will always be disjoint.

Observe that the sets below are *subsets* of $A_i$ and $B_i$. Also observe that if $C' \triangleleft D' \iff C \triangleleft \{ \alpha \} \triangleleft D$ then $D' \triangleleft C' \iff D \triangleleft \{ \alpha \} \triangleleft C$. Since the conditions are reversible, $C \triangleleft \{ \alpha \} \triangleleft D$ is forbidden by $(V3)$ if and only if $D \triangleleft \{ \alpha \} \triangleleft C$ is forbidden by $(V3)$. Because of that it is not necessary to check the reverse of any condition.

Based on the result in section 3.4 the following division into cases is appropriate. The figures show how the coatom $\alpha_j$ is obtained from $\Gamma$ by contracting or deleting an edge.
Case 1: This case is shown in Figure 1, where \( n, m \geq 2 \).

\[
\text{(V3c) } \{x_1, y_1^c\} \cap \{x_2^c, \ldots, x_n^c, y_1^c, \ldots, y_m^c\} \leftarrow \{x_1, y_1\} \cap \{y_0^c, \ldots, y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{y_1^c, \ldots, y_m^c\} \text{ which violates (V3b) or (V3c).}
\]

\[
\text{(V3c) } \{x_1, x_2^c\} \cap \{x_3^c, \ldots, x_n^c, y_1^c, \ldots, y_m^c\} \leftarrow \{x_1, x_2\} \cap \{y_0^c, \ldots, y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1, x_2, y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{y_1^c, \ldots, y_m^c\} \text{ which violates (V3b).}
\]

\[
\text{(V3c) } \{x_1, x_2^c\} \cap \{x_3^c, \ldots, x_n^c, y_1^c, \ldots, y_m^c\} \leftarrow \{x_1, x_2\} \cap \{y_0^c, \ldots, y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1, x_2, y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{y_1^c, \ldots, y_m^c\} \text{ which violates (V3d).}
\]

\[
\text{(V3d) Take } k \text{ such that } 0 \leq k \leq n:\n\]

\[
m = 2: \{y_1^c, \ldots, y_1^c, x_1^c, \ldots, x_n^c\} \cap \{y_2^c, x_{k+1}^c, \ldots, x_n^c\} \Rightarrow \{y_1^c, y_2^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1^c, y_2^c\} \cap \{y_1^c, \ldots, y_m^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
\text{Take } k \text{ such that } 0 \leq k \leq n-1, \text{ and } k = 0 \text{ if } n = 2:\n\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
\text{Take } k \text{ such that } 0 \leq k \leq n-1, \text{ and } k = 0 \text{ if } n = 2:\n\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
\text{Take } k \text{ such that } 0 \leq k \leq n-1, \text{ and } k = 0 \text{ if } n = 2:\n\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]

\[
m = 2: \{x_1^c, x_2^c, \ldots, x_n^c\} \cap \{y_1^c, \ldots, y_m^c\} \Rightarrow \{y_1, y_0\} \cap \{x_1^c, x_2^c\} \text{ which violates (V3b).}
\]
(V3d) Take $k$ and $\ell$ such that $1 \leq k + \ell \leq n + m - 1$:
\[ n \geq 3 \quad \{ x_1^c, x_2^c, \ldots, x_k^c, y_k^d, y_{k+1}^d, \ldots, y_m^d \} \]
\[ n \geq 3 \quad \{ x_1^c, x_2^c, \ldots, x_k^c, y_k^d, y_{k+1}^d, \ldots, y_m^d \} \cap \{ y_0^d \} \]
\[ \Rightarrow \{ y_0^d \exists \} \]

\[ (V3d) \text{ since } \ell \leq m - 1 \text{ can be assumed.} \]

![Figure 2: Case 2](image)

Case 2: This case is shown in figure 2, where $n, m \geq 3$.

(V3b) Suppose $x_0^d$ exists.
\[ n = 3 \quad \{ x_1^c, x_2^c \} \cap \{ x_0^d \} \]
\[ \Rightarrow \{ x_1^c, x_2^c \} \cap \{ x_0^d, x_0^d \} \text{ which violates (V3b).} \]

(V3c) \[ n \geq 4 \quad \{ x_1^c, x_2^c \} \cap \{ x_0^d, \ldots, x_n^d \} \Rightarrow \{ x_1^c, x_2^c \} \cap \{ x_0^d, x_0^d, \ldots, x_n^d \} \text{ which violates (V3c).} \]

(V3d) Take $k$ such that $1 \leq k \leq n - 1$:
\[ n \geq 4 \quad \{ x_1^c, x_2^c, \ldots, x_k^c, x_n^d \} \cap \{ x_1^c, x_2^c, \ldots, x_k^c, x_n^d \} \]
\[ \Rightarrow \{ x_0^d, \ldots, x_n^d \} \text{ exists} \Rightarrow \{ x_0^d, \ldots, x_n^d \} \text{ which violates (V3d).} \]

Case 3: This case is shown in figure 3, where $m = 2$ and $n \geq 2$. If $m \geq 3$, then the coatoms near $v$ in $\alpha_j$ are the “same” as the coatoms near $v'$ in $\Gamma$, and $\deg(v) = \deg(v')$.

(V3b) The coatoms $x_0^d$ do not exist. Suppose $x_0^d$ exists.
\[ n = 2 \quad \{ x_0^d, x_1^c \} \cap \{ x_0^d \} \]
\[ \Rightarrow \{ x_0^d, x_1^c \} \cap \{ y_1^d \} \text{ which violates (V3b).} \]
(V3c) \( \{x_1, x_2\} \triangleleft \{x_0^d, x_1^d, \ldots, x_n^d\} \)

\( n \geq 3 \) \( \{x_1, x_2\} \triangleleft \{y_1^d\} \triangleleft \{y_0^d, x_0^d, \ldots, x_n^d\} \Rightarrow [x_0^d = \{y_1^d, y_2^d\}] \Rightarrow \\
\{x_1, x_2\} \triangleleft \{x_0^d, x_1^d, \ldots, x_n^d\} \) which violates (V3c).

(V3d) Take \( k \) such that \( 0 \leq k \leq n - 1 \):

\( n \geq 3 \) \( \{x_0^d, x_0^c, \ldots, x_k^c, x_k^d, \ldots, x_n^c, x_n^d\} \triangleleft \{x_{k+1}^c, x_{k+1}^d, \ldots, x_n^c, x_n^d\} \)

\( \{x_0^d, y_0^c, x_1^c, \ldots, x_k^c, x_k^d, \ldots, x_n^c, x_n^d\} \triangleleft \{y_1^d\} \triangleleft \{x_{k+1}^c, x_{k+1}^d, \ldots, x_n^c, x_n^d\} \Rightarrow \\
[\text{since } x_0^d = \{y_1^d, y_2^d\}] \Rightarrow \\
\{x_0^d, \ldots, x_k^c, x_k^d, \ldots, x_n^c, x_n^d\} \triangleleft \{x_{k+1}^c, x_{k+1}^d, \ldots, x_n^c, x_n^d\} \\
which violates (V3d).

\( \begin{array}{c}
\text{Figure 3: Case 3} \\
\end{array} \)

\( \begin{array}{c}
\text{Case 4: This case is shown in figure 4, where } m = 3 \text{ and } n \geq 2. \text{ If } m \geq 4, \\
then the coatoms near } v \text{ in } \alpha_j \text{ are the "same" as the coatoms near } v' \text{ in } \Gamma, \\
\text{and } \deg(v) = \deg(v'). \\
\end{array} \)

(V3bc) \( \{x_1^c, x_2^c\} \triangleleft \{x_0^d, \ldots, x_n^d\} \)

\( \{x_1^c, x_2^c\} \triangleleft \{y_1^d\} \triangleleft \{x_0^d, \ldots, x_n^d\} \Rightarrow [V3c], y_0^d = \{x_1^c, x_2^c\} \)

\( \Rightarrow \begin{cases} 
\text{if } n = 2 : & \{y_0^d\} \triangleleft \{y_1^d\} \triangleleft \{x_0^d, \ldots, x_n^d\} \rightarrow [V3c], y_0^d = \{x_1^c, x_2^c\} \\
\text{if } n \geq 3 : & \{x_1^c, x_2^c, y_0^d\} \triangleleft \{y_1^d\} \triangleleft \{x_0^d, \ldots, x_n^d\} \rightarrow [V3c], y_0^d = \{x_1^c, x_2^c\} \\
\end{cases} \\
\{y_0^d, y_1^d\} \triangleleft \{y_0^d, y_1^d\} \) which violates (V3c).

(V3bc) \( \{x_1^c, y_0^d\} \triangleleft \{x_0^d, \ldots, x_n^d\} \)

\( \{x_1^c, y_0^d\} \triangleleft \{y_1^d\} \triangleleft \{x_0^d, \ldots, x_n^d\} \) which violates (V3bc).
Take $k$ such that $0 \leq k \leq n - 1$:

\[
\begin{align*}
(V3d) & \quad \text{(a)} \quad \{y_0^c, y_2^c, x_1^c, \ldots, x_k^c\} \triangleleft \{x_{k+1}^d, \ldots, x_n^d\} \
& \quad \{y_0^c, y_2^c, x_1^c, \ldots, x_k^c\} \triangleleft \{y_1^c\} \triangleleft \{x_{k+1}^d, \ldots, x_n^d\} \\
& \quad \Rightarrow [y_i^d \text{ exists}] \Rightarrow \\
& \quad \{y_0^d \text{ before } y_1^d : \{y_0^d, y_1^d, x_1^d, \ldots, x_k^d\} \triangleleft \\
& \quad \triangleleft \{x_{k+1}^d, \ldots, x_n^d\} \} \text{ which violates (V3d).} \\
& \quad \{y_0^d \text{ after } y_1^d : \{y_1^d, y_2^d\} \triangleleft \{y_1^c, y_2^c\} \} \text{ which violates (V3d).}
\end{align*}
\]

Thus $A_v \bowtie B_v$ for every interior vertex $v$ in $\alpha_j$.

\[\square\]

### 5.3 $A$ and $B$ are compatible with (V3)

**Definition 5.2.** If $\gamma$ is a coatom of $[0, \beta]$ which is obtained by contraction or deletion of an edge of the component $K$ in $\beta$, then this will be written $\alpha \in K$. 

The following lemma is needed in the proof of Lemma 3.16.

**Lemma 5.3.** Let $\alpha_1, \ldots, \alpha_j, \ldots, \alpha_t$ be a coatom ordering of $[0, \Gamma]$ satisfying (V3). Fix $j$, let $A = \{\gamma : \gamma < \alpha_k, \alpha_j \text{ and } k < j\}$ and let $B = \{\gamma : \gamma < \alpha_k, \alpha_j \text{ and } k > j\}$. (The sets $A$ and $B$ are disjoint since $S(X)$ is thin.) Then $A \bowtie B$.

**Proof:** From Lemma 3.13 follows that it suffices to show that $A \triangleleft B$ does not contradict the condition (V3a).

The $X$-forest $\alpha_j$ is obtained from $\Gamma$ by contracting or deleting an edge $x_0$. Let $K_1, \ldots, K_{t+1}$ be the components in $\Gamma$, and suppose $x_0 \in K_{t+1}$. Then $\alpha_j$ has the components $K_1', \ldots, K_{s+1}'$ where $s \geq 1$ and $K_{s+i}' = K_i$ for all $1 \leq i \leq t$.

First $\{\gamma : \gamma \in \bigcup_{i=1}^{t} K_{s+i}' \}$ and $\{\gamma : \gamma \in \bigcup_{i=s+1}^{t} K_{s+i}' \}$ for all $1 \leq k \leq s - 1$ will be shown. Here $s \geq 1$, why the following two cases arise:

![Figure 5: Case 1](image-url)
Case 1: This case is shown in figure 5. Here \( s = n \) and \( \alpha_j = x_0^d(\Gamma) \).

\[
\begin{align*}
n = 2 & \quad \{ \gamma \mid \gamma \in K_{t+1}' \} \cup \{ \gamma \mid \gamma \in K_{t+2}' \} \Rightarrow \\
& \quad \{ x_1^d, x_1^d \} \subseteq \{ x_2^d, x_2^d \} \\
& \quad \{ x_1^d, x_1^d, x_1^d \} \subseteq \{ x_0^d \} \subseteq \{ x_2^d, x_2^d, x_2^d \} \Rightarrow \\
& \quad \{ x_1^d, x_0^d \} \subseteq \{ x_2^d, x_2^d \} \text{ which violates (V3b).}
\end{align*}
\]

\[
\begin{align*}
n \geq 3 & \quad \{ \gamma \mid \gamma \in \bigcup_{k=1}^n K_{t+1}' \} \not\subseteq \{ \gamma \mid \gamma \in \bigcup_{k=1}^n K_{t+1}' \} \Rightarrow \\
& \quad \{ x_1^d, x_1^d, \ldots, x_k^d, x_k^d \} \not\subseteq \{ x_{k+1}^d, x_{k+1}^d, \ldots, x_n^d, x_n^d \} \iff \\
& \quad \{ x_1^d, x_1^d, \ldots, x_k^d, x_k^d \} \not\subseteq \{ x_0^d \} \subseteq \{ x_{k+1}^d, x_{k+1}^d, \ldots, x_n^d, x_n^d \} \Rightarrow \\
& \quad \text{[x_0^d exists]} \Rightarrow \\
& \quad \{ x_0^d, x_0^d, \ldots, x_k^d, x_k^d, x_0^d \} \not\subseteq \{ x_{k+1}^d, x_{k+1}^d, \ldots, x_n^d, x_n^d \} \text{ which violates (V3d).}
\end{align*}
\]

Case 2: This case is shown in figure 6. Here \( s = 2 \) and \( \alpha_j = x_0^d(\Gamma) \).

\[
\begin{align*}
n \geq 4 & \quad \{ \gamma \mid \gamma \in K_{t+1}' \} \cup \{ \gamma \mid \gamma \in K_{t+2}' \} \Rightarrow \\
m = 3 & \quad \{ x_1^d, x_1^d, x_0^d, x_0^d \} \not\subseteq \{ y_1^d, y_1^d, y_2^d, \ldots, y_3^d, (y_3^d, y_3^d) \} \iff \\
& \quad \{ x_1^d, x_1^d, x_0^d, x_0^d \} \not\subseteq \{ y_1^d, y_1^d, y_2^d, \ldots, y_3^d, (y_3^d, y_3^d) \} \Rightarrow \\
& \quad (V3c) \quad \{ x_1^d, x_1^d, \ldots, x_n^d, x_n^d \} \not\subseteq \{ y_1^d, y_1^d, \ldots, y_3^d, y_3^d \} \Rightarrow \\
& \quad \text{[x_0^d exists]} \Rightarrow \\
& \quad \{ x_0^d, x_0^d \} \not\subseteq \{ y_1^d, y_1^d, \ldots, y_3^d, y_3^d \} \text{ which violates (V3d).}
\end{align*}
\]

The cases \( n = 3, m = 3 \) and \( n \geq 4, m \geq 4 \) can easily be shown in similar ways.

Let \( C_i = \{ \gamma \mid \gamma \in K_i' \} \) for all \( 1 \leq i \leq \ell \), \( C_{t+1} = \{ \gamma \mid \gamma \in \bigcup_{i=1}^n K_{t+1}' \} \) and let \( D_i = \{ \gamma \mid \gamma \in K_i \} \) for all \( 1 \leq i \leq \ell + 1 \).

Let \( v_1 \) and \( v_2 \) be the vertices incident with \( x_0 \). Now if \( A \not\subseteq B \) contradicts (V3a) then the following is the case \((1 \leq k \leq \ell)\):
\[ \alpha_j = x_0^j \quad C_1 \cup \ldots \cup C_k \triangleleft C_{k+1} \cup \ldots \cup C_{t+1} \]
\[ \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k \triangleleft \{x_0^j\} \cup \mathcal{D}_{k+1} \cup \ldots \cup \mathcal{D}_{t+1} \setminus \{x_0^j, x_1^j\} \Rightarrow [(V3d)] \]
\[ \Rightarrow \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k \triangleleft \mathcal{D}_{k+1} \cup \ldots \cup \mathcal{D}_{t+1} \] which violates \((V3a)\).

\[ \alpha_j = x_0^j \quad C_1 \cup \ldots \cup C_k \triangleleft C_{k+1} \cup \ldots \cup C_{t+1} \]
\[ \text{deg}(v_1) = 4 \quad \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k \triangleleft \{x_0^j\} \cup \mathcal{D}_{k+1} \cup \ldots \cup \mathcal{D}_{t+1} \setminus \{x_0^j, x_1^j\} \Rightarrow [(V3c)] \]
\[ \Rightarrow \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k \triangleleft \mathcal{D}_{k+1} \cup \ldots \cup \mathcal{D}_{t+1} \setminus \{x_0^j, x_1^j\} \Rightarrow (V3d) \]
\[ \mathcal{D}_1 \cup \ldots \cup \mathcal{D}_k \triangleleft \mathcal{D}_{k+1} \cup \ldots \cup \mathcal{D}_{t+1} \] which violates \((V3a)\).

The cases \(\text{deg}(v_1) = 4, \text{deg}(v_2) = 4 \) and \(\text{deg}(v_1) \geq 5, \text{deg}(v_2) \geq 5\) can easily be shown in similar ways.

Thus \(A \triangleleft B\) does not contradict \((V3a)\), why \(A \triangleleft B\). \(\square\)

### 5.4 There is a coatom ordering satisfying \((V3)\) and \(A \triangleleft B\)

**Proof of Lemma 3.16:** Let \(v_1\) and \(v_2\) be adjacent interior vertices, and denote the edge between them \(x_1\). The coatoms near \(v_1\) which are also near \(v_2\) are the following:

- \(x_1^1\) and \(x_1^d\) if \(\text{deg}(v_1), \text{deg}(v_2) \geq 4\).
- \(x_1^1, x_1^1, x_1^d\) and \(x_1^d\) if \(\text{deg}(v_1) \geq 4\) and \(\text{deg}(v_2) = 3\).
- \(x_1^c, x_1^c, x_1^c, x_1^d\) and \(x_1^d\) if \(\text{deg}(v_1) = \text{deg}(v_2) = 3\).

Because of Lemmas 3.14 and 3.15 it is now possible to find a coatom ordering that satisfies \((V3)\) in the following way. Choose a component, and then choose an interior vertex \(v_p\) in that component. Take an order of the coatoms near that vertex that satisfies \((V3)\) and \(A_p \triangleleft B_p\). Then for each of the vertices \(v_i, i \in I\), adjacent to the first vertex, choose an order of the coatoms near \(v_i\) that does not contradict the earlier chosen order and satisfies \((V3)\) and \(A_i \triangleleft B_i\). Then continue in the same way with the vertices adjacent to them, etc., until all interior vertices in the component are dealt with. At last, combine the orderings of the coatoms in the components by choosing one of the components that have at least two coatoms in \(A\) (if such a component exists) and put the first coatom of its ordering first in \(A\) and the last coatom of its ordering last in \(A\). Do the same with \(B\). Together with Lemma 5.3 this implies that the result is a coatom ordering of \([0, \alpha_j]\) satisfying \((V3)\) and \(A \triangleleft B\). \(\square\)
6 Appendix: Some topological results

In this appendix we state some results from point-set topology that we use above. For basic terminology concerning topology see, for example, [6].

First we recall that if $Z$ is a compact topological space, and $f : Z \to Y$ is a continuous surjection with $Y$ Hausdorff, then $Y \cong Z/R$ (i.e., $Y$ is homeomorphic to the quotient space $Z/R$), where $R$ is the equivalence relation on $Z$ induced by $f$.

**Lemma 6.1.** Let $Z$ be a topological space, and let $B$ be a closed $d$–dimensional ball with boundary $S$. Suppose that $g : B \to Z$ is a continuous surjection whose restriction to $\text{Int}B$ is bicontinuous and injective. Suppose furthermore that $g(S)$ is homeomorphic to $S$, $g(S) \cap g(\text{Int}B) = \emptyset$, and that for each $q \in S$, $g^{-1}(q)$ is a contractible regular cell complex. Then $Z$ is homeomorphic to $B$.

**Proof:** $Z$ is homeomorphic to the quotient space $B/R_g$ where $R_g$ is the equivalence relation that identifies elements of $Z$ that are identified by $g$. Now, $g|_S$ extends to a continuous map $F : B \to B$ that maps $\text{Int}(B)$ injectively onto $\text{Int}(B)$ (Robert Daverman, personal communication). Thus $B$ is homeomorphic to $B/R_F$. Now, $R_F = R_g$ since $F$ and $g$ are both injective on $\text{Int}B$ and $F(\text{Int}B) \cap F(S) = g(\text{Int}B) \cap g(S) = \emptyset$ and $F|_S = g$. Consequently, $B \cong B/R_F = B/R_g \cong Z$, as required.

**Lemma 6.2.** For a finite index set $I$ let $Z_i$, $i \in I$, be compact, hausdorff topological spaces, and let $R_i$, $i \in I$, be corresponding equivalence relations. Let $R$ be the (product) equivalence relation on $Z := \prod_{i \in I} Z_i$ defined by $(z_i)_{i \in I} R (z'_i)_{i \in I}$ if and only if $z_iR_i z'_i$ for each $i \in I$. Then $Z/R \cong \prod_{i \in I} (Z_i/R_i)$.

**Proof:** Let $[(z_i)_{i \in I}]$ denote the $R$–equivalence class of $(z_i)_{i \in I}$ and $[z]_i$ denote the $R_i$–equivalence class of $z_i$. Then it is straight-forward to check that the map from $Z/R$ to $\prod_{i \in I} (Z_i/R_i)$ defined by

$$[(z_i)_{i \in I}] \mapsto ([z]_i)_{i \in I}$$

is a (well-defined) homeomorphism.

**References**


