Constitutive equations for concrete materials subjected to high rate of loading

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Abstract

Continuum mechanics is used to model the mechanical behaviour of concrete structures subjected to high rates of loading in defence applications. Large deformation theory is used and an isotropic elastic-plastic constitutive equation with isotropic hardening, damage and strain rate dependent loading surface. The hydrostatic pressure is governed by an equation of state. Numerical analysis is performed using the finite element method and the central difference method for the time integration.

Projectile penetration is studied and it is concluded that it is not suitable to use material description of the motion of both the target and the projectile together with an erosion criterion. Instead, the material description should be used only for the projectile and the spatial description for the target. In this way the need for an erosion criterion is eliminated. Also, in the constitutive model used it is necessary to introduce a scaling of the softening phase in relation to the finite element size, in order to avoid strain localization.

Drop weight testing of reinforced concrete beams are analysed, where a regularisation is introduced that renders mesh objectivity regarding fracture energy release. The resulting model can accurately reproduce results from material testing but the regularisation is not sufficient to avoid strain localization when applied to an impact loaded structure. It is finally proposed that a non-local measure of deformation could be a solution to attain convergence.

The third study presents the behaviour of a concrete constitutive model in a splitting test and a simplified non-local theory applied in a tensile test. The splitting test model exhibits mesh dependency due to a singularity. In the tensile test the non-local theory is shown to give a convergent solution. The report is concluded with a discussion on how to better model concrete materials.
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Thesis

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1 Introduction

The use of conventional weapons against fortifications gives rise to fast and large loadings, cf. [1]. In order to assess or optimize the protection level of a structure there are two possible methods; tests and mathematical modelling. Each one of them has its advantages and disadvantages but if used together they can render a powerful tool. The Swedish defence research agency, FOI, has been involved in experimental work for decades and there exist a great knowledge in this field. Different types of mathematical models have been used but it is only since 1994 that continuum mechanics together with numerical analysis have been employed more regularly. Today there exist several numerical tools with the techniques needed to solve the problems at hand. The part still not mastered in a satisfying manner is the constitutive modelling at high rates of loading. This work has been focused on the mechanical constitutive modelling of concrete material at high loading rates, in particular impact loading.

Research in the field of mechanical constitutive equations for concrete subjected to high loading rates started with the experimental investigation presented in Abrams [2]. Since then many studies have been devoted to this area. Most of the found knowledge on three-dimensional stressing of concrete is compiled in the European construction code, cf. [3]. This code is however only valid for constant strain rates in the range from 0 to $10^3\text{s}^{-1}$ in compression and $3\cdot10^2\text{s}^{-1}$ in tension. In penetration problems strain rate values of order $10^4\text{s}^{-1}$ occur. Important contributors to the content in this code are Kupfer [4], William and Warnke [5] and Ottosen [6] on the shape of the loading surface, Hillerborg [7] on the softening behaviour and Reinhardt [8] on the effects of loading rates. Fundamental work on the constitutive modelling of concrete subjected to high rate of loading can be found in Nilsson [9] and Nilsson and Oldenburg [10]. A comprehensive textbook on the modelling of concrete is Chen [11].

The mechanical behaviour of concrete materials is complex. The inelastic behaviour is not related to the motion of dislocations as for metallic materials. Instead, the fracture, buckling and crushing of the cement paste and aggregate microstructure are the main mechanism of inelasticity. In a uniaxial deformation the response is
approximately linearly elastic in a regime, during which micro cracks are developed. As the deformation increases the amount of cracks increases and they propagate through the material. In extensional deformation the crack planes are orthogonal to the load direction and in compression they are parallel to the load direction. During these two phases the material exhibits stable cracking or hardening. A peak stress is reached at a point where one goes into unstable cracking or softening. If hydrostatic pressure is present the material shows a residual strength. Concrete also displays dilation, i.e. volume change, in the inelastic range. For a triaxial test the development of cracks is restrained and the equation of state displays tree different phases: elastic, compaction and solidification. During the compaction phase the water and air filled pores in the material collapses and in the final solidification phase the material is approximately homogenous and the volumetric response is once again linearly elastic. Also, the strain rate influences the material response. Two mechanisms have been identified to explain this. In the lower range (<1-10 s⁻¹) it is the water filled pores that increase the strength through viscous effects. In the higher range the development of micro cracks is restrained due to inertia effects, i.e. the cracks do not have time to develop. Practical limitations make it difficult to model these two mechanisms explicitly and they are hence considered as discrete phenomena. Incorporation of these two discrete phenomena must then be done in the mathematical model through the constitutive equation.
2 Continuum mechanics

In physics there are two viewpoints in modelling the nature of matter, discrete and field theories. In field theories, or phenomenological theories, continuous fields represent matter, motion energy etc. Continuum mechanics is defined as the mechanics of deformable media without consideration of the internal material structure, c.f. Truesdell and Noll [12]. Continuum mechanics can be further subdivided into fluid mechanics and solid mechanics.

2.1 Kinematics

Kinematics is the study of motion and deformation of a body within a mathematical framework.

2.1.1 Motion

The motion of a body is described by a smooth mapping of the material, or reference, configuration onto the spatial, or current, configuration.

\[ x = x(X, t) \]  

A general motion consists of translation, rotation and deformation. The material particles \( X \) constituting the body are here identified with their position vector \( X \) in the material configuration schematically shown in Figure 1.
The smooth mapping $x(X,t)$ of the material (or reference) configuration $B_0$ onto the spatial (or current) configuration $B$.

The displacement of a material point is given by

$$u(X,t) = x(X,t) - X$$

and the velocity and acceleration of a material point respectively by

$$v(X,t) = \frac{\partial}{\partial t} x(X,t) = \frac{\partial}{\partial t} u(X,t)$$

$$a(X,t) = \frac{\partial^2}{\partial t^2} v(X,t) = \frac{\partial^2}{\partial t^2} u(X,t)$$

The material time derivative for spatial quantities is

$$\frac{D}{Dt} \cdot = \frac{\partial}{\partial t} \cdot + v(X,t) \nabla \cdot$$

where the last term on the right hand side is called the convective or transport term.

### 2.1.2 Deformation

The deformation of a body is characterized by the deformation gradient defined as

$$F = \nabla_x x(X,t)$$

where $\nabla_x$ is the gradient with respect to the material configuration

$$\nabla_x \cdot = \frac{\partial}{\partial X} \cdot$$

When the motion of a body gets larger, i.e. when the deformation gradient differs much from the identity tensor, we cannot use the linear measure of strain defined as
\[
\varepsilon = \frac{1}{2} \left[ \nabla_x u + (\nabla_x u)^T \right]
\]  
(8)

The inadequacy of this strain is that it is non-zero for an arbitrary rigid body rotation. The rate of deformation tensor \( \mathbf{D} \) is a spatial tensor that measures the rates of change in length of a spatial line segment and of their relative orientations. \( \mathbf{D} \) is defined as

\[
\mathbf{D} = \frac{1}{2} \left( \mathbf{L} + \mathbf{L}^T \right)
\]  
(9)

where

\[
\mathbf{L} = \frac{D \mathbf{F}}{D t} \mathbf{F}^{-1} = \left( \nabla_x \mathbf{v} \right)^T
\]  
(10)

The rate of deformation vanishes for any rigid body motion but it has another drawback, it is path dependent. If it is integrated in a closed deformation cycle it does not necessarily vanish when returning to the initial configuration violating the field equation for energy balance, see Section 2.2.1. However, if the elastic strain is small compared to the total strain and the dissipation is small, the error in elastic strain energy is negligible, cf. Belytschko et al. [13]. Also, for the applications at hand the loadings are mainly monotonic. The rate of deformation is the most commonly used measure of deformation in finite element codes, and it is also the basis for the constitutive model used in this study, see Section 2.2.2. The rate of deformation tensor is integrated in time to give the strain

\[
\mathbf{E} = \int_0^t \mathbf{D} \, dt
\]  
(11)

For uniaxial deformation this strain is equal to the logarithmic strain

\[
E_{\text{xx}} = \int_0^t D_{\text{xx}} \, dt = \log_e \left( \frac{L}{L_0} \right)
\]  
(12)

where \( L \) and \( L_0 \) are the reference and current length, respectively. This holds true for the multiaxial case only if the principal axes of deformation are fixed, cf. Belytschko et al. [13].

2.2 Dynamics

Dynamics is the study of the mathematical relations between loading of a body and the resulting deformations. The coupled system of partial differential equations to be solved is referred to as a boundary-initial value problem.
2.2.1 Field equations

The field equations of solid mechanics are here given in their local spatial form.

- Mass
  \[ \frac{D}{Dt} \rho + \rho \nabla \cdot \mathbf{v} = 0 \]  
  (13)
  where \( \rho \) is the density.

- Linear momentum
  \[ \nabla \cdot \mathbf{\sigma} + \rho \mathbf{b} - \rho \mathbf{a} = 0 \]  
  (14)
  where \( \mathbf{\sigma} \) is the true, or Cauchy, stress tensor and \( \mathbf{b} \) is the volume force per unit mass.

- Angular momentum
  \[ \mathbf{\sigma}^T = \mathbf{\sigma} \]  
  (15)

- Energy
  \[ \rho \frac{D}{Dt} e = \mathbf{\sigma} : \mathbf{D} \]  
  (16)
  where \( e \) is the specific internal energy per unit mass.

\[ \partial B = \partial B_c \cup \partial B_n \]

**Figure 2** A deformable body \( B \) with boundary \( \partial B \).

To arrive at a well-posed problem, initial- and boundary conditions have to be stated.

\[
\begin{align*}
\mathbf{v}(\mathbf{X},t) &= \mathbf{v}_c(\mathbf{X}) & \mathbf{X} \in \partial B_c \\
\mathbf{\sigma}(\mathbf{X},t) \mathbf{n} &= \mathbf{t}_n(\mathbf{X}) & \mathbf{X} \in \partial B_n
\end{align*}
\]  
  (17)
  Boundary conditions
\[
\varphi(X,0) = \varphi_0(X)
\]
\[
u(X,0) = \nu_0(X)
\]
\[
v(X,0) = \nu_0(X)
\]
\[
e(X,0) = e_0(X)
\]
\[X \in B \quad \text{Initial conditions} \quad (18)\]

The field equations do not allow for singular surfaces or jumps in a quantity, such as fracture and chock waves. However, chock waves are handled in numerical continuum mechanics using an artificial bulk viscosity, cf. Neumann and Richtmyer [14].

### 2.2.2 Constitutive equation

The rate of deformation tensor can be split additively into an elastic and an inelastic part as

\[
D = D^e + D^i
\]  

(19)

The model used in this study is based on hypoelasticity, cf. Truesdell and Noll [12], where the stress rate is a linear function of the rate of deformation. The material time derivative of the true stress tensor is a non-objective tensor, i.e. it is not invariant under an arbitrary change of frame of reference, cf. Ogden [15], and cannot be used directly as a measure of the stress rate. This problem is circumvented by the use of an objective rate. In this study the Jaumann rate, \( \sigma \), cf. Lubliner [16], has been used.

\[
\frac{D}{Dt}\sigma = \sigma + W\sigma + \sigma W^T = C : D^e + W \sigma + \sigma W^T
\]  

(20)

where

\[
W = \frac{1}{2} \left( L - L^T \right)
\]  

(21)

and \( C \) is the tensor of elasticity. The Jaumann rate has been shown to provide incorrect results for simple elastic shearing, cf. Belytschko et al. [13]. In Figure 3 to Figure 5 the resulting stresses for the isotropic linear elastic case are shown from three different objective stress rates with equivalent elastic material parameters. For the material and applications in this study the elastic deformations are negligible compared to the total deformation. Thus the Jaumann rate can be used.
The first mathematical models of the mechanical behaviour of concrete were based on isotropic, linear hyperelasticity combined with a failure criterion in tension, cf. Chen [11]. These models were restricted to problems where brittle failure in tension is prevailing and they soon showed to be inadequate for many problems.

Mechanical testing of concrete revealed that the strength of concrete depended on all three invariants of the stress tensor. To model this behaviour attention was turned towards the theory of plasticity, cf. Hill [17], from which the idea of an elastic domain in stress space bounded by a failure surface was adopted. One or combinations of new and existing functions, for example von Mises, Rankine, Mohr-Coulomb and Drucker-Prager, were used to describe the failure surface. Two of the most widely used functions are due to Ottosen [6] and William and Warnke [5], used for example in the CEB-FIP model code [3].

Refinement of the models, still within the ideas of plasticity theory, included the addition of an initial elastic domain bounded by a yield surface and a corresponding hardening rule. All of the functions mentioned so far are open surfaces in stress space, but from tests it was observed that the elastic domain should be closed. This has been modelled using either a separate function for the volumetric behaviour or closed functions. Examples of closed functions are the critical state function, the two surface cap, the generalized ellipsoidal, cf. Nilsson [9], and the Hoffman failure criterion used for example in Winnicki et al. [18].

Further testing of concrete, now under displacement control, showed the existence of a descending branch after the peak stress, a phenomenon commonly called softening. To model this the brittle failure models were abandoned and softening rules were introduced, cf. Hillerborg [19], still within the framework of plasticity theory.
The introduction of inelastic deformations in the constitutive relations made it necessary to separate the elastic and the inelastic strains. For small, or infinite, deformations there exists an intersubjective theory on the mathematical treatment, but not for large, or finite, deformations, cf. Ristinmaa and Ottosen [20]. But, the rate of deformation can always be additively partitioned and this is the basis for hypoelasticity that has been more used than hyperelasticity. The incremental deformation theory of plasticity has been used more extensively than the total deformation theory, cf. Nilsson [9].

Strength enhancement due to dynamic loading has been included in the models mainly through enhancement of the failure surface based on strain rate. Strictly, this is a contradiction since the theory of plasticity is the theory for time independent inelastic deformations. Viscoplasticity, the theory for time dependent inelastic deformations, cf. Perzyna [21], was used in Nilsson [9] but has since then not been used extensively. One of the more recent works is Winnicki et al. [18].

Among the state of the art models available in commercial finite element codes for different situations of dynamic loading of concrete structures are the RHT model from Riedel [22], the Winfrith model [23], the cap model by Schwer and Murray [24] and the JHC model, cf. Holmquist et al. [25].

The K&C concrete model, cf. Malvar et al. [26], is an enhanced version of the Pseudo tensor model available in LS-DYNA [27] and developed at the Lawrence Livermore National Laboratories, USA. It was developed and modified mainly to analyse concrete structures subjected to blast loading. It is a linear isotropic hypoelastic-plastic model with strain rate scaled elastic domain, a non-associated flow rule and non-linear anisotropic strain hardening and softening representing stable and unstable cracking. The deviatoric and isotropic parts of the response are uncoupled and the isotropic behaviour is governed by a compaction curve or equation of state.

**Elastic domain**
The deviatoric elastic domain is defined as

\[
E_r = \{ (\sigma, E^p) \in S \times R_+ \mid f(\sigma, E^p) < 0 \} \tag{22}
\]

where \(S\) is the six dimensional stress space with linear, symmetric and positive definite second order tensors. \(R_+\) is the space of positive real values and \(E^p\) is an internal history variable representing plastic straining. The isotropic criterion \(f\) is stated as

\[
f(\sigma, E^p) = J_2 - f_1 \tag{23}
\]

where

\[
J_2 = J_2(\sigma) \tag{24}
\]
\[ f_1 = f_1(p, \nu, E^p) \]  
(25)

and

\[ p = -\frac{1}{3} I_1(\sigma) \]  
(26)

\[ \cos(3\nu) = \frac{1}{27} \left( \frac{J_3(\sigma)}{J_2(\sigma)} \right)^3 \]  
(27)

\[
\begin{align*}
I_1(\sigma) &= \text{tr}(\sigma) \\
J_2(\sigma) &= \left[ \frac{3}{2} \text{dev}(\sigma) : \text{dev}(\sigma) \right]^{\frac{1}{2}} \\
J_3(\sigma) &= \left[ \frac{9}{2} \text{tr}(\sigma^3) \right]^{\frac{1}{3}} \\
\text{dev}(\sigma) &= \sigma - \frac{1}{3} I_1(\sigma)
\end{align*}
\]  
(28)

These forms on the invariants of the stress tensor are taken from Lemaitre and Chaboche [28]. The calculation of the modified effective plastic strain \( E^p \) is given in the section Inelastic domain. In the principal stress space this corresponds to a loading surface constructed as described in the following. The compressive and tensile meridians are defined as lines in the Rendulic stress space for which the angle \( \nu \) equals \( \pi/3 \) and 0, respectively, see Figure 6 and Figure 7. For hydrostatic pressures below one third of the compressive strength the meridians are piecewise linear functions connecting the points corresponding to triaxial extension, biaxial extension, uniaxial extension and uniaxial compression. For hydrostatic pressure exceeding one third of the compressive strength, the initial, quasi-static compressive meridians are given by the general relation

\[ f_1^c = f_1(p, \nu = \frac{\pi}{3}, E^p = 0) = a_0 + \frac{p}{a_1 + a_2 p} \]  
(30)

where \( a_s \) are scalar valued parameters that are chosen to fit data from material characterization tests. Three compression meridians are defined, one representing the initial elastic domain, one for the failure strength and one for the residual strength according to
\[ f_i^c = f_i(p,v = \frac{c_3}{\sqrt{2}}, E^p = 0) = a_0^i + \frac{p}{a_1^i + a_2^i p} \]  

(31)

\[ f_i^c = f_i(p,v = \frac{c_3}{\sqrt{2}}, E^p = E_i^p) = a_0^i + \frac{p}{a_1^i + a_2^i p} \]  

(32)

\[ f_i^c = f_i(p,v = \frac{c_3}{\sqrt{2}}, E^p \geq E_i^p) = \frac{p}{a_1^i + a_2^i p} \]  

(33)

from which the current compressive load meridian is interpolated as

\[ f_i^c = \begin{cases} f_i^c, & E^p \leq 0 \\ d(f_i^c - f_i^c) + f_i^c, & 0 \leq E^p \leq E_i^p \\ d(f_i^c - f_i^c) + f_i^c, & E_i^p \leq E^p \leq E_i^p \\ f_i^c, & E_i^p \leq E^p \\ \end{cases} \]  

(34)

where

\[ d = d(E^p) \mid d \in [0,1[ \]  

(35)

The minimum, i.e. tensile, pressure is interpolated as

\[ p_{\min} = \begin{cases} -f_t, & 0 \leq E^p \leq E_i^p \\ -f_t d, & E_i^p \leq E^p \end{cases} \]  

(36)

where \( f_t \) is the failure strength in tension. The tensile meridian is given as a fraction \( k(p) \) of the compressive meridian according to

\[ k(p) = \frac{r_t(p)}{r_i(p)} \]  

(37)

and the values on \( k(p) \) are set according to Table 1, where \( f_t \) and \( f_i \) is the compressive and tensile failure strength, respectively.
Table 1 Values on the piecewise linear function $k(p)$

<table>
<thead>
<tr>
<th>$p$</th>
<th>$\leq 0$</th>
<th>$\frac{1}{3}f_c$</th>
<th>$\frac{2\cdot 1.15}{3}f_c$</th>
<th>$3f_c$</th>
<th>$\geq 8.45f_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$k(p)$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1 + 3f_c}{2f_c}$</td>
<td>$\frac{1.15f_c}{d'_0 + \frac{2\cdot 1.15 f_c/3}{d'_2 + 2d'_2 1.15 f_c/3}}$</td>
<td>0.753</td>
<td>1</td>
</tr>
</tbody>
</table>

In Figure 6 and Figure 7 graphical representations are given of the meridians corresponding to a concrete material with compressive and tensile strength of 100 and 5.3 MPa, respectively.

![Figure 6](image1.png)  ![Figure 7](image2.png)

**Figure 6** Compressive ($\nu=\pi/3$) and tensile ($\nu=0$) meridians.  **Figure 7** Compressive ($\nu=\pi/3$) and tensile ($\nu=0$) meridians in the lower pressure range.

The generalisation to a three-dimensional stress space, i.e. to include the third invariant of the deviatoric stress tensor, is done using the function proposed in William and Warnke [5] through the following expression

$$r_c(p, \nu) = r_c \left( 2(k^2 - k^2) \cos \nu + \left( (2k - 1) \right) \left( \left( 1 - k^2 \right) \cos^2 \nu + 5k^2 - 4k \right) \right)$$

$$\frac{4(1 - k^2) \cos^2 \nu + (1 - 2k)^2}{4(1 - k^2) \cos^2 \nu + (1 - 2k)^2}$$

(38)

where $r_c$ is the distance from the hydrostatic axis to an arbitrary meridian. In Figure 8 a graphical representation of the ratio $r_c/r_c$ is given.
Figure 8 Ratio between the distances from the hydrostatic axis to an arbitrary and the compressive meridian, respectively.

Strength enhancement due to high rate of loading, see Section 1, is included through the factor

$$a = a(D)$$

where

$$D = \left( \frac{2}{3} \mathbf{D} : \mathbf{D} \right)^{\frac{1}{2}}$$

and carried out radially from the origin in the principal stress space. An example of such a relation is given in Figure 9.
The complete expression for the load function then becomes

\[ f_i = a r_i f_i^c \]  

(41)

**Inelastic domain**

In the deviatoric inelastic domain defined as

\[ \partial E_T = \left\{ (\sigma, E^p) \in S \times R_+ \mid f(\sigma, E^p) = 0 \right\} \]  

(42)

the evolution of the inelastic deformation is governed by an non-associated flow rule with non-linear anisotropic strain hardening and softening. The derivation starts with the standard relations for plasticity theory

\[ D^p = \eta \dot{\epsilon} \]  

(43)

\[ \dot{\epsilon} = 0 \]  

(44)

and a for this model, a modified effective plastic strain measure defined as

\[ E^p = h D^p \]  

(45)

where

\[ D^p = \left( \frac{2}{3} D^p : D^p \right)^{\frac{1}{2}} \]  

(46)
Here \( b_i \) is test data fitting parameters, \( f_i \) is the uniaxial tensile strength and \( a \) is a factor to include rate effects. An associated flow rule would have direction according to

\[
\mathbf{r} = \mathbf{r}_a = \frac{\partial f}{\partial \mathbf{a}} = \frac{\partial J_2}{\partial \mathbf{a}} - \frac{\partial f_1}{\partial \mathbf{p}} \frac{\partial \mathbf{p}}{\partial \mathbf{a}} - \frac{\partial f_1}{\partial \mathbf{v}} \frac{\partial \mathbf{v}}{\partial \mathbf{a}}
\]

and the relation stating that that a stress point in plastic loading must remain on the loading surface is

\[
J = J_2 - \frac{\partial f_1}{\partial \mathbf{p}} \mathbf{p} - \frac{\partial f_1}{\partial \mathbf{v}} \mathbf{v} - \frac{\partial f_1}{\partial E^p} E^p = 0
\]

or as

\[
E^p = \frac{\frac{\partial f_1}{\partial E^p}}{\partial f_1}
\]

Using equations 43, 45, 46 and 50 the plastic multiplier for the associated case is evaluated as

\[
\mathbf{n}_a = \frac{J_2 - \frac{\partial f_1}{\partial \mathbf{p}} \mathbf{p} - \frac{\partial f_1}{\partial \mathbf{v}} \mathbf{v}}{b \frac{\partial f_1}{\partial E^p} \left( \frac{2}{3} \mathbf{r}_a : \mathbf{r}_a \right)^\frac{1}{2}}
\]

In our application of this model the direction of the plastic deformation is assumed to be independent of \( \mathbf{p} \) and \( \mathbf{v} \), i.e. the non-associated direction

\[
\mathbf{r} = \frac{\partial J_2}{\partial \mathbf{a}} = \frac{3}{2 J_2} \text{dev} (\mathbf{\sigma})
\]
is used. Thus, the plastic multiplier

$$ m = \frac{J_2 - \frac{\partial f_1}{\partial \tilde{p}} \tilde{p} - \frac{\partial f_1}{\partial \tilde{v}} \tilde{v}}{b \frac{\partial f_1}{\partial \tilde{E}_p^p} \left( \frac{2}{3} \mathbf{r} : \mathbf{r} \right)^{1/2}} $$

forms the final form of the flow rule expressed as

$$ \mathbf{D}^p = \frac{3 \left( J_2 - \frac{\partial f_1}{\partial \tilde{p}} \tilde{p} - \frac{\partial f_1}{\partial \tilde{v}} \tilde{v} \right)}{2 J_2 b \frac{\partial f_1}{\partial \tilde{E}_p^p}} \text{dev} (\mathbf{\sigma}) $$

(54)

To include damage due to isotropic tensile stressing, a volumetric part is added to the damage

$$ E^p_v = b_3 f_d k_d (D^v - D^v_l) $$

(55)

where

$$ f_d = \begin{cases} 
1 - \left| \frac{J_2}{0.1 \tilde{p}} \right|, & 0 \leq \left| \frac{J_2}{\tilde{p}} \right| \leq 0.1 \\
0, & \left| \frac{J_2}{\tilde{p}} \right| \geq 0.1 
\end{cases} $$

(56)

Here $b_3$ and $k_d$ are scalar valued parameters and $D^v$ and $D^v_l$ are the current volumetric strain and the volumetric strain at the load surface, respectively. An example of a function for the scalar valued internal variable $d(E^p)$ is given in Figure 10. This damage curve is optimized for one finite element size and to make it independent of the spatial discretization. Thus, in order to get the correct fracture energy release for all element sizes in a model, it has to be scaled relative the current element size using

$$ s = \frac{V^+}{L_c} $$

(57)

and
\[ E_s = \begin{cases} E_p + E_v^p, & 0 < E_p < E^p_i \\ E_f^p + \left( E_v^p \right)^{E^p_i < E_p} & \end{cases} \tag{58} \]

where \( V \) is the current element volume and \( L_c \) is a reference length.

The volumetric material behaviour is governed by an equation of state that incorporates three phases: Elastic, compaction and solidification. In the compaction phase the air filled pores collapse and in the solidification phase all pores have collapsed and the material is solidified. An example of such a relation is given in Figure 11 where \( V_0 \) and \( V \) denote the initial volume and current volume, respectively.

**Figure 10** Example of the function \( d(E_p) \).

**Figure 11** Equation of state for concrete material with an uniaxial compressive strength of 100 MPa.
3 Numerical analysis

An analytical solution to the field equations of continuum mechanics can be derived only in special cases. To solve the general form one must rely on numerical analysis. The idea of numerical analyses is to efficiently calculate accurate approximations to the solution. For the applications at hand the Finite Element Method (FEM) is the chosen numerical tool.

A kinematically admissible velocity field \( \mathbf{v} \) is defined as

\[
\mathbf{v}(\mathbf{X}, t) \in S, \quad S = \left\{ \mathbf{v} \mid \mathbf{v} \in C^0(\mathbf{X}), \mathbf{v}(\partial B_e) = \mathbf{v}_e \right\}
\] (59)

The \( C^0 \)-condition assures that the functions are square integrable. This gives a residual equation for the linear momentum

\[
\nabla \cdot \mathbf{\sigma} + \varphi \mathbf{b} - \varphi \mathbf{a} = \mathbf{d}
\] (60)

where \( \mathbf{d} \) is the residual vector. The idea here is to minimize a weighted residual over the spatial domain

\[
\int_B \mathbf{d} \mathbf{w} dB = \int_B (\nabla \cdot \mathbf{\sigma} + \varphi \mathbf{b} - \varphi \mathbf{a}) \mathbf{w} dB = 0
\] (61)

using a weight function \( \mathbf{w} \), or variation, defined as

\[
\mathbf{w}(\mathbf{X}) \in V, \quad V = \left\{ \mathbf{w} \mid \mathbf{w} \in C^0(\mathbf{X}), \mathbf{w}(\partial B_e) = 0 \right\}
\] (62)

If the weight functions are taken the same as the trial functions the formulation is referred to as the Bubnov-Galerkin method and if they differ, the Petrov-Galerkin method. Applying integration by parts and using the natural boundary conditions results in the variation form of the linear momentum equation.
\[
\int_B \sigma : (\nabla w) dB - \int_{B_e} \varphi b w dB - \int_{\partial B_e} t_n w dB + \int_{B_e} \varphi a w dB = 0
\]  
(63)

This expression quantifies the principle of virtual power and it constitutes the basis for the Finite Element Method in solid mechanics.

### 3.1 Spatial discretization

The body considered is discretized into \( ne \) subdomains, or finite elements, \( B_e \) defined by their nodes \( i \).

\[
\sum_{e=1}^{ne} \left( \int_{B_e} \sigma : (\nabla w) dB - \int_{B_e} \varphi b w dB - \int_{\partial B_e} t_n w dB + \int_{B_e} \varphi a w dB \right) = 0
\]  
(64)

The motion and weight function in each finite element are approximated as

\[
x(X, t) = N^T(X)x^e(t)
\]  
(65)

\[
w(X, t) = N^T(X)w^e(t)
\]  
(66)

where \( N \) is a matrix containing the element shape functions, \( x^e \) is the element nodal co-ordinate vector and \( w^e \) is the element weight function. These approximations and that the principle of virtual power should hold for any \( w \), result in

\[
\sum_{e=1}^{ne} \left( \mathbf{M}' a^e - f^e_{\text{ext}} + f^e_{\text{int}} \right) = 0
\]  
(67)

where

\[
\mathbf{M}' = \int_{B_e} \varphi \mathbf{N} \mathbf{N}^T dB
\]  
(68)

\[
f^e_{\text{ext}} = \int_{B_e} \varphi b \mathbf{N}^T dB + \int_{\partial B_e} t_n \mathbf{N}^T dB
\]  
(69)

\[
f^e_{\text{int}} = \int_{B_e} \sigma : \nabla (\mathbf{N}^T) dB
\]  
(70)

To reduce CPU-costs and use an explicit time integration a lumped, or diagonalized, mass matrix is computed through row summation as
An assembly procedure, where the element tensors are scattered on global tensors, is then performed which yields the semi-discrete linear momentum equation for the system

$$M_d \mathbf{a} + f_{\text{int}} = f_{\text{ext}}$$  \hfill (71)

To avoid volumetric locking and to further reduce the CPU-costs, the volume integration is performed using single point Gaussian quadrature. This introduces rank deficiency, manifested as hourglass modes, which has to be controlled, cf. Belytschko et al. [13]. This control is done through the addition of a stabilisation vector $f_{\text{stab}}$

$$M_d \mathbf{a} + f_{\text{int}} = f_{\text{ext}} + f_{\text{stab}}$$  \hfill (72)

### 3.2 Temporal discretization

For the time integration of the semi-discrete linear momentum equation the central difference method is used, which is an explicit step-by-step method. The integration starts with the initial conditions and the force vectors at time $t_0$. Nodal accelerations are calculated at the current time step $t_n$

$$\mathbf{a}(t_n) = M_d^{-1}[f_{\text{ext}}(t_n) - f_{\text{int}}(t_n) + f_{\text{stab}}(t_n)]$$  \hfill (73)

Then the central, or mid, velocities at time $t_{n+\frac{1}{2}}$ are calculated as

$$\mathbf{v}(t_{n+\frac{1}{2}}) = \mathbf{v}(t_n) + \mathbf{a}(t_n)(t_{n+\frac{1}{2}} - t_n)$$  \hfill (74)

where

$$t_{n+\frac{1}{2}} = \frac{1}{2}(t_n + t_{n+1})$$  \hfill (75)

After this step the velocity boundary conditions are enforced and the displacement is updated as
\[ \mathbf{u}(t_{n+1}) = \mathbf{u}(t_n) + \mathbf{v}[t_{n+\frac{1}{2}}] \Delta t \]  

(76)

where

\[ \Delta t = t_{n+1} - t_n \]  

(77)

The external force vector is assembled from prescribed loading and contact forces. To calculate the internal force vector one needs the Cauchy stress tensor at \( t_{n+1} \). First the stress tensor at \( t_n \) is rotated into the configuration at \( t_{n+1} \) and the hydrostatic pressure is subtracted

\[ \text{dev}\left[\sigma'(t_n)\right] = \sigma(t_n) + \left[\sigma(t_n)\mathbf{W}[t_{n+\frac{1}{2}}]\right] + \sigma_n\mathbf{W}[t_{n+\frac{1}{2}}] \Delta t + p(t_n) \mathbf{I} \]  

(78)

secondly, the deviatoric increment from the constitutive routine is added

\[ \text{dev}\left[\sigma'(t_{n+\frac{1}{2}})\right] = \text{dev}\left[\sigma'(t_n)\right] + \text{dev}\left[\sigma'(t_{n+\frac{1}{2}})\right] \Delta t \]  

(79)

The hydrostatic pressure at \( t_{n+1} \) is obtained from the equation of state

\[ p = p(V, E) \]  

(80)

where \( V \) is the relative volume and \( E \) is the internal energy. The internal energy is updated as

\[ E(t_{n+1}) = E(t_n) + \Delta E \]  

(81)

where

\[ \Delta E = m \frac{D\epsilon}{Dt} \Delta t = mg^{-1} : \mathbf{D} \Delta t = \nu \text{dev}(\mathbf{\sigma}) : \mathbf{D} \Delta t - \nu p V \Delta t \]  

(82)

Here \( m \) and \( \nu \) are the current element volume and mass, respectively, and \( V \) is the relative volume. The temporal discretization of this equation is

\[ \Delta E = \frac{1}{2} \left[ \rho(t_n) + \rho(t_{n+1}) \right] \frac{1}{2} \left[ \text{dev}\left[\sigma'(t_n)\right] + \text{dev}\left[\sigma'(t_{n+1})\right] \right] : \mathbf{D}[t_{n+\frac{1}{2}}] \Delta t - \frac{1}{2} \Delta \nu p \]  

(83)

and an iterative procedure is performed according to

\[ p_n \rightarrow E^{n+1}_n \rightarrow p_{n+1}^* \rightarrow E^{n+1}_n \rightarrow p_{n+1} \]  

(84)

Finally the new pressure \( p(t_{n+1}) \) is added to the stress tensor
After computing the internal force vector the acceleration at time $t_{n+1}$ is given by
\[
a(t_{n+1}) = M_d^{-1} \left[ f_{\text{ext}}(t_{n+1}) - f_{\text{int}}(t_{n+1}) + f_{\text{stab}}(t_{n+1}) \right]
\] (86)
and the mid velocities are updated to time $t_{n+1}$ as
\[
v(t_{n+1}) = v(t_{n+\frac{1}{2}}) + a(t_{n+1})(t_{n+1} - t_{n+\frac{1}{2}})
\] (87)
Finally the energy balance is controlled and, unless the computation is terminated, the current time is updated and the procedure is repeated.

### 3.3 Shock waves

The presence of singular surfaces, cf. Truesdell and Toupin [29], results in multiple solutions to the field equations. Shock waves, defined as singular surfaces of first order with discontinuous deformation gradient and longitudinal velocity, can occur in materials where the sound velocity increases with increasing pressure. Shock waves are treated with bulk, or pseudo, viscosity that prohibits a shock wave to fully develop into a singular surface, cf. Neumann and Richtmyer [14]. The method consists in adding a hydrostatic pressure term,
\[
q = \begin{cases} 
\rho \nu^2 \left[ A \nu^2 \text{tr}(D)^2 - B \text{tr}(D) \right], & \text{tr}(D) < 0 \\
0, & \text{tr}(D) \geq 0
\end{cases}
\] (88)
where $A$ and $B$ are constants and $\nu$ is the material bulk sound speed, to the stress tensor in the field equations for linear momentum and energy.
4 Summary of appended publications

Numerical simulations of penetration and perforation of high performance concrete with 75mm steel projectile

The purpose of this study was to assess the ability to predict penetration depth or residual velocity with the chosen numerical methods and concrete constitutive model. The material description of the motion of both the targets and the projectiles was chosen together with a numerical erosion based on a shear strain criterion. The concrete material was modelled with the K&C concrete model and for the analysis LS-DYNA was used. For the perforation good agreement with test data was achieved but in the case of penetration, the results were not satisfying. The results were greatly influenced by the erosion criteria and the material model could not handle a discretized domain of finite element of different sizes. The conclusions were that the description of the softening behaviour had to be modified to render a fracture energy release that is independent of the spatial discretization. Also, it is not suitable to describe the target in a material reference frame, due to the need of an erosion criterion. Instead, the target should be described in a spatial reference frame, where the need for erosion is eliminated, while the material reference frame can be retained for the projectile.

Numerical simulations of the response of reinforced concrete beams subjected to heavy drop tests

The purpose of the work was to evaluate the ability of the chosen numerical method and material models to predict the material and structural response. The material model was modified to scale the softening behaviour relative the finite element sizes. The finite element analysis gave a different type of failure compared to the tests. In the test, the failure was mode I cracking combined with crushing in the impact zone. In the simulations, the failure was mainly due to mode II cracking. A material parameter analysis was performed but the results from the test could not be
reproduced. The conclusion is that the modified material model does not seem to be capable of correctly describing the problem, given the material properties and the numerical method of analysis. To handle the strain localization, that occurred in the problem, it is suggested that non-local measures of deformation should be used to attain a convergent solution.

Finite element analysis of the splitting test
The purpose with this study was to evaluate the possibility to use non-local measures of deformation to attain convergence when strain localization is present. A simplified non-local theory is used, where the local strain measure is weighted and integrated over an element neighbourhood and used to calculate the rate of evolution of the inelastic strain. The size of the neighbourhood in the non-local theory has to be determined through material characterization tests. The theory is applied to a splitting test and a tensile test for three different materials. The split test model shows mesh dependency due to a singularity. In the tensile test the non-local theory is shown to give a convergent solution. The conclusion is that it is possible to handle singularities with a non-local theory. The concrete material model will not be used in future work, due to the many problems encountered in this and previous studies. The report is concluded with a discussion on how to better model concrete material.
Bibliography


