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Upper gradients and Sobolev spaces on metric spaces

David Färm

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Applied Mathematics, Linköping University

David Färm

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Supervisor: Jana Björn,
Applied Mathematics, Linköping University

Examiner: Jana Björn,
Applied Mathematics, Linköping University

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All papers on this topic are written for an audience of fellow researchers and people with graduate level mathematical skills. In this thesis we give an introduction to the Newtonian spaces accessible also for senior undergraduate students with only basic knowledge of functional analysis. We also give an introduction to the tools needed to deal with the Newtonian spaces. This includes measure theory and curves in general metric spaces.

Many of the properties of ordinary Sobolev spaces also apply in the generalized setting of the Newtonian spaces. This thesis includes proofs of the fact that the Newtonian spaces are Banach spaces and that under mild additional assumptions Lipschitz functions are dense there. To make them more accessible, the proofs have been extended with comments and details previously omitted. Examples are given to illustrate new concepts.

This thesis also includes my own result on the capacity associated with Newtonian spaces. This is the theorem that if a set has \( p \)-capacity zero, then the capacity of that set is zero for all smaller values of \( p \).

Capacity, measure, metric space, Sobolev space, upper gradient.
Abstract

The Laplace equation and the related $p$-Laplace equation are closely associated with Sobolev spaces. During the last 15 years people have been exploring the possibility of solving partial differential equations in general metric spaces by generalizing the concept of Sobolev spaces. One such generalization is the Newtonian space where one uses upper gradients to compensate for the lack of a derivative.

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Keywords: Capacity, measure, metric space, Sobolev space, upper gradient.
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Contents

1 Introduction ............................................. 1
   1.1 The Laplace equation ................................. 1
   1.2 The Sobolev space associated with the Laplace equation .... 2
   1.3 The $p$-Laplace equation and its Sobolev space .............. 2
   1.4 The Sobolev space and how to generalize it .................. 2
   1.5 The aim of this thesis ............................... 3
   1.6 Outline of this thesis ................................ 3

2 Preliminaries .......................................... 5
   2.1 Metric spaces and norms .............................. 5
   2.2 Open and closed sets ................................ 5
   2.3 The extended reals ................................... 7
   2.4 A special constant .................................... 7
   2.5 Lipschitz functions ................................... 7
   2.6 Measure theory ...................................... 9
   2.7 The measure $\mu$ .................................... 10
   2.8 Integration ......................................... 11
   2.9 Equivalence relations ................................ 13
   2.10 Borel functions ..................................... 14

3 Curves .................................................. 17
   3.1 Rectifiable curves .................................... 17
   3.2 Arc length parameterization ............................ 18
   3.3 Modulus of curve families ............................. 21

4 Finding a replacement for the derivative ................. 27
   4.1 The upper gradient ................................... 27
   4.2 The minimal weak upper gradient ..................... 32
   4.3 Absolute continuity .................................. 35

5 Newtonian spaces – Sobolev spaces on metric spaces .... 39
   5.1 The Newtonian Space .................................. 39
   5.2 Capacity ............................................ 41
   5.3 Density of Lipschitz functions ....................... 50

6 Final remarks ........................................... 59

Färn, 2006.
Chapter 1

Introduction

1.1 The Laplace equation

Many processes in physics can be modelled by partial differential equations. Two examples are the heat equation

\[
\frac{\partial u}{\partial t} = \Delta u
\]

and the wave equation

\[
\frac{\partial^2 u}{\partial t^2} = \Delta u,
\]

where

\[
\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2}.
\]

If one considers the stationary case, both these equations reduce to the equation

\[
\Delta u = 0, \quad (1.1)
\]

known as the Laplace equation. Often, one needs to find a function that satisfies the Laplace equation in a region \( \Omega \) and equals a function \( f \) on the boundary \( \partial \Omega \). This is the Dirichlet problem.

One approach to the Dirichlet problem is to look for a function \( u : \Omega \to \mathbb{R} \) that minimizes the integral

\[
\int_{\Omega} |\nabla u(x)|^2 \, dx, \quad (1.2)
\]

among functions that satisfy \( u = f \) on \( \partial \Omega \). It can be shown that if we look for this minimizer among the right type of functions, we will indeed find a solution to the problem called the weak solution. The correct space to look in is the Sobolev space (see the following section). Note that as opposed to \( (1.1) \), in \( (1.2) \) we do not need the partial derivatives of \( u \). In fact, we only need the modulus \(|\nabla u|\) of the gradient \( \nabla u \). This will be important when studying this problem in a more general setting.

For proofs and further reading about partial differential equations, see [13].
1.2 The Sobolev space associated with the Laplace equation

Let \( u \) be locally integrable on an open set \( \Omega \subset \mathbb{R}^n \). A locally integrable function \( v \) on \( \Omega \) is a weak derivative of \( u \) in the direction \( x_i \) if

\[
\int_{\Omega} \phi(x)v(x) \, dx = - \int_{\Omega} u(x) \frac{\partial \phi(x)}{\partial x_i} \, dx,
\]

where \( x = (x_1, \ldots, x_n) \), for all infinitely differentiable functions \( \phi \) vanishing on the boundary \( \partial \Omega \). We write \( v = D_i u \).

The Sobolev space \( W^{1,2}(\Omega) \) is the space of all functions \( u \in L^2(\Omega) \) that have weak derivatives \( D_i u \in L^2(\Omega) \) in all directions. It is equipped with the norm

\[
\|u\|_{W^{1,2}(\Omega)} = \left( \int_{\Omega} |u(x)|^2 + \sum_{i=1}^n |D_i u(x)|^2 \, dx \right)^{1/2}.
\]

1.3 The \( p \)-Laplace equation and its Sobolev space

For fluids like water or alcohol the relation between stress and the rate of strain can be modelled as linear. Such fluids are called Newtonian (note that this has nothing to do with the Newtonian space introduced in Chapter 5). There are fluids that do not have this property and these are called non-Newtonian fluids. Examples of such fluids are blood and molten plastics. For more information on fluids and their properties, take a look in just about any introductory book on fluid mechanics, for example [15]. One equation that is used for modelling flows of non-Newtonian fluids is the equation

\[
\text{div}(|\nabla u|^{p-2} \nabla u) = 0, \quad 1 \leq p < \infty,
\]

known as the \( p \)-Laplace equation. We see that for \( p = 2 \) this equation reduces to the ordinary Laplace equation.

When solving the Dirichlet problem for the \( p \)-Laplace equation one can look for a solution by minimizing the integral

\[
\int_{\Omega} |\nabla v(x)|^p \, dx.
\]

Associated with this integral is the Sobolev space \( W^{1,p}(\Omega) \), the space of all functions \( u \in L^p(\Omega) \) that have weak derivatives \( D_i u \in L^p(\Omega) \) in all directions. It is equipped with the norm

\[
\|u\|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} |u(x)|^p + \sum_{i=1}^n |D_i u(x)|^p \, dx \right)^{1/p}.
\]

1.4 The Sobolev space and how to generalize it

As the index 1 in \( W^{1,p} \) suggests, there are spaces \( W^{k,p} \) for \( k > 1 \), but we will not deal with those spaces in this thesis.
1.5. The aim of this thesis

There has been a lot of work done about Sobolev spaces and their properties. During the last 15 years people have explored the possibility to solve partial differential equations in general metric spaces by generalizing the concept of Sobolev spaces. One such generalization is the Newtonian space, introduced by Shanmugalingam in [20] and [21]. The Newtonian space is defined not only for \( \mathbb{R}^n \) equipped with the Euclidean norm and the \( n \)-dimensional Lebesgue measure. Instead, it is defined for any metric space equipped with almost any measure. It has been shown that many of the properties of ordinary Sobolev spaces also apply for the generalized setting of the Newtonian spaces. Thus, some of the techniques used when solving partial differential equations in \( \mathbb{R}^n \) can be used when solving partial differential equations in general metric spaces.

1.5 The aim of this thesis

In this thesis we study the Newtonian spaces and some of their properties. Most papers on this topic are written for an audience of fellow researchers and people with graduate level mathematical skills. The aim of this thesis is to give an introduction to the Newtonian spaces accessible also for senior undergraduate students with only basic knowledge of functional analysis. We will therefore give an introduction to tools needed to deal with the Newtonian spaces. This includes measure theory and curves in general metric spaces.

When introducing the Newtonian spaces we will mostly follow the original definitions and proofs by Shanmugalingam. During the time since [20] and [21] was published, several other authors have developed the theory and rewritten some of the proofs. The main source of inspiration for this thesis has been [3]. To make them more accessible, most proofs have been extended with comments and details previously omitted. When possible, examples are given to illustrate new concepts.

In this thesis we also study the capacity, which is a tool associated with Newtonian spaces to measure sizes of sets. We also prove a new result about sets with capacity zero.

1.6 Outline of this thesis

To be able to follow the arguments when dealing with Sobolev spaces, an undergraduate student like myself needs to be introduced to some topics not always dealt with in undergraduate level mathematics, or at least to recall them. Thus, in Chapter 2 we give the definitions of concepts that we will use. We go through the notation and the settings that will be used in the rest of the thesis. A large part of the chapter is devoted to give an introduction to the measure theory needed in this thesis.

Curves play an important role in the Newtonian space, so in Chapter 3 we introduce the concept of curves in general metric spaces. There we define what we mean by a line integral and we describe a way to measure the size of a family of curves.

In ordinary Sobolev spaces we have a weak derivative, but in general metric spaces there is no such thing. Instead, in Chapter 4 we introduce and study the upper gradient. We show that it can be used to create something that works as
a substitute for the modulus of the gradient.

In Chapter 5 we define the Newtonian space that is a generalization of ordinary Sobolev spaces to general metric spaces. We show that some of the properties of ordinary Sobolev spaces also hold in the Newtonian space. The most important properties are that the Newtonian space is a Banach space and that under mild additional assumptions, Lipschitz functions are dense there.

In Section 5.2 we study the capacity associated with the Newtonian space. A new result about the capacity is proven. The capacity depends on a parameter \( p \). It is shown that if a set has capacity zero for a given value of \( p \), then it remains zero if \( p \) is decreased.

Chapter 6 contains some final remarks on this thesis and the theory in general.
Chapter 2

Preliminaries

2.1 Metric spaces and norms

In this thesis, $X$ will always denote a metric space with a metric $d$. This means that we have $d : X \times X \to \mathbb{R}$ such that for all $x, y \in X$ we have

- $d$ is finite and non-negative.
- $d(x, y) = 0$ if and only if $x = y$.
- $d(x, y) = d(y, x)$.
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $z \in X$.

We write $X = \{X, d\}$ with the metric $d$ omitted for simplicity.

As customary, $\| \|$ will be used to denote norms. In cases where it is not obvious we will use index to specify which metric or norm we mean.

In the norms used in this thesis we will often have a parameter $p$. This will be a number $1 \leq p < \infty$. The parameter $p$ will show up in other places as well.

2.2 Open and closed sets

To say that $A$ is a subset of $B$ we will write $A \subset B$ rather than writing $A \subseteq B$. Note that in our notation we get that $A \subset B$ includes the case $A = B$.

In some situations it is inconvenient to write $X \setminus A$ for the complement of a set $A$ in $X$ relative to $X$. In those cases, where there is little risk of confusion, we will write $A^c$.

We will sometimes use the characteristic function of a set.

**Definition 2.2.1.** The characteristic function of a set $E \subset Y$ is the function $\chi_E : Y \to \mathbb{R}$ such that

$$\chi_E(x) = \begin{cases} 1 & \text{when } x \in E; \\ 0 & \text{otherwise.} \end{cases}$$

If $a, b \in \mathbb{R}$, then $(a, b)$ denotes the open interval from $a$ to $b$, while $[a, b]$ denotes the closed interval from $a$ to $b$.

Färm, 2006.
Open balls will be used frequently in this thesis. By an open ball \( B(x, r) \), in a set \( \Omega \subseteq X \), we mean \( B(x, r) = \{ y \in \Omega : d(x, y) < r \} \). When we write \( \lambda B \) for an open ball \( B = B(x, r) \), we mean the open ball \( B(x, \lambda r) \).

We will use these definitions of open and closed sets.

**Definition 2.2.2.** (Open set) A set \( G \subseteq X \) is said to be open if for every \( x \in G \) there exists \( \epsilon > 0 \), such that \( B(x, \epsilon) \subseteq G \).

**Definition 2.2.3.** (Closed set) A set \( F \subseteq X \) is said to be closed if \( F^c \) is open.

The following fact about open sets in \( \mathbb{R} \) will be useful later on.

**Proposition 2.2.4.** Every open set \( G \subseteq \mathbb{R} \) can be written as \( G = \bigcup_{i=1}^{\infty} A_i \), where \( \{A_i\}_{i=1}^{\infty} \) is a sequence of pairwise disjoint open intervals.

**Proof.** For every \( x \in G \) there exists a \( \delta > 0 \) such that \( (x - \delta, x + \delta) \subseteq G \) since \( G \) is open. Thus, for every \( y \in Q \cap G \), we can create

\[
 a = \inf \{ x : (x, y) \subseteq G \}, \quad b = \sup \{ x : (y, x) \subseteq G \} \quad \text{and} \quad A = (a, b).
\]

We want to prove that \( A \subseteq G \). Let \( x \in A \) be arbitrary. Then:

- If \( x = y \), then \( x \in G \) since \( y \in G \).
- If \( x \in (a, y) \), then there exists some \( c \) such that \( a < c < x \) and \( x \in (c, y) \).
  
  By the definition of \( a \) we see that \( (c, y) \subseteq G \) and thus \( x \in G \).
- If \( x \in (y, b) \), then there exists some \( c \) such that \( x < c < b \) and \( x \in (y, c) \).
  
  By the definition of \( b \) we see that \( (y, c) \subseteq G \) and thus \( x \in G \).

Since \( x \in A \) was arbitrary we obtain \( A \subseteq G \). Now let \( \{y_i\}_{i=1}^{\infty} \) be an enumeration of \( Q \cap G \). For each \( y_i \), create the open interval \( A_i = (a_i, b_i) \) according to (2.1).

Then let

\[
 G^* = \bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (a_i, b_i).
\]

Given an arbitrary \( x \in G^* \) we see that \( x \in (a_i, b_i) \) for some \( i \). But \( (a_i, b_i) \subseteq G \) for all \( i \) so \( x \in G \). Thus \( G^* \subseteq G \).

Let \( x \in G \) be arbitrary. Since \( G \) is open there exists a \( \delta > 0 \) such that \( (x - \delta, x + \delta) \subseteq G \). We know that \( Q \) is dense in \( \mathbb{R} \) and thus we can find \( y_i \in Q \) such that \( y_i \in (x - \delta, x + \delta) \). Since \( (x - \delta, x + \delta) \subseteq G \) we know that \( a_i \leq x - \delta \) and \( b_i \geq x + \delta \). Thus \( x \in (x - \delta, x + \delta) \subseteq (a_i, b_i) \subseteq G^* \). Since \( x \in G \) was arbitrary this implies that \( G \subseteq G^* \). Thus \( G = G^* \).

So far we have shown that \( G \) can be written as \( G = \bigcup_{i=1}^{\infty} A_i \), where \( \{A_i\}_{i=1}^{\infty} \) is a sequence of open intervals. The remaining part of the proof is to make the intervals pairwise disjoint.

If we consider an interval \( A_i \) and compare it to the intervals before it in the sequence, only two situations can occur:

- We might get that \( A_i \cap A_j = \emptyset \) for all \( j \) such that \( 0 < j < i \). This means that \( A_i \) is already disjoint relative to the previous intervals and we do not need to do anything about it.
• On the other hand it may occur that \( A_i \cap A_j \neq \emptyset \) for some \( j \) such that \( 0 < j < i \). Since the intersection is non-empty there must exist some \( z \) such that \( z \in A_i \) and \( z \in A_j \). The interval between \( z \) and \( y_i \) as well as the interval between \( z \) and \( y_j \) is in \( G \), so we see that the interval between \( y_i \) and \( y_j \) must be in \( G \). We assume that \( y_i < y_j \), which implies \([y_i, y_j] \subset G\).

From (2.1) we know get

\[
a_i = \inf\{x : (x, y_i) \subset G\} = \inf\{x : (x, y_j) \subset G\} = a_j
\]

and

\[
b_i = \sup\{x : (y_i, x) \subset G\} = \sup\{x : (y_j, x) \subset G\} = b_j.
\]

The case \( y_i > y_j \) can be treated analogously, so \( A_i = A_j \). To make our sets pairwise disjoint we simply remove \( A_i \) and lower the index by one for every set previously placed after the removed set \( A_i \).

By carrying this out in order of index, starting with \( i = 2 \), we get a sequence \( \{A_i\}_{i=1}^{\infty} \) of pairwise disjoint open intervals. Since we have only altered the index of the sets and removed sets equal to other sets in the sequence, we still have

\[
G = \bigcup_{i=1}^{\infty} A_i
\]

and the proposition follows.

\[\square\]

2.3 The extended reals

Throughout this thesis we will use the extended real number system denoted by \( \mathbb{R} \). It is the ordinary reals completed with the symbols \( -\infty \) and \( \infty \) used as numbers with the obvious interpretation. For example, if something is larger than all \( n \in \mathbb{N} \), it is equal to \( \infty \).

2.4 A special constant

The letter \( C \) will denote a strictly positive constant with a value which we are not interested in. This allows us to use \( C \) freely, letting it change with each usage. The important thing is that there is a constant for which the equations hold, we do not care about the value of the constant as long as it is strictly positive. For example, if \( f \) is a function of some sort, then \( 3\pi f(x) < C \) for all \( x \), implies \( f(x) < C \) for all \( x \). Note that \( C \) is the only constant dealt with in this way.

2.5 Lipschitz functions

It is often useful to know how much the value of a function changes when the variable changes. For Lipschitz functions we have an upper bound for this change.
Definition 2.5.1. (Lipschitz function) Let $Y$ and $Z$ be metric spaces with metrics $d_Y$ and $d_Z$ respectively. A function $f : Y \rightarrow Z$, that satisfies

$$d_Z(f(a), f(b)) \leq Ld_Y(a, b)$$

for all $a, b \in Y$ is called a Lipschitz function with constant $L$. Functions satisfying this condition are often referred to as $L$-Lipschitz.

If one has a Lipschitz function $f : E \rightarrow \mathbb{R}$, where $E$ is a subset of a metric space $Y$, then it is possible to extend $f$ to a Lipschitz function on the entire space $Y$. One such extension is the McShane extension from [16]. The following proposition assures that this extension really is Lipschitz.

Proposition 2.5.2. Let $f : E \rightarrow \mathbb{R}$ be $L$-Lipschitz, where $E$ is a subset of a metric space $Y$. Then the function $\tilde{f} : Y \rightarrow \mathbb{R}$, defined by

$$\tilde{f}(x) = \inf_{y \in E} \{f(y) + Ld(x, y)\}$$

is $L$-Lipschitz on $Y$ and $\tilde{f} = f$ on $E$.

Proof. Let $x_1, x_2 \in Y$. Then, for every $\epsilon > 0$ there exists a point $x_3 \in E$ such that

$$f(x_3) + Ld(x_1, x_3) < \tilde{f}(x_1) + \epsilon.$$ 

For the other point, $x_2$, we then get

$$\tilde{f}(x_2) = \inf_{y \in E} \{f(y) + Ld(x_2, y)\}$$

$$\leq f(x_3) + Ld(x_2, x_3)$$

$$\leq f(x_3) + Ld(x_2, x_1) + Ld(x_1, x_3)$$

$$< \tilde{f}(x_1) + Ld(x_2, x_1) + \epsilon,$$

for all $\epsilon > 0$. By letting $\epsilon \rightarrow 0$ we get

$$\tilde{f}(x_2) - \tilde{f}(x_1) \leq Ld(x_1, x_2).$$

Analogously we get

$$\tilde{f}(x_1) - \tilde{f}(x_2) \leq Ld(x_2, x_1).$$

We have thus proven that $\tilde{f}$ is $L$-Lipschitz on $X$. We now continue by proving that $\tilde{f}(x) = f(x)$ on $E$. For $x \in E$, we easily get that

$$\tilde{f}(x) \leq f(x) + Ld(x, x) = f(x).$$

If $\tilde{f}(x) < f(x)$ there must be an $x' \in E$ such that $f(x') + Ld(x', x) < f(x)$. This implies that

$$Ld(x', x) < f(x) - f(x') \leq Ld(x', x),$$

which is a contradiction. Thereby we get $\tilde{f}(x) = f(x)$ for all $x \in E$. 

2.6 Measure theory

Measure theory involves the use of a couple of new concepts. This section is mostly a collection of definitions that we will use later on.

Definition 2.6.1. (σ-algebra) A collection \( \mathcal{M} \) of subsets of a set \( X \) is said to be a σ-algebra in \( X \) if \( \mathcal{M} \) has the following properties:

(a) \( X \in \mathcal{M} \).

(b) If \( A \in \mathcal{M} \) then \( A^c \in \mathcal{M} \).

(c) If \( A = \bigcup_{n=1}^{\infty} A_n \) and if \( A_n \in \mathcal{M} \) for \( n = 1, 2, 3, \ldots \), then \( A \in \mathcal{M} \).

Definition 2.6.2. (Measurability) Let \( \mathcal{M} \) be a σ-algebra in \( X \). Then \( X \) is called a measurable space, and the members of \( \mathcal{M} \) are called measurable sets in \( X \).

Let \( X \) be a measurable space, \( Y \) be a metric space and \( f : X \to Y \). Then \( f \) is said to be measurable provided that \( f^{-1}(V) \) is a measurable set in \( X \) for every open set \( V \) in \( Y \).

Often in this thesis we will begin with measurable sets and functions and manipulate them in different ways. After doing this we will still need them to be measurable.

For sets we will take complements, countable unions and countable intersections. As a σ-algebra is closed under complements and countable unions it is also closed under countable intersections. Thus, the sets we create from measurable sets in this way will be part of the same σ-algebra as the measurable sets, so they will be measurable.

For functions we will do many different things. That the sum, product, maximum, minimum, supremum, infimum, pointwise limit and modulus of measurable functions are measurable is proven in [19](Theorems 1.7, 1.8, 1.14 and corollaries). It is also proven there that the characteristic function of a measurable set is measurable and that continuous functions of measurable functions are measurable.

From now on we will consider all functions created in the ways just mentioned as measurable, without further notice. Sometimes we will come across other ways to create new functions or sets. In those cases we will prove that the sets or functions are measurable if we need them to be.

Let \( Y \) be a metric space. Then there exists a smallest σ-algebra \( \mathcal{B} \) in \( Y \) such that every open set in \( Y \) belongs to \( \mathcal{B} \). For a proof, see Theorem 1.10 in [19].

Definition 2.6.3. (Borel set) The smallest σ-algebra in \( X \), that contains every open set in \( X \), is called \( \mathcal{B} \). The members of \( \mathcal{B} \) are called Borel sets of \( X \).

The definition of Borel sets is somewhat complicated. In fact, Borel sets are open sets or sets created from open sets by successive countable unions and complements.

Definition 2.6.4. (Measure) A positive measure is a function \( \mu \), defined on a σ-algebra \( \mathcal{M} \), whose range is in \([0, \infty]\) and which is countably additive. This means that if \( \{A_i\}_{i=1}^{\infty} \) is a pairwise disjoint countable collection of members of \( \mathcal{M} \), then

\[
\mu \left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} \mu(A_i).
\]
By a measure space we mean a measurable space which has a positive measure defined on the \( \sigma \)-algebra of its measurable sets.

**Definition 2.6.5.** (Borel measure) A measure \( \mu \) defined on the \( \sigma \)-algebra \( B \) of all Borel sets in \( X \) is called a Borel measure on \( X \).

### 2.7 The measure \( \mu \)

When generalizing the Sobolev spaces to metric spaces we cannot use the Lebesgue measure, since it is only defined on \( \mathbb{R}^n \). Instead, we will assume as little as possible about the measure, so that our conclusions hold for as many measures as possible.

In this section we will describe the properties we need for the measure. Throughout the rest of this thesis, \( \mu \) will be a general measure satisfying the conditions described in this section.

We begin by letting \( \mu \) be a Borel measure on \( X \) that satisfies

\[
0 < \mu(B) < \infty, \text{ for all open balls } B \in X. \tag{2.2}
\]

Let \( B^* \) be the family of sets \( E \subset X \), for which there exist Borel sets \( A, D \in \mathcal{B} \), such that

\[
A \subset E \subset D \quad \text{and} \quad \mu(D \setminus A) = 0. \tag{2.3}
\]

![Figure 2.1: The Borel sets \( A \) and \( D \) have the same measure as \( E \).](image)

Complete \( \mu \) by letting

\[
\mu(E) = \mu(D),
\]

where \( E \in B^* \) and \( D \) is as in (2.3). Then \( B^* \) is a \( \sigma \)-algebra and \( \mu \) is a measure on \( B^* \) according to Theorem 1.36 in [19].

Now Theorem 2.2.2. in [5] implies that, for all \( \epsilon > 0 \),

- there is an open set \( V \supset D \) such that \( \mu(V \setminus D) < \epsilon \), and hence
  \[
  \mu(V \setminus E) \leq \mu(V \setminus D) + \mu(D \setminus E) < \epsilon;
  \]

- there is a closed set \( F \subset A \) such that \( \mu(A \setminus F) < \epsilon \), and hence
  \[
  \mu(E \setminus F) \leq \mu(E \setminus A) + \mu(A \setminus F) < \epsilon.
  \]
This holds for all $E \in B^*$. In the second statement, Federer [5] assumes that the set $A$ is contained in a union of countably many open sets with finite measure. This fact follows from our previous assumptions. Indeed, since $X$ is a metric space, we get

$$A \subset X = \bigcup_{j=1}^{\infty} B(x_0, j),$$

where $\mu(B(x_0, j)) < \infty$, by (2.2), for any $x_0 \in X$.

Through the rest of this thesis, $\mu$ will be a measure with these properties. We say that $\mu$ is a Borel regular measure. We have thus created a measure space $X$ with a Borel regular measure $\mu$ where the measurable sets are all the Borel sets in $X$ plus all sets $E$ satisfying

$$A \subset E \subset D \quad \text{and} \quad \mu(D\setminus A) = 0$$

for some Borel sets $A$ and $D$. We will write $X = \{X, d, \mu\}$ with the metric $d$ and the measure $\mu$ omitted for simplicity. If a property holds everywhere but on a set $E$ with $\mu(E) = 0$ we say that the property holds almost everywhere with respect to $\mu$, or $\mu$-a.e. for short. Since we almost always use the measure $\mu$, we will omit it in our notation. So, if we do not say otherwise, a.e. always refers to the measure $\mu$.

## 2.8 Integration

In this thesis we will deal with a lot of integrals. Since our functions will be maps from a general metric space we recall the definition of integrals over such spaces.
The definition uses simple functions, that is, functions whose range consists of only finitely many points. A simple function \( s : X \to \mathbb{R} \) can be written as
\[
s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},
\]
where \( \alpha_1, \ldots, \alpha_n \) are the values of \( s \) on the disjoint sets \( A_1, \ldots, A_n \) and where \( \chi_{A_i} \) is the characteristic function for \( A_i \). We see that a simple function is measurable if and only if the sets \( A_1, \ldots, A_n \) are measurable.

**Definition 2.8.1.** (Integration of positive functions) For a positive simple measurable function \( s : X \to [0, \infty) \) and a measurable set \( E \subset X \), we define
\[
\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E).
\]
If for some \( i \), \( \alpha_i = 0 \) and \( \mu(A_i \cap E) = \infty \), we end up with \( 0 \cdot \infty \). In this case we use the convention that \( 0 \cdot \infty = 0 \). For a positive measurable function \( f : X \to [0, \infty] \) and a measurable set \( E \subset X \), we define
\[
\int_E f \, d\mu = \sup_{0 \leq s \leq f} \int_E s \, d\mu,
\]
where we take the supremum over all positive simple measurable functions \( s \), such that \( s \leq f \).

Note that in this notation the variable, which we integrate with respect to, is omitted. Instead we end the integral with \( d\mu \) since the integral is dependent on the measure.

When dealing with integrals we often use the \( L^p \)-spaces. We recall that a function \( f : X \to \mathbb{R} \) belongs to \( L^p(X) \) if \( f \) is measurable and
\[
\int_X |f|^p \, d\mu < \infty.
\]
The space \( L^p(X) \) is equipped with the norm
\[
\|f\|_{L^p(X)} = \left( \int_X |f|^p \, d\mu \right)^{1/p}.
\]
A function \( f \in L^p(X) \) is often referred to as \( p \)-integrable on \( X \). Whether \( f \in L^P(X) \) or not depends of course on the measure \( \mu \), but since we always use this measure we omit it in our notation. We also omit the space \( X \) in our notation so that \( L^p = L^p(X) \). On the other hand, if \( f \) is \( p \)-integrable on \( A \subset X \) we write \( f \in L^p(A) \), since it is needed to avoid confusion.

There are many theorems about measures and integrals. We will state two of the most well known theorems for later use. The first one is Lebesgue’s Monotone Convergence Theorem, which is stated and proven as Theorem 1.26 in [19]. It states that one can switch limits and integrals under certain conditions.

**Theorem 2.8.2.** (Lebesgue’s Monotone Convergence Theorem) Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of measurable functions on \( X \). Suppose that the sequence is positive
2.9. Equivalence relations

and pointwise increasing. Furthermore suppose that \( f_n(x) \to f(x) \) a.e. as \( n \to \infty \). Then \( f \) is measurable and

\[
\int_X f_n \, d\mu \to \int_X f \, d\mu, \text{ as } n \to \infty.
\]

The second theorem is Lebesgue’s Dominated Convergence Theorem, which is stated and proven as Theorem 1.34 in [19]. This theorem states that if the conditions of Lebesgue’s Monotone Convergence Theorem are not met we can still switch limits and integrals as long as we can majorize our sequence pointwise with an integrable function.

**Theorem 2.8.3.** (Lebesgue’s Dominated Convergence Theorem) Let \( \{f_n\}_{n=1}^\infty \) be a sequence of measurable functions on \( X \) such that

\[
f(x) = \lim_{n \to \infty} f_n(x)
\]

exists a.e. If there exists a function \( g \in L^1 \) such that

\[
|f_n(x)| \leq g(x) \text{ for all } n = 1, 2, 3, \ldots, \text{ and all } x \in X,
\]

then \( f \in L^1 \),

\[
\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0
\]

and

\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.
\]

Sums are common when dealing with integrals. Often the question arises whether or not one can switch an integral and an infinite sum. In cases that we will be interested in this follows easily from monotone convergence (Theorem 2.8.2). The following theorem can be found, with a short proof, as Theorem 1.27 in [19].

**Theorem 2.8.4.** If \( f_i : X \to [0, \infty] \) is measurable for \( i = 1, 2, 3, \ldots \), then

\[
\int_X \sum_{i=1}^\infty f_i \, d\mu = \sum_{i=1}^\infty \int_X f_i \, d\mu.
\]

2.9 Equivalence relations

In \( L^p \)-spaces, the value of functions on sets of measure zero can be ignored. Thus, functions that are equal a.e. are regarded as equal in the sense of equivalence classes. When we create our generalized Sobolev spaces we will have another type of equivalence classes than those of \( L^p \) so we recall some definitions on this topic.

**Definition 2.9.1.** (Equivalence relation) An equivalence relation on a set \( M \) is a subset of \( M \times M \), that is, a collection \( S \) of ordered pairs of elements of \( M \), satisfying certain properties. We write \( x \sim y \) to say that \( (x, y) \) is an element of \( S \). The properties are:

(a) \( x \sim x \), for all \( x \in M \) (reflexivity);
(b) \( x \sim y \) implies \( y \sim x \), for all \( x, y \in M \) (symmetry);
(c) \( x \sim y \) and \( y \sim z \) imply \( x \sim z \), for all \( x, y, z \in M \) (transitivity).

**Definition 2.9.2.** (Equivalence class) Two elements \( x, y \) in a set \( M \), are in the same equivalence class if \( x \sim y \), where \( \sim \) is an equivalence relation.

**Definition 2.9.3.** (Quotient space) The quotient space \( M/\sim \) of \( M \) with respect to the equivalence relation \( \sim \) on \( M \), is the set of equivalence classes of elements in \( M \).

As mentioned before, we have an equivalence relation in the \( L^p \)-spaces. If \( f_1, f_2 \in L^p \), then \( f_1 \sim f_2 \) if and only if \( f_1 = f_2 \) a.e.

When dealing with equivalence classes of functions one often lets a function act as a representative of its class. Hence we talk about functions in \( L^p \) rather than classes of functions, even though \( L^p \) is a quotient space. We will do this with our generalized Sobolev spaces as well.

### 2.10 Borel functions

**Definition 2.10.1** (Borel function). A function \( f \) is called a Borel function if \( f^{-1}(V) \) is a Borel set for every open set \( V \).

The following proposition assures that there is at least one Borel function in every equivalence class in \( L^p \). This property will prove useful since much of the discussion in this thesis is carried out with Borel functions.

**Proposition 2.10.2.** If \( f : X \to \mathbb{R} \) is measurable, then there exists a Borel function \( \tilde{f} : X \to \mathbb{R} \) such that \( f = \tilde{f} \) a.e. In fact, there are Borel functions \( \tilde{f}_1 \) and \( \tilde{f}_2 \) such that \( \tilde{f}_1 \leq f \leq \tilde{f}_2 \) in \( X \) and \( \tilde{f}_1 = f = \tilde{f}_2 \) a.e.

**Proof.** Let \( f : X \to \mathbb{R} \) be measurable. For all \( r \in \mathbb{R} \), define

\[
E_r = \{ x \in X : f(x) \geq r \} = f^{-1}([r, \infty)).
\]

We see that \( E_r \) is the inverse image of the open set \( [-\infty, r) \) under a measurable function and thus measurable. This implies that \( E_r \) is measurable as well. For each \( q \in \mathbb{Q} \), change \( E_q \) to get Borel sets \( \tilde{A}_q \supset E_q \) with \( \mu(\tilde{A}_q \setminus E_q) = 0 \). That this is possible is assured by the assumptions made about \( \mu \) in Section 2.7. Define

\[
A_r = \bigcap_{q \in \mathbb{Q}, q < r} \tilde{A}_q.
\]

These sets are also Borel. Finally we define

\[
\tilde{f}(x) = \sup\{ r \in \mathbb{R} : x \in A_r \}.
\]

By the definition of \( \tilde{f}(x) \) we see that if \( x \in A_r \) then \( \tilde{f}(x) \geq r \). And also, if \( \tilde{f}(x) \geq r \) then \( x \in A_r \). Thus

\[
A_r = \{ x \in X : \tilde{f}(x) \geq r \} = \tilde{f}^{-1}([r, \infty)).
\]
According to Proposition 2.2.4, every open set \( O \subset \mathbb{R} \) can be seen as a countable union of open intervals. Any open interval can be created from closed sets of the form \([r, \infty]\) by

\[(a, b) = [-\infty, b) \cap (a, \infty] = (-\infty, b) \cup (a, \infty)^c = [b, \infty] \cup (a, \infty)^c.\]

where

\[(a, \infty] = \bigcup_{n=1}^{\infty} [a + 1/n, \infty].\]

The inverse image of \([r, \infty]\) under \(\tilde{f}\) is equal to \(A_r\) and thus Borel. Since every open set can be created from sets like \([r, \infty]\) by successive countable unions and complements, the inverse image of every open set under \(\tilde{f}\) can be created by successive countable unions and complements of Borel sets. The family of Borel sets is closed under countable unions and complements so the inverse image of every open set under \(\tilde{f}\) is Borel. Thus \(\tilde{f}\) is Borel.

We now need to show that \(f = \tilde{f}\) a.e. First we notice that

\[\tilde{f}(x) = \sup\{r \in \mathbb{R} : x \in \bigcap_{q \in Q, q < r} \tilde{A}_q\}. \tag{2.4}\]

By the definition of \(E_r\), we have similarly that

\[f(x) = \sup\{r \in \mathbb{R} : x \in \bigcap_{q \in Q, q < r} E_q\}. \tag{2.5}\]

Since \(E_q \subseteq \tilde{A}_q\) for all \(q \in Q\), we see that \(f(x) \leq \tilde{f}(x)\) for all \(x \in X\). If \(x \in X\) is such that \(f(x) < \tilde{f}(x)\) then we can find \(c\) such that \(f(x) < c < \tilde{f}(x)\). From (2.4) and (2.5) we then get

\[x \in \left( \bigcap_{q \in Q, q < c} \tilde{A}_q \right) \setminus \left( \bigcap_{q \in Q, q < c} E_q \right) \subseteq \bigcup_{q \in Q, q < c} (\tilde{A}_q \setminus E_q) \subseteq \bigcup_{q \in Q} (\tilde{A}_q \setminus E_q).\]

But \(\mu(\tilde{A}_q \setminus E_q) = 0\), for all \(q\), so \(f(x) = \tilde{f}(x)\) a.e. Thus, for every measurable function \(f\) there is a Borel function \(\tilde{f}\) such that \(f = \tilde{f}\) a.e.

By choosing Borel sets \(\tilde{A}_q\) such that \(\tilde{A}_q \supseteq E_q\) as we did in this proof, we got a Borel function \(\tilde{f}\) such that \(\tilde{f} \geq f\). To receive a Borel function with the same properties but smaller than or equal to \(f\), we can choose the Borel sets \(\tilde{A}_q\) such that \(\tilde{A}_q \subset E_q\) for all \(q \in Q\) and then continue as we did with \(\tilde{f}\).

This shows that in \(L^p\), we can omit the discussion about whether or not functions are Borel since there is a Borel function in each equivalence class. We might just as well always choose a Borel function to represent the equivalence class.
Chapter 3

Curves

3.1 Rectifiable curves

There will be a lot of line integrals in this thesis so we need to specify which types of curves that are permitted to integrate along and how we define the line integral. To this end, we will create a certain kind of parameterization that will prove useful. Note that all important results in this section, along with many more, can be found with proofs in [1].

Curves $\gamma$ in a space $X$, are continuous maps $\gamma: I \to X$, where $I \subset \mathbb{R}$ is an interval. A curve is called compact if $I$ is compact.

**Definition 3.1.1.** (Rectifiable curve) A compact curve $\gamma: [a, b] \to X$ is said to be rectifiable if it has finite length, where the length of a curve is as follows:

$$l_\gamma = \sup \sum_{i=1}^{k} d(\gamma(t_i), \gamma(t_{i-1})),$$

where the supremum is taken over all sequences of points satisfying

$$a = t_0 \leq t_1 \leq \cdots \leq t_k = b.$$

![Figure 3.1: A sequence of points dividing a curve and giving rise to a series of distances](image)

Färm, 2006. 17
Example 3.1.2. The map \( \gamma(t) = (t, t\sin(\pi/t)) \), for \( 0 \leq t \leq 1 \), is continuous from a compact interval to \( \mathbb{R}^2 \), so it is a compact curve. Its length is infinite, since by putting
\[
t_i = \frac{2}{2i + 1}, \quad i = 0, 1, \ldots, k
\]
we get
\[
d(\gamma(t_i), \gamma(t_{i-1})) \geq \frac{2}{2i + 1} + \frac{2}{2i - 1} = \frac{8i}{4i^2 - 1} > \frac{2}{i}
\]
and
\[
l_{\gamma(t)} \geq \sum_{i=1}^{k} \frac{2}{i} \to \infty, \text{ as } k \to \infty.
\]
Thus, it is not rectifiable.

Figure 3.2: The curve \( \gamma(t) = (t, t\sin(\pi/t)) \) is not rectifiable.

By a constant curve we mean a constant mapping. The image of a constant curve is thus a point in the space \( X \). In this thesis we will only be interested in non-constant compact rectifiable curves. So, from now on the word curve will always refer to a non-constant compact rectifiable curve.

3.2 Arc length parameterization

Our goal is to parametrize curves with respect to their length. For this purpose we need to know how the length increases along a curve. Thus, for a curve \( \gamma : [a, b] \to X \) we define the length function
\[
s_\gamma : [a, b] \to [0, l_\gamma]
\]
as
\[
s_\gamma(t) = l_{\gamma|_{[a,t]}},
\]
i.e. \( s_\gamma(t) \) is the length of the subcurve \( \gamma' : [a, t] \to X \) of \( \gamma \).

Lemma 3.2.1. The length function \( s_\gamma \) of a curve \( \gamma : [a, b] \to X \) is non-decreasing and continuous.
Proof. That $s_\gamma$ non-decreasing is quite obvious. Let $a \leq u < v \leq b$. If we let $a = t_0 < t_1 < \cdots < t_j = u < v = t_{j+1}$, we get
\[
\sum_{i=1}^{j} d(\gamma(t_i), \gamma(t_{i-1})) \leq \sum_{i=1}^{j+1} d(\gamma(t_i), \gamma(t_{i-1})) \leq s_\gamma(v).
\]
By taking the supremum over all sequences $a = t_0 < t_1 < \cdots < t_j = u$, as in the definition of the length function, we get
\[
s_\gamma(a) \leq s_\gamma(v).
\]
We prove the continuity by contradiction. Fix $a < T < b$. Because $s_\gamma$ is increasing, the one-sided limits $s_\gamma(T^-)$ and $s_\gamma(T^+)$ exist. Suppose that $s_\gamma(T) - s_\gamma(T^-) > \delta > 0$. Let $a < t_1 < T$. We note that
\[
l_{\gamma}(t_1, T) = s_\gamma(T) - s_\gamma(t_1) \geq s_\gamma(T) - s_\gamma(T^-) > \delta.
\]
Thus there are numbers $t_1 = a_0 < \cdots < a_k = T$ such that
\[
\sum_{j=1}^{k} d(\gamma(a_j), \gamma(a_{j-1})) > \delta.
\]
The curve $\gamma$ is continuous, so we can find $t_2$ such that $a_{k-1} < t_2 < a_k$ and
\[
\sum_{j=1}^{k-1} d(\gamma(a_j), \gamma(a_{j-1})) + d(\gamma(t_2), \gamma(a_{k-1})) > \delta. \tag{3.1}
\]
We have
\[
l_{\gamma}(t_1, t_2) = s_\gamma(t_2) - s_\gamma(t_1) > \gamma(T) - s_\gamma(T^-) > \delta.
\]
But by (3.1) we get
\[
l_{\gamma}(t_{i-1}, t_i) = s_\gamma(t_i) - s_\gamma(t_{i-1}) > \delta,
\]
so by induction we can find a sequence $\{t_i\}_{i=1}^{\infty}$, $t_1 < t_2 < \cdots < t_i < \cdots < T$, such that $l_{\gamma}(t_{i-1}, t_i) > \delta$ for all $i$. This implies
\[
l_{\gamma}(t_{i-1}, t_i) \geq l_{\gamma}(t_{(i-1)}, t_i) > (i-1)\delta
\]
for every $i = 1, 2, \ldots$, contradicting the fact that $\gamma$ is rectifiable. Thus $s_\gamma(T^-) = s_\gamma(T)$. By analogous arguments we can prove that $s_\gamma(T^+) = s_\gamma(T)$. We have proven that $s_\gamma$ is continuous on $(a, b)$. The continuity in $a$ and $b$ follows by a similar argument. \qed

Given one curve we can always create another curve with the same image as the first one by changing the parameterization. To get a unique curve associated with every image we use a special parameterization called the arc length parameterization.

**Definition 3.2.2.** (Arc length parameterization) The arc length parameterization of a curve $\gamma : [a, b] \to X$ is the curve $\gamma_s : [0, l_\gamma] \to X$ defined by
\[
\gamma_s(t) = \gamma(s_\gamma^{-1}(t)), \text{ where } s_\gamma^{-1}(t) = \sup\{s : s_\gamma(s) = t\}.
\]
Since \( s_\gamma \) does not always have an inverse, we have chosen \( s_\gamma^{-1} \) to be the right-sided inverse. By using this parameterization we get a unique curve satisfying
\[
\gamma(t) = \gamma_s(s_\gamma(t)).
\]

The rectifiable curves we are interested in are continuous, but do we get continuous functions when we arc length parameterize them? The following proposition assures that we do.

**Proposition 3.2.3.** The arc length parameterization of a curve \( \gamma : [a, b] \to X \) is 1-Lipschitz and thus continuous.

**Proof.** We begin by noticing that since \( s_\gamma \) is continuous and \( s_\gamma^{-1}(t) = \sup \{s : \gamma_s(s) = t\} \),
we have
\[
s_\gamma(s_\gamma^{-1}(t)) = t.
\]
For any \( u \) and \( v \) satisfying \( a \leq u < v \leq b \) we then get
\[
\frac{d(\gamma_u, \gamma_v)}{v - u} = \frac{l_{\gamma_s[u,v]}}{v - u} = \frac{l_{s_\gamma^{-1}(u), s_\gamma^{-1}(v)}}{v - u} = \frac{s_\gamma(s_\gamma^{-1}(v)) - s_\gamma(s_\gamma^{-1}(u))}{v - u} = \frac{v - u}{v - u} = 1,
\]
so \( \gamma_s \) is 1-Lipschitz. \( \square \)

In multivariate calculus, the line integral of a scalar function \( f \), along a curve \( l : [\alpha, \beta] \to \mathbb{R}^n \), is
\[
\int_l f(r) \, ds = \int_{\alpha}^{\beta} f(r(t)) \left| \frac{dr(t)}{dt} \right| \, dt,
\]
where \( r(t) \in \mathbb{R}^n \) is the image of \( l \). Using the arc length parameterization we can define a line integral analogously. For the length function \( \gamma_s : [0, l] \to X \), it can be proven that
\[
\lim_{u \to t, u \neq t} \frac{d(\gamma_s(t), \gamma_s(u))}{|t - u|} = 1
\]
for almost all \( t \in [0, l] \), in the sense of the Lebesgue measure. By (3.2) we have shown one of the two required inequalities. For a complete proof see Theorem 4.2.1 in [1].

**Definition 3.2.4.** (Line integral) Given a curve \( \gamma : [a, b] \to X \) and a non-negative function \( \rho : X \to [0, \infty] \), we define the line integral of \( \rho \) over \( \gamma \) by
\[
\int_\gamma \rho \, ds = \int_0^l \rho(\gamma_s(t)) \, dt
\]
whenever \( \rho \circ \gamma_s : [a, b] \to [0, \infty] \) is measurable.

Note that in the line integral we integrate with respect to a scalar variable \( t \in \mathbb{R} \), so we use the ordinary Lebesgue measure. If \( \rho \) is Borel, then \( \rho \circ \gamma_s \) is also Borel since \( \gamma_s \) is continuous. Thus, the line integral is defined for all non-negative Borel functions.

We see that \([0, l_\gamma] \subset \mathbb{R}\) together with the one-dimensional Lebesgue measure is an example of what our general metric measure space can be. Thus, we can use Theorem 2.8.4 to switch places on sums and integrals in the following remark.
3.3. Modulus of curve families

Remark 3.2.5. Given the curve \( \gamma : [a, b] \to X \) and functions \( g_i : [a, b] \to [0, \infty] \), it holds that
\[
\int_{\gamma} \sum_{i=1}^{\infty} g_i \, ds = \sum_{i=1}^{\infty} \int_{\gamma} g_i \, ds.
\]
as long as \( g_i \circ \gamma_s \) is measurable for each \( i = 1, 2, \ldots \).

3.3 Modulus of curve families

Many of the concepts in this thesis involve line integrals along curves. It will be useful to know along how many curves line integrals satisfy different conditions. Thus, we need some way to measure how large a family of curves is. The most important question is if a family of curves is small enough to be ignored. The following definition first appeared in [6].

Definition 3.3.1. (Modulus of curve family) Let \( \Gamma \) be a family of curves. We then define the \( p \)-modulus of \( \Gamma \) by
\[
\text{Mod}_p(\Gamma) = \inf_{\rho} \int_{X} \rho^p \, d\mu,
\]
where the infimum is taken over all non-negative Borel functions \( \rho \) such that \( \int_{\gamma} \rho \, ds \geq 1 \) for all \( \gamma \in \Gamma \).

Definition 3.3.2 (\( p \)-modulus almost every curve). If a property is true for all curves but a family of curves with \( p \)-modulus zero, it is said to be true for \( p \)-almost every curve, or \( p \)-a.e. curve for short. It is implicitly assumed that the property is defined for all curves but a family with \( p \)-modulus zero.

The modulus of a curve family is not easy to determine in general. But to give a hint on how this new concept works we give two examples where we can find a value for the modulus.

Example 3.3.3. Let \( X = \mathbb{R}^n \) equipped with the Euclidean norm and the \( n \)-dimensional Lebesgue measure. Let \( \Gamma \) be the family of all curves \( \gamma \) containing the segment between the points \((0,0,\ldots,0)\) and \((1,0,0,\ldots,0)\).

Now, let
\[
\rho' = \begin{cases} 
1 & \text{on the segment between } (0,0,\ldots,0) \text{ and } (1,0,0,\ldots,0); \\
0 & \text{elsewhere.}
\end{cases}
\]

We see that \( \int_{\gamma} \rho' \, ds \geq 1 \) for all \( \gamma \in \Gamma \) and that \( \int_{\mathbb{R}^n} \rho' \, d\mu = 0 \) if \( n > 1 \). Thus, \( \text{Mod}_p(\Gamma) = 0 \) for all \( p \) as long as \( n > 1 \).

On the other hand, if \( n = 1 \) this is not the case. The segment between \((0,0,\ldots,0)\) and \((1,0,0,\ldots,0)\) becomes the interval \([0,1]\) and that is the image of a curve \( \gamma' \) in \( \Gamma \). In order for a function \( \rho \) to satisfy \( \int_{\gamma} \rho \, ds \geq 1 \) for all curves \( \gamma \in \Gamma \), it has to do so for \( \gamma' \). By the Hölder inequality, with \( 1/p + 1/q = 1 \), we
then get

\[ 1 \leq \int_{\gamma'} \rho \, d\mu = \int_0^1 \rho(x) \, dx \]

\[ \leq \left( \int_0^1 \rho(x)^p \, dx \right)^{1/p} \left( \int_0^1 \rho^q \, dx \right)^{1/q} \]

\[ = \left( \int_0^1 \rho(x)^p \, dx \right)^{1/p} \]

\[ \leq \int_0^1 \rho(x)^p \, dx \]

\[ \leq \int_{\mathbb{R}} \rho^p \, d\mu \]

for all functions \( \rho \) in the definition of \( \text{Mod}_p(\Gamma) \). Thus, \( \text{Mod}_p(\Gamma) \geq 1 \) for all \( p \).

**Example 3.3.4.** Again, let \( X = \mathbb{R}^n \) equipped with the Euclidean norm and the \( n \)-dimensional Lebesgue measure. Let \( \Gamma \) be the family of all curves \( \gamma \) that pass through the point \( a = (0,0,0,\ldots,0) \).

If \( n = 1 \), the space is a line. The interval \([a - 1/2, a + 1/2]\) is the image of a curve \( \gamma' \in \Gamma \). In order for a function \( \rho \) to satisfy \( \int_{\gamma} \rho \, ds \geq 1 \) for all curves \( \gamma \in \Gamma \),
it has to do so for $\gamma'$. By using the Hölder inequality with $1/p + 1/q = 1$, we get
\[
1 \leq \int_{\gamma'} \rho \, ds = \int_{a-1/2}^{a+1/2} \rho(x) \, dx \\
\leq \left( \int_{a-1/2}^{a+1/2} \rho(x)^p \, dx \right)^{1/p} \left( \int_{a-1/2}^{a+1/2} 1^q \, dx \right)^{1/q} \\
\leq \int_{a-1/2}^{a+1/2} \rho(x)^p \, dx \\
\leq \int_{\mathbb{R}} \rho^p \, d\mu
\]
for all functions $\rho$ in the definition of $\text{Mod}_p(\Gamma)$. Thus, $\text{Mod}_p(\Gamma) \geq 1$ for all $p$.

Now, let $n > 1$. For $\epsilon > 0$, let
\[
\rho_\epsilon(x) = \begin{cases} 
\frac{1}{d(x,a)} & \text{for all } x \text{ such that } d(x,a) \leq \epsilon; \\
0 & \text{elsewhere.}
\end{cases}
\]
Since none of our curves are constant we know that any given curve $\gamma \in \Gamma$ must pass through a point $x_1 \neq a$. In some way, the curve must go between $a$ and $x_1$. Since $\rho_\epsilon$ only depends on the distance from $a$ we get the smallest value of the line integral by following the straight line between $a$ and $x_1$. Thus, if $d(x_1, a) \geq \epsilon$ we get
\[
\int_{\gamma} \rho_\epsilon \, ds \geq \left\lfloor \int_0^\epsilon \frac{dr}{r} \right\rfloor = \infty > 1.
\]
for all $\epsilon > 0$. If $d(x_1, a) = \epsilon' < \epsilon$ we get
\[
\int_{\gamma} \rho_\epsilon \, ds \geq \left\lfloor \int_0^{\epsilon'} \frac{dr}{r} \right\rfloor = \infty > 1,
\]
regardless of how small $\epsilon'$ is. This implies
\[
\text{Mod}_p(\Gamma) \leq \int_{\mathbb{R}^n} \rho_\epsilon^p \, d\mu \quad (3.3)
\]
for all $\epsilon > 0$.

If we consider a ball in $\mathbb{R}^n$, the generalized surface area of it is proportional to the radius to the power of $n - 1$. From this we get
\[
\text{Mod}_p(\Gamma) \leq \int_{\mathbb{R}^n} \rho_\epsilon^p \, d\mu = \int_0^\epsilon C_n r^{n-1} r^{-p} \, dr = C_n \int_0^\epsilon r^{n-p-1} \, dr,
\]
where $C_n$ is a constant only dependent on $n$. If $p < n$ we get
\[
C_n \int_0^\epsilon r^{n-p-1} \, dr = \frac{C_n \epsilon^{n-p}}{n-p} \rightarrow 0 \text{ when } \epsilon \rightarrow 0,
\]
which implies that $\text{Mod}_p(\Gamma) = 0$. 

We see that \( \rho \epsilon \) will not be \( p \)-integrable if \( p \geq n \) so from our computations we cannot tell if Mod\(_p\)(\( \Gamma \)) = 0 in general. But by using
\[
\rho_\epsilon'(x) = \begin{cases} \frac{1}{d(x,a)\ln(d(x,a))} & \text{for all } x \text{ such that } d(x,a) \leq \epsilon; \\ 0 & \text{elsewhere.} \end{cases}
\]

instead of \( \rho_\epsilon \), we can improve the result. For \( p = n > 1 \) we get that \( \int_\gamma \rho_\epsilon' ds = \infty > 1 \), for all \( \epsilon \), and that \( \int_{\mathbb{R}^n} \rho_\epsilon' p d\mu \to 0 \) as \( \epsilon \to 0 \). This proves that Mod\(_p\)(\( \Gamma \)) = 0 for all \( p \leq n \) as long as \( n > 1 \).

The modulus will be used frequently in the following chapters. Thus, it will be useful to have some properties of the modulus already figured out.

**Lemma 3.3.5.** The following properties hold:

(a) if \( \Gamma_1 \subset \Gamma_2 \), then Mod\(_p\)(\( \Gamma_1 \)) \leq Mod\(_p\)(\( \Gamma_2 \));

(b) Mod\(_p\)(\( \bigcup_{j=1}^{\infty} \Gamma_j \)) \leq \sum_{j=1}^{\infty} \text{Mod}\(_p\)(\( \Gamma_j \));

(c) if for every \( \gamma \in \Gamma \) there exists \( \gamma' \subset \gamma \), \( \gamma' \in \Gamma' \), then Mod\(_p\)(\( \Gamma \)) \leq Mod\(_p\)(\( \Gamma' \)).

**Proof.** (a) This is quite obvious. The modulus of a curve family is defined as the infimum over functions with line integrals greater than 1 over every curve in the curve family. Since all curves in \( \Gamma_1 \) are found in \( \Gamma_2 \), all functions satisfying the requirements for the modulus of \( \Gamma_2 \) will automatically satisfy the requirements for the modulus of \( \Gamma_1 \). Thus, the modulus of \( \Gamma_2 \) cannot be smaller than the modulus of \( \Gamma_1 \).

(b) For every \( j = 1, 2, 3, \ldots \) and every \( \epsilon > 0 \), let \( \rho_j \) be a Borel function such that
\[
\int_\gamma \rho_j ds \geq 1 \text{ for all } \gamma \in \Gamma_j \quad \text{and} \quad \int_X \rho_j^p d\mu \leq \text{Mod}\(_p\)(\( \Gamma_j \)) + \epsilon/2^j.
\]
Let \( \rho = \sup_j \{ \rho_j \} \) pointwise. We see that \( \rho \) satisfies the condition
\[
\int_\gamma \rho ds \geq 1 \text{ for all } \gamma \in \bigcup_{j=1}^{\infty} \Gamma_j.
\]

In the definition of the modulus, \( \rho \) needs to be a Borel function. Since \( \rho \) is the supremum of Borel functions it is not very difficult to show that it is indeed a Borel function. We do not need this here though. If \( \rho \) is not a Borel function, then Proposition 2.10.2 implies that there is a Borel function \( \tilde{\rho} \) such that \( \tilde{\rho} \geq \rho \) and \( \tilde{\rho} = \rho \) a.e. This new function \( \tilde{\rho} \) obviously also satisfies (3.4). Thus, we get
\[
\text{Mod}\(_p\)(\( \bigcup_{j=1}^{\infty} \Gamma_j \)) \leq \int_X \tilde{\rho}^p d\mu
\]
\[
= \int_X \rho^p d\mu
\]
\[
\leq \int_X \sum_{j=1}^{\infty} \rho_j^p d\mu
\]
\[
= \sum_{j=1}^{\infty} \int_X \rho_j^p d\mu \leq \sum_{j=1}^{\infty} \text{Mod}\(_p\)(\( \Gamma_j \)) + \epsilon.
\]
Here we have used Theorem 2.8.4 to switch the sum and integral. Letting $\epsilon \to 0$ yields

$$\text{Mod}_p \left( \bigcup_{j=1}^{\infty} \Gamma_j \right) \leq \sum_{j=1}^{\infty} \text{Mod}_p(\Gamma_j).$$

(c) Let $\rho$ be such that

$$\int_{\gamma'} \rho \, ds \geq 1, \text{ for all } \gamma' \in \Gamma'.$$

Since $\rho \geq 0$ and for every $\gamma \in \Gamma$ there is at least one $\gamma' \in \Gamma'$ with $\gamma' \subset \gamma$, we have that

$$\int_{\gamma} \rho \, ds \geq 1, \text{ for all } \gamma \in \Gamma.$$

We see that every candidate $\rho$ in the definition of $\text{Mod}_p(\Gamma')$ is also a candidate in the definition of $\text{Mod}_p(\Gamma)$. Thus $\text{Mod}_p(\Gamma) \leq \text{Mod}_p(\Gamma').$ \qed
Chapter 4

Finding a replacement for the derivative

In general metric spaces we lack a derivative, even in the weak sense of Sobolev spaces. The purpose of this chapter is to provide a replacement for the derivative, so that we can extend the theory of Sobolev spaces to general metric spaces.

4.1 The upper gradient

The following definition was introduced in 1998 by Heinonen-Koskela in [11].

**Definition 4.1.1. (Upper gradient)** A non-negative Borel function \( g \) on \( X \), is an upper gradient of an extended real-valued function \( f \) on \( X \), if for all curves \( \gamma : [0, l_\gamma] \to X \)

\[
|f(\gamma(0)) - f(\gamma(l_\gamma))| \leq \int_\gamma g \, ds,
\]

(4.1)

whenever both \( f(\gamma(0)) \) and \( f(\gamma(l_\gamma)) \) are finite, and \( \int_\gamma g \, ds = \infty \) otherwise.

We see in the definition that the upper gradient plays the role of a derivative in something that is similar to the fundamental theorem of calculus. The idea is that by defining the upper gradient in this way we can imitate many of the properties of ordinary Sobolev spaces even though we do not have derivatives.

**Example 4.1.2.** Let \( X = \mathbb{R}^n \) equipped with the Euclidean norm and the \( n \)-dimensional Lebesgue measure. Take a function \( u \in C^1 \) and an arc length parameterized curve \( \gamma : [0, l_\gamma] \to \mathbb{R}^n \) connecting two points \( x, y \in \mathbb{R}^n \). With
Chapter 4. Finding a replacement for the derivative

$x = \gamma(0)$ and $y = \gamma(l_\gamma)$, we get

\[
 u(y) - u(x) = \int_0^{l_\gamma} \nabla u(\gamma(t)) \frac{d\gamma}{dt} dt
\]  

\[
 \leq \int_0^{l_\gamma} \left| \nabla u(\gamma(t)) \right| \left| \frac{d\gamma}{dt} \right| dt
\]

\[
 \leq \int_0^{l_\gamma} \left| \nabla u(\gamma(t)) \right| |d\gamma| dt
\]

where $|d\gamma| = 1$ since $\gamma$ is arc length parameterized. Equality (4.2) holds for smooth curves according to Theorem 4.4 in [18]. But to require that $\gamma$ is smooth is more restrictive than necessary. It can be shown that (4.2) holds for all curves that are Lipschitz. Since our curves are Lipschitz, we see that $|\nabla u|$ is an upper gradient of $u$.

The inequality in (4.1), instead of an equality as in the fundamental theorem of calculus, allows several functions to be upper gradients of a function. For example, if $g_u$ is an upper gradient of $u$ then all functions $g \geq g_u$ are also upper gradients of $u$. In fact, all functions have $\infty$ as upper gradient. This makes the upper gradient quite bad at describing the variations of a function. To get a better description we would like to use the smallest possible upper gradient. In this way the upper gradient of a function $u$ would have a chance to coincide with the modulus, $|\nabla u|$, of the ordinary gradient, $\nabla u$, when it exists. A problem is that the set of upper gradients is not closed, so there might not be a minimal upper gradient as the following example shows.

**Example 4.1.3.** Let $X = \mathbb{R}^n$ equipped with Euclidean norm and the $n$-dimensional Lebesgue measure. Let $u : X \rightarrow \mathbb{R}$ be such that

\[
 u(x) = \begin{cases} 
 1 & \text{when } x = x_0; \\
 0 & \text{elsewhere}. 
\end{cases}
\]

For $\epsilon > 0$, let

\[
 \rho_\epsilon(x) = \begin{cases} 
 \frac{1}{d(x,x_0)} & \text{for all } x \text{ such that } 0 < d(x,x_0) \leq \epsilon; \\
 0 & \text{elsewhere}. 
\end{cases}
\]

By the same argument as in Example 3.3.4 we get

\[
 \int_\gamma \rho_\epsilon \, ds \geq 1
\]

for all curves $\gamma$ passing through $x_0$. This implies

\[
 |u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_\gamma \rho_\epsilon \, ds,
\]

so $\rho_\epsilon$ is an upper gradient of $u$. A minimal upper gradient $g_u$ should be smaller or equal to all other upper gradients. But $\rho_\epsilon(x) \rightarrow 0$ as $\epsilon \rightarrow 0$, for all $x$. We
then get $g_u = 0$, which is not an upper gradient of $u$. We see that in this case there is no minimal upper gradient.

From the example we see that if we take a constant function and change its value in just one point, then some of the upper gradient will be lost. We do not want our upper gradients to be this sensitive to changes on small sets. To solve this problem we introduce weak upper gradients.

**Definition 4.1.4 (Weak upper gradient).** A non-negative measurable function $g$ on $X$, is a $p$-weak upper gradient of an extended real-valued function $f$, if the inequality (4.1) is satisfied for $p$-almost every curve $\gamma : [0, l_{\gamma}] \rightarrow X$. Since the line integral is only defined when $g \circ \gamma$ is measurable, it is implicitly assumed that the line integral is defined for $p$-almost every curve. In fact, we will see in Corollary 4.1.9 that this is always true.

Since there is little risk of confusion we omit $p$, writing only weak upper gradient when we mean $p$-weak upper gradient.

Let $g_u$ and $g_v$ be such that they satisfy (4.1) along a curve $\gamma$, for $u$ and $v$ respectively. If $a, b \in \mathbb{R}$, we then get

$$\left| au(\gamma(0)) + bv(\gamma(0)) - au(\gamma(l_\gamma)) - bv(\gamma(l_\gamma))\right|$$
$$\leq \left| a|u(\gamma(0)) - u(\gamma(l_\gamma))| + |b|v(\gamma(0)) - v(\gamma(l_\gamma))|\right|$$
$$\leq |a| \int_\gamma g_u \, ds + |b| \int_\gamma g_v \, ds$$
$$= \int_\gamma (|a|g_u + |b|g_v) \, ds.$$

We see that $|a|g_u + |b|g_v$ satisfies (4.1) along $\gamma$ for $au + bv$.

**Remark 4.1.5.** Let $a, b \in \mathbb{R}$. If $g_u$ and $g_v$ are upper gradients of $u$ and $v$ respectively, then $|a|g_u + |b|g_v$ is an upper gradient of $au + bv$. Also, if $g_u$ and $g_v$ are weak upper gradients of $u$ and $v$ respectively, then $|a|g_u + |b|g_v$ is a weak upper gradient of $au + bv$.

Now we return to Example 4.1.3. We have already determined that the function

$$u(x) = \begin{cases} 
1 & \text{when } x = x_0; \\
0 & \text{elsewhere},
\end{cases}$$

does not have a minimal upper gradient. The reason was that the only function that is less than or equal to all upper gradients of $u$ is 0, which is not an upper gradient of $u$. From Example 3.3.4 we know that the modulus of all curves through $x_0$ is zero, as long as $p \leq n$ and $n > 1$. Thus, when looking for weak upper gradients we can ignore all those curves. Then, we see that 0 is a weak upper gradient of $u$.

We can conclude that when $p \leq n$ and $n > 1$, then there is a minimal weak upper gradient of $u$, something that was not true for the upper gradients. Later, in Section 4.2, we will prove a proposition that guarantees the existence of minimal weak upper gradients if there exists a $p$-integrable weak upper gradient.
In the example we did not have any $p$-integrable weak upper gradients for the case $p > n$. That is why we did not have a minimal weak upper gradient.

From the definitions we see that an upper gradient is also a weak upper gradient. In fact, if there exists a weak upper gradient $g \in L^p$, then there are upper gradients arbitrarily close to $g$ in the $L^p$-norm. To prove this we need a couple of lemmas. We will use the first lemma several times later in this thesis.

**Lemma 4.1.6.** Let $\Gamma$ be a family of curves in $X$. Then $\text{Mod}_p(\Gamma) = 0$ if and only if there is a non-negative function $\rho \in L^p$ such that $\int_\gamma \rho \, ds = \infty$ for all $\gamma \in \Gamma$.

**Proof.** Assume first that $\text{Mod}_p(\Gamma) = 0$. Then according to the definition of the modulus we can find non-negative $\tilde{\rho}$ such that $\|\tilde{\rho}\|_{L^p}$ is arbitrarily small and $\int_\gamma \tilde{\rho} \, ds \geq 1$ for all $\gamma \in \Gamma$.

Thus, for $n = 1, 2, \ldots$ we can choose $\rho_n$ such that $\|\rho_n\|_{L^p} \leq 2^{-n}$ and $\int_\gamma \rho_n \, ds \geq 1$ for all $\gamma \in \Gamma$.

Let $\rho = \sum_{n=1}^{\infty} \rho_n$. Then $\rho \in L^p(X)$, since $\|\rho\|_{L^p} \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty$.

We also know that $\int_\gamma \rho \, ds \geq \int_\gamma \sum_{n=1}^{m} \rho_n \, ds = \sum_{n=1}^{m} \int_\gamma \rho_n \, ds \geq m$, for all $m \geq 1$ and for all $\gamma \in \Gamma$. Thus $\int_\gamma \rho \, ds = \infty$ for all $\gamma \in \Gamma$.

If on the other hand there is a non-negative $\rho \in L^p(X)$ such that $\int_\gamma \rho \, ds = \infty$ for all $\gamma \in \Gamma$, then $\int_\gamma \frac{\rho \, ds}{n} = \frac{1}{n} \int_\gamma \rho \, ds = \infty \geq 1$ for all $\gamma \in \Gamma$.

Hence $\text{Mod}_p(\Gamma) \leq \|\rho/n\|_{L^p}^p = n^{-p}\|\rho\|_{L^p}^p \to 0$, as $n \to \infty$, since $\|\rho\|_{L^p}^p$ is finite. 

**Definition 4.1.7.** ($\Gamma^+_E$) Let $E \subset X$. $\Gamma^+_E$ is the family of all curves $\gamma$ such that $|\gamma^{-1}(\gamma \cap E)| \neq 0$, i.e. such that $\gamma^{-1}(\gamma \cap E)$ has strictly positive Lebesgue measure.

**Lemma 4.1.8.** Let $E \subset X$. If $\mu(E) = 0$, then $\text{Mod}_p(\Gamma^+_E) = 0$. 


4.1. The upper gradient

Proof. By the properties of the measure $\mu$ from Section 2.7 we know that there is a Borel set $F \supset E$ such that $\mu(F \setminus E) = 0$, and thus $\mu(F) = 0$. Let $\rho = \infty \chi_F$. For $\gamma \in \Gamma_E^+$, we have $|\gamma^{-1}(\gamma \cap F)| \neq 0$. Moreover $\gamma^{-1}(\gamma \cap F)$ is measurable, and thus

$$\int_\gamma \rho \, ds = \infty > 1 \quad \text{for all } \gamma \in \Gamma_E^+.$$ 

Hence, by the definition of the modulus

$$\text{Mod}_p(\Gamma_E^+) \leq \int_X \rho^p \, d\mu = 0. \qed$$

Corollary 4.1.9. Let $g$ and $\tilde{g}$ be non-negative measurable functions such that $g = \tilde{g}$ a.e. Then

$$\int_\gamma g \, ds = \int_\gamma \tilde{g} \, ds$$

for $p$-a.e curve $\gamma$. In particular, $\int_\gamma g \, ds$ is defined for $p$-a.e curve $\gamma$ (with a value in $[0, \infty]$).

Proof. By Proposition 2.10.2 there is a non-negative Borel function $g'$ which equals $g$ a.e. Let $E = \{x \in X : g(x) \neq g'(x)\}$. As $g'$ is a Borel function, $\int_\gamma g' \, ds$ is defined for every curve $\gamma$. For curves $\gamma \notin \Gamma_E^+$,

$$\int_\gamma g \, ds = \int_\gamma g' \, ds. \quad (4.3)$$

Since $\mu(E) = 0$, by Lemma 4.1.8 we have $\text{Mod}_p(\Gamma_E^+) = 0$, and thus (4.3) holds for $p$-a.e curve $\gamma$. By the same argument we may show that $\int_\gamma \tilde{g} \, ds = \int_\gamma g' \, ds$ for $p$-a.e. curve $\gamma$. Thus $\int_\gamma g \, ds = \int_\gamma \tilde{g} \, ds$ for $p$-a.e. curve $\gamma$. \qed

Remark 4.1.10. Let $g \in L^p$ be a weak upper gradient of $u$, and $h = \tilde{g}$ a.e. Then, Corollary 4.1.9 tells us that for $p$-a.e curve $\gamma$ the line integral of $h$ is equal to the line integral of $g$. Thus, $h$ is also a weak upper gradient of $u$. By Proposition 2.10.2 we know that there is a Borel function that is equal to $g$ a.e. Thus, Corollary 4.1.9 tells us that there is a Borel weak upper gradient $\tilde{g}$ of $u$ such that $\tilde{g} = g$ a.e.

Proposition 4.1.11. If $g \in L^p$ is a weak upper gradient of $u$, then there are upper gradients $g_j$ of $u$ such that $g_j \to g$ in $L^p$, as $j \to \infty$.

Proof. By Remark 4.1.10 we can find a Borel weak upper gradient $g'$ of $u$ such that $g' = g$ a.e. Let

$$\Gamma = \{\gamma : g' \text{ does not satisfy } (4.1) \text{ along } \gamma\}.$$ 

By assumption $\text{Mod}_p(\Gamma) = 0$, and hence by Lemma 4.1.6, there is a non-negative Borel function $\rho \in L^p$ such that $\int_\gamma \rho \, ds = \infty$ for all $\gamma \in \Gamma$. Let finally $g_j = g' + \rho/j$. We see that $g_j \in L^p$ since both $g' \in L^p$ and $\rho \in L^p$. That $g_j$ satisfies (4.1) for all curves $\gamma \in \Gamma$ is clear, since we have

$$\int_\gamma g_j \, ds \geq \frac{1}{j} \int_\gamma \rho \, ds = \infty,$$

for all $j$ and all $\gamma \in \Gamma$. Thus, $g_j$ is an upper gradient of $u$ for every $j$, and $g_j \to g$ in $L^p$, as $j \to \infty$. \qed
4.2 The minimal weak upper gradient

As noted earlier, a problem with the upper gradients is that there may not exist a minimal upper gradient. The $p$-integrable weak upper gradients on the other hand, have this property. To prove this we need two lemmas.

Lemma 4.2.1. Let $g_1, g_2 \in L^p$ be weak upper gradients of a function $u$. Then $g = \min \{g_1, g_2\}$ is also a weak upper gradient of $u$.

Proof. By Remark 4.1.10 there are Borel weak upper gradients of $u$, $g'_1$ and $g'_2$, such that $g'_1 = g_1$ a.e. and $g'_2 = g_2$ a.e. Let $\Gamma_\infty$ be the family of all curves $\gamma$ such that $\int_\gamma (g'_1 + g'_2)ds = \infty$. By Lemma 4.1.6, $\text{Mod}_p(\Gamma_\infty) = 0$. Thus, $\int_\gamma (g'_1 + g'_2)ds < \infty$ for $p$-almost all curves $\gamma$. Let $E = \gamma^{-1}\{x \in X : g'_1 < g'_2\}$ for an arbitrary curve $\gamma : [0,l_\gamma] \to X$ among these curves. $E$ is the inverse image of a Borel set under the Borel function $\gamma$, so it is Borel. By the properties of the measure $\mu$, from Section 2.7, we know that there exist open sets $U_1 \supset U_2 \supset \cdots \supset E$ such that $\mu(U_1 \setminus E) \to 0$, as $n \to \infty$. By Proposition 2.2.4 we can write $U_n$ as a pairwise disjoint union of open intervals for any $n$. We write $U_n = \bigcup_{i=1}^{\infty} I_i$, where $I_i = (a_i, b_i)$. We then get

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq |u(\gamma(0)) - u(\gamma(a_1))| + |u(\gamma(a_1)) - u(\gamma(b_1))| + |u(\gamma(b_1)) - u(\gamma(l_\gamma))|$$

$$\leq \int_{\gamma(I_1)} g'_1 ds + \int_{\gamma \setminus \gamma(I_1)} g'_2 ds$$

for the first interval. If $b_1 < a_2$ we get

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq |u(\gamma(0)) - u(\gamma(a_1))| + |u(\gamma(a_1)) - u(\gamma(b_1))| + |u(\gamma(b_1)) - u(\gamma(a_2))| + |u(\gamma(a_2)) - u(\gamma(b_2))| + |u(\gamma(b_2)) - u(\gamma(l_\gamma))|$$

$$\leq \int_{\gamma(I_1 \cup I_2)} g'_1 ds + \int_{\gamma \setminus (\gamma(I_1 \cup I_2))} g'_2 ds.$$  

Of course, we may have $b_2 < a_1$, but this case is handled analogously. By continuing in this way we get for all $j = 1, 2, \ldots$

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_{\gamma(U_{j-1} \cup I_j)} g'_1 ds + \int_{\gamma \setminus (\gamma(U_{j-1} \cup I_j))} g'_2 ds$$

$$= \int_{\gamma} \chi_{\gamma(U_{j-1} \cup I_j)} g'_1 ds + \int_{\gamma} \chi_{\gamma \setminus (U_{j-1} \cup I_j)} g'_2 ds.$$  

(4.4)

We see that

$$\lim_{j \to \infty} \chi_{\gamma(U_{j-1} \cup I_j)} g'_1 = \chi_{\gamma(U_n)} g'_1 \quad \text{and} \quad \lim_{j \to \infty} \chi_{\gamma \setminus (U_{j-1} \cup I_j)} g'_2 = \chi_{\gamma \setminus (U_n)} g'_2$$

pointwise. Furthermore

$$\chi_{\gamma(U_{j-1} \cup I_j)} g'_1 \leq g'_1 \quad \text{and} \quad \chi_{\gamma \setminus (U_{j-1} \cup I_j)} g'_2 \leq g'_2$$

for all $j$. Let $j \to \infty$ in (4.4). We then get

$$|u(\gamma(0)) - u(\gamma(l_\gamma))| \leq \int_{\gamma} \chi_{\gamma(U_n)} g'_1 ds + \int_{\gamma} \chi_{\gamma \setminus (U_n)} g'_2 ds$$  

(4.5)
4.2. The minimal weak upper gradient

by dominated convergence (Theorem 2.8.3). We see that

\[ \lim_{n \to \infty} \chi_{\gamma(U_n)} g'_1 = \chi_{\gamma(E)} g'_1 \quad \text{and} \quad \lim_{n \to \infty} \chi_{\gamma \setminus \gamma(U_n)} g'_2 = \chi_{\gamma \setminus \gamma(E)} g'_2 \]

pointwise. Furthermore

\[ \chi_{\gamma(U_n)} g'_1 \leq g'_1 \quad \text{and} \quad \chi_{\gamma \setminus \gamma(U_n)} g'_2 \leq g'_2 \]

for all \( n \). Let \( n \to \infty \) in (4.5). We then get

\[ |u(\gamma(0)) - u(\gamma(l))| \leq \int_\gamma \chi_{\gamma(E)} g'_1 \, ds + \int_\gamma \chi_{\gamma \setminus \gamma(E)} g'_2 \, ds \]

by dominated convergence (Theorem 2.8.3). But, by Corollary 4.1.9, the line integral of \( g'_1 \) and \( g_1 \) are equal along \( p \)-a.e. curve, and so are the line integrals of \( g'_2 \) and \( g_2 \). Thus, we get

\[ |u(\gamma(0)) - u(\gamma(l))| \leq \int_\gamma g_1 \, ds + \int_{\gamma \setminus \gamma(E)} g_2 \, ds = \int_\gamma g \, ds \]

for \( p \)-a.e. curve and the proof is complete.

In Lemma 4.2.1 it is essential that \( g_1 \) and \( g_2 \) are \( p \)-integrable as we see in the following example.

**Example 4.2.2.** Let \( X = \mathbb{R}^n \) equipped with Euclidean norm and the \( n \)-dimensional Lebesgue measure. Let

\[ u(x) = \begin{cases} 1 & \text{when } x = x_0; \\ 0 & \text{elsewhere.} \end{cases} \]

We will now create weak upper gradients of \( u \) by summing up step functions. For each \( i = 1, 2, \ldots \), let

\[ g_i(x) = \begin{cases} \infty & \text{when } 2^{-i} - 2^{-(i+2)} < d(x, x_0) < 2^{-i}; \\ 0 & \text{elsewhere} \end{cases} \]

and

\[ g'_i(x) = \begin{cases} \infty & \text{when } 2^{-(i+1)} < d(x, x_0) < 2^{-i} - 2^{-(i+2)}; \\ 0 & \text{elsewhere} \end{cases} \]

The functions

\[ g = \sum_{i=1}^{\infty} g_i \quad \text{and} \quad g' = \sum_{i=1}^{\infty} g'_i \]

will have infinite value on regions arbitrarily close to \( x_0 \). Every curve that passes through \( x_0 \) will pass through at least one of the regions where \( g = \infty \) and one where \( g' = \infty \). Even the shortest way through such a region has a positive length. Hence, the line integral will be infinite and we can conclude that both \( g \) and \( g' \) are upper gradients of \( u \).

Now, let \( g_u = \min\{g, g'\} \). Since always one of \( g \) and \( g' \) is zero, we see that \( g_u = 0 \). From Example 3.3.4 we know that the family of curves through a point
in $\mathbb{R}^n$ has non-zero modulus, if for example $n = p = 1$. Thus, in that case 0 cannot be a weak upper gradient of $u$. Therefore, Lemma 4.2.1 does not hold without the restriction that $g_1, g_2 \in L^p$.

To show the existence of a minimal weak upper gradient we will create a decreasing sequence of weak upper gradients. To be sure that the limit function is a weak upper gradient we need the following lemma. It can be found in [3] (Lemma 5.1) together with a proof. It is originally a result from [6] (Theorem 3).

**Lemma 4.2.3.** (Fuglede’s lemma) Assume that $g_j \to g$ in $L^p$, as $j \to \infty$. Then there is a subsequence $(\hat{g}_k)_{k=1}^{\infty}$ such that

$$\int_{\gamma} \hat{g}_k \, ds \to \int_{\gamma} g \, ds, \text{ as } k \to \infty,$$

for p-a.e. curve $\gamma$. Furthermore, all the integrals are well-defined and finite.

**Proposition 4.2.4.** Among the set of all $p$-integrable weak upper gradients to a function $u$, there is a smallest member in the $L^p$-norm, called the minimal weak upper gradient. The minimal weak upper gradient, $g_u$, is unique up to a set of measure zero and $g_u \leq g$ a.e. for all weak upper gradients $g$ of $u$.

**Proof.** To prove the existence of a minimal weak upper gradient, let

$$I = \inf\{\|f\|_{L^p} : g \text{ is a weak upper gradient of } u\}.$$ 

Let $g_j$ be a sequence of weak upper gradients of $u$ such that $\|g_j\|_{L^p} \to I$. By Lemma 4.2.1 we can create a pointwise decreasing sequence of weak upper gradients by taking $\hat{g}_j = \min\{g_1, \ldots, g_j\}$. Since this sequence is pointwise decreasing and non-negative it converges pointwise to a function $g_u$. We see that $\hat{g}_1(x) \geq \hat{g}_j(x)$ for all $x \in X$ and all $j$. Thus, dominated convergence (Theorem 2.8.3) implies that $\hat{g}_j \to g_u$ in $L^p$. By Fuglede’s lemma (Lemma 4.2.3), there is a subsequence $(\hat{g}_{j_i})_{i=1}^{\infty}$ such that for p-a.e. curve $\gamma : [0,1] \to X$, we have

$$|u(\gamma(0)) - u(\gamma(1))| \leq \int_{\gamma} \hat{g}_{j_i} \, ds \to \int_{\gamma} g_u \, ds$$

as $i \to \infty$. Thus, $g_u$ is a weak upper gradient of $u$. We now have

$$I \leq \|g_u\|_{L^p} \leq \lim_{j \to \infty} \|g_j\|_{L^p} = I$$

so $\|g_u\|_{L^p} = I$ and the minimality in the $L^p$-norm is secured.

Assume that $g_u'$ is a weak upper gradient of $u$ such that $g_u' < g_u$, on a set of positive measure. Then $g_{\min} = \min\{g_u', g_u\}$ is a weak upper gradient of $u$, according to Lemma 4.2.1. But then $\|g_{\min}\|_{L^p} < \|g_u\|_{L^p} = I$, which is a contradiction. Thus, $g_u \leq g$ a.e. for all weak upper gradients of $u$. In particular, if we have two minimal weak upper gradients $g_u'$ and $g_u''$ of $u$, then $g_u' \leq g_u''$ a.e. and $g_u'' \leq g_u'$ a.e. This implies $g_u' = g_u''$ a.e., so the minimal weak upper gradient is unique up to sets of measure zero.

Proposition 4.2.4 was the main goal of this section. Now we know that if a function has a $p$-integrable weak upper gradient it also has a minimal weak upper gradient. The minimal weak upper gradient will replace the modulus of the ordinary gradient in our generalized Sobolev Spaces.
Example 4.2.5. In Example 4.1.2 we noted that $|\nabla u|$ is an upper gradient of $u$ if $u \in C^1$, where $X = \mathbb{R}^n$ equipped with the Euclidean norm and $n$-dimensional Lebesgue measure. In fact, $|\nabla u|$ is the minimal weak upper gradient of $u$. This is shown in [3](Proposition 8.3).

We finish the section by proving a property of upper gradients that we will use in the proof of Lemma 5.2.6.

**Lemma 4.2.6.** Let $u_i$ be functions with upper gradients $g_i$, and let $u = \sum_{i=1}^{\infty} u_i$ and $g = \sum_{i=1}^{\infty} g_i$. Then $g$ is an upper gradient of $u$.

**Proof.** Given a curve $\gamma : [a, b] \to X$, we get

$$|u(\gamma(b)) - u(\gamma(a))| = |\sum_{i=1}^{\infty} u_i(\gamma(b)) - \sum_{i=1}^{\infty} u_i(\gamma(a))|$$

$$\leq \sum_{i=1}^{\infty} |u_i(\gamma(b)) - u_i(\gamma(a))|$$

$$\leq \sum_{i=1}^{\infty} \int_{\gamma} g_i \, ds$$

$$= \int_{\gamma} \sum_{i=1}^{\infty} g_i \, ds$$

$$= \int_{\gamma} g \, ds.$$  

Here we switched places on the sum and integral by using Remark 3.2.5.  

4.3 Absolute continuity

By definition, curves in this thesis are continuous. When dealing with line integrals of functions along curves, it is useful to know how well the functions behave along the curves. To deal with this we will use absolute continuity, which is a stronger property than ordinary continuity.

**Definition 4.3.1.** (Absolute continuity) A function $f : [a, b] \to \mathbb{R}$ is absolutely continuous on $[a, b]$ if for every $\epsilon > 0$ there is a $\delta > 0$ such that

$$\sum_{i=1}^{n} |f(b_i) - f(a_i)| < \epsilon$$

for any $n \in \mathbb{N}$ and any $a \leq a_1 < b_1 \leq a_2 < b_2 \leq \cdots \leq a_n < b_n \leq b$ such that

$$\sum_{i=1}^{n} (b_i - a_i) < \delta.$$

**Definition 4.3.2.** (ACC$_p$) A function $u$ is said to be ACC$_p$ or absolutely continuous on $p$-almost every curve if $u \circ \gamma : [0, l_\gamma] \to \mathbb{R}$ is absolutely continuous for $p$-almost every curve $\gamma$ in $X$. 
Example 4.3.3. Let \( X = \Omega \subset \mathbb{R}^n \) equipped with the Euclidean norm and the \( n \)-dimensional Lebesgue measure. Let \( u \in L^p(\Omega) \). In \([23]\) it is shown as Theorem 2.1.4. that \( u \in W^{1,p}(\Omega) \) (the ordinary Sobolev space described in Section 1.3), if and only if \( u \) has a representative \( \tilde{u} \) that is absolutely continuous on almost all line segments in \( \Omega \) parallel to the coordinate axes and whose directional derivatives belong to \( L^p(\Omega) \).

It is shown in \([22]\) Theorem 28.2, that for functions \( u : \Omega \to \mathbb{R} \), the \( \text{ACC}_p \)-property follows from being absolutely continuous on almost all line segments along the coordinate axes and having directional derivatives in \( L^p(\Omega) \).

Lemma 4.3.4. If a function \( u \) has a weak upper gradient in \( L^p \) then \( u \) is \( \text{ACC}_p \).

Proof. Since \( u \) has a weak upper gradient in \( L^p \), Proposition 4.1.11 implies that \( u \) has an upper gradient \( g \in L^p \). Let \( \Gamma \) be the collection of all curves \( \gamma \) such that \( \int_\gamma g \, ds = \infty \). Then \( \text{Mod}_p(\Gamma) = 0 \), according to Lemma 4.1.6. For \( \gamma \not\in \Gamma \), with \( a, b \in [0, l_\gamma] \), we get

\[
|\left( u \circ \gamma \right)(a) - \left( u \circ \gamma \right)(b) | \leq \int_{\gamma[a,b]} g \, ds < \infty. \tag{4.6}
\]

Assume that \( f = u \circ \gamma \) is not absolutely continuous on \([0, l_\gamma]\). Then there exist \( \epsilon > 0 \) such that for every \( j = 1, 2, \ldots \), there are \( 0 \leq a_{j,1} < b_{j,1} \leq \cdots \leq a_{j,n_j} < b_{j,n_j} \leq l_\gamma \) such that

\[
\sum_{i=1}^{n_j} (b_{j,i} - a_{j,i}) < \frac{1}{2^j} \quad \text{and} \quad \sum_{i=1}^{n_j} |f(b_{j,i}) - f(a_{j,i})| \geq \epsilon.
\]

Let \( I_j = \bigcup_{i=1}^{n_j} [a_{j,i}, b_{j,i}] \). Then by (4.6) we get

\[
\epsilon \leq \sum_{i=1}^{n_j} |f(b_{j,i}) - f(a_{j,i})| \leq \sum_{i=1}^{n_j} \int_{\gamma[a_{j,i}, b_{j,i}]} g \, ds = \int_{\gamma[I_j]} g \, ds = \int_0^{l_\gamma} g(\gamma(t)) \chi_{I_j}(t) \, dt. \tag{4.7}
\]

Since \( \mu(I_j) \to 0 \) as \( j \to \infty \), dominated convergence (Theorem 2.8.3) implies that the right hand side of (4.7) tends to zero as \( j \to \infty \). But \( \epsilon > 0 \), so this is a contradiction. We conclude that \( u \) is absolutely continuous on \( \gamma \). Since \( \gamma \not\in \Gamma \) was arbitrary, \( u \) is absolutely continuous on \( p \)-a.e. curve. \( \square \)

Now that we know that functions with weak upper gradients are absolutely continuous along \( p \)-a.e. curve we can use this to manipulate weak upper gradients.

Lemma 4.3.5. Let \( u \in L^p \) and let \( g_u \in L^p \) be a weak upper gradient of \( u \). Also, let \( k \in \mathbb{R} \) and \( E = \{ x \in X : u(x) > k \} \). Then, \( g = g_u \chi_E \) is a weak upper gradient of \( f = \max\{u, k\} \).
4.3. Absolute continuity

Proof. Since \( g_u \) is a weak upper gradient of \( u \) it is obvious from the definition of weak upper gradients that it is also a weak upper gradient of \( f \). Then, according to Lemma 4.3.4, both \( u \) and \( f \) are absolutely continuous along \( p \)-a.e. curve. Take a curve \( \gamma : [a, b] \to X \) such that \( u \) and \( f \) are absolutely continuous on \( \gamma \) and such that inequality (4.1) is satisfied by \( g_u \) together with \( u \) along \( \gamma \) and all its subcurves.

If \( \gamma \cap E = \emptyset \), then
\[
|f(\gamma(a)) - f(\gamma(b))| = |k - k| = 0.
\]

On the other hand, if \( \gamma \cap E \neq \emptyset \), let
\[
\alpha = \inf \{ t \in [a, b] : \gamma(t) \in E \} \quad \text{and} \quad \beta = \sup \{ t \in [a, b] : \gamma(t) \in E \}.
\]
Then \( a \leq \alpha \leq \beta \leq b \). If \( \gamma(\alpha) \notin E \) we get
\[
|f(\gamma(a)) - f(\gamma(\alpha))| = |k - k| = 0. \tag{4.8}
\]
If \( \gamma(\alpha) \in E \) we get
\[
|f(\gamma(a)) - f(\gamma(\alpha - \epsilon))| = |k - k| = 0
\]
and \( \gamma(\alpha - \epsilon) \notin E \) for arbitrarily small \( \epsilon > 0 \). Since \( f \) is continuous along \( \gamma \) we get \( f(\gamma(\alpha)) \to f(\gamma(\alpha)) \), as \( \epsilon \to 0 \) and (4.8) follows. Similarly
\[
|f(\gamma(\beta)) - f(\gamma(b))| = |k - k| = 0.
\]
It is not sure that \( \gamma(\alpha), \gamma(\beta) \in E \), but by the definition of \( \alpha \) and \( \beta \) there exist arbitrarily small \( \epsilon_1, \epsilon_2 > 0 \), such that \( \gamma(\alpha + \epsilon_1), \gamma(\beta - \epsilon_2) \in E \). Since \( f \) is continuous along \( \gamma \) we get \( f(\gamma(\alpha + \epsilon_1)) \to f(\gamma(\alpha)) \) and \( f(\gamma(\beta - \epsilon_2)) \to f(\gamma(\beta)) \) as \( \epsilon_1, \epsilon_2 \to 0 \) Thus
\[
|f(\gamma(\alpha)) - f(\gamma(\beta))| = \lim_{\epsilon_1, \epsilon_2 \to 0} |u(\gamma(\alpha + \epsilon_1)) - u(\gamma(\beta - \epsilon_2))| \\
\leq \lim_{\epsilon_1, \epsilon_2 \to 0} \int_{[\alpha + \epsilon_1, \beta - \epsilon_2]} g_u \, ds \\
\leq \int_{\gamma} g \, ds.
\]
Now, by the triangle inequality, we get
\[
|f(\gamma(a)) - f(\gamma(b))| \\
\leq |f(\gamma(a)) - f(\gamma(\alpha))| + |f(\gamma(\alpha)) - f(\gamma(\beta))| + |f(\gamma(\beta)) - f(\gamma(b))| \\
\leq \int_{\gamma} g \, ds.
\]
We see that inequality (4.1) is satisfied by \( g \) together with \( f \) along \( p \)-a.e. curve, so \( g \) is a weak upper gradient of \( f \).

\[\square\]

Corollary 4.3.6. Let \( u \in L^p \) and let \( g_u \in L^p \) be a weak upper gradient of \( u \). If \( u = 0 \) on a set \( E \subset X \), then \( g = g_u \chi_X \setminus E \) is a weak upper gradient of \( u \).

Proof. First, we observe that \( u = \max\{u, 0\} - \max\{-u, 0\} \). By Remark 4.1.5, \( g_u \) is a weak upper gradient of \( -u \). By Lemma 4.3.5, \( g_u \chi_{\{u > 0\}} \) is a weak upper gradient of \( \max\{u, 0\} \) and \( g_u \chi_{\{u < 0\}} \) is a weak upper gradient of \( \max\{-u, 0\} \). Finally, Remark 4.1.5 implies that \( g_u \chi_X \setminus E = g_u \chi_{\{u > 0\}} + g_u \chi_{\{u < 0\}} \) is a weak upper gradient of \( u = \max\{u, 0\} - \max\{-u, 0\} \).

\[\square\]
Chapter 5

Newtonian spaces – Sobolev spaces on metric spaces

In the previous chapters we have introduced new concepts and proven properties about them. Now we are ready to define a generalization of Sobolev Spaces to general metric spaces.

5.1 The Newtonian Space

This definition was introduced by Shanmugalingam [20].

Definition 5.1.1. (Newtonian space) Whenever \( u \in L^p \), let

\[
\|u\|_{N^{1,p}(X)} = \left( \int_X |u|^p d\mu + \int_X g_u^p d\mu \right)^{1/p},
\]

where the \( g_u \) is the minimal weak upper gradient of \( u \). The Newtonian space on \( X \) is the quotient space

\[
N^{1,p}(X) = \{ u : \|u\|_{N^{1,p}(X)} < \infty \}/\sim,
\]

where \( u \sim v \) if and only if \( \|u - v\|_{N^{1,p}(X)} = 0 \).

We see by the definition that the Newtonian space is made up by equivalence classes of \( p \)-integrable functions with \( p \)-integrable weak upper gradients. Note that if there are no \( p \)-integrable weak upper gradients of a function \( u \), then \( u \notin N^{1,p}(X) \).

Analogously with the space \( L^p \), we will omit \( X \) in our notation writing \( N^{1,p} \) for the Newtonian space \( N^{1,p}(X) \).

The Newtonian space would not be a very good generalization of the ordinary Sobolev space \( W^{1,p} \) if it did not coincide with it in \( \mathbb{R}^n \).

Example 5.1.2. Let \( X = \Omega \) be \( \mathbb{R}^n \), or a subset of \( \mathbb{R}^n \), equipped with the Euclidean norm and the \( n \)-dimensional Lebesgue measure. In [17] (Theorem 4.4.2 and Corollary 4.4.6 ) it is shown that if \( u \in W^{1,p}(\Omega) \), then there is a function \( \bar{u} \) such that \( \bar{u} = u \) a.e. and \( \bar{u} \in N^{1,p}(\Omega) \). It is not that easy to follow the proofs since there are definitions involved that we do not use in this thesis.
In [20](Theorem 4.5) it is shown that if $u \in N^{1,p}(\Omega)$, then $u \in W^{1,p}(\Omega)$.

Put together, these two results show that $N^{1,p}$ and $W^{1,p}$ are equivalent as function spaces, i.e. to each equivalence class in $W^{1,p}$ there is an associated equivalence class in $N^{1,p}$.

The definition of Newtonian spaces would be quite useless if it did not give the Newtonian spaces at least some of the properties of ordinary Sobolev spaces. First and foremost we need to prove that the Newtonian norm really is a norm.

**Proposition 5.1.3.** The Newtonian space is a normed vector space. That is, the following conditions are met:

(a) If $u, v \in N^{1,p}$ then $au + bv \in N^{1,p}$, for all $a, b \in \mathbb{R}$.

(b) $\|u\|_{N^{1,p}} \geq 0$, for all $u \in N^{1,p}$.

(c) $\|u\|_{N^{1,p}} = 0 \iff u \sim 0$, for all $u \in N^{1,p}$.

(d) $\|cu\|_{N^{1,p}} = |c|\|u\|_{N^{1,p}}$, for all $u \in N^{1,p}$ and for all $c \in \mathbb{R}$.

(e) $\|u + v\|_{N^{1,p}} \leq \|u\|_{N^{1,p}} + \|v\|_{N^{1,p}}$, for all $u$ and $v$ in $N^{1,p}$.

**Proof.** (a) $au + bv \in L^p$ since $L^p$ is a vector space. If $g_u$ and $g_v$ are weak upper gradients of $u$ and $v$ respectively, then $|a|g_u + |b|g_v$ is a weak upper gradient of $au + bv$ by Remark 4.1.5. Since $L^p$ is a vector space we have $|a|g_u + |b|g_v \in L^p$. Thus $au + bv \in N^{1,p}$.

(b) $\|u\|_{N^{1,p}} \geq 0$ since both $|u|$ and its minimal weak upper gradient are non-negative.

(c) $\|u\|_{N^{1,p}} = 0 \iff u \sim 0$ follows directly from the definition of the equivalence classes in $N^{1,p}$. A function $u \in N^{1,p}$ is in the same equivalence class as the zero function if and only if $\|u\|_{N^{1,p}} = \|u - 0\|_{N^{1,p}} = 0$.

(d) For $c = 0$ the property is obviously true, so let $c \neq 0$. We know that $\|cu\|_{L^p} = |c|\|u\|_{L^p}$, since $L^p$ is a vector space.

Let $g_u$ be the the minimal weak upper gradient of $u$ and let $g_{cu}$ be the minimal weak upper gradient of $cu$. Then, by Remark 4.1.5 we know that $|c|g_u$ is a weak upper gradient of $cu$ and $g_{cu}/|c|$ is a weak upper gradient of $u$. Hence, we get

$$\|g_{cu}\|_{L^p} \leq \|c|g_u\|_{L^p} = |c|\|g_u\|_{L^p} \leq |c|\frac{g_{cu}}{|c|} \|L^p = \|g_{cu}\|_{L^p}.$$ 

We see that $\|g_{cu}\|_{L^p} = |c|\|g_u\|_{L^p}$, which implies that $|c|g_u$ is the minimal weak upper gradient of $cu$. From this we get

$$\|cu\|_{N^{1,p}} = \left(|c|^p\|u\|_{N^{1,p}}^p + |c|^p\|g_u\|_{N^{1,p}}^p\right)^{1/p} = |c|\|u\|_{N^{1,p}}.$$ 

(e) Let $g_u, g_v$ and $g_{u+v}$ be the minimal weak upper gradients of $u, v$ and $u + v$, respectively. Then $g_u + g_v$ is a weak upper gradient of $u + v$ by Remark 4.1.5. We then get

$$\|u + v\|_{N^{1,p}} = \|u + v\|_{L^p} + \|g_{u+v}\|_{L^p}^{1/p} \leq \left(\|u\|_{L^p} + \|v\|_{L^p}\right)^{1/p} + \left(\|g_u\|_{L^p} + \|g_v\|_{L^p}\right)^{1/p}.$$  \hspace{1cm} (5.1)
5.2. Capacity

Now, let \( t = \{ t_i \}_{i=1}^\infty \) be the sequence

\[
\| u \|_{L^p}, \| g_u \|_{L^p}, 0, 0, 0, \ldots,
\]

and let \( s = \{ s_i \}_{i=1}^\infty \) be the sequence

\[
\| v \|_{L^p}, \| g_v \|_{L^p}, 0, 0, 0, \ldots.
\]

Both these sequences obviously are in \( l^p \), the space of all sequences \( l = \{ l_i \}_{i=1}^\infty \) such that

\[
\sum_{i=1}^\infty |l_i|^p < \infty
\]

with the norm

\[
\| l \|_{l^p} = \left( \sum_{i=1}^\infty |l_i|^p \right)^{1/p}.
\]

A proof of the fact that \( l^p \) is indeed a normed space can be found for example in [14] Section 1.2-3. Thus, by the triangle inequality for \( l^p \) we get from (5.1) that

\[
\| u + v \|_{N^1,p} \leq \left( \left( \| u \|_{L^p} + \| v \|_{L^p} \right)^p + \left( \| g_u \|_{L^p} + \| g_v \|_{L^p} \right)^p \right)^{1/p} = \| t + s \|_{l^p} \leq \| t \|_{l^p} + \| s \|_{l^p} = \left( \| u \|_{L^p}^p + \| g_u \|_{L^p}^p \right)^{1/p} + \left( \| v \|_{L^p}^p + \| g_v \|_{L^p}^p \right)^{1/p} = \| u \|_{N^1,p} + \| v \|_{N^1,p}
\]

and we are done.

In ordinary Sobolev spaces, a useful property is that the minimum and maximum of two functions are still in the Sobolev space. A space with this property is called a lattice.

**Proposition 5.1.4.** The Newtonian space space \( N^{1,p} \) is a lattice, i.e. if \( u, v \in N^{1,p} \), then \( f_1 = \min\{ u, v \} \) and \( f_2 = \max\{ u, v \} \) are also in \( N^{1,p} \).

**Proof.** We note that

\[
f_1 = \max\{ u, v \} = u + \max\{ 0, v - u \}.
\]

We know that \( u \in N^{1,p} \), and since \( N^{1,p} \) is a vector space, so is \( u - v \). That \( \max\{ 0, v - u \} \) has a weak upper gradient in \( L^p \) is implied by Lemma 4.3.5 so it is also in \( N^{1,p} \). Since \( N^{1,p} \) is a vector space we get that \( f_1 \in N^{1,p} \). That \( f_2 \in N^{1,p} \) can be shown by a similar argument. \( \square \)

5.2 Capacity

In the \( L^p \)-spaces, as well as in the ordinary Sobolev spaces, the equivalence classes are given by the measure. In the Newtonian spaces the equivalence classes are smaller. It is not enough that two functions are equal a.e. to be regarded as equivalent. The capacity in the following definition is a better tool in the Newtonian spaces.
Definition 5.2.1. (Capacity) The capacity of a set \( E \subset X \) is the number
\[
C_p(E) = \inf \| u \|_{N^{1,p}(X)}^p,
\]
where the infimum is taken over all \( u \in N^{1,p}(X) \) such that \( u(x) \geq 1 \), for all \( x \in E \).

A property that holds everywhere but on a set of capacity zero is said to hold quasi-everywhere or q.e. for short. The capacity is obviously dependent on \( p \), but we will most often omit it in cases where it is obvious which \( p \) that is used.

In the definition of the capacity we only need to take the infimum among functions \( u \in N^{1,p}(X) \) such that \( 0 \leq u \leq 1 \) in \( X \) and \( u = 1 \) in \( E \). To see this we do the following:

Let \( u \in N^{1,p} \) with a minimal weak upper gradient \( g_u \). By Lemma 4.3.5, the function \( u^+ = \max\{u, 0\} \) has a weak upper gradient \( g_u \chi_{\{u>0\}} \). We now create the function \( u' = \min\{u^+, 1\} = -\max\{-u^+, -1\} \). By Remark 4.1.5 we know that \( g_u \chi_{\{u>0\}} \) is a weak upper gradient of \( -u \) and Lemma 4.3.5 then implies that \( g_u \chi_{\{u>0\}} \chi_{\{u<1\}} = g_u \chi_{\{0<u<1\}} \) is a weak upper gradient of \( -u' \). Finally, by Remark 4.1.5 we get that \( g_u \chi_{\{0<u<1\}} \) is a weak upper gradient of \( u' \).

We have now created a function \( u' \) that is the truncation of \( u \) at the levels 0 and 1. We have shown that \( g_u \chi_{\{0<u<1\}} \) is a weak upper gradient of \( u' \). This implies
\[
\| u \|_{N^{1,p}} = \left( \| u \|_{L^p}^p + \| g_u \|_{L^p}^p \right)^{1/p} \geq \left( \| u' \|_{L^p}^p + \| g_u \chi_{\{0<u<1\}} \|_{L^p}^p \right)^{1/p} \geq \| u' \|_{N^{1,p}}.
\]
Thus, we only need to look at functions with values between 0 and 1 when we are interested in the capacity.

To indicate of how the capacity works, we give a very simple example.

Example 5.2.2. We let \( X = \mathbb{R}^1 \) and \( E = [0, 1] \). We use the Euclidean norm and the one-dimensional Lebesgue measure. When trying to find the capacity of \( E \), a function giving a value close to the infimum in (5.2) will characteristically be 1 on \( E \) and outside \( E \) it will decrease in a nice way towards 0. It may look something like Figure 5.1.

![Figure 5.1: A function giving a value close to the infimum in (5.2)](image)

Capacities are not easy to calculate in general. Before we give any examples of sets for which we can find the capacity, we prove some properties of the capacity. Some of these will make the calculation in our examples easier and some will be useful later on.

Theorem 5.2.3. Let \( E, E_1, E_2, \ldots \) be arbitrary subsets of \( X \). Then
5.2. Capacity

(a) \( C_p(\emptyset) = 0 \);
(b) \( \mu(E) \leq C_p(E) \);
(c) if \( E_1 \subset E_2 \), then \( C_p(E_1) \leq C_p(E_2) \);
(d) the capacity is countably subadditive, that is
\[
C_p \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \sum_{i=1}^{\infty} C_p(E_i).
\]

Although we will not refer to it in that way, a set function satisfying conditions (a), (c) and (d) in Theorem 5.2.3 is called an outer measure. In order to prove part (d) we need a lemma.

**Lemma 5.2.4.** Let \( u_i \) be functions with upper gradients \( g_i \), and let \( u = \sup u_i \) and \( g = \sup g_i \). Then \( g \) is an upper gradient of \( u \).

**Proof.** Let \( \gamma : [a, b] \rightarrow X \) be a curve. Then
\[
|u(\gamma(a)) - u(\gamma(b))| \leq \sup_i |u_i(\gamma(a)) - u_i(\gamma(b))| \leq \sup_i \int_{\gamma} g_i \, ds \leq \int_{\gamma} g \, ds,
\]
so \( g \) is an upper gradient of \( u \).

**Proof of Theorem 5.2.3.** (a) Since the set in question is empty we take infimum over all functions in \( N_{1,p} \) in the definition of the capacity. But \( 0 \in N_{1,p} \), so the capacity is zero.
(b) For all functions \( u \) satisfying \( u \geq 1 \) on \( E \), we have
\[
\|u\|_{L^p} \geq \|\chi_E\|_{L^p} \quad \text{and} \quad \|g\|_{L^p} \geq 0
\]
for all weak upper gradients, \( g \), of \( u \). This implies
\[
C_p(E) = \inf_{u \geq \chi_E \text{ on } E} \|u\|_{N^{1,p}}^p \geq \|\chi_E\|_{L^p}^p = \int_X \chi_E^p \, d\mu = \int_E 1 \, d\mu = \mu(E).
\]
(c) Every function \( u \) satisfying \( u = 1 \) on \( E_2 \) also satisfies \( u = 1 \) on \( E_1 \). Thus, in the definition of \( C_p(E_1) \) we take infimum over a larger collection of functions than in the definition of \( C_p(E_2) \). This implies \( C_p(E_1) \leq C_p(E_2) \).
(d) Let \( \epsilon > 0 \). Choose \( u_i \geq \chi_{E_i} \) with upper gradients \( g_i \) such that
\[
\|u_i\|_{L^p}^p + \|g_i\|_{L^p}^p < C_p(E_i) + \frac{\epsilon}{2^n}.
\]
This is possible since the capacity is the infimum of norms over such functions as \( u_i \) and since the minimal weak upper gradients can be approximated in \( L^p \) by upper gradients. Let \( u = \sup_i u_i \) and \( g = \sup_i g_i \). By Lemma 5.2.4, \( g \) is an
upper gradient of $u$. Clearly $u \geq 1$ on $\bigcup_{i=1}^{\infty} E_i$. Hence
\[
C_p \left( \bigcup_{i=1}^{\infty} E_i \right) \leq \|u\|_{N^{1,p}}^{p} \leq \int_X \left( \sup_i u_i \right)^p d\mu + \int_X \left( \sup_i g_i \right)^p d\mu \\
\leq \sum_{i=1}^{\infty} \left( \int_X u_i^p d\mu + \int_X g_i^p d\mu \right) \\
\leq \sum_{i=1}^{\infty} \left( C_p(E_i) + \epsilon \right) \\
= \epsilon + \sum_{i=1}^{\infty} C_p(E_i).
\]

Letting $\epsilon \to 0$ completes the proof. \qed

**Definition 5.2.5.** $(\Gamma_E)$ Let $E \subset X$. $\Gamma_E$ is the family of curves $\gamma$ such that $\gamma \cap E \neq \emptyset$.

**Lemma 5.2.6.** Let $E \subset X$. Then, $C_p(E) = 0$ if and only if $\mu(E) = \text{Mod}_p(\Gamma_E) = 0$.

**Proof.** Assume that $\mu(E) = \text{Mod}_p(\Gamma_E) = 0$ and let $u = \chi_E$. Then $u = 0$ on $p$-a.e. curve, and thus $g = 0$ is a weak upper gradient of $u$. We then get
\[
C_p(E) \leq \|u\|_{N^{1,p}}^{p} \leq \|u\|_{L^p}^{p} + \|g\|_{L^p}^{p} = 0.
\]

Assume conversely that $C_p(E) = 0$. From Theorem 5.2.3 it follows that $\mu(E) \leq C_p(E) = 0$. Since $C_p(E) = 0$, for every $j = 1, 2, 3, \ldots$ there exists a function $v_j \in N^{1,p}$ with an upper gradient $g_j$ such that
\[
\|v_j\|_{L^p} < 2^{-j}, \quad \|g_j\|_{L^p} = 2^{-j} \quad \text{and} \quad v_j \geq \chi_E.
\]
Let $u = \sum_{j=1}^{\infty} v_j \in L^p$ and $g = \sum_{j=1}^{\infty} g_j \in L^p$. By Lemma 4.2.6, $g$ is an upper gradient of $u$. Let
\[
F = \{ x \in X : u(x) = \infty \}.
\]
Since $v_j \geq \chi_E$ for all $j = 1, 2, \ldots$, we get $E \subset F$. Let also
\[
\Gamma = \{ \gamma : \int_{\gamma} g \, ds = \infty \}.
\]
Since $g \in L^p$, Lemma 4.1.6 implies that $\text{Mod}_p(\Gamma) = 0$. We know that $u \in L^p$, so we have $\mu(F) = 0$. Then, Lemma 4.1.8 implies that $\text{Mod}_p(\Gamma_F) = 0$. In particular, on $p$-a.e. curve there is a point $y$ such that $u(y) \neq \infty$.

Let $\gamma : [0, l_\gamma] \to X$ be a curve such that $\gamma \notin \Gamma$ and such that there is a point $y$ that satisfies $u(y) \neq \infty$. Note that this is true for $p$-a.e. curve. For all $x \in \gamma$
we then get
\[ u(x) = \lim_{k \to \infty} \sum_{j=1}^{k} v_j(x) \leq \lim_{k \to \infty} \left( \sum_{j=1}^{k} v_j(y) + \sum_{j=1}^{k} |v_j(x) - v_j(y)| \right) \]
\[ \leq u(y) + \lim_{k \to \infty} \sum_{j=1}^{k} \int_{\gamma} q_j \, ds = u(y) + \lim_{k \to \infty} \int_{\gamma} \sum_{j=1}^{k} q_j \, ds \]
\[ \leq u(y) + \lim_{k \to \infty} \int_{\gamma} g \, ds = u(y) + \int_{\gamma} g \, ds < \infty. \]

We see that \( x \notin F \) which implies \( \gamma \cap F = \emptyset \), and thus \( \text{Mod}_p(\Gamma_F) = 0 \). Since \( E \subset F \) we have \( \Gamma_E \subset \Gamma_F \), and thus \( \text{Mod}_p(\Gamma_E) \leq \text{Mod}_p(\Gamma_F) = 0 \).

**Example 5.2.7.** For a set \( E \) with measure zero, Lemma 5.2.6 implies that \( \text{Mod}_p(\Gamma_E) = 0 \) is equivalent to \( C_{p}(E) = 0 \). In Example 3.3.4 we found that the capacity of all curves through a point in \( \mathbb{R}^n \) is zero if \( p \leq n \) and \( n > 1 \). Thus, \( C_{p}(\{x_0\}) = 0 \) for all \( x_0 \in \mathbb{R}^n \) if \( p \leq n \) and \( n > 1 \). By part (d) in Theorem 5.2.3 the capacity of countable unions of sets with capacity zero is zero. This implies that all countable sets have capacity zero if \( p \leq n \) and \( n > 1 \).

**Example 5.2.8.** What is more interesting is that there are spaces where a single point will have non-zero capacity. In Example 3.3.4 we saw that if \( p = n = 1 \), then the modulus of all curves through a point is non-zero. This implies that the capacity of a point is non-zero. From this we can conclude that every equivalence class consists of only one function, so no functions will be regarded as equivalent.

**Example 5.2.9.** Let \( E \subset X \) be a non-empty open set. Since \( E \) is open, there exists a point \( x \in E \) and a radius \( r \) such that \( B = B(x, r) \subset E \). From Section 2.7 we know that the measure \( \mu \) is non-zero for all open balls. Part (b) of Theorem 5.2.3 then implies \( C_{p}(E) \geq \mu(E) \geq \mu(B) > 0 \). The conclusion is that no open non-empty sets have zero capacity.

As mentioned earlier most capacities are difficult to compute, but in Section 2.11 in [10] the capacity for the one-point set and certain balls are computed. There, another kind of capacity is used, but Theorem 2.38 in [10] shows that it is comparable to the capacity used in this thesis.

The concept of capacity is very useful when dealing with Newtonian spaces. The sets of capacity zero play a similar role in the Newtonian space as the sets of measure zero play in \( L^p \). To reach that conclusion we need to do some preliminary work.

**Proposition 5.2.10.** If \( u, v \in \text{ACC}_p \) and \( u = v \text{ a.e.} \), then \( u = v \text{ q.e.} \).

*Proof.* Let \( w = u - v \) and \( E = \{ x : w(x) \neq 0 \} \). Since \( \mu(E) = 0 \), Lemma 4.1.8 implies that \( \text{Mod}_p(\Gamma_E^+) = 0 \). Thus, the Lebesgue measure of \( \gamma^{-1}(\gamma \cap E) \) is zero for \( p \text{-a.e. curve } \gamma \).

We know that \( u, v \in \text{ACC}_p \) so it is obvious from the definition of absolute continuity that \( w \in \text{ACC}_p \) as well. By Lemma 3.3.5, for \( p \text{-a.e. curve } \gamma \) we have
that \( w \) is continuous along \( \gamma \) and that the Lebesgue measure of \( \gamma^{-1}(\gamma \cap E) \) is zero. On such curves \( w = 0 \) a.e. with respect to the arc length. But since \( w \) is continuous along the curve it must be zero everywhere on it. Hence \( \Gamma_E = \Gamma_E^+ \), and by Lemma 5.2.6, \( C_p(E) = 0 \), which implies \( u = v \) q.e.

**Proposition 5.2.11.** Let \( u : X \to \overline{\mathbb{R}} \) be a function. Then \( \|u\|_{N^{1,p}} = 0 \) if and only if \( C_p(\{x \in X : u(x) \neq 0\}) = 0 \).

*Proof.* Let \( E = \{x \in X : u(x) \neq 0\} \). Assume first that \( u \in N^{1,p} \), and \( v = 0 \) a.e. Then \( u \in N^{1,p} \), and \( v \in AC \). Thus, by Proposition 5.2.10, \( u = 0 \) q.e and thus, \( C_p(E) = 0 \).

Assume conversely that \( C_p(E) = 0 \). Then by Lemma 5.2.6 we get \( \mu(E) = 0 \) and \( \text{Mod}_p(\Gamma_E) = 0 \). Curve families of modulus zero can be ignored when looking for weak upper gradients. Thus, 0 is a weak upper gradient of \( u \). This implies

\[
\|u\|_{N^{1,p}} \leq \left(\|u\|_{L^p}^p + \|0\|_{L^p}^p\right)^{1/p} = 0,
\]

and we are done.

This shows that two functions \( u, v \in N^{1,p} \) are in the same equivalence class if and only if \( u = v \) q.e. In other words, the equivalence classes in \( N^{1,p} \) are up to sets of capacity zero.

Although we have not assumed much about neither the metric nor the measure, the Newtonian space will be complete. Together with previously proven properties, this results in the fact that the Newtonian space is a Banach space.

**Proposition 5.2.12.** The Newtonian space is a Banach space.

*Proof.* We have already proven that \( N^{1,p} \) is a normed vector space so we only need to prove completeness. Let \( \{u_j\}_{j=1}^\infty \) be a Cauchy sequence in \( N^{1,p}(X) \). We need to prove that it converges. Now, let \( x \) be such that there is a \( k \) such that

\[
|u_{j+1}(x) - u_j(x)| < 2^{-j} \text{ for all } j \geq k. \tag{5.3}
\]

Obviously \( \{u_j(x)\}_{j=1}^\infty \) is a Cauchy sequence in \( \mathbb{R} \) and thus convergent there. This property will be useful, so we start out by proving that the set of all \( x \) not having this property has capacity zero.

The sequence \( \{u_j\}_{j=1}^\infty \) is a Cauchy sequence. By passing to a subsequence, if necessary, we can assume that \( \|u_{j+1} - u_j\|_{N^{1,p}(X)} \leq 2^{-j(p+1)/p} \). Let

\[
E_j = \{x \in X : |u_{j+1}(x) - u_j(x)| \geq 2^{-j}\}.
\]

The function \( f(x) = 2^j|u_{j+1}(x) - u_j(x)| \geq 1 \) on \( E_j \), so

\[
C_p(E_j) \leq (2^j\|u_{j+1} - u_j\|_{N^{1,p}(X)})^p \leq 2^{jp}2^{-j(p+1)} = 2^{-j}.
\]

Let

\[
F_k = \bigcup_{j=k}^\infty E_j \quad \text{and} \quad F = \bigcap_{k=1}^\infty F_k.
\]

Then according to Theorem 5.2.3,

\[
C_p(F_k) \leq \sum_{j=k}^\infty C_p(E_j) < \sum_{j=k}^\infty 2^{-j} = 2^{1-k},
\]

Chapter 5. Newtonian spaces – Sobolev spaces on metric spaces
and thus $C_p(F) = 0$. Let $x \in X \setminus F$, then $x \in X \setminus \bigcup_{j=k}^{\infty} E_j$ for some $k$ and $|u_{j+1}(x) - u_j(x)| < 2^{-j}$ for all $j \geq k$. Hence $F$ is the set of all $x$ lacking the desired property (5.3) and $F$ has capacity zero.

For $x \in X \setminus F$, the sequence $\{u_j(x)\}_{j=1}^{\infty}$ is convergent so we can define

$$u(x) = \lim_{j \to \infty} u_j(x) = u_k(x) + \sum_{j=k}^{\infty} (u_{j+1}(x) - u_j(x)).$$

The function $u$ is defined q.e. and

$$\|u - u_k\|_{L^p(X)} \leq \sum_{j=k}^{\infty} \|u_{j+1} - u_j\|_{L^p} \leq \sum_{j=k}^{\infty} \|u_{j+1} - u_j\|_{N^1,p} \leq 2^{-j}\sum_{j=k}^{\infty} 2^{-j} = 2^{1-k}.$$ 

We see that for all $x \in X \setminus F$

$$u(x) - u_k(x) = \sum_{j=k}^{\infty} (u_{j+1}(x) - u_j(x)). \quad (5.4)$$

As $C_p(F) = 0$, $p$-a.e. curve in $X$ has empty intersection with $F$ according to Lemma 5.2.6. Thus, equation (5.4) holds on $\gamma$ for $p$-a.e. curve $\gamma$ in $X$. Let $\gamma$ be one such curve, connecting $x$ and $y$. Then

$$\|(u - u_k)(x) - (u - u_k)(y)\| \leq \sum_{j=k}^{\infty} \|(u_{j+1} - u_j)(x) - (u_{j+1} - u_j)(y)\|$$

$$\leq \sum_{j=k}^{\infty} \int_{\gamma} g_j = \sum_{j=k}^{\infty} \int_{\gamma} g_j ds,$$

where $g_j$ is an upper gradient of $u_{j+1} - u_j$. Here we have used Remark 3.2.5 to switch places on the sum and integral. Hence $\tilde{g}_k = \sum_{j=k}^{\infty} g_j$ is a weak upper gradient of $u - u_k$. But

$$\|u_{j+1} - u_j\|_{N^1,p} \leq 2^{-j(p+1)} < 2^{-jp},$$

and Proposition 4.1.11 implies that there are upper gradients arbitrarily close to every weak upper gradient, including the minimal weak upper gradient. Thus, we can choose the upper gradient $g_j$ such that $\|g_j\|_{L^p} < 2^{-j}$. This gives us $\|\tilde{g}_k\|_{L^p} \leq 2^{1-k}$. It follows that

$$\|u - u_k\|_{N^1,p} \leq (\|u - u_k\|_{L^p}^p + \|\tilde{g}_k\|_{L^p}^p)^{1/p} \to 0, \text{ as } k \to \infty.$$ 

Thus, every Cauchy sequence is convergent, so the Newtonian space is complete.

We end this section by studying what happens with the capacity if the parameter $p$ changes. As far as I know, no results have been published about this in the setting of general metric measure spaces.
For a given set, we are most often only interested in if the capacity is zero or not. Assume that a set has capacity zero for some $p$. Does it still have capacity zero when $p$ changes? To prove that it does in some cases, we first notice a fact about the $L^p$-norm.

Assume that $A \subset X$ is such that $\mu(A) < \infty$ and $u \in L^p(A)$. With $p,p',q' \in (1,\infty)$ and $1/p' + 1/q' = 1$, the Hölder inequality implies

$$
\|u\|_{L^p(A)} = \left( \int_A |u|^p \, d\mu \right)^{\frac{1}{p}} \leq \left( \int_A |u|^{pp'} \, d\mu \right)^{\frac{1}{pp'}} \left( \int_A 1^{pq'} \, d\mu \right)^{\frac{1}{pq'}} = \left( \int_A |u|^{pp'} \, d\mu \right)^{\frac{1}{pp'}} \mu(A)^{\frac{1}{pq'}} = \|u\|_{L^{pp'}(A)} \mu(A)^{\frac{1}{pq'}}.
$$

Since $p' \in (1,\infty)$ was arbitrary, we get

$$
\|u\|_{L^p(A)} \leq \|u\|_{L^{p'}(A)} \mu(A)^{\frac{1}{p'}}
$$

for all $s \in (p,\infty)$. We will need this fact in the next section as well, so we put it in a remark for later reference.

**Remark 5.2.13.** Let $A \subset X$ be such that $\mu(A) < \infty$ and let $u \in L^p(A)$. It then holds that $\|u\|_{L^p(A)} \leq \|u\|_{L^s(A)} \mu(A)^{\frac{1}{p} - \frac{1}{s}}$, for all $s \in (p,\infty)$.

**Proposition 5.2.14.** Let $p \in (1,\infty)$. Let $X$ be such that $\mu(X) < \infty$ and let $C_p(E) = 0$. Then $C_{p'}(E) = 0$ for all $p' \in (1,p)$.

**Proof.** Let $u \in L^p$. Then, by Remark 5.2.13 we get

$$
\|u\|_{L^{p'}} \leq \|u\|_{L^p} \mu(X)^{\frac{1}{p'} - \frac{1}{p}},
$$

so $u \in L^{p'}$. The minimal weak upper gradient depends on the parameter $p$, but it can be approximated by upper gradients and these do not depend on $p$. Let $g_u \in L^p$ be an upper gradient of $u$. By Remark 5.2.13 we get

$$
\|g_u\|_{L^{p'}} \leq \|g_u\|_{L^p} \mu(X)^{\frac{1}{p'} - \frac{1}{p}}.
$$

Since $C_p(E) = 0$, there exists $u \in N^{1,p}$ such that $u \geq 1$ on $E$ and such that $\|u\|_{N^{1,p}}$ is arbitrarily small. For each $u \in N^{1,p}$ there is an upper gradient arbitrarily close to the minimal weak upper gradient in the $L^p$-norm. Thus, for every $\epsilon > 0$ there exists $u \in N^{1,p}$ with an upper gradient $g_u \in L^p$ such that $u \geq 1$ on $E$ and such that

$$
\|u\|_{L^p} < \epsilon \quad \text{and} \quad \|g_u\|_{L^p} < \epsilon.
$$

By (5.5) and (5.6) this implies

$$
C_{p'}(E) \leq \|u\|_{N^{1,p'}} \leq \|u\|_{L^{p'}} + \|g_u\|_{L^{p'}} \leq \left( \|u\|_{L^p} + \|g_u\|_{L^p} \right) \mu(X)^{1 - \frac{p'}{p}} \leq (\epsilon + \epsilon) \mu(X)^{1 - \frac{p'}{p}}.
$$

By letting $\epsilon \to 0$, we get $C_{p'}(E) = 0$. \qed
In fact, the statement of Proposition 5.2.14 holds even if \( \mu(X) = \infty \).

**Theorem 5.2.15.** Let \( p \in (1, \infty) \). If \( C_p(E) = 0 \), then \( C_{p'}(E) = 0 \) for all \( p' \in (1, p) \).

**Proof.** First, assume that \( E \) is a bounded set and \( C_p(E) = 0 \). Since \( E \) is bounded there is a ball \( B_1 = B(x_0, r) \) such that \( E \subset B_1 \), where \( x_0 \in X \) and \( r \in (0, \infty) \). Let \( \eta \) be a function such that \( \eta = 1 \) on \( B_1 \) and such that \( \eta \) decreases linearly with the distance from \( x_0 \) on \( B_2 \setminus B_1 \), so that it is zero on \( X \setminus B_2 \). It is obvious that \( \eta \) is 1-Lipschitz, so it has \( g_\eta \) as an upper gradient.

Let \( u \in N^{1,p} \) and \( u \geq 1 \) on \( E \). It is clear that \( u\eta \geq 1 \) on \( E \), and by Remark 5.2.13 we get

\[
\|u\eta\|_{L^{p'}} \leq \|u\|_{L^{p'}(B_2^c)} \leq \|u\|_{L^{p'}(B_2)} \mu(B_2)^{\frac{1}{p'} - \frac{1}{p}}. \tag{5.7}
\]

If \( g_u \) is an upper gradient of \( u \), then according to Lemma 1.7 in [4]

\[
g = (|\eta| + \epsilon)g_u + (|u| + \epsilon)g_\eta
\]

is an upper gradient of \( u\eta \), for every \( \epsilon > 0 \). But since \( u\eta = 0 \) outside \( B_2 \), Corollary 4.3.6 implies that \( g' = g\chi_{B_2} \) is a weak upper gradient of \( u\eta \), independently of the values of \( p \) and \( p' \). By Remark 5.2.13 we get

\[
\|g'\|_{L^{p'}} \leq \|(|\eta| + \epsilon)g_u\|_{L^{p'}(B_2)} + \|(|u| + \epsilon)g_\eta\|_{L^{p'}(B_2)}
\]

\[
\leq (1 + \epsilon)\|g_u\|_{L^{p'}(B_2)} + \|u\|_{L^{p'}(B_2)} + \epsilon \mu(B_2)^{1/p'}
\]

\[
\leq ((1 + \epsilon)\|g_u\|_{L^{p}(B_2)} + \|u\|_{L^{p}(B_2)} + \epsilon \mu(B_2)^{1/p}) \mu(B_2)^{\frac{1}{p'} - \frac{1}{p}} \tag{5.8}
\]

As in the proof of Proposition 5.2.14 we get that for every \( \epsilon > 0 \) there exists \( u \in N^{1,p} \) with an upper gradient \( g_u \in L^p \) such that \( u \geq 1 \) on \( E \) and such that

\[
\|u\|_{L^p} < \epsilon \quad \text{and} \quad \|g_u\|_{L^p} < \epsilon.
\]

By (5.7) and (5.8) this implies

\[
C_{p'}(E) \leq \|u\eta\|_{N^{1,p'}} \leq \|u\eta\|_{L^{p'}} + \|g'\|_{L^{p'}}
\]

\[
\leq \left( \|u\|_{L^p(B_2)} + \|g_u\|_{L^p(B_2)} + \|u\|_{L^p(B_2)} + \epsilon \mu(B_2)^{\frac{1}{p'}} \right) \mu(B_2)^{1 - \frac{1}{p'}}
\]

\[
< (\epsilon + (1 + \epsilon + \epsilon + C) \mu(B_2)^{1 - \frac{1}{p'}})
\]

By letting \( \epsilon \to 0 \), we get \( C_{p'}(E) = 0 \) and the proposition is proven for bounded sets.

Assume now that \( E \) is unbounded. Since \( E \subset X = \bigcup_{i=1}^{\infty} B(x, 2^i) \) for any \( x \in X \), we get that

\[
E = \bigcup_{i=1}^{\infty} \left( E \cap B(x, 2^i) \right).
\]

Since \( E \cap B(x, 2^i) \subset E \) we have

\[
C_p\left( E \cap B(x, 2^i) \right) = 0.
\]
We have already proven that the proposition holds for bounded sets, so since all balls are bounded we get

\[ C_p(E \cap B(x, 2^i)) = 0. \]

Since this holds for \( i = 1, 2, 3, \ldots \), we get \( C_p(E) = 0 \) by the subadditivity of the capacity, and we are done.

A natural question is whether the capacity remains zero or not, when \( p \) is increased. There are examples that prove that this is not the case in general. An example can be found in Section 2.11 in [10], where the capacity for the one-point set and certain balls are computed. As mentioned earlier, another kind of capacity is used there, but Theorem 2.38 in [10] shows that it is comparable to the capacity used in this thesis.

### 5.3 Density of Lipschitz functions

Supplied with only a metric space and a measure with the properties of section 2.7, it has been possible to show that the Newtonian space is a Banach space, just as ordinary Sobolev spaces.

Another property of ordinary Sobolev spaces is the fact that smooth functions are dense. In general metric spaces there are no derivatives so we cannot talk about smooth functions there. On the other hand, we do have Lipschitz functions. To be able to prove that Lipschitz functions are dense in Newtonian spaces one needs to know more about the metric space and the measure. The following two properties are sufficient for our purposes. With them one can prove many other things about the Newtonian space as well, but we will settle with the density of Lipschitz functions.

The first property limits the growth of the measure when the size of the measured set increases. In this way, one can find an upper bound for the measure of a set \( E \), by using the measure of subsets of \( E \).

**Definition 5.3.1.** (Doubling measure) A measure \( \mu \) is doubling if there exists a constant \( C_\mu > 0 \) such that

\[ \mu(2B) \leq C_\mu \mu(B), \]  

for all balls \( B = B(x_0, r) \) in \( X \), where \( \lambda B = B(x_0, \lambda r) \).

**Example 5.3.2.** The n-dimensional Lebesgue measure in \( \mathbb{R}^n \), equipped with the Euclidean norm, is doubling since for a ball \( B = B(x_0, r) \) we have

\[ \mu(2B) = \mu(B(x_0, 2r)) = 2^n \mu(B(x_0, r)) = 2^n \mu(B). \]

**Example 5.3.3.** Let \( x_0 \) be a point in \( X \) and

\[ \mu(E) = \begin{cases} 
1 & \text{if } x_0 \in E \\
0 & \text{otherwise}
\end{cases} \]

for all measurable sets \( E \subset X \). Assume that there is another point \( x_1 \in X \), such that \( x_0 \neq x_1 \). For \( r = 2d(x_0, x_1)/3 \) and \( B = B(x_1, r) \), we get \( \mu(B) = 0 \) and \( \mu(2B) = 1 \), so \( \mu \) is not doubling.
The second property deals with the upper gradients. It assures that if a function differs much from its mean value on a ball, then the upper gradients will be large there. For a metric space with a doubling measure this is not true in general, as this simple example shows.

**Example 5.3.4.** Let \( X = \Omega \subset \mathbb{R}^2 \) equipped with the Euclidean norm, where \( \Omega \) consists of two separated squares \( S_1 \) and \( S_2 \), of the same size. Let \( \mathbb{R}^2 \) be equipped with the 2-dimensional Lebesgue measure \( \mu_{\mathbb{R}^2} \), while the measure on \( \Omega \) is the restriction of \( \mu_{\mathbb{R}^2} \) to \( \Omega \) which we denote by \( \mu_\Omega \). For all points \( x_0 \in \Omega \) we have

\[
\frac{1}{4} \mu_{\mathbb{R}^2}(B(x_0, r)) \leq \mu_\Omega(B(x_0, r)) \leq \mu_{\mathbb{R}^2}(B(x_0, r)),
\]

for all \( r > 0 \). Since \( \mu_{\mathbb{R}^2} \) is doubling we see that \( \mu_\Omega \) is also doubling. Now, let

\[
u = \begin{cases} 1 & \text{on } S_1; \\ 0 & \text{on } S_2. \end{cases}
\]

Then, \( g = 0 \) is an upper gradient of \( \nu \) since \( \nu \) is constant on \( S_1 \) and \( S_2 \), and since those sets are separated. On any ball, \( B \), that has non-empty intersections with both \( S_1 \) and \( S_2 \), the mean value of \( \nu \) will be neither 1 or 0. Thus, \( \nu \) will differ from its mean value on \( B \) while having zero as an upper gradient.

In the literature, Poincaré inequalities can be found in different forms and with slightly different names. The one below is often referred to as a weak Poincaré inequality.

**Definition 5.3.5.** (Poincaré inequality) We say that \( X \) supports a \( p \)-Poincaré inequality if there exist constants \( C > 0 \) and \( \lambda \geq 1 \), such that for all balls \( B = B(x, r) \subset X \), all measurable functions \( f \) on \( X \) and all upper gradients \( g \) of \( f \), it holds that

\[
\frac{1}{\mu(B)} \int_B |f - f_B| \, d\mu \leq C r \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p \, d\mu \right)^{1/p},
\]

(5.10)

where \( f_B = \int_B f \, d\mu / \mu(B) \).

In fact, if \( X \) supports a \( p \)-Poincaré inequality it supports an \( s \)-Poincaré inequality for all \( s > p \) as well, since by using Remark 5.2.13, we get

\[
\left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p \, d\mu \right)^{\frac{1}{p}} \leq \mu(\lambda B)^{-\frac{1}{p}} \| g \|_{L^p(\lambda B)}
\]

\[
\leq \mu(\lambda B)^{-\frac{1}{p}} \| g \|_{L^p(\lambda B)} \mu(\lambda B)^{\frac{1}{p} - \frac{1}{s}}
\]

\[
= \left( \frac{1}{\mu(\lambda B)} \int_{\lambda B} \rho^s \, d\mu \right)^{\frac{1}{s}}
\]

for all balls \( B = B(x, r) \subset X \) and all \( s \in (p, \infty) \).

In Example 5.3.4, \( X \) was disconnected, but there are cases where \( X \) is connected and \( \mu \) is doubling but where the \( p \)-Poincaré inequality is not satisfied.

**Example 5.3.6.** Let \( X = X_+ \cup X_- \subset \mathbb{R}^n \) equipped with the Euclidean norm, where \( X_+ \) is the part of \( \mathbb{R}^n \) where all coordinates are non-negative and \( X_- \)
is the part where all coordinates are non-positive. Let $\mathbb{R}^n$ be equipped with the $n$-dimensional Lebesgue measure $\mu_{\mathbb{R}^n}$, while on $X$ we have the restriction of $\mu_{\mathbb{R}^n}$ to $X$, denoted $\mu_X$. For all points $x_0 \in X$ we have $\frac{1}{2^n} \mu_{\mathbb{R}^n}(B(x_0, r)) \leq \mu_X(B(x_0, r)) \leq \mu_{\mathbb{R}^n}(B(x_0, r))$, for all $r > 0$. We see that $\mu_X$ is doubling since $\mu_{\mathbb{R}^n}$ is doubling.

We recall the result from Example 3.3.4 that the family of all curves through a point has zero modulus as long as $p \leq n$ and $n > 1$. In $X$, the only point connecting $X_+$ and $X_-$ is the origin. Thus, when it comes to weak upper gradients, $X_+$ and $X_-$ can be regarded as separated from each other. Let

$$u = \begin{cases} 1 & \text{on } X_+; \\ 0 & \text{on } X_-.
\end{cases}$$

We see that $g = 0$ is a weak upper gradient of $u$. With zero as a weak upper gradient, the Poincaré inequality (5.10) is not satisfied since the right hand side will be zero, while the left hand side will not.

A space that does indeed satisfy the $p$-Poincaré inequality is $\mathbb{R}^n$ equipped with Euclidean norm and the $n$-dimensional Lebesgue measure. For a proof of this fact, see [7], where it can be found as (7.45), on page 157. Note that the inequality looks quite different since the notation used there is different from the one used in this thesis.

In the remaining part of this chapter we will assume that we have a doubling measure and a space that satisfies a $p$-Poincaré inequality.

In ordinary Sobolev spaces smooth functions, and thereby continuous functions, are dense. Under our assumptions we will now be able to prove that Lipschitz functions are dense in the Newtonian space. Since Lipschitz functions are continuous, this can be seen as a generalization of the property of the ordinary Sobolev spaces. As a tool for the proof we introduce yet another definition.

**Definition 5.3.7.** (Maximal function) For a function $f \in L^1$, the non-centered maximal function is

$$M^*(f)(x) = \frac{1}{\mu(B)} \sup_B \int_B |f| d\mu,$$

where the supremum is taken over all balls $B$ in $X$, containing $x \in X$.

**Lemma 5.3.8.** For a function $f \in L^1$, the non-centered maximal function $M^*f(x)$ satisfies

$$\mu(E_\tau) \leq \frac{C}{\tau} \int_{E_\tau} |f| d\mu \quad \text{and} \quad \lim_{\tau \to \infty} \tau \mu(E_\tau) = 0,$$

where $E_\tau = \{ x \in X : M^*f(x) > \tau \}$.

To prove this, we need a property about sets called the 5-covering lemma. The proof given below of the 5-covering lemma follows the proof of Theorem 1.2 in [12].
5.3. Density of Lipschitz functions

**Lemma 5.3.9** (5-covering lemma). Let $\mathcal{F}$ be a family of open balls in $X$ such that the radii of all balls in $\mathcal{F}$ have a supremum $K < \infty$. Then there exists a countable subfamily $\mathcal{G}$ of $\mathcal{F}$ such that the balls in $\mathcal{G}$ are pairwise disjoint and such that

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B.$$

**Proof.** Let $\mathcal{F}$ be a family satisfying the conditions of the lemma. Let $\Omega$ be the set of all subfamilies $\omega$ of $\mathcal{F}$ such that the elements in $\omega$ are pairwise disjoint and such that $\omega$ satisfies the following property:

If an open ball $B_1 \in \mathcal{F}$ has a non-empty intersection with an open ball $B_2 \in \omega$, then $B_1$ has a non-empty intersection with an open ball in $\omega$ with a radius that is at least half that of $B_1$.

We see that $\Omega$ is non-empty since the family $\omega = \{B\}$, with $B$ having a radius $r > K/2$ is in $\Omega$.

Take a sequence of subfamilies $\{\mathcal{F}_i\}_{i=1}^\infty$, such that $\mathcal{F}_i \subset \mathcal{F}_{i+1}$ and $\mathcal{F}_i \in \Omega$ for all $i = 1, 2, 3, \ldots$. Then $\omega_0 = \bigcup_{i=1}^\infty \mathcal{F}_i$ belongs to $\Omega$ since:

- If $B_1, B_2 \in \omega_0$ then there is an $i \in \mathbb{N}$ such that $B_1, B_2 \in \mathcal{F}_i$. Thus, $B_1$ and $B_2$ are disjoint since $\mathcal{F}_i$ is a pairwise disjoint subfamily of $\mathcal{F}$.
- If a ball $B \in \mathcal{F}$ has a non-empty intersection with a ball in $\omega_0$, $B$ has a non-empty intersection with a ball in some $\mathcal{F}_i$ and thus $B$ has a non-empty intersection with a ball in $\mathcal{F}_i$ with a radius of at least half that of $B$.

This shows that every sequence in $\Omega$, that increases in size like $\{\mathcal{F}_i\}_{i=1}^\infty$, has an upper bound, i.e. a subfamily in $\Omega$ that includes the elements of all $\mathcal{F}_i$. Now Zorn’s Lemma, found as 4.1-6 in [14], implies that $\Omega$ has a maximal element, that is, an element $\mathcal{G}$ such that if $\mathcal{G} \subset \mathcal{A}$ for some $\mathcal{A} \in \Omega$, then $\mathcal{G} = \mathcal{A}$.

Next, we show that every ball in $\mathcal{F}$ has non-empty intersection with some ball in $\mathcal{G}$.

Let $\mathcal{F}'$ be the family of balls in $\mathcal{F}$ that have empty intersections with all balls in $\mathcal{G}$. If $\mathcal{F}' \neq \emptyset$, then pick a ball $B_0 \in \mathcal{F}'$ such that the radius of $B_0$ is larger than half that of any ball in $\mathcal{F}'$. We create a new family of balls $\mathcal{G}' = \mathcal{G} \cup \{B_0\}$. The balls in $\mathcal{G}'$ are pairwise disjoint and if a ball $B$ in $\mathcal{F}$ has a non-empty intersection with a ball in $\mathcal{G}'$, then $B$ has a non-empty intersection with a ball in $\mathcal{G}'$ with a radius of at least half that of $B$. Thus, $\mathcal{G}' \in \Omega$. But $\mathcal{G}$ was maximal, so we have a contradiction. Hence, $\mathcal{F}'$ is empty and every ball $B \in \mathcal{F}$ has a non-empty intersection with a ball in $\mathcal{G}$, with a radius of at least half that of $B$.

Let $x \in \bigcup_{B \in \mathcal{F}} B$ be arbitrary. Then $x \in B(x_0, r)$, for some ball $B(x_0, r) \in \mathcal{F}$. We know that $B(x_0, r)$ has non-empty intersection with a ball $B(y_0, r') \in \mathcal{G}$ such that $r' \geq r/2$. We then get

$$d(x, y_0) \leq d(x, x_0) + d(x_0, y_0) < r + r + r' \leq 2r' + 2r + r' = 5r'.$$

This shows that $x \in B(y_0, 5r')$ and thus

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B.$$

The thing left to prove is that $\mathcal{G}$ is countable. By the properties of $\mu$ from Section 2.7 we know that $0 < \mu(B) < \infty$, for all balls $B \subset X$. Then, the measure of
uncountably many disjoint balls must be infinite. This implies that there are at most countably many disjoint balls in any ball \(B(x, 2^i)\), where \(x \in X\) and \(i \in \mathbb{N}\). We know that \(X = \bigcup_{i=1}^{\infty} B(x, 2^i)\) for any \(x \in X\). Since countable unions of countable sets are countable sets, we conclude that there can be at most countably many disjoint balls in \(X\). Since \(G\) is a family of pairwise disjoint balls in \(X\), it must thereby be countable.

Proof of Lemma 5.3.8. Consider first the restricted non-centered maximal function \(M_R^* f\) for which the supremum is only taken over balls of radius less than \(R\). Let

\[
E_R^* = \{ x \in X : M_R^* f(x) > \tau \}.
\]

By the definition of \(E_R^*\), for every \(x \in E_R^*\) there is a ball \(B_x\) such that \(x \in B_x\) and such that

\[
\frac{1}{\mu(B_x)} \int_{B_x} |f| \, d\mu > \tau.
\]  (5.11)

Let \(y \in B_x\) be arbitrary. Then \(B_x\) is among the balls in the supremum when calculating \(M_R^* f(y)\). But then (5.11) implies that \(y \in E_R^*\). Since \(y \in B_x\) was arbitrary we get \(B_x \subset E_R^*\). Thus

\[
E_R^* = \bigcup_{x \in X} B_x.
\]

By the 5-covering lemma, there exist pairwise disjoint balls \(B_{x_j} \subset E_R^*, j = 1, 2, \ldots,\) among the balls \(B_x\), such that \(E_R^* \subset \bigcup_{j=1}^{\infty} 5B_{x_j}\). Using the doubling property and the balls \(B_{x_j}\) we get

\[
\tau \mu(E_R^*) \leq \tau \sum_{j=1}^{\infty} \mu(5B_{x_j}) \leq C \tau \sum_{j=1}^{\infty} \mu(B_{x_j}).
\]

By using (5.11) and the fact that the balls \(B_{x_j} \subset E_R^*\) are pairwise disjoint, we get

\[
C \tau \sum_{j=1}^{\infty} \mu(B_{x_j}) \leq C \sum_{j=1}^{\infty} \int_{B_{x_j}} |f| \, d\mu \leq C \int_{E_R^*} |f| \, d\mu.
\]

We have thus shown that

\[
\tau \mu(E_R^*) \leq C \int_{E_R^*} |f| \, d\mu.
\]  (5.12)

We note that the constant \(C\) in (5.12) originates from the doubling property and does not depend on \(R\). Thus, by letting \(R \to \infty\) we get the first inequality in the statement of the lemma, i.e.

\[
\mu(E_\tau) \leq \frac{C}{\tau} \int_{E_\tau} |f| \, d\mu.
\]  (5.13)

But since \(E_\tau\) does not expand with \(\tau\), the integral in (5.13) will not grow when \(\tau\) increases. Thus, it follows that \(\mu(E_\tau) \to 0, \text{ as } \tau \to \infty\). We now get

\[
\tau \mu(E_\tau) \leq C \int_{E_\tau} |f| \, d\mu = C \int_X \chi_{E_\tau} |f| \, d\mu.
\]  (5.14)

As \(\chi_{E_\tau} |f| \leq |f| \in L^1\) for all \(\tau\), dominated convergence (Theorem 2.8.3) implies \(\int_X \chi_{E_\tau} |f| \, d\mu \to 0, \text{ as } \tau \to \infty\). From (5.14) we get that \(\tau \mu(E_\tau) \to 0, \text{ as } \tau \to \infty\), and we are done. \(\square\)
Now we are almost ready for the main result of this section, the density of Lipschitz functions in the Newtonian space. Yet, we need one more thing, Lebesgue points.

**Definition 5.3.10.** Let $u : X \to \mathbb{R}$ be locally integrable, i.e. around every point $x \in X$ there is a ball, $B$, such that $u$ is integrable on $B$. Then, $x_0 \in X$ is a Lebesgue point of $u$ if

$$
\lim_{r \to 0} \frac{1}{\mu(B(x_0, r))} \int_{B(x_0, r)} |u - u(x_0)| \, d\mu = 0.
$$

In $\mathbb{R}^n$, equipped with the Euclidean norm and the $n$-dimensional Lebesgue measure, it is true that for locally integrable functions, almost every point is a Lebesgue point. This result follows from Theorem 7.7 in [19]. This also holds in general metric measure spaces, as long as the measure is doubling. The result follows from Theorem 14.8 in [8].

**Remark 5.3.11.** Let $X$ be a metric measure space equipped with a doubling measure. For all locally integrable functions $u : X \to \mathbb{R}$, and thus all functions in $N^{1,p}$, almost all points are Lebesgue points.

Finally it is time for the theorem. To make our notation in the following proof less cumbersome, we write

$$u_B = \frac{1}{\mu(B)} \int_B u \, d\mu,$$

for $u \in N^{1,p}$ and balls $B$ in $X$.

**Theorem 5.3.12.** If $\mu$ is a doubling measure and $X$ satisfies a $p$-Poincaré inequality, then Lipschitz functions are dense in $N^{1,p}$.

**Proof.** Let $u \in N^{1,p}$. Assume first that $u$ is bounded and let $g \in L^p$ be an upper gradient of $u$. For $\tau > 0$, we let $E_\tau = \{ x \in X : M^* g^p(x) > \tau^p \}$, where $M^* g^p$ is the non-centered maximal function of $g^p$. Let $x, y \in X \setminus E_\tau$, $r > 0$ and $\rho \in [r/2, r]$ be such that $B(x, r) \supset B(y, \rho) \supset B(x, r/2^n)$ for some $n \in \mathbb{N}$. We then get

$$\mu(B(x, r)) \leq C^n \mu(B(x, r/2^n)) \leq C^n \mu(B(y, \rho))$$

by the doubling property of $\mu$. Since in our notation $C^n = C = 1/C$ etc., this implies

$$|u_B(y, \rho) - u_B(x, r)| = \left| \frac{1}{\mu(B(y, \rho))} \int_{B(y, \rho)} u \, d\mu - u_B(x, r) \right|$$

$$\leq \frac{1}{\mu(B(y, \rho))} \int_{B(y, \rho)} |u - u_B(x, r)| \, d\mu$$

$$\leq C \frac{1}{\mu(B(x, r))} \int_{B(x, r)} |u - u_B(x, r)| \, d\mu. \quad (5.15)$$

By the $p$-Poincaré inequality we get from (5.15) that

$$|u_B(y, \rho) - u_B(x, r)| \leq C r \left( \mu(\lambda B(x, r)) \int_{\lambda B(x, r)} \lambda^p \, d\mu \right)^{1/p}$$

$$\leq C r (M^* g^p(x))^{1/p}$$

$$\leq Cr. \quad (5.16)$$
The last inequality follows from the fact that \( x \not\in E_\tau \). In particular, with \( x = y \in X \setminus E_\tau \), \( \rho \in (0, r) \) and a suitable \( k \in \mathbb{N} \), we have

\[
|u_{B(x,r)} - u_{B(x,\rho)}| \\
\leq |u_{B(x,r)} - u_{B(x,\rho/2)}| + |u_{B(x,\rho/2)} - u_{B(x,\rho/4)}| + \cdots + |u_{B(x,\rho/2^k)} - u_{B(x,\rho)}| \\
\leq C\tau(r + \rho/2 + \rho/4 + \cdots) = 2C\tau r = C\tau r.
\]

(5.17)

It follows that every sequence \( \{u_{B(x,r_j)}\}_{j=1}^\infty \) with \( r_j \to 0 \), is a Cauchy sequence in \( \mathbb{R} \), and thus the limit

\[
\bar{u}(x) = \lim_{r \to 0} u_{B(x,r)}
\]

exists for all \( x \in X \setminus E_\tau \). We shall show that the function \( \bar{u} \) is \( C\tau \)-Lipschitz on \( X \setminus E_\tau \). Let \( x, y \in X \setminus E_\tau \) be arbitrary. We then get

\[
|\bar{u}(x) - \bar{u}(y)| \leq |\bar{u}(x) - u_{B(x,4d(x,y))}| + |u_{B(x,4d(x,y))} - \bar{u}(y)|.
\]

By (5.17) we get

\[
|u_{B(x,4d(x,y))} - \bar{u}(x)| = |u_{B(x,4d(x,y))} - \lim_{j \to \infty} u_{B(x,2^{-j}d(x,y))}| \leq 4C\tau d(x,y).
\]

We see that \( B(x,4d(x,y)) \supset B(y,2d(x,y)) \supset B(x,d(x,y)) \), so by (5.16) and (5.17) we get

\[
|u_{B(x,4d(x,y))} - \bar{u}(y)| \leq |u_{B(x,4d(x,y))} - u_{B(y,2d(x,y))}| + |u_{B(y,2d(x,y))} - \bar{u}(y)| \\
\leq 4C\tau d(x,y) + |u_{B(y,2d(x,y))} - \lim_{j \to \infty} u_{B(y,2^{-j}d(x,y))}| \\
\leq 4C\tau d(x,y) + 2C\tau d(x,y).
\]

Put together, this implies

\[
|\bar{u}(x) - \bar{u}(y)| \leq 4C\tau d(x,y) + 6C\tau d(x,y) = C\tau d(x,y),
\]

which proves that \( \bar{u} \) is \( C\tau \)-Lipschitz on \( X \setminus E_\tau \).

Since \( \bar{u} \) is \( C\tau \)-Lipschitz on \( X \setminus E_\tau \), Proposition 2.5.2 implies that it can be extended to a Lipschitz function \( \bar{u}_\tau \) on the whole of \( X \), where

\[
\bar{u}_\tau(x) = \inf_{y \in X \setminus E_\tau} \{ \bar{u}(y) + Ld(x,y) \}.
\]

Let \( u_\tau \) be the function obtained by truncating \( \bar{u}_\tau \) at the levels \( \tau \) and \( -\tau \). Note that the Lipschitz property is not affected by truncation. At Lebesgue points for \( u \) in \( X \setminus E_\tau \) we have \( u = \bar{u} \). But since \( u \) is bounded we have for large \( \tau \) that \( u_\tau = \bar{u}_\tau = \bar{u} = u \) at all Lebesgue points for \( u \) in \( X \setminus E_\tau \). By Remark 5.3.11, almost every point in \( X \setminus E_\tau \) is a Lebesgue point for \( u \), so we get

\[
\|u - u_\tau\|_{L^p} \leq \left( \int_{E_\tau} |u - u_\tau|^p d\mu \right)^{1/p} \\
\leq \left( \int_{E_\tau} |u|^p d\mu \right)^{1/p} + \left( \int_{E_\tau} |u_\tau|^p d\mu \right)^{1/p} \\
\leq \left( \int_{E_\tau} |u|^p d\mu \right)^{1/p} + \tau \mu(E_\tau)^{1/p} \\
= \left( \int_X |u|^p \chi_{E_\tau} d\mu \right)^{1/p} + \tau \mu(E_\tau)^{1/p}.
\]

(5.18)
According to Lemma 5.3.8, we get \( \tau^p \mu(E_\tau) \to 0 \), as \( \tau \to \infty \). This implies that both the second expression in (5.18) and \( \mu(E_\tau) \) tend to zero as \( \tau \to \infty \). Now, by dominated convergence (Theorem 2.8.3), the first expression in (5.18) tends to zero as \( \tau \to \infty \).

Let \( E \) be the set of all non-Lebesgue points for \( u \). For large \( \tau \) we then have \( u - u_\tau = 0 \) on \( X \setminus (E \cup E_\tau) \). Remark 4.1.5 and Corollary 4.3.6 then show that \( g_{u - u_\tau} = (g + C\tau) \chi_{E_\tau} \cup E \) is a weak upper gradient of \( u - u_\tau \). Since \( \mu(E) = 0 \) by Remark 5.3.11, Lemma 5.3.8 implies

\[
\|g_{u - u_\tau}\|_{L^p} \leq \|g\|_{L^p(E_\tau)} + \|C\tau\chi_{E_\tau} \cup E\|_{L^p} + C\tau \mu(E_\tau)^{1/p} \to 0,
\]
as \( \tau \to \infty \). We have used dominated convergence (Theorem 2.8.3) to get \( \|g\|_{L^p(E_\tau)} = \|g\chi_{E_\tau}\|_{L^p} \to 0 \), as \( \tau \to \infty \). These results lead to the conclusion that

\[
\|u - u_\tau\|_{N^{1,p}} \leq \left(\|u - u_\tau\|_{L^p}^p + \|g_{u - u_\tau}\|_{L^p}^p\right)^{1/p} \to 0,
\]
as \( \tau \to \infty \), so the theorem is proven for bounded functions.

If \( u \) is unbounded we can approximate \( u \) by its truncations \( u_k \) at levels \( k \) and \( -k \). Clearly, \( u_k \to u \) in \( L^p \) as \( k \to \infty \). Corollary 4.3.6 shows that \( g\chi_{\{u < -k\} \cup \{u > k\}} \) is a weak upper gradient of \( u - u_k \). Since \( u \in L^p \) we know that \( \mu(\{u < -k\} \cup \{u > k\}) \to 0 \), as \( k \to \infty \). By dominated convergence (Theorem 2.8.3), this implies

\[
\|g\chi_{\{u < -k\} \cup \{u > k\}}\|_{L^p} \to 0,
\]
as \( k \to \infty \) and hence

\[
\|u - u_k\|_{N^{1,p}} \leq \left(\|u - u_k\|_{L^p}^p + \|g\chi_{\{u < -k\} \cup \{u > k\}}\|_{L^p}^p\right)^{1/p} \to 0,
\]
as \( k \to \infty \). This completes the proof. \( \square \)
Chapter 6

Final remarks

The theory of Sobolev spaces on metric spaces has developed greatly since it was introduced. Many more properties than the ones mentioned in this thesis have been shown to hold.

People can now solve the Dirichlet problem in this general setting and it has been shown that there exists a kind of minimum principle. Solving partial differential equations on metric spaces is still an active area of research where the theory is currently under development.

Several texts covering large parts of the established theory are being written. The paper [9] is already available on the internet. I have used it to find some of the sources for this thesis. The text [3], which was the main source of inspiration for this thesis, is still under development. It is based on a course in potential theory, given in Linköping during the fall of 2005.

In this thesis we spent a lot of time and effort on introducing concepts and results that we needed to deal with the upper gradients and the Newtonian spaces. To carry the discussion further, one would probably need to include more theory in the preliminaries, although I believe that most of the necessary preliminaries are already included in this thesis. Nevertheless, the theory of Newtonian spaces and its applications is far too extensive to be included in a thesis at this level.
Bibliography


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