Performance estimation of a ducted fan UAV

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This thesis aims to estimate the performance of the concept of the ducted fan UAV. The company where this thesis has been written, DST Control AB, is currently investigating the economical possibilities to continue the development of this kind of UAV. This thesis shall provide DST Control AB with a theoretical as well as experimental ground for the investigation by estimation the lift capacity, position accuracy and wind tolerance.

A ducted fan UAV prototype and a mathematical model for that UAV have been developed by DST Control AB and a student project at Linköping University. The model is constructed through pure physical modeling. Several noise sources have been added to better fit the reality. Several experiments have been conducted to validate the model with satisfying results. Experiments to determine the lift capacity of the craft have also been conducted. These experiments showed a slightly smaller lift capacity than the theoretically calculated lift capacity. The wind tolerance has not been tested in experiments because of the lack of available wind tunnels but simulations have given an estimation of this tolerance.

To estimate the position accuracy, two different control systems have been implemented. The simplest control system is a system consisting of several PID controllers. The system is divided into two separate subsystems connected in cascade. The inner subsystem takes the pitch, roll and yaw angle as inputs and gives the rudder angles as outputs. The outer subsystem takes the inertial position as input and gives roll, pitch and yaw as outputs. Together, the two subsystems can be used to control the entire craft. The inner subsystem has also been replaced with a small LQ Compensator. An LQ Compensator for the entire system is also implemented giving about as good performance as the PID controller and better performance than the PID/LQ combination.

Keywords: ducted fan, UAV, LQ, PID
Abstract

The ducted fan UAV is an unmanned aerial vehicle consisting mainly of a propeller enclosed in a open ended tube. The UAV has the same basic functions as an ordinary helicopter UAV but has several advantages to the same.

This thesis aims to estimate the performance of the concept of the ducted fan UAV. The company where this thesis has been written, DST Control AB, is currently investigating the economical possibilities to continue the development of this kind of UAV. This thesis shall provide DST Control AB with a theoretical as well as experimental ground for the investigation by estimation the lift capacity, position accuracy and wind tolerance.

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## Notation

\( x_i, y_i, z_i \) Coordinates in inertial reference frame \([\text{m}]\)

\( x, y, z \) Coordinates in body fixed reference frame \([\text{m}]\)

\( u, v, w \) Velocities in the body fixed reference frame \([\text{m/s}]\)

\( p, q, r \) Angular velocities about the axis in the body fixed reference frame \([\text{rad/s}]\)

\( \phi, \theta, \psi \) Euler angles \([\text{rad}]\)

\( X, Y, Z \) Total forces in body fixed reference frame (except gravitational forces) \([\text{N}]\)

\( L, M, N \) Total momentum about the axis in the body fixed reference frame \([\text{Nm}]\)

\( I_a \) Moment of inertia about the arbitrary axis \( a \) \([\text{kgm}^2]\)

\( C_{D,x}, C_D \) Drag coefficient for the body \( x \) \([-]\)

\( C_{L,x}, C_L \) Lift coefficient for the body \( x \) \([-]\)

\( F_D \) Drag force from wing \([\text{N}]\)

\( F_L \) Lift force from wing \([\text{N}]\)

\( T \) Total lift force \([\text{N}]\)

\( P \) Total power \([\text{W}]\)

\( F_a \) Force in the direction of \( a \) \([\text{N}]\)

\( \rho \) Density \([\text{kg/m}^3]\)

\( h \) Lever length \([\text{m}]\)

\( U \) Arbitrary wind speed \([\text{m/s}]\)

\( U_R \) Radial wind speed \([\text{m/s}]\)

\( U_T \) Tangential wind speed \([\text{m/s}]\)

\( U_P \) Out-of-plane wind speed (in Blade Element Theory) \([\text{m/s}]\)

\( V_c \) Wind speed above the propeller caused by the crafts axial motion \([\text{m/s}]\)

\( v_i \) Induced wind speed at the propeller \([\text{m/s}]\)

\( v_c \) Wind speed far below the propeller \([\text{m/s}]\)

\( v_{w,m} \) Magnitude of the noise wind \([\text{m/s}]\)

\( \gamma \) Introduced angle for integration \([\text{rad}]\)

\( v_w \) Noise wind, \( v_w = v_{w,m} \cos(\gamma) \) \([-]\)
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_x$</td>
<td>Area of body $x$</td>
<td>$[m^2]$</td>
</tr>
<tr>
<td>$c$</td>
<td>Cord</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$R_w$</td>
<td>Rudder length</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$r_w$</td>
<td>Distance from the center of the craft to the blade element</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$N_b$</td>
<td>Number of propeller blades</td>
<td>$[-]$</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Effective angle of attack</td>
<td>$[\text{rad}]$</td>
</tr>
<tr>
<td>$\theta_\alpha$</td>
<td>Angle between wing and plane of motion</td>
<td>$[\text{rad}]$</td>
</tr>
<tr>
<td>$\phi_\alpha$</td>
<td>Difference between $\alpha$ and $\theta_\alpha$</td>
<td>$[\text{rad}]$</td>
</tr>
<tr>
<td>$\theta_1 - \theta_4$</td>
<td>Constants</td>
<td>$[\text{rad}]$</td>
</tr>
<tr>
<td>$c_{L0} - c_{L2}$</td>
<td>Constants</td>
<td>$[-]$</td>
</tr>
<tr>
<td>$c_{D0} - c_{D2}$</td>
<td></td>
<td>$[-]$</td>
</tr>
<tr>
<td>$\lambda_h$</td>
<td>Dimensionless quantity used for lift calculations</td>
<td>$[-]$</td>
</tr>
<tr>
<td>$V$</td>
<td>Local wind speed</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$S$</td>
<td>Surface</td>
<td>$[m^2]$</td>
</tr>
<tr>
<td>$\dot{m}$</td>
<td>Mass flow rate</td>
<td>$[kg/s^2]$</td>
</tr>
<tr>
<td>$C_x \cos(x)$, $x$ is an arbitrary angle. Note that $x$ can not be $L, l, D$ or $d$ because $C_L, C_D$ and $C_{D,x}$ are lift and drag coefficients.</td>
<td>$[-]$</td>
<td></td>
</tr>
<tr>
<td>$S_x \sin(x)$, $x$ is an arbitrary angle.</td>
<td>$[-]$</td>
<td></td>
</tr>
<tr>
<td>$T_x \tan(x)$, $x$ is an arbitrary angle.</td>
<td>$[-]$</td>
<td></td>
</tr>
<tr>
<td>$m_x$</td>
<td>Mass of component $x$</td>
<td>$[kg]$</td>
</tr>
<tr>
<td>$a_i$</td>
<td>Constant describing the relation between $x$ and $y$ on propeller blade $i$</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$H_G$</td>
<td>Angular momentum</td>
<td>$[Js]$</td>
</tr>
<tr>
<td>$\omega_p$</td>
<td>Rotational speed of the propeller</td>
<td>$[\text{rad/s}]$</td>
</tr>
<tr>
<td>$g$</td>
<td>Gravitational constant</td>
<td>$[m/s^2]$</td>
</tr>
<tr>
<td>$b_w$</td>
<td>Width of wing or rudder</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$l_w$</td>
<td>Length of wing or rudder</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Angle of attack of the noise wind</td>
<td>$[\text{rad}]$</td>
</tr>
<tr>
<td>$\varphi_j$</td>
<td>Angle between the noise wind and rudder $j$</td>
<td>$[\text{rad}]$</td>
</tr>
<tr>
<td>$G$</td>
<td>Biplane gap</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$D$</td>
<td>Biplane stagger</td>
<td>$[m]$</td>
</tr>
<tr>
<td>$V_c$</td>
<td>Velocity vector</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$\omega$</td>
<td>Angular velocity vector</td>
<td>$[\text{rad/s}]$</td>
</tr>
<tr>
<td>$F$</td>
<td>Force vector</td>
<td>$[N]$</td>
</tr>
<tr>
<td>$T$</td>
<td>Torque vector</td>
<td>$[Nm]$</td>
</tr>
<tr>
<td>$I$</td>
<td>Moment of inertia vector</td>
<td>$[\text{kgm}^2]$</td>
</tr>
<tr>
<td>$F_{\text{aero}}$</td>
<td>Vector containing the aerodynamic forces</td>
<td>$[N]$</td>
</tr>
<tr>
<td>$W$</td>
<td>Lyapunov function</td>
<td>$[-]$</td>
</tr>
<tr>
<td>$\xi$</td>
<td>Virtual control quantity</td>
<td>$[-]$</td>
</tr>
<tr>
<td>$\mathbf{u}_T$</td>
<td>Angular velocity vector</td>
<td>$[\text{rad/s}]$</td>
</tr>
<tr>
<td>$\mathbf{u}_V$</td>
<td>Velocity control vector</td>
<td>$[m/s]$</td>
</tr>
<tr>
<td>$\Phi, \Psi$</td>
<td>Nonlinearities</td>
<td>$[-]$</td>
</tr>
</tbody>
</table>
\[ A, B, C,\] System matrices
\[ D, G,\]
\[ A_L, B_L, C_L,\] Linearized system matrices
\[ x\] System state vector
\[ \hat{x}\] Estimated state vector
\[ \tilde{x}\] Difference between state vector and estimated state vector
\[ x_{\text{int}}\] States introduced to obtain integral action
\[ \bar{x}\] Difference between the states at the operating point and the real states.
\[ y\] System out signal vector
\[ \hat{y}\] Estimated out signal vector
\[ \tilde{y}\] Difference between output vector and estimated output vector
\[ u\] System input vector
\[ \omega\] System input noise vector
\[ \nu\] System measurement noise vector
\[ Q_{\omega\omega}, Q_{\nu\nu}, Q_{\omega\nu}\] Noise weight matrix
\[ R_{\omega\omega}, R_{\nu\nu}, R_{\omega\nu}\] Noise intensity matrix
\[ K_e\] Estimator gain matrix
\[ R_{xx}, R_{uu}, R_{xu}\] LQ weight matrices
\[ L\] Feedback gain matrix
\[ L_r\] Reference gain matrix
\[ M_{bi}\] Conversion matrix from body fixed reference frame to inertial reference frame
\[ X\] Time prediction state vector
\[ U\] Time prediction output vector
\[ H\] System matrix
\[ S\] System matrix
\[ \tilde{C}\] Matrix with system matrices \( C \) in the diagonal
\[ \tilde{Q}\] Matrix with weight matrices \( Q \) in the diagonal
\[ R\] Time prediction reference vector
\[ T_S\] Sample time
\[ T_d\] Derivative time
\[ T_i\] Integration time
\[ K\] Proportional gain
\[ \zeta\] Design parameter in Taylor expansion

Observe that these notations do not apply at Appendix A.
Chapter 1

Introduction

1.1 Purpose

DST Control AB is currently investigating the commercial potential of an UAV based on a ducted fan construction. The purpose of this thesis is to theoretically and with experiments estimate the performance limits of a ducted fan UAV. The performance estimation will be used by DST Control AB to determine whether the concept should be further developed or not.

1.2 Goals

The goal of this project is to estimate the performance of a ducted fan UAV. The performance of the system is determined by the lift capacity, the position accuracy and the wind tolerance. To test the performance a model and a control system have to be implemented. The estimation of the performance elements will result in suggested improvements as well as a solid ground for DST Controls AB’s decision regarding the future development of the ducted fan UAV.

1.3 Extent and Limitations

A prototype ducted fan UAV exists and has been used in the previous development work. To this prototype a test rig, a mathematical model and a basic control system are available. The existing mathematical model will be recalculated and thereafter validated. The existing control system must be completely rebuilt in order to function. The position accuracy can be divided into two parts: accuracy of the position in the air and accuracy of the orientation of the craft. The accuracy of the orientation as well as the lift capacity will be evaluated both theoretically, using the model, and with experiments. The accuracy of the position in the air and the wind tolerance will only be evaluated theoretically.

To enable sufficient testing of model and control system a new test rig is constructed, based on the basic structure of the existing test rig. Some physical
modifications of the craft, like rudder or shroud modifications, might be necessary to obtain the wanted performance and to investigate possible future modifications.

1.4 Background

The ducted fan technology as well as a patented invention by Prof. em. Fritz Hjelte is currently under evaluation by DST Control AB. The goal is to create a low-weight cost effective ducted fan UAV that is safe and easy to use even for minimally trained operators.

The need for a simple low cost UAV was discovered at a time when only heavy expensive UAV:s for advanced users were available on the market. The company responsible for the earlier part of the development, Scandicraft AB, found that the simple ducted fan UAV would complement their existing range of products. The project has now moved from Scandicraft AB to DST Control AB where a prototype has been developed in cooperation with Linköping University.

The ducted fan technology was at the start of the project uncommon in civilian applications and was regarded as promising. The key features making the ducted fan technology promising were the simplicity and safety of the design. The need for only a single rotor makes the construction simple to produce and the duct protects the surrounding environment from the rotating blades.

1.5 Available Hardware and Software

The project is carried out entirely at DST Control AB where earlier a prototype has been developed for testing. The prototype is basically ready to fly with control card and sensor board mounted but the sensor board does at this time not contain any sensors. A test rig also exists equipped with sensors measuring the yaw, pitch and roll angles defined in Section 2.4, force sensor, current sensor and a sensor measuring the rotational velocity of the propeller. The motor on Bombus is powered by two 12 V batteries and the sensors and control card are powered by an external 24 V power supply.

The tools used to create the model and control system are the programs MatrixX and SystemBuild. When the control system has been uploaded to the control card on the craft, RTC Browser is used as an interface to the control system. RTC Browser is developed by DST Control AB and allow the user to change parameters and inputs in real-time as well as logging and plotting inputs and outputs.

In addition to the above mentioned hardware and software there exists a large arsenal of conventional tools and materials for construction of for example a test rig or for rudder modifications.

1.6 Structure of this Document

The document is built up by chapters all discussing a specific part of the theory and the resulting conclusions. The sections discussing a part of the theory where
1.6 Structure of this Document

experiments have been made to test the accuracy of the theory, will also contain
the experimental results.

The first three chapters contain an introduction to the problem, the craft and
the test rig. The following three chapters discuss different theories and present
experimental results all leading to a mathematical model at the end of Chapter 7.
This is followed by a chapter where different controllers are discussed, evaluated
and tested. The final chapters consists of results and conclusions for the entire
project and suggestions for future development.
Chapter 2

Theory

2.1 Unmanned Aerial Vehicle

An Unmanned Aerial Vehicle, or UAV, is basically an aircraft controlling itself without the aid of a human. UAV:s have been used for several years where the mission for some reason is considered inconvenient for humans. The reason for choosing an UAV instead of a manned aircraft can be everything from a dangerous environment to a dull mission. In military applications UAV:s have been used mostly for surveillance until recently when unmanned combat aerial vehicles, or UCAVs, have been developed.

Bombus is a production name of the civilian ducted fan UAV discussed in this thesis. The main usage for Bombus is surveillance or information acquisition in situations not requiring a more advanced UAV. Even if the intended use is civilian, Bombus could be used in several military applications like surveillance from a military ship. The current version of Bombus is designed to be tethered to the ground giving it power and data connection through a cable. The tethered design simplifies the construction since the altitude can be trivially controlled and the craft does not need to bring its own power supply. This means that the available lift capacity can be used to lift mission critical pay load, for example a camera.

2.2 Ducted Fan

Bombus is a prototype based on the ducted fan principle, meaning that it is basically a fan mounted in a open ended tube. This setup has several advantages compared to the normal helicopter. The ideal duct has its smallest diameter where the propeller is placed and the diameter increases after the propeller forcing the air passing through the duct to accelerate. An ideal duct will increase the lift capacity of the engine and propeller [2]. On Bombus, the duct has the same diameter at the outlet as it has at the propeller, reducing the duct’s addition to the total lift. The lift of the duct will be discussed in Chapter 4. Another advantage of the ducted fan is that the duct protects people and equipment close to the craft from
the moving propeller blades.

2.3 Vectored Thrust

The direction of flight of Bombus is controlled by four rudders placed in the air stream from the duct. The craft uses a principle called vectored thrust to steer. Vectored thrust, illustrated in Figure 2.1, divert the exhaust from the engine (or in the case of Bombus, the air accelerated by the propeller) to generate a force in the desired direction. Vectored thrust is used in hover crafts and on water jet engines and in recent years also on advanced combat aircraft. On Bombus, the vectored thrust is used to control the roll, yaw and pitch angles (defined in Section 2.4) and thereby also the position.

![Figure 2.1. Vectored thrust.](image)

2.4 Definitions

Before a mathematical model of the craft can be constructed, reference frames and angles need to be defined. Two reference frames will be used, the inertial reference frame and the body fixed reference frame. The inertial reference frame has its origin fixed to the ground and the body fixed reference frame has its origin fixed in the center of gravity of the craft. Both reference frames are shown in Figure 2.2, where the coordinates with index $i$ are the coordinates in the inertial reference frame. Since the craft is almost symmetrical about the $z$ axis, a direction for one of the $x$ or $y$ axes has to be defined to obtain a completely defined body fixed reference frame. The $x$ axis is defined as the axis between the center of the craft and one of the control card and the $z$ axis is the axis pointing downwards.

The interesting angles when creating the control system for the craft are the angles in the body fixed reference frame. The body fixed angles are called pitch, roll and yaw and are the angles usually used when the rotation of an aircraft is described. Pitch is the angle originating from a rotation about the $y$ axis, roll is the angle originating from a rotation about the $x$ axis and yaw is the angle
originating from a rotation about the \( z \) axis \([18]\). The definitions of the angles are shown in Figure 2.3.
Chapter 3

Rig

3.1 Modifying the Test Rig

A test rig is a construction where the craft can be placed to simulate a natural environment and to perform controlled tests. A wish from DST Control AB is the construction of a new test rig allowing more advanced testing of the craft.

3.1.1 Test Rig Alpha

Since Bombus is a prototype without known flight ability, tests can not be performed in any of the situations Bombus was designed for. The control system, the rudders and the motor are at this time not sufficiently tested to perform free flight tests. Therefore, a test rig must be used to test the control system, the rudders, lift capacity, stability etc. The existing test rig, Test Rig Alpha (see Figure 3.1), only allows Bombus to rotate limited angles around the \( x \) and \( y \) axes but it allows unlimited rotation around the \( z \) axis. The limited rotations are not around the center of mass but around a point below the center of mass giving an unnatural rotation resulting in measurements differing from those obtained in free flight. There are several possible solutions that would solve this problem in a satisfactory way.

Sensors in Test Rig Alpha

Bombus contains only a propeller revolution sensor, \( i.e., \) almost all measurements have to be made with sensors manufactured with the test rig. One requirement is that the data from sensors on a rig in some way can represent data measured with sensors that could be placed on the craft, like the angles in the body fixed reference frame. Test Rig Alpha contained four sensors: a yaw sensor, a current sensor, an angle sensor (a joystick turned upside down) and a sensor to measure the lift capacity of the craft. The joystick rotated with the craft and measured the roll and pitch angles defined in Figure 2.3. In the new test rig some or all of the sensors have to be replaced or modified. The joystick, for example, was the
reason that the craft did not rotate around its center of mass in Test Rig Alpha. The craft rotated around the point where the joystick was fixed in the craft. The yaw sensor was placed at the end of the joystick, under the craft, and measured the rotation of the stick. This means that it only measured the actual yaw angle when both roll and pitch angles were zero. To get the actual yaw angle at all times the values from the joystick must be combined with the yaw sensor in some way.

### 3.1.2 Tethers with slack

One solution would be to connect the craft to tethers with slack so that it could move freely but a limited distance. The solution has several disadvantages making it less suitable as test rig at this time. Bombus is designed to have angle and velocity sensors mounted on the duct but these sensors are at this time not available. The sensors currently used must be placed on the test rig. Mounting high precision sensors on a test rig based on tethers with slack is close to impossible. Another disadvantage is the relative lack of safety. The craft would in this test rig be loosely tethered which puts high demands on the control system and the rudder performance. The tethered test rig also has some advantages like a natural test situation.

### 3.1.3 Gimbal

Another solution is to place the craft in the center of a gimbal with the gimbal’s axes through the center of mass of the craft. In a gimbal construction there are three rotations normally made possible by two or three rings. In this case two rings will be used. The first rotation is the outer ring rotating around one inertial axis. The second rotation is the inner ring rotating around the axle rotating with the
outer ring. The last rotation is the craft rotating around the axle rotating with the inner ring. All three rotations are perpendicular to the rotation closest in order. This construction will allow the craft to rotate freely around all its axes and still be enough tied down to avoid crashing or flying away if the control system fails to keep it stable. The gimbal construction has been used in this way by NASA to train astronauts [7]. The astronauts were strapped were the craft is planned to be placed and then rotated around all axes to simulate, among other things, a space craft spin. The only clear drawback with the gimbal test rig is that at one point the number of degrees of freedom is two, instead of three, occuring when the two rings are aligned. Hopefully this will not cause any problems since the control system should prevent the craft from turning to large angles.

Sensors in the Gimbal Test Rig

If all the rings in the gimbal are horizontal there should not be a problem to measure lift capacity. The construction can be done in a way that allows the craft to move vertically within a boundary, when that is required (for example, the axle can be attached to the craft in vertical tracks allowing the craft to move freely when it is not fixed). With the gimbal rings fixed the lift capacity can be measured with for example a Newton meter or the existing lift capacity sensor.

There is no easy way to measure the roll, pitch and yaw angles directly for a craft mounted in a gimbal. However, there is no problem measuring the rotations of the gimbal rings using for example potentiometers measuring the rotation about the three axes. The three dimensions of rotation of the craft can be described as a quaternion calculated from the gimbal angles [13]. From the quaternion the body fixed angles are obtained as

\[
\begin{pmatrix}
\text{roll} \\
\text{pitch} \\
\text{yaw}
\end{pmatrix} = \begin{pmatrix}
\arctan \left( \frac{2(q_0 q_1 + q_2 q_3)}{q_0^2 - q_1^2 - q_2^2 + q_3^2} \right) \\
\arcsin \left( \frac{2(q_0 q_2 - q_1 q_3)}{q_0^2 + q_1^2 - q_2^2 - q_3^2} \right) \\
\arctan \left( \frac{2(q_0 q_3 + q_1 q_2)}{q_0^2 + q_1^2 + q_2^2 + q_3^2} \right)
\end{pmatrix}
\] (3.1)

where \(q_0, q_1, q_2\) and \(q_3\) represents the coefficients in the quaternion \(Q = q_0 + iq_1 + jq_2 + kq_3\) [22]. Appendix A gives a detailed description of how to get the roll, pitch and yaw angles from gimbal angles using quaternions.

3.2 Building Test Rig Beta

The approach chosen in this project is the gimbal because it is fairly easy to manufacture and allows a relative large number of degrees of freedom. It will also allow the craft to rotate around its center of gravity. The craft is placed with its center of mass in the center of two rings, the outer somewhat larger than the inner. The outer ring is attached to the basic structure from Test Rig Alpha, the inner ring is attached to the outer ring on the top and on the bottom and the craft is attached to the inner ring, see Figure 3.2. This allows the craft to rotate around all axes and thereby simulate the conditions of free flight.
3.2.1 Sensors in Test Rig Beta

The new rotation sensors are placed on the axle between the outer ring and the basic structure and on the axle between the inner ring and the craft. These sensors are ordinary rotational potentiometers with low rotational friction. On the axle between the outer and inner ring a high precision digital rotation sensor used as yaw sensor is placed. This sensor is the same sensor used as yaw sensor in Test Rig Alpha. The current sensor is left unmodified and the sensor for measuring the lift capability is only slightly modified to fit the new rig.
Chapter 4

Lift capacity

One of the most important characteristics of an UAV is its lift capacity. Without sufficient lift capacity, the UAV will not be able to lift any equipment. Bombus was designed primarily for surveillance which often means that a camera has to be lifted. In addition to this, Bombus has to lift the tethering cable, including both power supply cable and possibly also a data cable.

4.1 Optimal Lift

The lift calculated in this section is optimal in that sense that it only considers the area of the propeller disc and the power of the engine and not the design of the propeller blades. This corresponds to the lift generated by an optimal propeller.

4.1.1 Initial Calculations

These calculations are based on basic aerodynamics and fluid mechanics. A fundamental rule is that the mass flow into the control volume in Figure 4.1 [14] must be the same as the mass flow out of it:

$$\oint_S \rho \mathbf{V} \cdot d\mathbf{S} = 0$$

where \( \mathbf{V} \) is the local velocity of the air, \( \rho \) is the density of air and \( S \) is the surface enclosing the propeller in Figure 4.1, including \( A_0 \) and \( A_\infty \). In the following calculations \( \mathbf{V} = |\mathbf{V}| \cdot (0, 0, 1)^T \) since the air speed can be assumed to have only a component downwards. The interesting property to calculate is the lift in hover, meaning that \( V_C \) in Figure 4.1 is zero. Some other assumptions are made to simplify the calculations: the fluid (here the gas air) is incompressible, inviscid, one dimensional and the flow is in a quasi-steady state. The assumption that the air is incompressible results in a constant air density. This assumption can be made since the air can move free and will not be compressed against anything. The assumption that the fluid is one dimensional means that the properties of the
fluid is constant along any plane parallel to the rotor disc and only changes along the axis perpendicular to the rotor disc. In Figure 4.1 the air passing through the propeller is assumed to pass only through $A_0$ and $A_\infty$ and not through any other part of $S$.

![Airflow through the rotor of a helicopter.](image)

Because of the steady state the mass flow in through the area at $A_2$ must be the same as the flow out of the surface at $A_\infty$:

$$\int\int_{A_\infty} \rho \mathbf{V} \cdot d\mathbf{S} = \int\int_{A_2} \rho \mathbf{V} \cdot d\mathbf{S} = \dot{m}$$  \hspace{1cm} (4.2)$$

where $\dot{m}$ is the mass flow rate through either of the surfaces. The difference in rate of change of momentum between the surfaces $A_2$ and $A_\infty$ equals the lift force, $T$, produced by the rotor.

$$T \dot{z} = \int\int_{A_\infty} \rho \mathbf{V} \cdot d\mathbf{S} \mathbf{V} - \int\int_{A_0} \rho \mathbf{V} \cdot d\mathbf{S} \mathbf{V}$$  \hspace{1cm} (4.3)$$

Since this calculation assumes hover $\mathbf{V} = V_c \dot{z} = 0$ at $A_0$, (4.3) becomes

$$T \dot{z} = \int\int_{A_\infty} \rho (\mathbf{V} \cdot d\mathbf{S}) \mathbf{V} = \dot{m} \mathbf{V}$$  \hspace{1cm} (4.4)$$

where $\mathbf{V}$ is as before the local air speed. The mass flow rate is

$$\dot{m} = \rho A_\infty v_c \Rightarrow$$
4.1 Optimal Lift

\[ T\dot{z} = \dot{m}V = \rho A_\infty v_e V = \rho A_\infty v_e^2 \dot{z} = \dot{m}v_e \dot{z} \quad (4.5) \]

\( A_\infty \) and \( v_e \) are the area and air velocity of the control surface at \( A_\infty \), respectively. The power of the rotor is calculated as \( P = T|V| \) giving \( P = T v_i \) at the propeller.

\[ P = T v_i = \iint_{A_\infty} \frac{1}{2} \rho (V \cdot dS)V^2 - \iint_{A_0} \frac{1}{2} \rho (V \cdot dS)V^2 \quad (4.6) \]

Like above the last integral in (4.6) is zero, because the rotor is in hover. This, together with (4.5), gives

\[ P = T v_i = \iint_{A_\infty} \frac{1}{2} \rho (V \cdot dS)V^2 = \frac{1}{2} \dot{m}v_e^3 = \frac{1}{2} T v_e \Rightarrow v_i = \frac{1}{2} v_e \quad (4.7) \]

Because of the symmetry in Figure 4.1, \( \rho A_\infty w = \rho A v_i \). This combined with the result in (4.7) gives

\[ \rho A_\infty w = \rho A v_i \Rightarrow A_\infty w = A \frac{1}{2} w = 2A_\infty = A \quad (4.8) \]

Since \( A = \pi R^2 \) it follows that \( T = 2\rho \pi R^2 v_i^2 \) and \( P = 2\rho \pi R^2 v_i^3 \)

4.1.2 Calculating the Lift

The rotor disc is the two dimensional surface the air has to pass through to pass the rotor. This area is \( A = \pi R^2 \), where \( R \) is the radius of the rotor. Since the motor mounted on Bombus is an electric motor the power is calculated from \( P = UIe \) where \( U \) is the voltage, \( I \) is the current and \( e \) is the efficiency of the motor. The previous section resulted in

\[ T = 2\rho \pi R^2 v_i^2 \quad (4.9) \]

\[ P = 2\rho \pi R^2 v_i^3 \quad (4.10) \]

Calculating \( v_i^2 \) from (4.10) gives

\[ v_i^2 = \left( \frac{P}{2\rho \pi R^2} \right)^\frac{2}{3} \quad (4.11) \]

Inserting (4.11) in (4.9) results in an expression for the lift, \( T \), independent of \( v_i \)

\[ T = 2\rho \pi R^2 \left( \frac{P}{2\rho \pi R^2} \right)^\frac{2}{3} = (2\rho R^2 P^2)^\frac{1}{3} \quad (4.12) \]

This result allows the calculation of \( v_i \) which can be useful when calculating for example the effect of the guiding vanes and the effect of the rudders used for steering.
4.2 Theoretical Lift with Consideration to the Propeller

One commonly used method to mathematically describe the characteristics of a propeller is the Blade Element Theory (BET). BET divides the propeller blade into a infinite number of sections, the characteristics of each section is calculated and finally all sections are added to get the characteristics of the entire blade.

All the interesting definitions can be seen in Figure 4.2 [14]. There \(dF_L\) and \(dF_D\) are lift and drag forces from the section of the blade. The resulting forces in a cartesian reference frame is denoted \(dF_x\) and \(dF_z\). The variable \(U_T = \omega y\) is the velocity of the blade in the direction of the rotation, while \(U_P = V_c + v_i\) is the wind velocity (\(V_c\) is the wind velocity caused by the upward motion and \(v_i\) is the wind velocity induced by the propeller) and \(U_R\) is the radial velocity induced by the blade. Vector summation gives \(U = \sqrt{U_T^2 + U_P^2}\), assuming \(U_R = 0\). The resulting lift and drag for each section are

\[
dF_L = \frac{1}{2} \rho U^2 c C_L dy \quad dF_D = \frac{1}{2} \rho U^2 c C_D dy
\]

(4.13)

where \(\rho\) is the air density and \(C_L\) and \(C_D\) are lift and drag coefficients. The geometry in Figure 4.2 gives

\[
dF_x = dF_L \sin(\phi) + dF_D \cos(\phi) \quad dF_z = dF_L \cos(\phi) - dF_D \sin(\phi)
\]

(4.14)

To simplify the calculations some assumptions can be made.

- \(V_c\) is zero since all calculations are in hover. This means that \(U_P = v_i\).
- \(U_P\) is usually much smaller than \(U_T\) which gives the approximation \(U = U_T\).
- The drag is much smaller than the lift so \(dF_D \sin(\phi)\) can often be neglected.
4.2 Theoretical Lift with Consideration to the Propeller

The thrust from a propeller with \( N_b \) blades is the \( z \) part of the force in (4.14) times the number of blades. Applying the assumptions above on this gives

\[
\dot{T} = N_b dF_z \approx N_b dF_L \cos (\phi) \approx N_b dF_L \tag{4.15}
\]

Equations (4.13) and (4.15) combined will result in

\[
\dot{T} = \frac{1}{2} N_b \rho U^2 c C_L \, dy \tag{4.16}
\]

Integrating (4.16) gives

\[
T = \int_{r_a}^{r_1} \frac{1}{2} N_b \rho U^2 c C_L \, dy \tag{4.17}
\]

The calculations above results in an expression for the thrust, or the total lift of the rotor. Some future calculations will also need an expression for the total drag of the rotor. This can easily be obtained by replacing \( C_L \) in (4.16) with \( C_D \). This gives

\[
D_T = \int_{r_a}^{r_1} \frac{1}{2} N_b \rho U^2 c C_D \, dy \tag{4.18}
\]

4.2.1 Obtaining \( C_L \)

Unfortunately not only \( U^2 = \omega_p y \) depends on \( y \) but \( C_L \) depends on the angle of attack, \( \alpha \), which in turn depends on \( y \) (and \( v_i \)). The coefficient \( C_L \)'s dependence on \( y \) exists because the propeller blades on Bombus are twisted. There are difficulties calculating \( C_L \), especially for the relative high angles of attack existing on the Bombus propeller. The American National Advisory Committee for Aeronautics (NACA), the predecessor to NASA, made wind tunnel tests on a large number of different airfoils [18]. These airfoils were initially intended for airplane wings but since propeller blades are based on the same basic principle, the tests can be used here too. The difference in \( C_L \) between different but similar airfoils is small and therefore it is not critical in this project which of the airfoils tested by NACA is used, as long as it is not to far away from the real airfoil [10].

To get good values of \( C_L \) for the different angles of attack on the blade, the test result in [15] is approximated with a polynomial of order 2 around the interesting angles.

\[
C_L(\alpha) = c_{L0} \alpha^2 + c_{L1} \alpha + c_{L2} \tag{4.19}
\]

where \( c_{L0} - c_{L2} \) are constants calculated from the result in [15]. To be able to use this in (4.17) \( C_L(\alpha(y)) \) must be available. To obtain \( C_L \) the angle \( \alpha \) is defined as a function of \( y \) and inserted in (4.19). According to Figure 4.2, \( \alpha = \theta - \phi \) where \( \theta \) is approximately changing with \( y \) according to (4.21).

\[
\theta = \theta_0 + \theta_1 e^{-\left(\frac{\theta_2 y + \theta_3}{\theta_4}\right)} \tag{4.20}
\]

This gives a two dimensional mathematical model of the propeller blade. An illustration of this model is shown in Figure 4.3.
The variables $\theta_0 - \theta_3$ in (4.20) are constants calculated from measurements on the propeller. To get a good approximation of $\phi_\alpha$, $v_i$ can be approximated as $v_i = \lambda_h \omega_y$. This fact together with the relation $U_T = \omega_y$ and geometry from Figure 4.2 gives

$$\phi_\alpha = \arctan \lambda_h$$

Even though $\lambda_h$ at this time is unknown it can be written as $\lambda_h = \sqrt{\frac{C_T}{T}}$. This does not help since $C_T$ has not yet been calculated and depends on $T$. To solve the problem all calculations have to be iterated and will then converge to the correct value.

With help from (4.19), $\alpha = \theta_\alpha - \phi_\alpha$, (4.20) and (4.21), the lift can be calculated.

$$T = \frac{1}{2} N_b \rho \omega_p^2 C \int_{r_0}^{r_1} y^2 C_L dy =$$

$$= \frac{1}{2} N_b \rho \omega_p^2 C \int_{r_0}^{r_1} y^2 \left( c_{L0}(\theta_0 + \theta_1 e^{-\theta_2 y + \theta_3}) - \arctan(\lambda_h) \right)^2 +$$

$$+ c_{L1}(\theta_0 + \theta_1 e^{-\theta_2 y + \theta_3}) - \arctan(\lambda_h)) + c_{L2}) dy$$

(4.22)

To calculate the total drag $C_D$ is used instead of $C_L$. The NACA wind tunnel tests also gave values for $C_D$ which can be approximated by a linear function for the interesting angles. This means that $C_D(\alpha) = l_1 \alpha + l_2$. Using this in (4.20) – (4.22) gives an expression for the total drag.
4.2.2 Theoretical Lift Capacity of the Duct

The duct itself can actually increase the lift capacity of the craft. This is partly due to the fact that the design of the duct can accelerate the air through the propeller adding to the upward force. The duct also allows the propeller to maintain its lift all the way out to the tip of the blade. An open propeller will lose some of its lift due to vortices at the tips.

There are two characteristics in a duct affecting its ability to accelerate the air through the propeller and thereby increasing the lift of the duct [16]. The angle of convergence is the first one. Positive angle of convergence means that the air is diffused when its entering the duct and is converged when leaving the duct. This decreases the lift. Negative convergence is of course the other way around. With zero convergence, which is the case on Bombus, the air is neither converged nor diffused in the inlet and the outlet. This means that the angle of convergence on Bombus will not have any effect on the lift capacity. The second characteristic of the duct that affects its lift capacity is the duct’s camber angle. The camber angle of a duct is the same as the camber angle of a wing if we see the cross section of the duct as a cross section of a wing. Both angles can be seen in Figure 4.4. Bombus has no significant camber angle so this will not effect its lift capacity.

![Figure 4.4. Angle of convergence and camber angle.](image)

4.3 Lift Experiments

In Test Rig Beta it is not as trivial to test the lift capacity of Bombus as it was in Test Rig Alpha. Bombus has to be disconnected from the gimbal and from the pitch and roll sensors and be tightly tethered and connected to the force sensor (the same sensor that measured lift in Test Rig Alpha). The tethers keep the craft from falling down while the connection to the force sensor keeps it from lifting off during the experiment.

During the implementation of the experiment, the angular velocity of the rotor was increased in small steps and the output from the force sensor and the angular velocity of the propeller were logged.
4.4 Comparing Theory and Experiments

In the sections above, the theoretical lift of the craft has been calculated in two ways. To be able to decide which of the two theoretical lifts to implement in the mathematical model both theoretical results have to be compared to experimental results. The result of the lift experiments compared to the theoretical results are shown in Figure 4.5. The dots are the result from the experiments, the dashed line is the optimal lift and the solid line is the theoretical lift with consideration to the propeller. The reason that the experimental results show a higher but constant lift for low engine speeds is that the lift force sensor only registers forces larger than the force required for the craft to take off.

4.4.1 Comparison for Optimal Lift

The calculations of the optimal lift resulted in a larger lift capacity than the measured one. There can be several reasons for this difference, like the optimal propeller in the calculations is better than the propeller on Bombus in many ways. For example, the theoretical propeller does not have the limitation of a finite number of blades. Despite the limitations, the difference between the theoretical and measured lift capacity is not very large which indicates that the propeller on Bombus, in combination with the duct, is rather effective.

![Figure 4.5. Lift force. Experimental results as dots. Optimal lift force as a dashed line and theoretical lift force with consideration to the propeller as a solid line.](image)

4.4.2 Comparison for Lift with Consideration to the Propeller

The theoretical lift is just like in the case of optimal lift, somewhat larger than the experimental results. This difference is due to several reasons. For example,
the tools used to measure sizes and angles are rather poor and have a resolution of about 0.005 m. Other things that may affect the experimental lift is drag due to angle of attack of the rudders and other things in the airflow’s way. In Figure 4.5, the lines representing the theoretical results show a much smaller lift for low engine speeds. This is due to the force sensor and the problem is explained in the beginning of Section 4.4.

4.5 Alternative Engines

The electric motor used in Bombus is a brushless electric DC motor intended primarily for model aircraft. A motor better suited for Bombus must above all have a higher power to weight ratio. The Bombus motor has a maximum power to weight ratio of about 1.5 \( \frac{kW}{kg} \). Many of the more powerful electric motors have significant higher power to weight ratio but they need a higher voltage to operate which puts high demands on the power supply arrangement. One alternative is to use electric AC motors. These also put high demands on the power supply units but can have a power to weight ratio of over 2. The high performance AC motors are large and heavy and are therefore not suited for Bombus.

Another alternative is a combustion engine which can have a power to weight ratio higher than 2 but which have some other drawbacks. Combustion engines use some kind of fuel which must be lifted by the craft. This fact will limit the flight time to that allowed by the fuel tank and reducing the crafts ability to carry equipment other than fuel. With an electric engine the craft can potentially have a infinite flight time if the power cable is connected to an infinite power source.
Chapter 5

Rudder Performance

An important quality needed for an UAV like Bombus is the ability to control tilting in any direction, i.e., roll or pitch, while airborne. If Bombus should tilt during flight it would cause it to change the direction of the force induced by the propeller. This would not only cause the craft to lose lift force but it would also result in a horizontal force. This is not desirable unless the craft is supposed to move in a horizontal direction or if there is an external force, caused by for example wind, to be counteracted.

Bombus should also be able to avoid rotating around the vertical axis, yaw rotation. The propeller blades do not only generate vertical "lift" forces but also horizontal "drag" forces. These forces induces a torque which should result in a yaw rotation of the craft. This rotation would be contrary to the propeller’s rotation and cause the propeller to rotate slower in the inertial frame. This would lead to reduced lift force due to lower velocity of the propeller blades. See Chapter 4.

Bombus is equipped with eight rudders assembled to the craft like a star covering the outlet of the duct. These rudders are fixed in a small angle to generate tangential forces counteracting the torque from the propeller blades drag forces. These rudders will be referred to as the guiding vanes.

Bombus is also equipped with four controllable rudders. These rudders are placed under the craft. The controllable rudders are oriented in pairs at a right angle to eachother, like a cross. This means that there are two rudders for controlling roll and two for controlling pitch. Obviously all four rudders can be used together to generate tangential forces in a similar manner as the guiding vanes.

5.1 Theoretical Force Generated by a Rudder

The theories of rudder forces is the same theories as used in Chapter 4 but simpler in some aspects. The rudders are not twisted at all, that is, the angle of attack is constant over the rudders when the craft is not rotating. The craft not rotating can be compared to hovering in lift force calculations while rotation can be compared to climb or descent operations. When this occurs the rudders can be divided into
a infinite number of sections, all behaving like wings climbing or descending with a velocity proportional to the distance to the rotation center.

Figure 5.1. Forces acting on an infinitesimal part of a rudder. (Observe similarities with Figure 4.2.)

5.1.1 Air Flow at the Rudders

In Figure 5.1, $U_P$ is the out-of-plane component and will in these discussions only contain the velocity due to the craft’s rotation around its vertical axis.

$$U_P = rr_w$$  \hspace{1cm} (5.1)

where $r$ is the rudder element’s distance to the rotation axis and $r_w$ is the crafts angular velocity around the same. Looking at Figure 5.1 gives with equation (5.1) the resultant velocity at the rudder element.

$$U = \sqrt{U_T^2 + U_P^2} = \sqrt{v_i^2 + r^2 r_w^2}$$  \hspace{1cm} (5.2)

where $U_T$ is the in-plane component which is known from earlier as the velocity of the wind induced by the propeller, $v_i$. From (4.9), $v_i$ can be calculated as

$$v_i = \sqrt{\frac{T}{2\rho A}}$$  \hspace{1cm} (5.3)

where $T$ is the thrust from the propeller. In the model $T$ is calculated with consideration to the propeller like in Section 4.2. The induced wind velocity calculated in (5.3) is the wind an optimal propeller would induce while generating the lift $T$. 
5.1 Theoretical Force Generated by a Rudder

5.1.2 Rudder Angles

The variable $\theta_\alpha$ represents the rudders angle relative the vertical plane. The effective angle of attack is denoted $\alpha$. As seen in Figure 5.1 the relation is

$$ \alpha = \theta_\alpha - \phi_\alpha $$ (5.4)

where

$$ \phi_\alpha = \tan^{-1} \left( \frac{U_P}{U_T} \right) $$ (5.5)

Here $U_P$ is expected to be small compared to $v_i$ which lets (5.5) be approximated as

$$ \phi_\alpha = \frac{U_P}{U_T} $$ (5.6)

Combining (5.1), (5.4) and (5.6) gives an expression for the effective angle of attack.

$$ \alpha = \theta_\alpha - \frac{r_w r}{v_i} $$ (5.7)

5.1.3 Rudder Forces using Blade Element Theory

In analogy with Chapter 4 (see in particular (4.13)), the resulting lift and drag for each section of the rudder are

$$ dF_L = \frac{1}{2} \rho U^2 c C_L dr \quad dF_D = \frac{1}{2} \rho U^2 c C_D dr $$ (5.8)

using the rather awkward notation $F_L$ (lift) for the force pulling the rudder sideways and $F_D$ (drag) for the force working more close to vertical than horizontal. This notation is used because of the metaphoric image of the rudder element as a wing. In (5.8) $C_L$ and $C_D$ are the rudder’s lift and drag coefficients as a wing. The parameter $\rho$ is the air density and $c$ is the cord of the rudder. Again in analogy with Chapter 4 looking at Figure 5.1 gives the horizontal and vertical components of the force working at the rudder section.

$$ dF_z = dF_L \sin(\phi_\alpha) - dF_D \cos(\phi_\alpha) \quad dF_x = dF_L \cos(\phi_\alpha) - dF_D \sin(\phi_\alpha) $$ (5.9)

Combining this with (5.2), (5.6) and (5.8) gives

$$ dF_z = \frac{1}{2} \rho c [v_i^2 + r_w^2] \left[ C_L \sin \left( \frac{r_w}{v_i} \right) - C_D \cos \left( \frac{r_w}{v_i} \right) \right] dr $$ (5.10)

$$ dF_x = \frac{1}{2} \rho c [v_i^2 + r_w^2] \left[ C_L \cos \left( \frac{r_w}{v_i} \right) - C_D \sin \left( \frac{r_w}{v_i} \right) \right] dr $$ (5.11)

$C_L$ and $C_D$ can for the interesting angles be approximated as

$$ C_L = c_{L0} \alpha = c_{L0} \left( \theta_\alpha - \frac{r_w}{v_i} \right) $$ (5.12)
and

\[ C_D = cD_0 + cD_1 \alpha + cD_2 \alpha^2 = cD_0 + cD_1 \left( \theta_\alpha - \frac{r r_w}{v_i} \right) + cD_2 \left( \theta_\alpha - \frac{r r_w}{v_i} \right)^2 \] (5.13)

Now \( dF_z \) and \( dF_x \) can be integrated over the rudder. With no rotation, \( r = 0 \), in particular

\[ dF_z = \frac{1}{2} \rho c v_i^2 \left[ C_L \sin(0) - C_D \cos(0) \right] dr \Rightarrow F_z = \frac{1}{2} \rho c v_i^2 (-C_D) \int_{r_0}^{r_1} dr \] (5.14)

\[ dF_x = \frac{1}{2} \rho c v_i^2 \left[ C_L \cos(0) - C_D \sin(0) \right] dr \Rightarrow F_x = \frac{1}{2} \rho c v_i^2 C_L \int_{r_0}^{r_1} dr \] (5.15)

According to [8] the force perpendicular to the wind attacking the rudder is

\[ F_L = \frac{1}{2} \rho c v_i^2 S C_L(\alpha) \] (5.16)

which without rotation and when \( S \) is the area of the rudder equals \( F_x \) in (5.15).

### 5.2 Theoretical Torque Generated by Rudders

This section discusses the theoretical torque generated by rudders.

#### 5.2.1 Guiding Vanes

Every guiding vane generates a torque acting on the vertical central axis of the craft. From (5.15) the torque generated from every rudder section can be calculated as

\[ dM_r = rdF_x = \frac{1}{2} \rho c v_i^2 C_L r dr \Rightarrow M_r = \frac{1}{4} \rho c v_i^2 C_L (r_1^2 - r_0^2) \] (5.17)

The total torque from all guiding vanes is supposed to counteract the torque generated from the propellers drag forces. This torque can be calculated using (4.18).

\[ M = \frac{1}{2} N_b \rho \omega_p^2 c_{prop} \int_{\text{blade}} y^3 C_{D, prop} dy \] (5.18)

where \( \omega_p \) is the angular velocity of the fan. This gives the equation

\[ N_b \frac{1}{4} \rho c v_i^2 C_L(\alpha_{opt})(r_1^2 - r_0^2) = \frac{1}{2} N_b \rho \omega_p^2 c_{prop} \int_{\text{blade}} y^3 C_{D, prop} dy \] (5.19)

where \( \alpha_{opt} \) is the angle of attack that generates a total torque from the guiding vanes with the same size as the torque generated by the propellers drag forces.
5.3 Rudder Experiments

Looking at (5.3) gives $v_i^2 = \frac{T}{\frac{1}{2} \rho A}$ and as seen in (4.22) the thrust from the propeller $T \sim \omega_p^2$. This means that $C_L(\alpha_{opt})$ is independent of $\omega_p$. In (5.12) we see that this means that $\alpha_{opt}$ is independent of the propellers angular velocity as well (actual angle of the rudder is the same as effective angle of attack due to no yaw rotation). This angle turns out to be small enough not to effect the lifting force noticeably. In more detailed discussions $C_D$ in (5.14) is small for the optimal angle of attack.

5.2.2 Controllable Rudders

The purpose of the controllable rudders is to give the control system ability to control torques at all three axes running through the rotational center of the craft. The torque at the vertical axis can be calculated in analogy with (5.17).

$$M_{z, \text{rudder}} = \frac{1}{4} \rho c v_i^2 C_L (r_1^2 - r_0^2) \quad (5.20)$$

Every rudder generates torques at both horizontal axes. The $x$ component of the rudder force generates a torque at the axis parallel with the rudder’s shaft.

$$dM_{||, \text{rudder}} = \frac{1}{2} \rho c v_i^2 C_L h dr \Rightarrow M_{||, \text{rudder}} = \frac{1}{2} \rho c v_i^2 C_L h (r_1 - r_0) \quad (5.21)$$

where $h$ is the distance from the rudder’s aerodynamic centrum to the axis running through the rotational center of the craft. This is the only difference from (5.15) and (5.21) can be rewritten as

$$M_{||, \text{rudder}} = h F_x \quad (5.22)$$

The $z$-component on the other hand generates a torque at the axis perpendicular to the rudder’s shaft. Every rudder section’s force in $z$ direction from (5.14) have the distance $r$ as lever, giving

$$dM_{\perp, \text{rudder}} = \frac{1}{4} \rho c v_i^2 (-C_D) r dr \Rightarrow M_{\perp, \text{rudder}} = \frac{1}{4} \rho c v_i^2 (-C_D) (r_1^2 - r_0^2) \quad (5.23)$$

But using $r_1^2 - r_0^2 = (r_1 - r_0)(r_1 + r_0)$ and (5.14) one realise that the torque is the rudder’s vertical force with the lever $(r_1 + r_0)/2$.

$$M_{\perp, \text{rudder}} = \frac{1}{2}(r_1 + r_0) F_z \quad (5.24)$$

5.3 Rudder Experiments

This section describes the experiments made to validate the theoretical torque generated by the rudders.

5.3.1 Testing performance of guiding vanes

The guiding vanes are fixed at the optimal angle obtained in (5.19). There are no advanced experiments made to test this configuration. However, it has been
obvious in all other experiments that these rudders serve their purpose. There has been no rotation around the body fixed $z$-axis other than when changing angular velocity of the fan. This rotation is caused by gyral torque (see Section 7.3.3 on page 44).

5.3.2 Testing Performance of Controllable Rudders

The controllable rudder’s main task is to control the torque about the $x$ and $y$ axes which will be referred to as tilt torque.

Measuring Tilt Torque in Test Rig Beta

To understand this section the reader should have read Chapter 3 thoroughly. A method for measure the torque around one of the horizontal axes was sought. This was obtained by locking the gimbal rings in Test Rig Beta to their initial position. In this way the craft was only allowed to rotate around the axis in which it is assembled to the inner gimbal ring. The sensor earlier used to measure lift forces was repositioned to be able to measure the vertical force in one leg of the craft. This force levered with the radius of the duct is the torque at the vertical axis. After some testing it was obvious that the sensor reacted when changing fan speed even if the rudders were in their initial position. Because of the lack of stiffness in the rig the sensor also measured lift force. The rig was tied down in the craft attachments to avoid this to a certain degree. An illustration of the test rig with these modification is showed in Figure 5.2.

Figure 5.2. Illustration of Test Rig Beta modified for measuring torque about the $y$-axis.
5.3 Rudder Experiments

**Experiment to obtain Characteristics of $C_L$**

In the theories used, $C_L$ is linear for angles smaller than an optimum (see (5.12)). To test these theories the craft’s fan is rotating at a constant speed enough for hovering according to Chapter 4. The angles of the two rudders generating the measurable torque are varied from a negative angle to a maximal angle far greater than the theoretical optimum. A positive angle at these rudders generates a negative torque about the body fixed $y$-axis. In Figure 5.3 the torque is plotted versus angle of the rudders. The measurement of the torque is rather noisy and it is hard to decide whether the relation actually is linear for small angles. However the experiments reveal an optimum that agree with the theories. The noise is probably caused by vortices in the air at the rudders and a rather bad measuring method. The noise will be considered when modeling the rudders in Chapter 7.

![Torque vs. Angle of Attack](image)

**Figure 5.3.** Torque with two rudders varying angle and fan at constant angular velocity.

**Varying propeller speed to verify torque model**

To validate the model of the torque generated by the two rudder’s perpendicular to the torque’s axis the rudders are fixed at the optimal angle. The torque is measured at different propeller speeds. Figure 5.4 shows the result from the experiment and the simulation in the model. The propeller is equipped with two magnets opposing each other and the fan speed sensor measure the time between one magnet’s passing to next (half period time). Due to a technical issue the sensor did not always register a magnet passing, resulting in a way too low measured value of the propeller speed. This explains all the dots in Figure 5.4 at low engine speed but high torque. The reader should ignore these values. Furthermore the experimental torque seems to be quite low and rather noisy.
Figure 5.4. Torque with two rudders fixed at angle generating most horizontal force. Dots are experimental while the solid line is theoretical torque. The measured torque far smaller than the theoretical torque for low engine speed is due to a poorly functioning rotational sensor.
Chapter 6

Noise

Bombus is, as almost every physical system, affected by some kind of noise. The sources of the noise are often many but in most cases just a few of them affect the system enough to be needed in a mathematical model. Weather phenomenon, like wind and rain, effects from the ground, like a pull on the tether, wind vortices below the fan and electrical disturbance in wires and control units can all be regarded as sources of noise in this project.

6.1 Weather

The largest source of noise is the natural wind. Therefore, a mathematical model of the wind will be inserted into the model of the system. The wind will be modeled as a wind speed from a certain angle. The angle will only be in the $xy$-plane of the inertial reference frame. This approximation means that the model cannot describe wind from above or below. The model of the wind speed will have a certain mean value and then low pass filtered white Gaussian noise with mean value zero will be added to it. This results in a more nature like wind compared to a model with constant wind speed. Figure 6.1 shows the noise wind and the different angles. Here $\beta$ is the angle from which the wind arrives, $\varphi$ is the angle to the rudder considered in the calculation and $v_w$ is the speed of the noise wind. Both angles equals zero at $\hat{x}$. Denote the rudders with numbers, $j = 1 \ldots 4$. Rudder $j = 1$ is the rudder at $\beta = 0$. This gives an angle to each rudder, $\varphi_j = (j - 1)\pi/2$. The noise wind at each rudder becomes

$$v_{w,j} = v_w \cos (\beta - (j - 1)\pi/2) \quad (6.1)$$

The noise wind used in the simulations can be seen in Figure 6.2.

Noise sources such as rain and snow will not be modeled since they only occur in rare situations.
Figure 6.1. Model of the noise wind.

Figure 6.2. Noise wind. The distance from the origin represents the magnitude of the wind and the angle represents the angle of attack.

6.2 Vortices

When the fan rotates vortices are created in the air below. These vortices will affect the wind passing the rudders. This will decrease the rudders ability to produce a force. The vortices can easily be modeled by changing the wind velocity at the rudders in the model. By adding white Gaussian noise with mean value zero,
the vortices can be modeled in a natural way that fits the experimental results well. Figure 6.3 shows the effect the modeled noise have on the rudder torque experiments in Chapter 5. The dots are measurements done on the craft and the line is the modeled torque with white Gaussian noise added to the wind speed. If no noise was modeled the simulation would only generate a sloping, nonoscillating line (see Figure 5.4). It is clear that the modeled noise improve the model.

![Figure 6.3](image)

**Figure 6.3.** Torque with two rudders fixed at the angle generating most horizontal force. Dots are experimental torque while the solid line is theoretical torque with noise added to windflow at all rudders. The measured torque far smaller than the theoretical torque for low engine speed is due to a poorly functioning rotational sensor.

### 6.3 Air Loss

The air moved downward by the fan can not escape from the duct except at the bottom. When the air reaches the bottom it will no longer continue in the same direction as before. Instead the air will disperse lowering the air velocity at the rudders and changing the angle of attack. The angle change is small enough to be neglected but the change in air velocity is not. The model of this noise is only a decrease of the wind speed at the rudders.

This is a noise which easily can be removed in the physical situation. If the duct is made longer, so that it covers the rudders, the decrease in wind speed will be much smaller. The extension of the duct must not be too long because then it will disturb the air that has just passed over the rudders, but if it is made the correct length it will increase the rudder’s efficiency.
Chapter 7

Theoretical Model

To be able to do simulations and create a control system, a mathematical model of the craft is needed. Creating this model is no trivial task since the total system that has do be modeled is rather large and complicated. Usually this means that it is preferable to use an existing model and modify it to suit the situation instead of building a model from scratch.

Usually, the model described in [18] is used as a simple model for conventional aircraft. This model is based on the equations

\[
X - mgS_{\theta} = m(\dot{u} + qw - rv) \\
Y + mgC_{\theta}S_{\phi} = m(\dot{v} + ru - pw) \\
Z + mgC_{\theta}C_{\phi} = m(\dot{w} + pv - qu)
\]

\[
L = I_x\dot{\phi} - I_{xz}\dot{\psi} + qr(I_z - I_y) - I_{xz}pq \\
M = I_y\dot{q} + rp(I_x - I_z) + I_{xz}(p^2 - r^2) \\
N = -I_{xz}\dot{\phi} + I_z\dot{\psi} + pq(I_y - I_x) + I_{xz}qr
\]

\[
p = \dot{\phi} - \dot{\psi}S_{\theta} \\
q = \dot{\theta}C_{\phi} + \dot{\psi}C_{\theta}S_{\phi} \\
r = \dot{\psi}C_{\theta}C_{\phi} - \dot{\theta}S_{\phi}
\]

\[
\dot{\theta} = qC_{\phi} - rS_{\phi} \\
\dot{\phi} = p + qS_{\phi}T_{\theta} + rC_{\phi}T_{\theta} \\
\dot{\psi} = (qS_{\phi} + rC_{\phi})(\frac{1}{C_{\theta}})
\]

where \(X, Y\) and \(Z\) are the forces, \(u, v\) and \(w\) are the velocities and \(p, q\) and \(r\) are the angular velocities in the directions \(\hat{x}, \hat{y}\) and \(\hat{z}\) in the body fixed reference.
Torques about \( \hat{x} \), \( \hat{y} \) and \( \hat{z} \) in the body fixed reference frame are denoted \( M, L \) and \( N \) respectively. The Euler angles are denoted \( \phi, \theta \) and \( \psi \), \( m \) is the total mass, \( g \) is the gravitational constant and \( C_\theta, S_\phi \) and \( T_\psi \) are \( \cos(\theta) \), \( \sin(\phi) \) and \( \tan(\psi) \) respectively. Finally, \( I_a \) is the moment of inertia about the axis \( a \) where \( a \) can be \( x \), \( y \) or \( z \) in the body fixed reference frame.

7.1 Forces; \( X, Y \) and \( Z \)

There are four kinds of forces acting on the craft. The first one is the gravity, the second one is the lift, the third one is the forces generated by the rudders and the final one is the drag generated by the rudders and the duct when the noise wind acts on them. The general equation for the forces are therefore

\[
X = X_{r_2,r_4} + X_{\text{duct drag}} + X_{\text{rudder drag}} \\
Y = Y_{r_1,r_3} + Y_{\text{duct drag}} + Y_{\text{rudder drag}} \\
Z = -T
\]

(7.2)

where \( X_{r_2,r_4} \) is the forces generated by the rudders, \( X_{\text{duct drag}} \) is the drag from the duct and \( X_{\text{rudder drag}} \) is the drag from the rudders. The reason for not including the gravitational force in the third equation in (7.2) is that it is already modeled in (7.1). \( X_{r_2,r_4} \) and \( Y_{r_1,r_3} \) can easily be obtained by calculating the force from each rudder, using equation (5.15), and then summarizing all forces acting in each direction.

The second term in (7.2) originates from the drag of the duct. According to [8] the drag of a structure can be calculated as

\[
X_{\text{duct drag}} = \frac{1}{2} \rho v^2 \cos(\beta) A_d C_{D,d}
\]

(7.3)

where \( A_d \) is the area of the duct facing the wind, \( C_{D,d} \) is the drag coefficient of the duct and \( \beta \) is the angle from the \( x \)-axis to the current propeller blade. According to [16] \( C_D \), of a smooth cylinder with the same dimensions as Bombus is about 1.

The final term in (7.2) is the drag force from two of the rudders. The equation for the drag of the rudders is the same as the equation for the drag of the duct (7.3), but the value of \( C_D \) is different and depends on the angle of attack of the rudder. An illustration of a rudder and the interesting angles can be seen in Figure 7.1. In the figure it is clear that \( \alpha_2 = \pi/2 - \theta_\alpha \) where \( \theta_\alpha \) is the angle of attack used to calculate the rudder forces in Chapter 5 and \( \alpha_2 \) is the angle of attack used in (7.4). Since \( \theta_\alpha \) is less or equal to the angle of attack that gives the largest lift, \( \theta_\alpha,\text{max} \), \( \alpha_2 \) is limited to \( \frac{\pi}{2} - \theta_\alpha,\text{max} \leq \alpha_2 \leq \frac{\pi}{2} + \theta_\alpha,\text{max} \). In this area \( C_D \) can be approximated by a polynomial of order 2. We get

\[
C_D,\text{r}(\alpha_2) = k_1 \alpha_2^2 + k_2 \alpha_2 + k_3 \Rightarrow \\
C_D,\text{r}(\theta_\alpha) = k_1(\pi/2 - \theta_\alpha)^2 + k_2(\pi/2 - \theta_\alpha) + k_3 = \\
k_1 \theta_\alpha^2 - (k_2 + k_1 \pi) \theta_\alpha + k_3 \frac{\pi^2}{4} + k_2 \frac{\pi}{2} + k_3
\]

(7.4)
This results in a force

\[ X_{\text{rudder drag}} = 2 \frac{1}{2} \rho v_w^2 \cos (\beta) A_r C_{D,r}(\theta_\alpha) \]  

(7.5)

where \( C_{D,r}(\theta_\alpha) \) is described in equation (7.4).

If the force from rudder \( x \) is denoted \( F_{r,x} \) the total rudder forces can be written as

\[
\begin{align*}
X_{r_2,r_4} &= -F_{r_2} - F_{r_4} \\
Y_{r_1,r_3} &= F_{r_1} - F_{r_3}
\end{align*}
\]  

(7.6)

The sign of the forces in (7.6) depend only on the definition of positive angle for each rudder as implemented in the software on the craft. The resulting forces becomes

\[
\begin{align*}
X &= -F_{r_2} - F_{r_4} + \frac{1}{2} \rho v_w^2 \cos (\beta) A_d C_{D,d} + 2 \frac{1}{2} \rho v_w^2 \cos (\beta) A_r C_{D,r}(\theta_\alpha) \\
Y &= F_{r_1} - F_{r_3} + \frac{1}{2} \rho v_w^2 \sin (\beta) A_d C_{D,d} + 2 \frac{1}{2} \rho v_w^2 \sin (\beta) A_r C_{D,r}(\theta_\alpha) \\
Z &= -T
\end{align*}
\]  

(7.7)

### 7.2 Moment of Inertia, \( I_\alpha \)

One important characteristic of the craft is its moment of inertia. The moment of inertia describes how hard it is to change the crafts velocity. It depends only on the weight and shape of the body and is calculated as
Theoretical Model

\[ I_a = \int r^2 dm = \int r^2 \rho d\nu \]
\[ I_{ab} = \int d_a d_b dm = \int d_a d_b \rho d\nu \]  \hspace{1cm} (7.8a)
\hspace{1cm} (7.8b)

where \( a \) is the axis about which the moment of inertia is calculated, \( \nu \) is the volume, \( \rho \) is the density of the material in the volume, \( m \) is the mass, \( r \) is the perpendicular distance to \( a \) and \( d_a \) is the distance in the direction of \( a \). This means in practice that the equations are

\[ I_x = \int (y^2 + z^2) \rho d\nu \quad I_{xy} = I_{yx} = \int xy \rho d\nu \]
\[ I_y = \int (x^2 + z^2) \rho d\nu \quad I_{xz} = I_{zx} = \int xz \rho d\nu \]
\[ I_z = \int (x^2 + y^2) \rho d\nu \quad I_{yz} = I_{zy} = \int yz \rho d\nu \]  \hspace{1cm} (7.9)

The total moment of inertia on matrix form is

\[ I = \begin{pmatrix}
I_x & I_{xy} & I_{xz} \\
I_{yx} & I_y & I_{yz} \\
I_{zx} & I_{zy} & I_z
\end{pmatrix} \]  \hspace{1cm} (7.10)

Most parts of the craft can be approximated by two or three dimensional bodies. This simplifies the calculations since both [19] and [20] has lists of different bodies and their moment of inertia. To get the total moment of inertia about one axis in the body fixed reference frame the moment of inertia for all the different parts about the axis are added.

### 7.3 Torques; \( M \), \( L \) and \( N \)

There are several factors affecting the total torque on Bombus. The rudders, described further in Section 5.2.2, will create torques about all three axis. Another factor affecting the total torque is the wind noise described in Chapter 6. This wind will have two different effects. First it will increase the lift on one side of the craft and decrease it on the other and second, it will generate a drag force from the rudders resulting in a torque. It will also create a drag force on the duct itself but since the center of gravity of the duct is placed in the center of the duct, this will not create any torque. One further factor affecting the total torque is the torque created by the gyroscopic effect of the propeller. The torques of the craft are

\[ L = L_{r2+r4} + L_{D(r1,r3,v_w)} + L_{\text{lift}} + L_{\text{gyro}} \]
\[ M = M_{r1+r3} + M_{D(r2,r4,v_w)} + M_{\text{lift}} + M_{\text{gyro}} \]
\[ N = N_{r1,r2,r3,r4} + N_{\text{gyro}} \]  \hspace{1cm} (7.11)
7.3 Torques; \(M, L\) and \(N\)

where \(r_n\) is rudder number \(n\), \(v_w\) is the noise wind and \(D\) refers to the drag of the rudders. The quantities \(L_{r_2+r_4},\ M_{r_1+r_3}\) and \(N_{r_1,r_2,r_3,r_4}\) are calculated in Section 5.2.2 and the rest will be calculated below.

### 7.3.1 Drag Torque

The general expression for the drag torque can be written \(O = O(r_a, r_b, h, v_w)\) where \(h\) is the perpendicular distance from the craft's center of gravity to the theoretical location of the force. Using (5.8) the drag from the propeller becomes

\[
F_{D,r} = \frac{1}{2} \rho v_w^2 A_r C_{D,r}
\]

where \(A_r\) is the rudder area and \(C_{D,r}\) is the drag coefficient. The drag coefficient \(C_{D,r}\) is determined by the airfoil and the angle of attack. The correct angle is approximately \(\alpha_{r_D} = \pi - \alpha\) where \(\alpha\) is the angle of the rudder from its original position. From Chapter 6 we have \(v_w = v_w (\cos(\beta)\hat{x} + \sin(\beta)\hat{y})\) which together with the fact that two rudders work in the same direction and (7.12) gives

\[
L_{D,(r_2,r_4,v_w)} = 2h_v \frac{1}{2} \rho v_w^2 A_r C_D \sin(\beta)
\]

\[
M_{D,(r_2,r_4,v_w)} = 2h_v \frac{1}{2} \rho v_w^2 A_r C_D \cos(\beta)
\]

In the equations above, \(h_v\) is the lever for the torque. It is the perpendicular vertical distance between the center of gravity of the craft and the aerodynamic center of the rudder.

### 7.3.2 Torque due to Difference in Lift

The third terms in (7.11), that is, \(L_{\text{lift}}\) and \(M_{\text{lift}}\), are not as straightforward to calculate as the previous terms. The torques \(L_{\text{lift}}\) and \(M_{\text{lift}}\) exist due to the fact that the noise wind is added to the induced wind on one side of the propeller and subtracted from the induced wind on the opposite side. This gives a larger lift on one of the sides which in turn creates a torque about the \(x\) and/or \(y\) axis. In this case (4.3) will not result in equation (4.4) as in the hover case. If the definitions in Figure 4.1 are used, the lift can be calculated as

\[
T = \rho A_{\infty} v_e (v_e - V_c)
\]

Since \(v_i = \frac{v_e + v_w}{2}\) [16] we get

\[
T = \rho A_r (V_c + v_i)2v_i
\]

As an example, we calculate the torque about the \(x\)-axis, \(L\), with the noise wind blowing from the direction where \(y = 0\). In this case the force is

\[
F_T = T_{F+} - T_{F-} = 2\rho A_r (v_w + v_i)2v_i - 2\rho A_r (-v_w + v_i)2v_i = 4\rho A_r v_w v_i
\]
where \( T_{F+} \) is the force on the side where the noise wind is added to the induced wind and \( T_{F-} \) is the side where the noise wind is subtracted from the induced wind. To calculate the total torque from this force we introduce an angle \( \gamma \) as the angle to an arbitrary part of the rotor disc. Using \( \gamma \) to describe the effect of the wind noise on the induced wind results in

\[
v_{\text{tot}} = v_i + v_{w,m} \cos(\gamma)
\]

(7.17)

where \( v_{w,m} \) is the magnitude of the noise wind. Since the noise wind is added to one side and subtracted to the other side, the total lift will not be affected. This means that the total lift still is \( T = 2\rho \pi R^2 v_i^2 \). To calculate the length of the lever, the lift per area unit, \( \rho L(r, \gamma) \), first has to be calculated. The lift per area unit depends only on the radius, \( r \), and the introduced angle, \( \gamma \), and can be calculated from

\[
\int_0^R \int_0^{2\pi} \rho L(r, \gamma) d\gamma dr = 2\rho \pi R^2 v_i^2 \Rightarrow \\
\Rightarrow \rho L(r, \gamma) = 2\rho v_i (v_i + v_{w,m} \cos(\gamma))
\]

(7.18)

The term \( v_{w,m} \cos(\gamma) \) in \( \rho L(r, \gamma) \) could be any constant times \( \cos(\gamma) \), but from (7.17) it follows that \( v_{w,m} \) is the correct constant.

The lever is calculated in the same way as the center of mass is calculated in [21],

\[
\bar{r} = \frac{\int_A r \rho L(r, \gamma) dA}{\int_A \rho L(r, \gamma) dA} = \\
= \frac{\int_0^R \int_0^{2\pi} 2\rho r^2 v_i (v_i + v_{w,m} \cos(\gamma)) d\gamma dr}{\int_0^R \int_0^{2\pi} 2\rho r v_i (v_i + v_{w,m} \cos(\gamma)) d\gamma dr} = \\
= \frac{\frac{2}{3} 2\pi \rho R^3 v_i^2}{\frac{2}{3} 2\pi \rho R^2 v_i^2} = \frac{2}{3} R
\]

(7.19)

Combining (7.16) and (7.19) gives

\[
L_{\text{lift}} = \frac{4}{3} \rho \pi R^3 v_w v_i \sin(\beta)
\]

\[
M_{\text{lift}} = \frac{4}{3} \rho \pi R^3 v_w v_i \cos(\beta)
\]

(7.20)

### 7.3.3 Torque due to the Rotation of the Propeller

The final term of the equations for \( L \) and \( M \) in (7.11) are the torques generated by the rotation of the propeller. These torques can be calculated with the moment of inertia of the rotating parts. According to the definitions in Section 7.2 the moment of inertia for the popeller about the z-axis is

\[
I_z = \int_{\nu} (x^2 + y^2) d\nu
\]

(7.21)
7.3 Torques; $M$, $L$ and $N$

The propeller in Figure (7.2) has an arbitrary number of blades. They are approximated as two dimensional areas with the length $l_w$ and the width $b_w$ and are placed at the angle $\gamma$ from the $x$-axis. The moment of inertia for the $i$:th blade is calculated as

$$I_{z,i} = \int_0^{\cos(\gamma)l_w} \int_{a_i x - b_w}^{a_i x + b_w} (x^2 + y^2) \rho d\nu =$$

$$= \frac{1}{3}(2b_w + 2a_i^2 b_w)(\cos(\gamma)l_w)^3 - \frac{1}{3}b_w^2 \cos(\gamma) L \rho =$$

$$= \frac{1}{3}(2 + 2a_i^2)(\cos(\gamma))^3 l_w^2 - \frac{1}{3}b_w \cos(\gamma))m_i$$

(7.22)

where $a_i$ is a constant that together with the width of the blade, $b_w$, describe the relation between $x$ and $y$ for blade $i$ as $y = a_i x + b_w$ or $y = a_i x - b_w$ for the two edges of the blade. This constant only depends on the angle from the $x$-axis to the current blade. The mass of blade $i$ is $m_i = \rho l_w b_w$. The total moment of inertia about the $z$-axis is the sum of the moment of inertia for each blade.

$$I_z = \sum_{i=1}^{N_b} I_{z,i}$$

(7.23)

The moments of inertia about the $x$- and $y$-axis are, in the same way as the moment of inertia about the $z$-axis, dependent on the angle between the $x$-axis and the blade. Since the propeller rotates in the $xy$-plane of the body fixed reference...
frame the blade that points in the direction of the $x$-axis at time $\tau = 0$ will point in the direction of the $y$-axis at time $\tau = \pi/2$ ($\omega_p$ is the angular velocity of the propeller versus the craft). This means that $I_x = I_y$ when the propeller is rotating. The cross terms of the moment of inertia, $I_{xy}$, $I_{xz}$ and $I_{yz}$ are zero. The term $I_{xy}$ is zero because the propeller is symmetrical around the $z$-axis and $I_{xz}$ and $I_{yz}$ are zero because $z = 0$ in the plane of the propeller. The moments of inertia above will result in a torque calculated as

$$\sum M = \left(\frac{dH_G}{dt}\right)_{yz} + \Omega \times H_G$$ (7.24)

where $H_G = I\omega$ is the angular momentum about the ducts center of gravity, $\Omega$ is the angular velocity of the craft and $\omega_p$ is the angular velocity of the propeller. This means that

$$\Omega = \begin{pmatrix} p \\ q \\ r \end{pmatrix}, \quad \omega = \begin{pmatrix} p \\ q \\ r + \omega_p \end{pmatrix}$$ (7.25)

Inserting (7.25) in (7.24) gives

$$\sum M = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} \dot{p} \\ \dot{q} \\ \dot{r} + \dot{\omega}_p \end{pmatrix} + \begin{pmatrix} p \\ q \\ r \end{pmatrix} \times \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix} \begin{pmatrix} p \\ q \\ r + \omega_p \end{pmatrix} =$$

$$= \begin{pmatrix} I_x \dot{p} - I_y qr + I_z q(r + \omega_p) \\ I_y \dot{q} + I_z pr - I_x p(r + \omega_p) \\ I_z (\dot{r} + \dot{\omega}_p) + I_y pq - I_x pq \end{pmatrix}$$ (7.26)

When $I_x = I_y$ is inserted we get

$$\sum M = \begin{pmatrix} I_x \dot{p} - I_y qr + I_z q(r + \omega_p) \\ I_y \dot{q} + I_z pr - I_x p(r + \omega_p) \\ I_z (\dot{r} + \dot{\omega}_p) \end{pmatrix}$$ (7.27)

This means that

$$L_{gyro} = I_x \dot{p} - I_y qr + I_z q(r + \omega_p)$$
$$M_{gyro} = I_y \dot{q} + I_z pr - I_x p(r + \omega_p)$$
$$N_{gyro} = I_z (\dot{r} + \dot{\omega}_p)$$ (7.28)
7.3.4 Resulting Torques

The calculations in the sections above result in the following equations for the torques

\[ L = h(-F_{r_1} - F_{r_3}) + 2h_v \frac{1}{2} \rho v_u^2 A_r C_D \sin(\beta)^2 - \]
\[ - \frac{2}{3} \rho \pi R^3 v_w v_i \sin(\beta) + I_x \dot{r} - I_y qr + I_z q(r + \omega_p) \]
\[ M = h(F_{r_1} + F_{r_3}) + 2h_v \frac{1}{2} \rho v_u^2 A_r C_D \cos(\beta)^2 + \]
\[ + \frac{2}{3} \rho \pi R^3 v_w v_i \cos(\beta) + I_y \dot{\psi} + I_z pr - I_z p(r + \omega_p) \]
\[ N = h_h (F_{r_1} + F_{r_2} - F_{r_3} - F_{r_4}) + I_z (\dot{\psi} + \dot{\omega}_p) \]

In the equations above, \( h_h \) is the lever for the torque about the z-axis. This is the horizontal perpendicular distance between the center of gravity of the duct and the aerodynamic center of the rudder. In most cases, \( \dot{\omega}_p \) will be zero but it will result in a torque that is too big to be neglected when it is not zero. Especially when \( \dot{r} \) is zero, which is the case in most test environments, an \( \dot{\omega}_p \) different from zero will be clearly noticeable.

7.4 The Complete Model

In the sections above everything but the velocities, angular velocities and Euler angles have been explained. The variables \( u, v \) and \( w \) are the velocities of the craft in the body fixed reference frame and \( p, q \) and \( r \) are the angular velocities in the same reference frame. Finally, \( \phi, \theta \) and \( \psi \) are the Euler angles. The complete
model in (7.1) can now be rewritten as

\[
\begin{align*}
\dot{u} &= \frac{X}{m} - qw + rv - g \sin(\theta) \\
\dot{v} &= \frac{Y}{m} - ru + pw + g \sin(\phi) \cos(\theta) \\
\dot{w} &= \frac{Z}{m} - pv + qu + g \cos(\phi) \cos(\theta)
\end{align*}
\]

\[
\begin{align*}
\dot{p} &= \frac{1}{I_x}(L - qr(I_z - I_y)) \\
\dot{q} &= \frac{1}{I_y}(M - pr(I_x - I_z)) \\
\dot{r} &= \frac{1}{I_x}(N - pq(I_y - I_x))
\end{align*}
\] (7.30)

\[
\begin{align*}
\dot{\phi} &= p + \tan(\theta)q + \tan(\phi)r \\
\dot{\theta} &= \cos(\phi)q - \sin(\phi)r \\
\dot{\psi} &= \frac{\sin(\phi)}{\cos(\theta)}q + \frac{\cos(\phi)}{\cos(\theta)}r
\end{align*}
\]

It is obvious that three equations have disappeared compared to equations (7.1). This is because the six last equations in (7.1) have been combined to create the three last equations in (7.30). Another mathematical relation used when calculating the model is the relation between the velocities in the inertial reference frame and the velocities in the body fixed reference frame, given by

\[
\begin{pmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt} \\
\frac{dz}{dt}
\end{pmatrix} =
\begin{pmatrix}
C_{\theta}C_{\psi} & S_{\phi}S_{\theta}C_{\psi} - C_{\phi}S_{\psi} & C_{\phi}S_{\theta}C_{\psi} + S_{\phi}S_{\psi} \\
C_{\phi}S_{\psi} & S_{\phi}S_{\theta}S_{\psi} + C_{\phi}C_{\psi} & C_{\phi}S_{\theta}S_{\psi} - S_{\phi}C_{\psi} \\
-S_{\theta} & S_{\phi}C_{\theta} & C_{\phi}C_{\theta}
\end{pmatrix}
\begin{pmatrix}
u \\
v \\
w
\end{pmatrix}
\] (7.31)

### 7.5 Improving the Model

The model calculated in the sections above corresponds fairly good to experiments carried out. However, strange behaviour showed up in longer simulations.

#### 7.5.1 Air Resistance

The most obvious was that a movement of the craft would continue until a force from the rudders or the noise wind counteracted it, making the system an unstable system. Such behaviour is on the real system opposed by air resistance. The air resistance will have the same effect on the craft as a noise wind, generating both the forces and torques the noise wind does. If the noise wind speed, \(v_w\), and the angle of attack of the noise wind, \(\beta\), is modified with the information about the
velocity of the craft only small changes have to be made to the model equations. The equations in (7.7) with the modifications becomes

\[ X = -F_{r_2} - F_{r_4} + \frac{1}{2} \rho (v^2_w \cos(\beta) - u) A_d C_{D,d} + 2 \frac{1}{2} \rho (u^2 + v^2_w \cos(\beta)) A_r C_{D,r}(\alpha_1) \]

\[ Y = F_{r_1} - F_{r_3} + \frac{1}{2} \rho (v^2_w \sin(\beta) - v) A_d C_{D,d} + 2 \frac{1}{2} \rho (v^2 + v^2_w \sin(\beta)) A_r C_{D,r}(\alpha_1) \]

where \( u \) and \( v \) are the \( x \) and \( y \) terms of the craft velocity in the body fixed reference frame.

### 7.5.2 Twin Rudders

Extensive simulations were made to determine the systems resistance to disturbances, the most important source of disturbance being the noise wind. The system reacted as expected but the forces generated by the rudders was not large enough to counteract even the slightest noise wind. When two rudders worked at maximum angle to counteract a noise wind of small magnitude the best result achieved was a state where the craft was stable but with a constant pitch/roll caused by the noise wind. The rudders were unable to force the craft back to a horizontal position. The insufficient rudder forces generated indicated that the craft, in particular the rudders, had to be modified to enable the craft to withstand a larger noise wind.

#### Modification of the Rudders

There are several ways to modify the craft so that the rudders generate a greater force and torque. If the lever was to be prolonged the torque would increase but the force generated by the rudders would not. A better solution is to find a way to increase the force without prolonging the lever.

To generate a greater torque by enlarging the lever the rudders must be placed lower or the weight distribution of the craft must be altered so that the center of gravity of the craft is located higher. Both these solutions result in unwanted effects like larger torque generated by the noise wind. Placing the rudders lower will increase the wanted torque generated by the movement of the rudders but it will also, and for the same reason, increase the torque generated by the drag force of the rudders caused by the noise wind. The increased noise wind torque is of course unwanted. If the weight distribution of the craft instead is altered so that the center of gravity is moved upwards the same effect on both the wanted and unwanted rudder torques is obtained. In addition to these two effects the movement of the center of gravity will result in the need for a more complicated mathematical model since the modification will cause the center of gravity to be placed at a point separated from the vertical center of the duct. If the center of gravity is somewhere else but at the vertical (and horizontal) center of the duct a torque caused by the noise wind’s effect on the duct will occur. This is of course unwanted.
To generate a greater torque without enlarging the lever would require larger rudder forces. The force of a rudder is calculated as

\[ F_L = \frac{1}{2} \rho v_i^2 A_r C_L(\alpha) \]

in Section 5.1.3. This equation means that the force generated by the rudder depends on the air density \( \rho \), the wind speed induced by the propeller \( v_i \), the area of the rudder \( A_r \), the lift coefficient \( C_L(\alpha) \) and through the lift coefficient also on the angle of attack \( \alpha \). The air density is difficult to increase, as is the wind speed induced by the propeller. Both the air density and the air speed is of course possible to change but the simplest way to change any of them would be to change the engine. Changing the engine would require an expensive new engine and many hours of installation work. The angle of attack is the simplest parameter to increase but \( C_L \) does not have a linear dependence of \( \alpha \) and the maximum angle of attack allowed at this point is the angle of attack generating the largest \( C_L \) (and thereby the largest force). There is one angle of attack that for some airfoils might give a greater force then the angle the rudders now are restricted to. This angle is more than twice as big as the angle of restriction and the larger angle only generates a few per cent more lift, if any. A larger angle have several disadvantages compared to the restriction angle now used. Since the servos controlling the rudders have a constant maximum turning velocity it will take longer time for the rudders to reach the larger angle. If two neighboring rudders were to be placed at their maximum angle (the maximum angle being the larger of the angles discussed) towards each other one rudder would disturb the other both by touch and by disturbing the wind moving over the other rudder with its own wind.

The only logical way to increase the rudder forces is to increase the rudder area (this might as a secondary effect increase or decrease \( C_L \)). Increasing the rudder area will in most cases increase the area exposed to the noise wind which in turn will increase the drag force due to the noise wind. The increased force and torque due to the noise wind is unwanted and must be avoided. To increase the rudder area without increasing the area exposed to the noise wind a twin rudder setup can be constructed. A schematic description of the twin rudder setup is shown in Figure 7.3. Twin rudders as it looks on Bombus is shown in Figure 7.4.

Even though the twin rudder doubles the rudder area without noticeable increasing the drag of the rudder, the twin rudder will not generate twice the force of the single rudder. This is because of something called the biplane effect. The name of this effect of course originates from biplanes using the same setup for their main wings. The biplane effect is the loss of lift due to the interference between the two wings (or in this case rudders) [17]. To decrease the biplane effect the gap \( G \) and the stagger \( D \) (see Figure 7.3) can be increased but to completely remove the effect the gap has to be greater than twice the cord, \( c \). On the current set up on Bombus the gap is about 30% of the cord and the stagger varies with the angle of attack but peaks at about 10% of the cord at the maximum angle of attack. In Figure 7.5, the theoretical \( C_L \) can be seen for the monoplane, the upper biplane wing, the lower biplane wing and the total biplane cellule. Note that Figure 7.5 is
Twin rudders without angle of attack  Twin rudders with angle of attack

Figure 7.3. Double rudders.

Figure 7.4. Bombus modified with twin rudders.
not an exact figure but shows the general appearance of the lift coefficient graph. The lift coefficient graph will vary some when parameters like gap, stagger, airfoil and cord varies. Since $C_L$ for the biplane is somewhat smaller than $C_L$ for the monoplane and the biplane has a twice as big wing area as the monoplane the theoretical lift force for the biplane is somewhat less than twice that of the monoplane just as predicted by the biplane effect.

**Experimental Performance of Twin Rudders**

To determine if the theory regarding the twin rudders and the comparison between the twin rudders and the biplane is correct, experiments had to be carried out. The twin rudder experiments were the same as the torque experiments in Section 5.3 but with twin rudders instead of single rudders. The results from one of the experiments are shown in Figure 7.6. The light pluses are the results from the experiment with a twin rudders and the dark dots are the results from the experiments with single rudders. It is clear that the torque when using double rudders is about twice as large as the torque when using single rudders. With the double rudders implemented in the mathematical model the simulations show a significant increase in disturbance resistance as well as a behavior more like the desired when a control system is connected and works to steer the craft to a specific angle or position. An even better effect can be achieved with a larger number of rudders in the same assembly, for example three instead of two, but then the biplane effect is increased resulting in a decrease increased torque to increased weight ratio. Increasing the number of rudders in the same assembly will also decrease the maximum angle of the rudder assembly before the assembly touches the neighboring rudder assembly.

**Figure 7.5.** Theoretical appearance of the lift coefficient graph.
Figure 7.6. Experiment with double rudders (light pluses) compared to similar experiment with original configuration (dark dots).
Chapter 8

Control

Almost every modern aircraft uses an advanced control system to assist the pilot. In some cases, like in modern fighter aircraft, the aircraft itself is unstable so the control system is necessary in order to give the pilot ability to fly the aircraft. Other aircraft are designed to be stable making a control system less necessary even if a control system still can assist the pilot with tasks like wind correction. Since an UAV has no pilot, the performance of the control system is of large importance. If the control system fails an accident is almost inevitable.

To obtain a good control system for Bombus several different control systems will be tested. Since a mathematical model has been carefully derived a control system based on this model can be implemented. However, a PID controller independent of the model will also be implemented.

All controllers will have the same inputs and generate the same outputs. The inputs are the coordinates in the inertial reference frame and the outputs are the angles of the rudders and the engine speed. If the craft were to be controlled by an operator it is possible that instead of using the three inertial coordinates as references, the corresponding velocities can be used. This gives the operator a more natural feeling.

8.1 PID

The PID controller can, despite its simplicity, in many cases be sufficiently effective. The problem when using PID controllers in large systems is that a large number of independent controllers have to be combined to make up the complete control system. All of these controllers have to be individually tuned to obtain maximum performance.

The PID controllers are implemented on a computer and are therefore discrete. The general appearance of the independent discrete PID controller is

\[
u(k) = K \left( e(k) + \frac{T_S}{T_i} \sum_{n=0}^{k} e(n) + T_d \frac{e(n) - e(n-1)}{T_S} \right)
\]

(8.1)
where \( u(k) \) is the output, \( e(k) \) is the input (usually the controlled quantity minus the reference value) and \( K, T_i \) and \( T_d \) are design parameters and \( T_S \) is the sample time. There are several methods for obtaining a PID controller with desired behaviour, all giving approximative values for the design parameters. For even better tuning, experience and a knowledge about how the different design parameters affect the system are the best tools. When several PID controllers are cascaded like on Bombus the best way to obtain good performance is to tune the entire system. There are several methods for tuning of systems with cascaded PID controllers.

8.1.1 PID Realization

Since the Bombus control system only has six input signals and five output signals the system is small enough for a PID controller system to be implemented. To implement the control system the main method of moving the craft from one point in space to another had to be determined. The simplest method for moving the craft is to first determine the direction of the movement in the body fixed reference frame, tilt the craft in the desired direction and let the force generated by the fan move the craft to the desired position (observe that when \( x \) and \( y \) are not bold they represents position, not states). The method just explained means that a change in the reference signal for the inertial \( x_i \) or \( y_i \) will result in an increase or decrease in reference signal for roll and/or pitch angles. If the reference signal for the inertial \( z \) is increased or decreased the speed of the fan will be increased or decreased. The structure of the PID control system can be seen in Figures 8.1 and 8.2.

The blocks labeled PID in Figures 8.1 and 8.2 are the individually tuned PID controllers. It is clear that in order to change the velocity along the body fixed \( x \)-axis (\( \hat{x} \)) the pitch angle should be changed. This change is made by changing the angle of rudder one and rudder three. In the same way, to change the velocity in the body fixed \( y \), the roll angle and therefore also the angle of rudders two and four should be changed. To change the yaw angle all rudders have to be moved. This dependency complicates the design of the control system as well as the tuning of the individual PID controllers.

8.1.2 Inner PID Performance

Tests of the inner control system’s performance is made with steps in the reference signal of roll or pitch or both.

Simulations with Inner PID controller

The experiments presented here are rolling operations. Roll is the hardest angle to control due to the direction of the wind. In Figure 8.3, the reference signal of roll makes steps from 0 rad to 0.1 rad at 15 s and back to 0 rad at 30 s. This behaviour of the reference signal is repeated. Pitch and yaw reference signals are constant zero and these angles behave as desired and stays around zero. The PID controller runs into some problems getting the roll angle equal to the reference
value. The same problem occurs on the way back to zero. The PID controller is tuned to perform best when a path is followed (see Figure 8.8(a)). To obtain better performance in the simulation with a rolling operation the proportional part can be raised. This will result in overshoots in pitch or roll angle in the simulations when a path is followed. An overshoot in one of these angles will result in a horizontal force far larger than the required one. This will result in large overshoots in the horizontal position.

**Inner PID Performance at the Actual System**

The PID controller is transferred to the actual system and similar experiments as the simulations are made. Figure 8.4 shows an experiment where the reference signal of pitch makes a step from 0 rad to 0.2 rad at 33 s and back to 0 rad at 42 s. Roll and yaw reference signals are constant equal to zero. Unfortunately, the gimbal rings in the rig suffer from a large lack of stiffness. This leads to a relatively large oscillation in the rig. Most of the oscillations that can be seen in Figure 8.4 are due to the lack of stiffness of the gimbal rings. This lack of stiffness results in more oscillations in the sensors placed between the rings than in the craft itself. The craft actually oscillates far less than the measurements show. However, Figure 8.4 shows the rudders ability to get the pitch angle to follow its reference signal.
Figure 8.2. The inner PID controller.

Figure 8.3. Simulated experiment with the inner PID controller. Steps are made in the roll reference signal.
8.1 PID

8.1.3 Replacing inner PID with LQ Controller

The inner PID can be replaced with an LQ Compensator, obtained by using MatrixX tools on a simplified model and with the same input and output signals as the inner PID control system. How to obtain an LQ Compensator is discussed more thoroughly in Sections 8.4 – 8.6 but there for the complete model. To obtain an LQ Compensator corresponding to the inner PID the calculations have to be done with a model with only the body fixed angles as output signals. The complete control system can be seen in Figure 8.1, but instead of the inner PID block the LQ controller in Figure 8.5 is used.

Simulations with Inner LQ controller

To be able to do good comparison between the different control systems, similar simulations and experiments to those done in Sections 8.1.2 and 8.1.2 are performed. Recall that the roll reference signal makes steps from 0 rad to 0.1 rad at 15 s, back to 0 rad at 30 s and repeats this behaviour. Figure 8.6 presents the result of the inner LQ controller simulation. If Figure 8.6 is compared with Figure 8.3, it can be seen that the LQ controller does not manage to control the angles as good as the PID controller. This can be explained by the linearization which is made at an operating point where all angles are zero. To avoid this problem the system can be linearized in every time step resulting in an estimator gain matrix varying with the linearized system. This type of estimator design is called Extended Kalman Filter [11].
Figure 8.5. Inner LQ-controller.

Figure 8.6. Simulated experiment with the inner LQ controller. Steps are made in the roll reference signal.
Inner LQ Performance at the Actual System

As for the PID controller, the LQ control system is transferred to the actual system and experiments similar to the simulated ones are made. Figure 8.7 shows an experiment where the reference signal of pitch makes a step from 0 rad to -0.2 rad at about 25 s and back to 0 rad at 36 s. Roll and yaw reference signals are constants equal to zero. Figure 8.7 can be compared to Figure 8.4. The oscillation problems discussed in the end of Section 8.1.2 are present here too and it is hard to decide whether the LQ controller performs better than the PID controller for the actual system or not.

8.1.4 Outer PID Performance

The outer PID controller uses the inner control system to keep the craft at, or moving it to, a desired position. Only simulated experiments are made with the outer controller. In these simulations the same wind model as in the simulations described in Sections 8.1.2 and 8.1.3 has been used. In Figure 8.8 the performance of the complete PID control system is shown. The reference signals in $x$ and $y$ makes a route starting at origo moving right and following the dashed line in Figure 8.8. The time between each change of direction is 50 s. The reference signal in $z_i$ is kept constant at zero which result in a slight increase in the fan speed during pitch and/or roll in order to compensate for the decrease in lift force due to the change of direction of the thrust. The simulation with a PID controller as the inner controller shows good performance. The performance decreases when the inner PID controller is replaced by an LQ Compensator. This can be explained by the fact that the linearization is made in a point where all angles are zero. To move
the craft, the outer PID controller changes the angle references and thereby moving
the LQ controller out of the area where the linearization is a good approximation
of the system.

Even if the control system with an outer and a inner PID controller shows
good performance more advanced control methods, like LQ control, will be imple-
mented and tested.
8.2 Linearization

8.1.5 Integral Windup for PID Controller

A single PID controller might suffer from integral windup if the control signal can be saturated. The algorithm for introducing anti-windup for a discrete-time PID controller, using conditional integration, is [4]

\[
\begin{align*}
\text{if not (control signal saturated)} & \quad \text{then } I_n = I_{n-1} + K \frac{T_d}{T_i} e(k) \quad \text{else } I_n = I_{n-1} \\
 u(k) &= Ke(k) + I_n + K \frac{T_d}{T_S} (e(k) - e(k-1)) \\
 u(k) &= \begin{cases} 
 u_{\text{max}} & \text{if } u(k) > u_{\text{max}} \\
 u(k) & \text{if } u_{\text{min}} < u(k) < u_{\text{max}} \\
 u_{\text{min}} & \text{if } u(k) < u_{\text{min}} 
\end{cases}
\end{align*}
\] (8.2)

When several PID controllers are connected in cascade some information have to be sent between them to prevent integral windup. If the inner subsystem’s control signal is saturated and the outer subsystem’s control signal is not, the increment of the outer controller’s integrating part has to be stopped as well. If it is not stopped a similar effect as windup for a single PID controller can occur. The PID controllers implemented on Bombus do not show any signs of integral windup. Therefore anti-windup is not implemented.

8.2 Linearization

Even if there exist control methods for nonlinear systems many modern control tools, like LQ control, require a linear model. Time-continuous nonlinear systems like the model of Bombus are represented by the following differential and output equations

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= g(x, u)
\end{align*}
\] (8.3)

where \( x \) is the state vector, \( u \) is the input vector and \( y \) is the output vector. In the complete model of Bombus, the output vector contains all measurable magnitudes, position and body fixed angles of the craft. The input vector contains fan speed and angles of the four rudders. The states in the model are positions, velocities, angles and angular velocities of the craft, but also, for example, rudder angles.

If the nonlinearities have small affect on the system, the simplest way to linearize the system is to simply remove the nonlinear parts. In the Bombus model the nonlinearities affect the model too much for this method to be used. Another way to linearize the model without removing the nonlinearities is to use the first-order term in the Taylor expansion. A Taylor expansion can be written as

\[
\begin{align*}
f(x) &= f(x_0) + \frac{1}{1!} f'(x_0)(x-x_0) + \frac{1}{2!} f''(x_0)(x-x_0)^2 + \ldots \\
&\quad + \frac{1}{n!} f^{(n)}(x_0)(x-x_0)^n + \frac{1}{(n+1)!} f^{(n+1)}(\zeta)(x-x_0)^{(n+1)}
\end{align*}
\] (8.4)
where \( \zeta \) is a point between \( x \) and \( x_0 \). The first-order part of (8.4) is the second term on the right side of the equality. Applying the first-order Taylor expansion to (8.3) results in

\[
\dot{\tilde{x}} = A\tilde{x} + B\tilde{u} + f_0
\]

\[
\dot{\tilde{y}} = C\tilde{x} + D\tilde{u}
\]

(8.5)

where \( \tilde{x} = x - x_0, \tilde{u} = u - u_0, \tilde{y} = y - y_0 \) and \( f_0 = f(x_0, u_0) \) and

\[
A = \left. \frac{\partial f(x, u)}{\partial x} \right|_{x=x_0, u=u_0}
\]

\[
B = \left. \frac{\partial f(x, u)}{\partial u} \right|_{x=x_0, u=u_0}
\]

\[
C = \left. \frac{\partial g(x, u)}{\partial x} \right|_{x=x_0, u=u_0}
\]

\[
D = \left. \frac{\partial g(x, u)}{\partial u} \right|_{x=x_0, u=u_0}
\]

(8.6)

If (8.5) and (8.6) are applied to Bombus with \( x_0 = 0, u_0 = 0 \) and \( f_0 = 0 \) the linearized system becomes

\[
\begin{bmatrix}
\dot{x} \\
y
\end{bmatrix} =
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
x \\
u
\end{bmatrix}
\]

(8.7)

where the matrices \( A, B, C \) and \( D \) are defined in (8.6) but with \( x_0 = 0 \) and \( u_0 = 0 \).

To be able to linearize the model, a simulation is done to put the system in a certain operating point represented by \( x_0 \) and \( u_0 \). At the selected operating point the fan runs at hovering speed and the rudders are all in initial position straight down. The matrices \( A, B, C \) and \( D \) are obtained with the function \texttt{lin} after running a simulation with the function \texttt{sim} in MatrixX SystemBuild. The linearization described above is described in the MatrixX manual as the linearization used by the function \texttt{lin}.

### 8.3 Discretizing

The linearized model obtained in the preceding section is time-continuous. However, the control system is supposed to be implemented in a computer and has to be time-discrete. There are two different solutions to this problem.

One alternative is to create all necessary control tools from the continuous system and, at last, discretize the complete control system. Doing this will involve an assumption that sample time, \( i.e. \), the time between two consecutive measurements from the sensors, is very small. An advantage of this alternative is that the discretizing is simple since there will be no derivatives in the complete control system.

The other alternative is to discretize the system right after the linearization and then create all control tools adapted to the discrete system. Using this method removes the assumption of very short time between measurements, but causes the discretizing process to be harder. However, the discretizing can be done by using the tool \texttt{discretize} in MatrixX.
The discrete system is expressed in state-space form as

\[
\begin{bmatrix}
    x_{k+1} \\
    y_k
\end{bmatrix} =
\begin{bmatrix}
    A & B \\
    C & D
\end{bmatrix}
\begin{bmatrix}
    x_k \\
    u_k
\end{bmatrix}
\] (8.8)

### 8.4 Estimator

To be able to use state-feedback control all the states must be available. However, in real systems only the measurements are available and often corrupted by noise. An observer has to be constructed in order to get estimates of all the system states. Figure 8.9 illustrates the structure of the system with an observer. The following equations describe the complete actual system:

\[
x(t + 1) = Ax(t) + Bu(t) + G\omega(t)
\]

\[
y(t) = Cx(t) + \nu(t)
\] (8.9)

where \( \omega \) is the input noise with the noise gain matrix \( G \). The measurement noise is denoted \( \nu \). The intensity of the noise is described by

\[
\begin{bmatrix}
    R_{\omega\omega} & R_{\omega\nu} \\
    R_{\omega\nu} & R_{\nu\nu}
\end{bmatrix}
\]

where \( R_{\omega\omega} \) represents the intensity of \( \omega \), \( R_{\omega\nu} \) represents the intensity of \( \nu \) and \( R_{\omega\nu} \) represents the cross correlation between \( \omega \) and \( \nu \). \( R_{\nu\nu} \) must be positive definite and \( \tilde{R}_{\omega\omega} = R_{\omega\omega} - R_{\omega\nu}R_{\nu\nu}^{-1}R_{\nu\omega}^T \) must be positive semi-definite.

The goal is to find an estimate of the states which minimizes the error between the actual state vector, \( \bar{x} \), and the estimated state vector, \( \hat{x} \). The quadratic expression to be minimized is

\[
\sum_{0}^{\infty} \begin{bmatrix}
    \bar{x} \\
    \bar{y}
\end{bmatrix}^T \begin{bmatrix}
    \tilde{R}_{\omega\omega} & \tilde{R}_{\omega\nu} \\
    \tilde{R}_{\omega\nu}^T & \tilde{R}_{\nu\nu}
\end{bmatrix} \begin{bmatrix}
    \bar{x} \\
    \bar{y}
\end{bmatrix}
\] (8.10)
where $\bar{x} = x - \hat{x}$, $\bar{y} = y - \hat{y}$ and $\hat{y}$ is the output vector computed using the estimated state vector. The covariance matrices are

$$
\bar{R}_{\omega\omega} = GR_{\omega\omega}G^T \\
\bar{R}_{\omega\nu} = GR_{\omega\nu} \\
\bar{R}_{\nu\nu} = R_{\nu\nu}
$$

where $G$ is defined by (8.9).

One kind of observer minimizing the estimate error is called a Kalman filter [6]. The difference between the outputs computed with the estimated states and the actual outputs is fed back through an estimator gain $K_e$. The Kalman filter is given by

$$
\hat{x}(t+1|t) = A\hat{x}(t|t-1) + Bu(t) + K_e(y(t) - C\hat{x}(t|t-1)) \tag{8.11}
$$

where

$$
K_e = (APC^T + R_{\omega\nu}^T)(CPC^T + R_{\nu\nu})^{-1} \tag{8.12}
$$

where $P$ is the symmetric positive semi-definite solution to the algebraic Riccati equation

$$
P = APA^T + R_{\omega\omega} - (APC^T + R_{\omega\nu}^T)(CPC^T + R_{\nu\nu})^{-1}(APC^T + R_{\omega\nu}^T)^T \tag{8.13}
$$

Figure 8.10 shows a diagram over the Kalman filter representation.

![Diagram over the Kalman filter representation.](image)

**Figure 8.10.** Diagram over the Kalman filter representation.

Obviously the hardest task while constructing an optimal estimator is to solve the Ricatti equation given the process, the measurement, and the cross-weighting noise matrices. The software used, MatrixX, includes tools for this (for example the commands estimator and riccati) allowing the designer to spend more time trying out different design matrices.
8.5 Linear Quadratic Controller

The Linear Quadratic Control System is a feedback controller with the goal to keep the states within acceptable levels using acceptable amounts of control.

Using the system (8.9) an optimal feedback gain $L$ minimizes the quadratic expression [6]

$$\sum_{0}^{\infty} \begin{bmatrix} x^T \\ u \end{bmatrix} \begin{bmatrix} Q_{xx} & Q_{xu} \\ Q_{xu}^T & Q_{uu} \end{bmatrix} \begin{bmatrix} x \\ u \end{bmatrix}$$  \hspace{1cm} (8.14)

The matrix $Q_{xx}$ is real, symmetric and positive-semidefinite and is the weighting of the state vector elements. The matrix which determines the weight of the control inputs is denoted $Q_{uu}$ and is also real and symmetric, but positive-definite. The matrix $Q_{xu}$ is the weight of the product of state vector elements and the control input elements. In many cases $Q_{xu}$ is set to zero, disregarding the products. In the Bombus controller this matrix is set to zero. The expression (8.14) establishes the trade-off between the permissible state fluctuation and amount of control required to move the states.

With $Q_{xu} = 0$ the feedback law which minimizes (8.14) is

$$L = (B^T S B + Q_{uu})^{-1} B^T S A$$  \hspace{1cm} (8.15)

where $S$ solves the Riccati equation

$$S = A^T S A + Q_{xx} - A^T S B (B^T S B + Q_{uu})^{-1} B^T S A$$  \hspace{1cm} (8.16)

giving, together with the Kalman filter from Section 8.4, the optimal controller

$$\dot{x}(t+1) = A\dot{x}(t) + B u(t) + K_c (y(t) - C\dot{x}(t))$$
$$u(t) = -L\dot{x}(t)$$  \hspace{1cm} (8.17)

The controller explained above will steer the states to zero. This is not always the desired case on Bombus. Therefore a reference signal, $r$, has to be introduced. The system (8.9) can be extended with

$$z = M x$$  \hspace{1cm} (8.18)

where $z$ contains the states desired to control. The matrix $M$ extracts the interesting states from $x$. This means that it is $z - r$ that must be steered to zero. Introducing the reference signal in (8.17) gives

$$\dot{x}(t+1) = A\dot{x}(t) + B u(t) + K_c (y(t) - C\dot{x}(t))$$
$$u(t) = -L\dot{x}(t) + L_r r(t)$$  \hspace{1cm} (8.19)

The matrix $L_r$ must be chosen so that the gain of the closed-loop system is equal to one [6] which in a discrete system in the static case means

$$M(I + BL - A)^{-1} BL_r = I$$  \hspace{1cm} (8.20)

If $\dim(z) = \dim(u)$, $L_r$ can be obtained by ordinary inversion. If $\dim(z) < \dim(u)$ the reference gain can be obtained with the least-square method.
8.6 The Linear Quadratic Gaussian Compensator

The two preceding sections describe two separate tasks. The first task is to find an optimal state estimator, the Kalman filter, while the second task is to find the optimal feedback gain matrix for the estimated states. The combination of these technics is referred to as Linear Quadratic Compensator or Linear Quadratic Gaussian Compensator if the noises are Gaussian.

\begin{equation}
\hat{x}_{k+1} = A\hat{x}_k + Bu + K_c(y - C\hat{x}_k)
\end{equation}

\begin{equation}
u = -L\hat{x}_k + L_r r
\end{equation}

Figure 8.11. Linear Quadratic Gaussian Compensator (in the gray rectangle).

8.6.1 Integral Action

To avoid stationary errors an integrating part can be introduced into the Linear Quadratic Compensator.

Integral Action by Introduction of Noise

One way to introduce the integrating part is to "tell" the system that there exist much noise at a very low frequency corresponding to a stationary error [6]. This is made by introducing the noise

\begin{equation}
w(t) = \frac{1}{p + \delta}v(t)
\end{equation}

If \( \delta \) is chosen small, (8.21) will represent a noise source with much noise energy at low frequencies. Including (8.21) in the original model results in

\begin{equation}
\dot{x} = \begin{pmatrix} A & 0 \\ 0 & -\delta I \end{pmatrix} x + \begin{pmatrix} B \\ 0 \end{pmatrix} u + \begin{pmatrix} N \\ I \end{pmatrix} v
\end{equation}

\begin{equation}
y = \begin{pmatrix} C \\ I \end{pmatrix} x
\end{equation}
This requires that the system is discretized again (Section 8.3) and that a new Linear Quadratic Controller (Section 8.5) is computed using $\tilde{A}, \tilde{B}, \tilde{N}$ and $\tilde{C}$. The new closed loop system will not have a stationary error.

**Integral Action by Introducing extra States**

Another way to introduce an integrating part into the system is to introduce states representing the errors. The additional states

$$x_{\text{int}} = r - y = r - Cx$$

(8.23)

can be introduced. Note that $x_{\text{int}}$ is a state vector with some dimension. The additional states inserted in the original system will result in the augmented system

$$
\begin{pmatrix}
\dot{x} \\
\dot{x}_{\text{int}}
\end{pmatrix} =
\begin{pmatrix}
A & 0 \\
-C & 0
\end{pmatrix}
\begin{pmatrix}
x \\
x_{\text{int}}
\end{pmatrix} +
\begin{pmatrix}
B \\
0
\end{pmatrix} u +
\begin{pmatrix}
F \\
0
\end{pmatrix} v +
\begin{pmatrix}
0 \\
I
\end{pmatrix} r
$$

(8.24)

If the original system is controllable and the systems transfer function is not zero in the origin it can be shown that the augmented system (8.24) is also controllable [5]. The state feedback which in the original system was

$$u = -Lx$$

becomes

$$u = -Lx - l_{\text{int}} x_{\text{int}}$$

(8.25)

where $l_{\text{int}}$ should be chosen such that the closed loop system

$$
\begin{pmatrix}
\dot{x} \\
\dot{x}_{\text{int}}
\end{pmatrix} =
\begin{pmatrix}
A - BL & -B l_{\text{int}} \\
-C & 0
\end{pmatrix}
\begin{pmatrix}
x \\
x_{\text{int}}
\end{pmatrix} +
\begin{pmatrix}
F \\
0
\end{pmatrix} v +
\begin{pmatrix}
0 \\
I
\end{pmatrix} r
$$

(8.26)

is asymptotically stable, meaning that the eigenvalues of

$$
\begin{pmatrix}
A - BL & -B l_{\text{int}} \\
-C & 0
\end{pmatrix}
$$

should be inside the unit circle. With constant control input and reference signal the system will converge to $y = Cx$, eliminating the stationary error.

**Integral windup for Linear Quadratic Gaussian Compensator**

Both methods for introducing integral action might suffer from integral windup. Integral windup occurs when the control signal is saturated and the reference value of the state representing the saturated control signal is not fulfilled. This situation will let the integrator in the controller grow which can result in large overshoots and slow return when the state has reached its reference value [1]. To solve this problem the increase of the integrating part must be stopped when the control signal is saturated. To stop the integrating part from increasing the condition

if $u_i < u_{i,\text{min}}$ or $u_i > u_{i,\text{max}}$
then $x_i(k + 1) = x_i(k)$
else $x_i(k + 1) = A_i x + B_i u$
where $A_i$ and $B_i$ is the part of the $A$ and $B$ matrices affecting $x_i$, can be used. In the experiments and simulations made on Bombus no integral windup has been detected. Because of this and lack of time, the solution to the problem explained above will not be implemented.

8.6.2 LQ Realization

The complete time-continuous model is linearized at a hovering operating point using the tools available in MatrixX and SystemBuild. Before the linearization, the input signals are balanced. This means that the fan speed control signal is resized to about the size of the rudder control signals. The linearization gives a statespace system with 19 states, three of them uncontrollable.

Selecting Suitable Weighting Matrices

The estimator matrix $\hat{R}_{\omega\omega}$ and $\hat{R}_{\nu\nu}$ are tuned to describe the noise well and give good estimates of the states. The cross-weighting matrix $\hat{R}_{\omega\nu}$ is set to zero.

The weighting matrix $Q_{xx}$ will only contain zeros except for the diagonal elements. Every element in the diagonal of the matrix describes the cost of one particular state variable. To create the Linear Quadratic Compensator it is not necessary that all states are controllable but the system must be stabilizable. The uncontrollable states have to be taken under special consideration when the weighting matrices are chosen.

The weighting matrix $Q_{uu}$ will also be a diagonal matrix. Since the control variables from the controller are the signals to the four rudders all the elements will be equal. The values of the elements are trimmed to give good performance.

Introducing an Integrating Part

To remove possible stationary errors an integrating part is added to the model. The solution used is the method of adding low frequency noise to certain states described in the first part of Section 8.6.1. Before the introduction a stationary error was present in several states but it was in the $z$-state where it was most obvious. The introduction of the low frequency noise completely removed this stationary error.

Creating the Observer and the Controller

After selecting the weighting matrices the observer and the controller are obtained with the commands `estimator` and `regulator` in MatrixX. The feedback gain matrix $K_r$ is obtained and the reference gain matrix $L_r$ is calculated using (8.20).

8.6.3 LQ Performance

The simulated performance of the LQ-controller can be seen in Figure 8.12. In the simulation a reference flight path like the one used to test the PID controller is used (see Section 8.1.2), represented by the dashed line. The flight path of the
craft is the solid line. As noise a wind as described in Section 6 is used. This means
that the wind speed is white Gaussian noise with a mean value and the angle from
which the wind arrives varies from $\pi/6$ rad to $-\pi/6$ rad. When the wind arrives
from 0 rad it is arriving from negative $x$. The performance of the LQ-controller
is acceptable with a position error of at most 0.5 m.

![Figure 8.12. The Linear Quadratic Compensator performance.](image)

The kind of surveillance for which Bombus is primarily designed only requires
the craft to hold one position. Therefore, simulations were made to determine the
control systems ability to hold the craft at one point in space. The result can
be seen in Figure 8.13. The wind is the same as in the simulations above. It is
clear that the control system is unable to keep the craft completely motionless but
the performance of the control system is satisfactory since the maximum position
error is no larger than the diameter of the craft. The reference value for $z_i$ is also
kept to zero resulting in Figure 8.14.

To explain the fact that the control system is unable to keep the craft at the
desired position the estimations from the observer can be studied. Even if all states
are available in the simulations only the states available in the physical system are
used. To analyse the Linear Quadratic Compensator the estimated states can be
compared to the actual states available in the simulations. The difference between
some of the estimated states and the corresponding actual states for the simulation
with a constant reference signal equal to zero can be seen in Figure 8.15. The figure
shows that the measured states $x$ and $y$ are well estimated. However, $z$, roll, pitch
and yaw are measured as well, but are not estimated as good as $x$ and $y$. The
explanation is that the rudders are positioned slightly off from the initial position
to counteract the wind. In the model used for computing the Estimator the wind
is not included more than as a Gaussian noise. The Estimator estimates the states
at the next time point knowing only the present value of the states and the signals sent to the rudders and the fan. These estimations will differ from the actual states since the actual states will also be affected by the mean value wind. The Estimator minimizes the variance of the estimation error.
Figure 8.15. Estimation errors of interesting states.

8.7 Other Controllers

The above mentioned controllers have been implemented, simulated and some of them have been tested on the physical system. The potential performance of some alternative controllers have been discussed. These controllers have because of lack of time never been implemented and their performance have therefore not been compared to the performance of the previously discussed controllers.

8.7.1 Model Predictive Control

Model predictive control, MPC, is a modern control method that uses knowledge of the future state of the system to calculate an optimal control law. MPC has only been available in commercial applications for about 20 years [4]. This is due to the fact that MPC requires a large number of calculations each sample and the computers have not been fast enough until recently. Over the last few years, computers have become fast enough to implement MPC in systems even with very fast dynamics. In very slow systems, like for example the processes in an oil refinery, MPC has been used for quite some time.

MPC has several advantages compared to LQ-control. The main advantage is MPC’s ability to handle constraints on the output signals. The LQ-control sys-
tem can have limited signals but the control system will not take this limitations under consideration. The MPC control system, on the other hand, will take the limitations under consideration and compute an optimal control law. The main disadvantage of MPC compared to LQ is that MPC require more online calculations.

MPC Method
The MPC algorithm is [1]

1. Measure/estimate the current states.
2. Calculate a control sequence using the optimization described below.
3. Apply the first element of the control sequence to the system.
4. Time update \( k = k + 1 \).
5. Go to step one.

The procedure to calculate the control sequence will now be described. A discrete model can be written as

\[
\begin{align*}
x(k + 1) &= Ax(k) + Bu(k) \\
y(k) &= Cx(k)
\end{align*}
\]

where \( x, y, \) and \( u \) can be vectors if the system is multidimensional. As the name of the method implicates a prediction of the future has to be made. A prediction of the future system is calculated as

\[
x(k + 2) = Ax(k + 1) + Bu(k + 1) = A^2x(k) + ABu(k) + Bu(k + 1)
\]

These equations can be written as

\[
X = Hx(k) + SU
\]

where

\[
X = \begin{pmatrix}
x(k + 1) \\
x(k + 2) \\
\vdots \\
x(k + M_{MPC})
\end{pmatrix}
\]

\[
H = \begin{pmatrix}
A \\
A^2 \\
\vdots \\
A^{M_{MPC}}
\end{pmatrix}
\]

\[
U = \begin{pmatrix}
u(k) \\
u(k + 1) \\
\vdots \\
u(k + M_{MPC} - 1)
\end{pmatrix}
\]
For MPC to be effective the model has to have a prediction horizon. The prediction horizon is the number of time samples into the future the states are predicted. The choice of prediction horizon is important to get the desired behaviour of the control system and should be made so that the horizon covers an ordinary settling \[1\]. If the prediction horizon is chosen to short the controller will not be able to predict the entire settling and thereby not include enough of the model dynamics in the computations to be able to obtain a good enough control law. On the other hand, if the prediction horizon is chosen too large, the number of calculations needed will be large. The prediction horizon is denoted \(M_{\text{MPC}}\).

The control horizon, denoted \(N_{\text{MPC}}\), is often \(M_{\text{MPC}} - 1\) but if it is possible, an as small as possible \(N_{\text{MPC}}\) is desired since this would lessen the complexity of the calculations.

The function

\[
\sum_{j=0}^{\infty} ||x(k + j + 1)||_{\tilde{Q}_1}^2 + ||u(k + j)||_{\tilde{Q}_2}^2
\]

is the function minimized in a LQ-controller. Replacing \(\infty\) with the prediction horizon minus one will give the function to minimize in MPC. Introducing the matrices

\[
\tilde{Q}_1 = \begin{bmatrix} Q_1 & Q_1 & \cdots & Q_1 \\ Q_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & Q_1 \end{bmatrix}, \quad \tilde{Q}_2 = \begin{bmatrix} Q_2 & \cdots & \cdots & Q_2 \\ \vdots & \ddots & \ddots & \vdots \end{bmatrix}
\]

The matrices \(Q_1\) and \(Q_2\) are design parameters or weight matrices indicating the importance of the different \(x\) and \(u\).

If the prediction horizon is inserted in (8.29) it can be rewritten as

\[
\sum_{j=0}^{M_{\text{MPC}} - 1} ||x(k + j + 1)||_{\tilde{Q}_1}^2 + ||u(k + j)||_{\tilde{Q}_2}^2
\]

\[
= X^T \tilde{Q}_1 X + U^T \tilde{Q}_2 U
\]

\[
= (Hx(k) + SU)^T \tilde{Q}_1 (Hx(k) + SU) + U^T \tilde{Q}_2 U
\]
Introducing a reference signal, $r$, in (8.30) gives

$$\sum_{j=0}^{M_{\text{MPC}}-1} ||y(k+j+1) - r(k+j+1)||_Q_1^2 + ||u(k+j)||_Q_2^2 =$$

$$= (\tilde{C}(Hx(k) + SU) - R)^T \tilde{Q}_1 (\tilde{C}(Hx(k) + SU) - R) + U^T \tilde{Q}_2 U =$$

$$= (\tilde{C}Hx + \tilde{C}SU - R)^T \tilde{Q}_1 (\tilde{C}Hx + \tilde{C}SU - R) + U^T \tilde{Q}_2 U =$$

$$= (\tilde{C}Hx - R)^T \tilde{Q}_1 (\tilde{C}Hx - R) + 2(\tilde{C}Hx - R)^T \tilde{Q}_1 \tilde{C}SU +$$

$$+ U^T S^T \tilde{C}^T \tilde{Q}_1 \tilde{C}SU + U^T \tilde{Q}_2 U$$

where

$$\tilde{C} = \begin{bmatrix} C & C & \ldots & C \\ \vdots & \ddots & \cdots & \vdots \\ C & \ldots & \ldots & C \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} r(k+1) \\ r(k+2) \\ \vdots \\ r(k + M_{\text{MPC}}) \end{bmatrix}$$

and $C$ is the matrix from (8.27).

A standard Quadratic Programming, or QP, problem can be written as

$$\min_x \frac{1}{2} x^T Hx + f^T x$$

subject to $Ax \leq b$ (8.32)

The expression (8.31) can be rewritten to fit the form of (8.32) by removing terms not depending on $U$ and dividing by two.

$$\min_U \frac{1}{2} U^T (S^T \tilde{C}^T \tilde{Q}_1 \tilde{C}S + \tilde{Q}_2) U + (\tilde{Q}_1 \tilde{C}^T S^T (\tilde{C}Hx - R)) U$$

subject to $U \leq U_{\text{limit}}$ (8.33)

If the constraints are

$$-y_{i,\text{max}} \leq y_i \leq y_{i,\text{max}} \quad i = 1, 2, \ldots, n$$

(8.34)

the constraints for the entire system can easily be expressed. Since $Y = \tilde{C}(Hx(k) + SU)$ according to (8.28) and (8.30) the constraints in (8.34) can be written as

$$\begin{bmatrix} I & -I \end{bmatrix} \tilde{C}(Hx(k) + SU) \leq \begin{bmatrix} Y_{\text{max}} \\ Y_{\text{max}} \end{bmatrix} \Rightarrow$$

$$\begin{bmatrix} I & -I \end{bmatrix} \tilde{C}SU \leq \begin{bmatrix} Y_{\text{max}} \\ Y_{\text{max}} \end{bmatrix} - \begin{bmatrix} I \\ -I \end{bmatrix} \tilde{C}Hx(k) \Rightarrow$$

$$A_y U \leq b_y$$

(8.35)

This will give the optimization

$$\min_U \frac{1}{2} U^T (S^T \tilde{C}^T \tilde{Q}_1 \tilde{C}S + \tilde{Q}_2) U + (\tilde{Q}_1 \tilde{C}^T S^T (\tilde{C}Hx - R)) U$$

subject to $A_y U \leq b_y$ (8.36)
8.7 Other Controllers

MPC for Bombus

The Bombus model consists of nineteen states. Apart from the nine states in Section 7.4 the roll, pitch and yaw angles and the rudder angles are also used as states. The model is nonlinear and since only linear MPC will be investigated a linearization (see Section 8.2) has to be made. Denote the linearized discrete model

\[ x(k + 1) = A_L x(k) + B_L u(k) \]
\[ y(k) = C_L x(k) \] (8.37)

The matrices in the linearized discrete model above can be computed using MatrixX. The optimization problem will look like (8.36), but with \( A_L, B_L \) and \( C_L \) instead of \( A, B \) and \( C \).

There are no physical constraints on the output signals for Bombus. Since the model is linearized, constraints can be put on the signals to limit the movement of the craft in order to keep it within an area where the linearized model is acceptable. The pitch and roll angles, for example, can be restrained to \( -\pi/18 \leq \text{roll} \leq \pi/18 \). If only the restraints on roll and pitch are considered the matrix \( [I, -I]^T \) in (8.35) must be modified so that all terms except those multiplied with roll and pitch are zero.

The MPC-controller puts high demands on the calculation capability of the computer since the QP-problem (8.36) has to be solved in every time sample. It is possible that the processor used on Bumbus does not have the required calculation capacity. With a new faster processor installed it is not unlikely that the MPC-controller will perform better than the LQ and PID controllers in many ways. Another solution to the problem with the lack of computational capability is Explicit MPC or EMPC. EMPC performs some of the required calculations offline reducing the number of calculations needed to be performed online [3]. EMPC will not be further investigated in this report.

8.7.2 Backstepping

In the late 1980’s a new method for nonlinear control was introduced. The method is based in on the Lyapunov theory and it is called backstepping. As the name suggests backstepping uses a recursive algorithm to obtain a control law as well as a proof of stability. The recursive algorithm gives a systematic way of calculating a Lyapunov function used to determine this stability. Unlike most other control methods for nonlinear systems, backstepping does not necessarily try to remove or linearize nonlinearities. Instead, backstepping uses the stabilizing properties of the nonlinearities to obtain a control law that might require less control effort compared to a method linearizing the nonlinearities.

Lyapunov Theory

Lyapunov functions are used to determine stability for nonlinear systems by examining the distance between \( x \) and a stable equilibrium \( x_0 \). First we must consider
a system $\dot{x} = f(x)$. A function

$$W(x_0) = 0 \quad W(x) > 0, \ x \neq x_0 \quad W_x(x)f(x) \leq 0$$  \hspace{1cm} (8.38)

where $W_x = (\partial W/\partial x_1, \ldots, \partial W/\partial x_n)$ can be interpreted as the distance between $x$ and $x_0$, is called a local Lyapunov function. If the Lyapunov function also satisfies

$$\dot{W} = W_x f(x) < 0, \ x \neq x_0 \quad \text{and} \quad W(x) \to \infty, \ |x| \to \infty$$  \hspace{1cm} (8.39)

the point $x = x_0$ is said to be a globally asymptotically stable equilibrium [6, 9]. If a new nonlinear system

$$\dot{x} = f(x, u)$$  \hspace{1cm} (8.40)

is considered, a control law

$$u = k(x)$$  \hspace{1cm} (8.41)

can be constructed for the system. To determine whether the control law is a globally stabilizing control law, a Lyapunov function for the system using that control law can be constructed. This Lyapunov function is called a control Lyapunov function, or a clf. A Lyapunov function satisfying (8.38) and

$$\inf_u W_x f(x, u) < 0, \ x \neq 0$$  \hspace{1cm} (8.42)

is a clf [9]. If a clf can be constructed the control law is globally stabilizing and if the control law is globally stabilizing a clf can always be constructed.

Control using Backstepping

The Lyapunov control described above requires that the clf is known. Below backstepping as described in [9] will be used to obtain a control law and a clf. Consider the system

$$\dot{x} = f(x, \xi)$$
$$\dot{\xi} = u$$  \hspace{1cm} (8.43)

where $x$ and $\xi$ are state variables and $u$ is the control input. A virtual control law can be constructed

$$\xi = \xi^{\text{des}}(x)$$  \hspace{1cm} (8.44)

making $x = 0$ a globally asymptotically stable equilibrium. The stability can be shown with a clf, $W_1(x)$. The clf should be choosen such that (8.43) and (8.44) satisfies

$$\dot{W}_1_{|\xi = \xi^{\text{des}}} = W_{1,x}(x)f(x, \xi^{\text{des}}) < 0, \ x \neq 0$$  \hspace{1cm} (8.45)

A residual can be constructed as

$$\tilde{\xi} = \xi - \xi^{\text{des}}(x)$$  \hspace{1cm} (8.46)
and the control input \( u \) can be used to steer \( \dot{\xi} \) to zero which in turn results in the desired \( \xi = \xi^{\text{des}} \). If (8.46) is inserted into the system (8.43) the result is

\[
\dot{x} = f(x, \dot{\xi} + \xi^{\text{des}}) \triangleq f(x, \xi^{\text{des}}(x)) + \Psi(x, \dot{\xi})\dot{\xi} \\
\dot{\xi} = u - \frac{\partial \xi^{\text{des}}}{\partial x}(x)f(x, \dot{\xi} + \xi^{\text{des}}(x))
\] (8.47)

where \( f(x, \xi^{\text{des}}) \) is the desired dynamics of the system and

\[
\Psi(x, \dot{\xi}, \dot{\xi}) = \frac{f(x, \xi^{\text{des}}(x)) - f(x, \xi^{\text{des}})}{\dot{\xi}}
\]

For the augmented system (8.47) a newclf can be obtained using \( W_1(x) \) and weighting \( \dot{\xi} \). In [9]

\[
W_2(x, \dot{\xi}) = W_1(x) + \frac{1}{2} \dot{\xi}^2
\] (8.48)

is used. According to (8.39), \( \dot{W}_2 < 0 \) for the control law to be a globally stabilizing control law. Differentiating (8.48) with help from (8.47) gives

\[
\dot{W}_2 = W_{1, x}(x) \left[ f(x, \dot{\xi} + \xi^{\text{des}}) + \frac{\partial \xi^{\text{des}}}{\partial x}(x)f(x, \dot{\xi} + \xi^{\text{des}}(x)) \right] + \xi \left[ u - \frac{\partial \xi^{\text{des}}}{\partial x}(x)f(x, \dot{\xi} + \xi^{\text{des}}(x)) \right] = \\
= W_{1, x}(x) f(x, \xi^{\text{des}}(x)) + \dot{\xi} \left[ W_x(x)\Psi(x, \dot{\xi}) + u - \frac{\partial \xi^{\text{des}}}{\partial x}(x)f(x, \dot{\xi} + \xi^{\text{des}}(x)) \right]
\] (8.49)

The term \( W_{1, x}(x) f(x, \xi^{\text{des}}(x)) \) in (8.49) is according to (8.45) negative if \( x \neq 0 \). To satisfy (8.39), \( u \) in (8.49) has to be carefully chosen. By selecting

\[
u = -W_{1, x}(x)\Psi(x, \dot{\xi}) + \frac{\partial \xi^{\text{des}}}{\partial x}(x)f(x, \dot{\xi} + \xi^{\text{des}}(x)) - \dot{\xi}
\] (8.50)

all terms inside the last brackets in (8.49) will be cancelled out except \(-\dot{\xi}\). This results in

\[
\dot{W}_2 = W_{1, x}(x) f(x, \xi^{\text{des}}) - \dot{\xi}^2 < 0, \ x \neq 0, \ \dot{\xi} \neq 0
\] (8.51)

Now both a globally stabilizing control law, \( u \), and a Lyapunov function, \( W_2 \), has been calculated and can be used for control of the nonlinear system.

**Backstepping applied to Bombus**

Backstepping for rigid bodies as explained in [9] can easily be applied to the Bombus model. The first six equations in (7.30) in Section 7.4 can be rewritten to fit the form presented in [9]. Then the model can be written as

\[
\dot{V}_c = -\omega \times V_c + \frac{1}{m} F \\
I\ddot{\omega} = -\omega \times I\omega + T
\] (8.52)
where $V_c$ is the velocity vector, $\omega$ is the angular velocity vector, $T$ is the torque vector, $F$ is the force vector and $I$ is a matrix containing the moment of inertia about the three axes. This means that

$$V_c = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, \quad \omega = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

$$I = \begin{pmatrix} I_x & 0 & 0 \\ 0 & I_y & 0 \\ 0 & 0 & I_z \end{pmatrix}, \quad T = \begin{pmatrix} L \\ M \\ N \end{pmatrix} \tag{8.53}$$

To perform the backstepping procedure $F$ has to be divided into two parts

$$F = m(F_{\text{aero}}(V_c) + u_V \dot{V}_c) \tag{8.54}$$

where

$$F_{\text{aero}} = \begin{pmatrix} X \\ Y \\ Z \end{pmatrix}, \quad \dot{V}_c = \frac{1}{|V_c|} V_c$$

and $u_V$ is a control parameter. The last part of (8.54) is an approximation of the thrust of an engine in the direction of $V_c$. For simplicity the effect of gravity has been included in $X Y$ and $Z$ in $F_{\text{aero}}$. If (8.52) – (8.55) are combined, the equations in Section 7.4 are obtained.

To perform the backstepping algorithm for the rigid body a steady state, described by $V_0$ and $\omega_0$, is required. In the Bombus case, the trivial steady state $V_0 = (0 0 0)^T$ and $\omega_0 = (0 0 0)^T$ is the the one best suited. To start the backstepping algorithm the second equation in (8.52) can be rewritten as

$$\dot{\omega} = I^{-1}(T - \omega \times I \omega) = u_T \tag{8.56}$$

resulting, together with (8.52) and (8.54), in

$$\dot{V}_c = -\omega \times V_c + F_{\text{aero}}(V_c) + u_V \dot{V}_c$$

$$\dot{\omega} = u_T$$

In [9], a virtual control law $\omega = \omega^{\text{des}}$ is introduced together with the Lyapunov function

$$W_1 = \frac{1}{2}(V_c - V_0)^T (V_c - V_0) \tag{8.58}$$

Combining the suggested Lyapunov function, the suggested virtual control law and (8.57) gives

$$\dot{W}_1 = (V_c - V_0)^T (-\omega^{\text{des}} \times V_c + F_{\text{aero}}(V_c) + u_V \dot{V}_c) \tag{8.59}$$

To cancel out $F_{\text{aero}}(V_c)$ and to achieve $\dot{W} \leq 0$ the choices

$$u_V = k_1 (V_0 - V_c)^T \dot{V}_c - \dot{V}_c^T F_{\text{aero}}(V_c)$$
are logical, giving
\[
\dot{W}_1 = -k_1((V_0 - V_c)^T \dot{V}_c)^2 - k_2 |V_0 \times V_c|^2 \triangleq -U(V_c, V_0) \leq 0 \quad (8.61)
\]

The next step in the backstepping method is to construct the residual
\[
\xi = \omega - \omega^\text{des}
\]

Inserting the above residual in the system (8.57) results in
\[
\begin{align*}
\dot{V}_c &= -\omega^\text{des} \times V_c + F_a\text{ero}(V_c) + u_v \dot{V} - \xi \times V_c \\
\dot{\xi} &= u_T + \Phi(V_c, \xi) 
\end{align*}
\]
(8.62)

where
\[
\Phi(V_c, \xi) = \frac{d}{dt} \omega^\text{des}
\]

In the first equation of (8.62) only the last cross product is not a part of the desired dynamics of the system. Like in (8.48) the existing Lyapunov function now has to be extended to
\[
W_2 = k_3 W_1 + \frac{1}{2} \xi^T \xi
\]
giving the derivative
\[
\begin{align*}
\dot{W}_2 &= \\
&= k_3 (V_c - V_0)^T (-\omega^\text{des} \times V_c + F_a\text{ero}(V_c) + u_v \dot{V} - \xi \times V_c) \\
&\quad + \xi^T (u_T - \Phi(V_c, \xi)) 
\end{align*}
\]
(8.63)

If the control law
\[
u_T = k_3 (V_0 \times V_c) - k_4 \xi + \Phi(V_c, \xi)
\]
is used (8.63) becomes
\[
\dot{W}_2 = -k_3 U(V_c, V_0) - k_4 \xi^T \xi \leq 0
\]
where \(U(V_c, V_0)\) is defined in (8.61). The last equality in (8.63) is calculated through major simplifications. These simplifications and the complete calculations from the section above can be seen in [9]. The control laws now calculated together with the Lyapunov function makes the system converge to \(V_c = V_0\) and \(\xi = 0\).

Backstepping is well suited for use in airplane control and can also seem to be well suited for Bombus. When calculating the backstepping equations for Bombus one realize soon that the first equation in (8.60) will cause a singularity as the magnitude of the velocity \(|V_c|\) approaches the desired steady state \(V_0 = (0 \ 0 \ 0)^T\). This problem has to be solved before backstepping can be used on Bombus.
8.8 Comparing Controller Performance

From the discussions about the performance of the different control systems it is easy to see a clear difference. The inner PID/outer PID control system performs good with a position error of only a few decimeters while the inner LQ/outer PID control systems perform poorly with a position error of sometimes almost one meter. The LQ Compensator performs satisfactory with a position error of less than one meter.

8.8.1 Reasons for Choosing the LQ Controller

Both the MPC and backstepping controllers have the potential to give a good performance for Bombus. Both controllers also have disadvantages in their use on Bombus compared to LQ and PID. The main reason for choosing the LQ and PID controllers instead of the other controllers is the simplicity to implement them. Several commands exist in MatrixX simplifying the implementation of both an LQ and a PID controller. Simplicity was preferred to possible better performance both because of the limited time available and because of the wishes from DST Control AB.
Chapter 9

Results

The goals of this thesis were to examine the performance of a ducted fan UAV regarding lift capacity, position accuracy and wind tolerance. The performance was to be tested with help of a control system that had to be constructed and implemented. A mathematical model should be constructed to facilitate the design of the control system. This chapter will discuss whether the goals are achieved or not.

9.1 Conclusions

The overall conclusion is that the goals are achieved. A mathematical model has been developed and used to design two different control systems. The model is verified with different experiments, for example lift force experiments, which also gave an opinion on the lift force performance. Simulations are made with the two control systems at the model to examine position accuracy and wind tolerance.

9.1.1 Discussion regarding Lift Force

Lift force experiments have been made while designing the mathematical model. The conclusion regarding lift force is that the craft needs more lift force to be able to lift cables, mission equipment and its own mass. Chapter 4 gives a detailed investigation of the lift force.

9.1.2 Discussion regarding the Model

The mathematical model has been developed using physical modeling. Most of the physical relations used in the model are either trivial or has been verified in experiments with the prototype in a test rig. The test rig has been developed during this thesis as well. We believe that the mathematical model describes the ducted fan UAV good at free flight, but not as good in the test rig. No friction or moment of inertia of the test rig has been modeled. However the test rig has
been very useful in getting an understanding of the UAV’s performance. Chapter 7 gives a detailed description of the mathematical model.

9.1.3 Discussion regarding the Control Systems

Two different control systems have been designed. One of them is constructed as a cascade of two control systems, outer and inner. The other control system uses Linear Quadratic control tools.

Cascaded Controllers

One of the control systems uses straightforward PID methods. In this case two of the rudders are controlled by a PID regulator controlling roll, the other two rudders are controlled by another PID regulator controlling pitch. The signals from the two mentioned PID regulators are added together with the signal from a third PID regulator controlling yaw rotation. This control system manages to both keep the system stable and follow a reference signal with satisfactory performance.

An additional PID controller is designed to control the reference signals in roll, pitch and yaw. In this way the position can be controlled. For example, a negative pitch angle change the direction of the force generated by the propeller and a component in positive x direction occurs. In a similar way roll can be used to regulate the position in y. When the force generated by the propeller gets a horizontal component the lift force is reduced. Another PID regulator is designed to regulate the position in z by varying the fan speed. This means the fan speed is increased during horizontal movement to keep the craft from descending. This system of PID regulators is referred to as the outer PID controller.

The cascade system of the outer and the inner PID controllers has been tested in simulations with a good result which can be seen in Section 8.1.4.

An alternative control system for controlling roll, pitch and yaw is designed using Linear Quadratic control tools. The performance of this control system is worse than the PID control system especially when a reference signal is followed. The worse performance is explained by the fact that the constant estimator matrix is chosen based on a linearization made at operating point with all angles zero. At larger angles the linearization is not as concordant with the model as for small angles. This means that the LQ controller is good at keeping the craft stable but since movement is achieved by pitching or rolling the craft the controller performs worse when a reference signal is followed. The result of a simulated flight can be seen in Section 8.1.4.

Unfortunately the test rig is in a rather bad shape which generates some problems validating the control systems at the actual system. However, disregarding the problems caused by the test rig, experiments show that the actual system follows a reference signal at roll or pitch quite good. Only the inner control systems can be tested at the actual system, since there are no sensors measuring position and the test rig is constructed to keep the craft fixed in space, only allowing rotations.
9.2 Summary

Linear Quadratic Compensator

The other control system is designed using tools from MatrixX to generate an estimator and a regulator. This control system takes only the three inertial position coordinates as reference and tries to keep the rest of the states to zero. This means that this control system only can be tested in simulations. This has been done with quite good results. Simulated experiments with varying reference positions has shown good performance.

Chapter 8 gives a detailed description of the outer and inner control systems and also discusses alternatives.

9.1.4 Discussion regarding Wind Tolerance

The wind tolerance has not been tested in experiments at the actual system. A model of the wind has been implemented in the mathematical model and in the simulated experiments a slight wind has been included.

It has been shown that the systems original configuration was not able to compensate for any wind at all. To get the ability to keep the system stable even with a wind disturbance, the forces from the rudders had to be increased without increasing their wind exposure. The actual system has been equipped with twin rudders to make the rudder area exposed for the wind induced by the propeller (vertical) larger but leaving the rudder area exposed for the disturbance wind (horizontal) the same. This modification of the UAV was implemented in the model as well and the ability to compensate for disturbance wind was increased.

The size of the wind used in the simulated experiment is still not as large as desired and raising it makes the system unstable. There is no solution to this problem with the current configuration. Increasing all dimensions in a future prototype would probably also increase the wind tolerance as well as the lift force.

Chapter 6 more deeply discusses wind and other noises affecting the UAV.

9.2 Summary

The goals of the thesis are achieved with varying, mostly positive, results. A continued development of the concept will be considerable facilitated by the guide lines given by this report.
Chapter 10

Future work

If DST Control AB decides to continue the development of the ducted fan UAV there are several different areas where improvements can be made.

10.1 The Craft

The current prototype is sufficient for the experiments conducted in this project but if more advanced testing, like free flight tests, shall be made the construction of a new prototype is recommended. The design of the new prototype ought to be developed with a more solid theoretical ground than the current prototype. The duct can be constructed in a way that increases the lift without sacrificing any of the benefits of the current duct.

Both the propeller and the rudders on the current prototype are far from optimal. A theoretical study could result in a propeller with the ability to generate larger lift without a larger radius and rudders with the ability to generate larger force without a larger area.

10.2 Sensors

New sensors for the prototype is essential for more advanced experiments with a more advanced test rig or in free flight. Most of the sensors currently used have a low resolution and are very sensitive to noise. The pitch and roll sensors are low price rotational potentiometers placed on the test rig in a way that decreases the accuracy further. A set of sensors consisting of high precision gyros and accelerometers are intended for the final product. These sensors would give better measurements but require some calculations to measure the correct angles.

10.3 Test Rig

Test Rig Beta was built with a minimal budget resulting in a test rig constructed by inferior materials and with poor size precision. This has resulted in a gimbal
with far too flexible rings and axles which are not centered. The flexibility of the rig has affected the experimental results in a negative way which can clearly be seen on the extensive oscillations of the angles in Section 8.1.3. A new test rig with stiffer gimbal rings made of light weight metal could improve the experimental results considerably.

Another option is to construct a completely different test rig. Some suggestions for alternative test rig are given in Chapter 3.

10.4 Controller

Even though two of the resulting controllers implemented in this project perform satisfactory in most cases, better controllers can, without too much effort, be implemented. Another model based controller, like EMPC or backstepping discussed earlier, might perform better than both the LQ and PID controllers. The controller structure might also be improvable to achieve a simpler and more effective structure.

Modifications can be made to the existing controllers to achieve better performance. The current LQ control system uses only a single linearization. Using an Extended Kalman Filter to linearize in every time sample would improve the performance of the controller but it would require more calculation capacity from the processor. The currently used processor might not have the required calculation capacity and would have to be replaced by a more powerful processor.
Bibliography


Appendix A

Getting Euler angles from gimbal angles

In Test Rig Beta the sensors for measuring roll and pitch of the craft actually measure rotation of the outer gimbal ring according to earth, rotation of the inner gimbal ring according to the outer gimbal ring and rotation of the craft according to the inner gimbal ring. These three angels will be called gimbal angles. A way of getting Euler angles from gimbal angels will be found and the method used will be described in this appendix. Observe that notations from the rest of the report does not apply at this Appendix.

A.1 Euler rotation in gimbals

The Euler angles are the three angles describing the craft’s rotation around three axes of the frame $X_b = \{\hat{x}_b, \hat{y}_b, \hat{z}_b\}$ following the movements of the craft. This frame rotates around one axis of the frame $X_{ig} = \{\hat{x}_{ig}, \hat{y}_{ig}, \hat{z}_{ig}\}$ following the movements of the inner gimbal ring. In a similar way this frame rotates around one axis of the frame $X_{og} = \{\hat{x}_{og}, \hat{y}_{og}, \hat{z}_{og}\}$ following the movements of the outer gimbal ring and this frame rotates around one axis in the inertial frame $X = \{\hat{x}_i, \hat{y}_i, \hat{z}_i\}$.

In Test Rig Beta the outer gimbal rotates in $X$ around $\hat{x}$, the inner gimbal in $X_{og}$ around $\hat{z}_{og}$ and the craft in $X_{ig}$ around $\hat{y}_{ig}$. As said earlier this is the rotations measured by the sensors in the rig.

It is easy to believe that the crafts rotation in $X_{ig}$ around $\hat{y}_{ig}$ actually is the Euler angle $\theta$. This is the case only when the gimbals are in their initial position, with all rotation axes perpendicular to each other. Imagine Test Rig Beta with all angles having their initial value. Rotating the craft with the gimbals fixed will now be the same as pitching the craft. Pitch the craft 45° in any direction. Now keep the outer gimbal fixed, fix the craft at 45° related to the inner gimbal and release this gimbal. By rotating this gimbal will now change all Euler angles including pitch despite the craft being fixed to the inner gimbal. The rotation of the craft according to the inner gimbal is not the same as the Euler angle pitch at all time.
With similar imagination one realize that none of the gimbal angles is an Euler angle.

The approach is to first get the rotation of the craft related to earth and second get the Euler angles from this.

### A.2 Quaternions

The following quote comes from [12]:

> The science named Quaternions by its illustrious founder, Sir William Rowan Hamilton, is the last and the most beautiful example of extension by the removal of limitations.

In ordinary complex calculus there is a real part in one dimension and an imaginary part in one dimension. The reader will soon realize that an imaginary part in three dimensions will be useful. The introduction to quaternions in this section of the appendix is written with big help from [12].

Let the imaginary numbers $i$, $j$ and $k$ be unit vectors along Cartesian axes $\hat{x}$, $\hat{y}$ and $\hat{z}$ respectively. See Figure A.1.

![Figure A.1. $i$, $j$ and $k$ as unit vectors in Cartesian frame.](image)

Multiplication will be defined similarly to the definition of a vector product. That is that multiplication of $i$ into $j$ or $ij$ is defined as the turning of $j$ through a right angle in the plane perpendicular to $i$ and in the positive direction. This will give us $ij = k$. All products of $i$, $j$ and $k$ can be defined in the same way.

\[
ij = k \quad \text{(A.1)}
\]
\[
ji = -k \quad \text{(A.2)}
\]
\[
ik = -j \quad \text{(A.3)}
\]
\( ki = j \)  \hspace{1cm} (A.4)
\( jk = i \)  \hspace{1cm} (A.5)
\( kj = -i \)  \hspace{1cm} (A.6)

The \textit{associative law of multiplication} is assumed to be applicable on this aspect of multiplication. This means that it is indifferent in what way operations are grouped as long as the order is not changed. For example \( i(jk) = (ij)k = ijk \). Since \( ij = k \), we have \( i(ij) = ik = -j \). From this the square of a unit vector can be defined by

\[
i(ij) = (ii)j = i^2 j
\]
\[
\therefore i^2 = -j
\]

Similarly
\[
j^2 = -1 \hspace{1cm} (A.8)
\]
\[
k^2 = -1 \hspace{1cm} (A.9)
\]

Any vector can be separated into the product of a tensor (scalar gain) and a unit vector. Let \( \alpha \) be a vector parallel to \( i \) containing \( a \) units and \( \beta \) a vector parallel to \( j \) containing \( b \) units. The traditional laws of numerical multiplication will be used whenever numerical quantities are mixed with vectors.

\[
\alpha = ai \\
\beta = bj \\
\alpha\beta = abij \\
\alpha^2 = (ai)(ai) = a^2i^2 = -\alpha^2
\]  \hspace{1cm} (A.10)

The equations above show the possibility to introduce tensors whenever wanted. Therefore it is enough to direct the attention to unit vectors.

Now let \( \alpha, \beta \) be unit vectors not perpendicular to each other. Let \( \theta \) be the angle from \( \alpha \) to \( \beta \). Let \( \gamma \) be the unit vector perpendicular to \( \alpha \) and \( \beta \) and in the plane defined by \( \alpha \) and \( \beta \). Let \( \epsilon \) be the unit vector perpendicular to \( \alpha, \beta \) and \( \gamma \) as in Figure A.2. In Figure A.2 let \( OA \) and \( OB \) be the vectors from origo to \( A \) and \( B \) respectively. Since \( \alpha \) and \( \beta \) both are unit vectors we have \( OA = \alpha \cos \theta \) and \( OB = \gamma \sin \theta \). With \( \gamma, \beta \) and \( \epsilon \) corresponding to \( j, i \) and \( -k \) respectively the following is given:

\[
\alpha = \beta \cos \theta + \gamma \sin \theta \\
\alpha\beta = (\beta \cos \theta + \gamma \sin \theta)\beta = \\
= \beta^2 \cos \theta + \gamma \beta \sin \theta
\]  \hspace{1cm} (A.11)
But according to (A.2) and (A.7) \( \gamma \beta = \epsilon \) and \( \beta^2 = -1 \) so

\[
\alpha \beta = -\cos \theta + \epsilon \sin \theta \tag{A.12}
\]

If \( \alpha \) and \( \beta \) are not unit vectors, but contain \( a \) and \( b \) units respectively, the product is according to (A.10)

\[
\alpha \beta = ab(-\cos \theta + \epsilon \sin \theta) \tag{A.13}
\]

The quotient \( \frac{\beta}{\alpha} \) is defined such that when it operates on \( \alpha \) it produces \( \beta \).

\[
\frac{\beta}{\alpha} = \beta \tag{A.14}
\]

The quotient \( \frac{\beta}{\alpha} \) is therefore the multiplier which, operating on \( \beta \cos \theta + \gamma \sin \theta \) (\( \alpha \) according to (A.11)), produces \( \beta \). But with \( \cos \theta + \epsilon \sin \theta \) operating on \( \alpha \) we have

\[
(\cos \theta + \epsilon \sin \theta)(\beta \cos \theta + \gamma \sin \theta) = \beta \cos^2 \theta + (\gamma + \epsilon \beta) \sin \theta \cos \theta + \epsilon \gamma \sin^2 \theta \tag{A.15}
\]

According to (A.4) and (A.6) \( \epsilon \beta = -\gamma \) and \( \epsilon \gamma = \beta \). This inserted into (A.15) gives

\[
(\cos \theta + \epsilon \sin \theta)(\beta \cos \theta + \gamma \sin \theta) = \beta \cos^2 \theta + \beta \sin^2 \theta = \beta \tag{A.16}
\]

Combining this with (A.14) gives

\[
\frac{\beta}{\alpha} = \cos \theta + \epsilon \sin \theta \tag{A.17}
\]

Since \( \frac{\beta}{\alpha} \) operating on \( \alpha \) causes it to become \( \beta \) it is clear that it is an operator of the same character as \(-k\) or \( \epsilon \). While \(-k\) or \( \epsilon \) would turn \( \alpha \) through a right angle, \( \frac{\beta}{\alpha} \) or \( \cos \theta + \epsilon \sin \theta \) turns it in the same direction, only through the angle \( \theta \). The expression \( \cos \theta + \epsilon \sin \theta \) is defined as the *versor* through the angle \( \theta \).
Looking at (A.10) again gives the quotient of two non unit vectors $\beta$ and $\alpha$ containing $b$ and $a$ units respectively.

$$\frac{\beta}{\alpha} = \frac{b}{a} (\cos \theta + \epsilon \sin \theta) \quad (A.18)$$

Apparently the product or quotient of two vectors may be expressed as the product of a tensor and a versor. Sir William Hamilton named this product a Quaternion.[12]

Every quaternion, $q$, can be written in polar form.

$$q = \sqrt{N_q} (\cos \theta + \hat{q} \sin \theta) \quad (A.19)$$

where $\hat{q}$ is a unit vector and $N_q$ is the norm of $q$. The basic algebraic form for a quaternion, $q$, is though

$$q = 1 q_0 + i q_1 + j q_2 + k q_3 \quad (A.20)$$

The following equations describes the relation between the two forms.

$$N_q = q_0^2 + q_1^2 + q_2^2 + q_3^2, \quad \cos \theta = \frac{q_0}{\sqrt{N_q}}, \quad \sin \theta = \frac{\sqrt{q_1^2 + q_2^2 + q_3^2}}{\sqrt{N_q}}$$

If $N_q = 1$ the quaternion is called a unit quaternion. Hamilton defined $S(q) = q_0$ as the scalar part and $V(q) = i q_1 + j q_2 + k q_3$ as the vector part of the quaternion $q$. That is

$$q = S(q) + V(q) \quad (A.21)$$

By analogy with complex numbers the conjugate of $q$ is

$$\overline{q} = S(q) - V(q) \quad (A.22)$$

If a quaternion, $q$ has a nonzero norm, $N_q$ its inverse is defined as

$$q^{-1} = \frac{\overline{q}}{N_q} \quad (A.23)$$

which gives that

$$qq^{-1} = \frac{(S(q) + V(q))(S(q) - V(q))}{N_q}$$

$$= \frac{S(q)^2 - V(q)^2}{N_q}$$

$$= \frac{q_0^2 - (i q_1 + j q_2 + k q_3)^2}{N_q} = \ldots$$

$$= \frac{q_0^2 - (-q_1^2 - q_2^2 - q_3^2)}{N_q} = 1 \quad (A.24)$$
A.3 Quaternions and three-dimensional rotations

In this section it will be shown that the transformation

$$\phi_q(x) \equiv x' = qxq^{-1} \quad (A.25)$$

with

$$x = \sqrt{N_x} (\cos \phi + \hat{x} \sin \phi)$$

and

$$q = \sqrt{N_q} (\cos \theta + \hat{q} \sin \theta)$$

in Figure A.3 geometrically describes a rotation of the vector part of $x$ about the vector part of $q$ through an angle $2\theta$. This section is written with great inspiration from [22]. If

![Figure A.3.](image)

$V(q)$ is the axis of rotation it is left fixed by the transformation

$$\phi_q(V(q)) = qV(q)q^{-1}$$

$$= (S(q) + V(q)) \left[ V(q) \frac{1}{N_q} (S(q) - V(q)) \right]$$

$$= \frac{1}{N_q} (S(q) + V(q)) \left[ (S(q) - V(q))V(q) \right]$$

$$= (S(q) + V(q)) \frac{1}{N_q} (S(q) - V(q))V(q)$$

$$= qq^{-1}V(q) = V(q) \quad (A.26)$$

It is also easy to show that the norm and the scalar part of $x$ is conserved.

$$N_{x'} = N_{qxq^{-1}} = N_xN_x^{-1} = N_x \quad (A.27)$$

since $N_{q^{-1}} = \frac{1}{N_q}$.

$$S(x') = S(q(xq^{-1})) = S((xq^{-1})q) = S(x) \quad (A.28)$$

using $S(qp) = S(pq)$. This gives

$$qxq^{-1} = q(S(x) + V(x))q^{-1} = qS(x)q^{-1} + qV(x)q^{-1} = S(x) + qV(x)q^{-1} \quad (A.29)$$
and with (A.28) this means that \( S(qV(x)q^{-1}) = 0 \) which gives that \( qV(x)q^{-1} \) is a pure quaternion. That is

\[
V(qxq^{-1}) = qV(x)q^{-1}
\]

(A.30)

The vector \( V(x') \) is parallel to \( q\hat{x}q^{-1} \equiv \hat{x}' \). Let \( \hat{p} \) be a unit pure quaternion in the plane with normal \( \hat{q} \). See Figure A.4. With \( \lambda \) being the angle between \( \hat{q} \) and \( \hat{p} \), \( \hat{x}' = \hat{q} cos \lambda + \hat{p} sin \lambda \)

(A.31)

\( \hat{x} \) has a unit norm since

\[
\hat{x}\hat{x} = (\hat{q} cos \lambda + \hat{p} sin \lambda)(-\hat{q} cos \lambda - \hat{p} sin \lambda)
= -\hat{q}^2 cos^2 \lambda - \hat{p}^2 sin^2 \lambda - sin \lambda cos \lambda (\hat{p}\hat{q} + \hat{p}\hat{q})
= cos^2 \lambda + sin^2 \lambda = 1
\]

(A.32)

using \( \hat{q} = \hat{q}^{-1} = -\hat{q} \), \( \hat{p} = \hat{p}^{-1} = -\hat{p} \) and \( \hat{p}\hat{q} = -\hat{q}\hat{p} \) as \( \hat{p} \) and \( \hat{q} \) are perpendicular.

With this it can be said that

\[
\hat{x}' = q\hat{x}q^{-1} = q(\hat{q} cos \lambda + \hat{p} sin \lambda)q^{-1}
= q\hat{q}q^{-1} cos \lambda + q\hat{p}q^{-1} sin \lambda
\]

(A.33)

It will be shown that \( q\hat{q}q^{-1} = \hat{q} \) and \( q\hat{p}q^{-1} \) is a pure quaternion which revolves through an angle \( 2\theta \) about \( \hat{q} \). See Figure A.4 again. The first part is not very hard to show.

\[
V(V(q)\hat{q}) = \frac{1}{2}[V(q)\hat{q} + \hat{q}V(q)] = \frac{1}{2}[V(q)\hat{q} - \hat{q}V(q)] = 0
\]

(A.34)

because \( V(q) \) and \( \hat{q} \) is parallel. Therefore

\[
q\hat{q}q^{-1} = (S(q) + V(q))[S(q)\hat{q} - \hat{q}V(q)] \frac{1}{N_q}
\]

\[
= \frac{1}{N_q} \{\hat{q}S^2(q) + S(q)[V(q)\hat{q} - \hat{q}V(q)] - V(q)\hat{q}V(q)\} - V(q)\hat{q}V(q)
\]

\[
= \frac{1}{N_q} \{\hat{q}S^2(q) - V(q)\hat{q}V(q)\}
\]

(A.35)
But \( V^{-1}(q) = -\frac{V(q)}{N_{V(q)}} \) which gives
\[
V(q)\dot{q}V(q) = -N_{V(q)}V(q)\dot{q}V^{-1}(q) = -N_{V(q)}\dot{q}
\] (A.36)

This with (A.35) gives the wanted relation
\[
qq^{-1} = \frac{1}{N_q}[S^2(q) + N_{V(q)}]\dot{q} = \dot{q}
\] (A.37)

The second part will be shown by using a similar approach.
\[
qq^{-1} = (S(q) + V(q))\dot{p}(S(q) - V(q))\frac{1}{N_q}
\] (A.38)

\[
= \frac{1}{N_q}\dot{p}S^2(q) + S(q)V(q)\dot{p} - \dot{p}V(q) - V(q)\dot{p}V(q)
\]

But \( V(q) \) and \( \dot{p} \) are perpendicular and \( V(q)\dot{p} = -\dot{p}V(q) \).
\[
V(q)\dot{p}V(q) = N_{V(q)}\dot{p}
\] (A.39)

\[
\therefore qq^{-1} = \frac{1}{N_q}[(S^2(q) - N_{V(q)})\dot{p} + S(q)V(q)\dot{p} - \dot{p}V(q)]
\] (A.40)

Showing that \( \ddot{p} = qq^{-1}\dot{q} \) is perpendicular to \( \dot{q} \) is the same as showing that \( \dot{p} \) has been rotated about \( \dot{q} \). To achieve this \( S[q(q^{-1})\dot{q}] = 0 \) has to be shown. Our own assumption of \( \dot{p} \) gives that \( S(\dot{p}\dot{q}) = -S(\dot{p}\dot{q}) = 0 \). So all we need to show is that
\[
S[(V(q)\dot{p} - \dot{p}V(q))\dot{q}] = 0
\] (A.41)
equivalently
\[
S[(\ddot{q})\dot{p} - \dot{p}\ddot{q}]\dot{q} = 0 \quad \text{or} \quad S(\dot{q}\dot{p}) = 0
\] (A.42)

But \( S(\dot{q}\dot{p}) = S(\dot{q}^2\dot{p}) = 0 \) which means that \( \ddot{p} \) is perpendicular to \( \dot{q} \). The angle of rotation can also be determined using
\[
\cos \phi = \frac{S(\dot{p}\dot{q})}{\sqrt{\dot{p}^2\dot{q}}\sqrt{\dot{p}}} = S(\dot{p}\dot{q}) = -S(\dot{p}^2) = -S[qqq^{-1}\dot{p}]
\]
\[
= -\frac{1}{N_q}(S^2(q) - N_{V(q)})S(\dot{p}^2) - \frac{S(q)}{N_q}S[V(q)\dot{p} - \dot{p}V(q)]
\]
\[
= \frac{1}{N_q}(S^2(q) - N_{V(q)}) - \frac{S(q)}{N_q}S(-V(q) - V(q))
\]
\[
= \frac{1}{N_q}(S^2(q) - N_{V(q)})
\]
\[
= \frac{1}{N_q}(N_q \cos^2 \theta - N_q \sin^2 \theta)
\]
\[
= \cos 2\theta
\]
\[
\therefore \phi = 2\theta
\] (A.44)
Finally

\[
\dot{x}' = \dot{q} \cos \lambda + \dot{p}' \sin \lambda
\]

where \(\dot{x}' = \dot{p} \cos 2\theta + \dot{q} \dot{p} \sin 2\theta\). Note that \(\dot{q} \dot{p}\) is perpendicular to both \(\dot{q}\) and \(\dot{p}\). This completes the proof of \(qxq^{-1}\) describing a 3-dimensional rotation of the vector part of \(x\) through an angle \(2\theta\) about the vector part of \(q\).

\section*{A.4 Quaternions in Test Rig Beta}

Let the craft’s initial attitude be described as the vector part of the quaternion \(x_0\). If the craft’s angle according to the inner gimbal ring is \(\xi\) the quaternion describing the craft’s attitude can be expressed \(q_\xi x_0 q_\xi^{-1}\) where

\[
q_\xi = \sqrt{N_{q_\xi}} \left( \cos \frac{\xi}{2} + \hat{q}_\xi \sin \frac{\xi}{2} \right)
\]

and

\[
\hat{q}_\xi = \begin{pmatrix} 0 \\ j \end{pmatrix}
\]

(A.46)

(A.47)

Here \(\hat{q}_\xi\) represents the axis running through the inner gimbal ring, i.e. the craft’s rotational axis. This axis rotates with the inner gimbal ring around the perpendicular axis running through the outer gimbal ring. If the angle between the gimbal rings is \(\zeta\) the craft’s attitude can be expressed \(q_\zeta q_\xi x_0 q_\zeta^{-1} q_\xi^{-1}\) where

\[
q_\zeta = \sqrt{N_{q_\zeta}} \left( \cos \frac{\zeta}{2} + \hat{q}_\zeta \sin \frac{\zeta}{2} \right)
\]

and

\[
\hat{q}_\zeta = \begin{pmatrix} 0 \\ 0 \\ k \end{pmatrix}
\]

(A.48)

(A.49)

The axis running through the outer gimbal ring perpendicular to \(\hat{q}_\xi\) is \(\hat{q}_\zeta\). In analogy with this using \(\vartheta\) as the angle of the outer gimbal ring in the inertial frame the craft’s attitude can be expressed \(q_\vartheta q_\zeta q_\xi x_0 q_\zeta^{-1} q_\xi^{-1} q_\vartheta^{-1}\) where

\[
q_\vartheta = \sqrt{N_{q_\vartheta}} \left( \cos \frac{\vartheta}{2} + \hat{q}_\vartheta \sin \frac{\vartheta}{2} \right)
\]

and

\[
\hat{q}_\vartheta = \begin{pmatrix} i \\ 0 \\ 0 \end{pmatrix}
\]

(A.50)

(A.51)

Obviously \(\hat{q}_\vartheta\) represents the inertial \(x\)-axis. Using \(x_0 = 1\) total rotation of the craft can be expressed as a rather large quaternion expression.
With the quaternion $Q = q_0 + iq_1 + jq_2 + kq_3$ defining the rotation of the craft [13] gives the rather simple transformation to roll, pitch and yaw.

\[
\tan(\text{roll}) = \frac{2(q_0q_1 + q_2q_3)}{q_0^2 - q_1^2 - q_2^2 + q_3^2} \quad (A.52)
\]
\[
\sin(\text{pitch}) = -2(q_1q_3 - q_0q_2) \quad (A.53)
\]
\[
\tan(\text{yaw}) = \frac{2(q_1q_2 + q_0q_3)}{q_0^2 + q_1^2 - q_2^2 - q_3^2} \quad (A.54)
\]
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