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Weak barriers in nonlinear potential theory

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Abstract. We characterize regular boundary points for p -harmonic functions using weak barriers. We use this to obtain some consequences on boundary regularity.

The results also hold for $\mathcal{A}$ -harmonic functions under the usual assumptions on $\mathcal{A}$, and for Cheeger p -harmonic functions in metric spaces.

Key words and phrases: $\mathcal{A}$ -harmonic, barrier, boundary regularity, metric space, nonlinear, p -harmonic, potential theory, regular, weak barrier.

Mathematics Subject Classification (2000): Primary: 31C45; Secondary: 35J65.

1. Introduction

In nonlinear potential theory barriers have been used to characterize regularity. In this paper we characterize regularity using weak barriers.

Let us be more precise. We let $1 < p < \infty$ and let μ be a p -admissible weight on $\mathbf{R}^n$, $n \geq 2$, see Chapters 1 and 20 in Heinonen–Kilpeläinen–Martio [12]. We also assume that $\mathcal{A}$ satisfies the degenerate ellipticity conditions (3.3)–(3.7) on p. 56 of [12], and refer to [12] for various definitions (including the definitions of $\mathcal{A}$ -harmonic and $\mathcal{A}$ -superharmonic functions). See also Section 2 for some definitions.

The following is our main result.

Theorem 1.1. *Let $\Omega \subset \mathbf{R}^n$ be a nonempty open set. Then $x_0 \in \partial\Omega$ is a regular boundary point if and only if there is a weak barrier at x_0 .*

This weak barrier characterization was conjectured in Conjecture 6.3 in Björn–Björn [5]. We use this characterization to obtain the following result.

Theorem 1.2. *Let $\Omega \subset \mathbf{R}^n$ be a nonempty open set. Then $x_0 \in \partial\Omega$ is regular with respect to Ω if and only if it is regular with respect to every component $G \subset \Omega$ such that $x_0 \in \partial G$.*

Note that a particular consequence is the following result.

Corollary 1.3. *Let $\Omega \subset \mathbf{R}^n$ be a nonempty open set. Let $x_0 \in \partial\Omega$ and assume that there is no component G of Ω such that $x_0 \in \partial G$. Then x_0 is regular.*

Note that even if Ω is connected, Theorem 1.2 and Corollary 1.3 can be of use if Ω is not locally connected at x_0 , as regularity is a local property, see Proposition 9.9 in [12].

The proofs of Theorem 1.1 in linear potential theory all seem to use the linearity in a fundamental way and thus do not generalize to the nonlinear case. On the other hand the proof of Theorem 1.2 below is essentially the same as in Armitage–Gardiner [1], Theorem 6.6.7.

In Section 2 we prove Theorems 1.1 and 1.2, and in Section 3 we use the weak barrier characterization to improve the understanding of the boundary behaviour

of $\mathcal{A}$ -harmonic functions. Finally, in Section 4, we show that the results in this paper generalizes to Cheeger p -harmonic functions on metric spaces. Observe that Cheeger p -harmonic functions and p -harmonic functions based on upper gradients (as defined in [5]) coincide whenever p -harmonic functions can be defined using a vector-valued structure, e.g. on $\mathbf{R}^n$, manifolds, graphs, the Heisenberg group and other Carnot–Carathéodory spaces, see, e.g., A. Björn [3] for examples of metric spaces for which this theory applies.

It is worth mentioning that there are several different definitions of barriers in the literature. What we call barriers are called barriers in Doob [10], Kellogg [13] and Wermer [19], as well as throughout the nonlinear potential theory, e.g. in [5] and [12]. What we call weak barriers are called barriers in Axler–Bourdon–Ramey [2] and weak barriers in Doob [10] and Wermer [19]. In Tsuji [18] a barrier is something that is locally a barrier in our sense. In many modern treatments of linear potential theory a barrier is locally a weak barrier in our sense, see Armitage–Gardiner [1], Constantinescu–Cornea [9], Helms [11] and Landkof [15].

It should be observed that in the linear potential theories the existence of any type of barrier is equivalent to regularity and therefore it is more a matter of taste which definition one uses. As we show this is so also for nonlinear potential theory on $\mathbf{R}^n$ and for Cheeger p -harmonic functions on metric spaces. (Observe that regularity has been shown to be a local property, see Proposition 9.9 in [12] and Theorem 4.2 in [5].)

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2. The weak barrier characterization

Definition 2.1. A function u is a *barrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if

- (i) u is $\mathcal{A}$ -superharmonic in Ω ;
- (ii) $\liminf_{\Omega \ni y \rightarrow x} u(y) > 0$ for every $x \in \partial\Omega \setminus \{x_0\}$;
- (iii) $\lim_{\Omega \ni y \rightarrow x_0} u(y) = 0$.

A function u is a *weak barrier* (with respect to Ω) at $x_0 \in \partial\Omega$ if u is a positive $\mathcal{A}$ -superharmonic function in Ω and $\lim_{\Omega \ni y \rightarrow x_0} u(y) = 0$.

If Ω is unbounded we consider ∞ to belong to $\partial\Omega$. By the strong minimum principle a barrier is always nonnegative, only $\mathbf{R}^n$ has a barrier which takes the value zero (viz. $u \equiv 0$ is a barrier for $\mathbf{R}^n$).

Definition 2.2. A point $x_0 \in \partial\Omega$ is *regular* if

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = f(x) \quad \text{for all } f \in C(\partial\Omega),$$

where Hf is the Perron solution of the Dirichlet problem with f as boundary values. The point x_0 is *irregular* if it is not regular.

We refer the reader to Section 9 in Heinonen–Kilpeläinen–Martio [12] for the definition of Perron solutions. A boundary point x_0 is regular if and only if it has a barrier, see Theorem 9.8 in [12].

In order to prove Theorem 1.1 we will use thinness and the Wiener criterion.

Definition 2.3. A set E is *thin* at x_0 if

$$W(E, x_0) := \int_0^1 \left(\frac{\text{cap}_{p,\mu}(E \cap B(x_0, t), B(x_0, 2t))}{\text{cap}_{p,\mu}(B(x_0, t), B(x_0, 2t))} \right)^{1/(p-1)} \frac{dt}{t} < \infty.$$

If E is not thin at x_0 , then E is *thick* at x_0 .

Here $\text{cap}_{p,\mu}$ is the variational capacity defined on p. 27 in [12], and $B(x_0, t) = \{x \in \mathbf{R}^n : |x - x_0| < t\}$.

The Wiener criterion says that $\mathbf{R}^n \setminus \Omega$ is thick if and only if x_0 is regular, see Maz'ya [16] and Kilpeläinen–Malý [14] in the unweighted case, and Mikkonen [17] in the weighted case.

Proof of Theorem 1.1. Assume first that $x_0 = \infty$. Let $G = \{x \in \mathbf{R}^n : |x| > 1\}$, $f(x) = 1$ for $|x| = 1$ and $f(\infty) = 0$. Let further $u = H_G f$ in G and $u(x) = 1$ for $|x| \leq 1$. It is then easy to see that u is a positive barrier at x_0 (with respect to Ω). Thus x_0 is regular by the barrier characterization, Theorem 9.8 in [12], and we are done.

In the rest of the proof we assume that $x_0 \neq \infty$. Assume first that x_0 is regular. By Theorem 9.8 in [12] there exists a barrier u at x_0 . The strong minimum principle gives directly that u must be positive and hence is a weak barrier.

Conversely, let u be a weak barrier at x_0 . Assume that x_0 is not regular. The Wiener criterion shows that $E := \mathbf{R}^n \setminus (\Omega \cup \{x_0\})$ is thin at x_0 . By Lemma 12.11 in [12], there is an open set $U \supset E$ which is thin at x_0 . By the Kellogg property, Theorem 9.11 in [12], $\text{cap}_{p,\mu}(\{x_0\}, B(x_0, t)) = 0$ for all $t > 0$, which shows that also the set $U' := U \cup \{x_0\}$ is thin at x_0 .

Let next $F = \{x \in U' : \text{dist}(x, E \cup \{x_0\}) \leq \text{dist}(x, \mathbf{R}^n \setminus U')\}$. Then F is closed and thin at x_0 , and thus $x_0 \in \partial F$. By the Wiener criterion, x_0 is irregular with respect to $G := \mathbf{R}^n \setminus F$. By construction, $\overline{G} \subset \Omega \cup \{x_0\}$ and $x_0 \in \partial G$. For $x \in \partial G \setminus \{x_0\}$ we have $\liminf_{G \ni y \rightarrow x} u(y) \geq u(x) > 0$, by the lower semicontinuity of u . Thus u is a (strong) barrier at x_0 with respect to G , and hence x_0 is regular with respect to G , by the barrier characterization, a contradiction. Thus x_0 must be regular with respect to Ω . $\square$

Proof of Theorem 1.2. The equivalence is trivial if $\Omega = \mathbf{R}^n$, assume therefore that $\Omega \neq \mathbf{R}^n$. The necessity follows directly from Corollary 9.16 in [12], so let us turn to the sufficiency.

Let $G_1, G_2, \dots$, be the components of Ω (either finitely or countably many). If $x_0 \in \partial G_j$, then let u'_j be a weak barrier at x_0 with respect to G_j and $u_j = \min\{u'_j, 1/j\}$. On the other hand, if $x_0 \notin \partial G_j$, then let $u_j \equiv 1/j$ in G_j . Let further $u : \Omega \rightarrow \mathbf{R}$ be defined by letting $u = u_j$ in G_j . Then u is a weak barrier at x_0 , and thus x_0 is regular, by Theorem 1.1. $\square$

3. Boundary regularity

Using the weak barrier characterization, we can improve the understanding of boundary regularity.

Definition 3.1. The point x_0 is *semiregular* if it is irregular and the limit

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y)$$

exists for all $f \in C(\partial\Omega)$.

The point x_0 is *strongly irregular* if it is irregular and for all $f \in C(\partial\Omega)$ there is a sequence $\{y_j\}_{j=1}^\infty$ such that

$$\Omega \ni y_j \rightarrow x_0 \text{ and } Hf(y_j) \rightarrow f(x_0), \quad \text{as } j \rightarrow \infty.$$

By Theorem 2.1 in A. Björn [4], an irregular boundary point is either semiregular or strongly irregular. Moreover, by Theorem 5.3 in [4], if x_0 is strongly irregular, then the sequence $\{y_j\}_{j=1}^\infty$ can always be chosen independently of f .

Proposition 3.2. *Let $\Omega \subset \mathbf{R}^n$ be a connected open set and $x_0 \in \partial\Omega$. Let $f \in C(\partial\Omega)$ be nonnegative and such that $f(x_0) = 0$ and $C_{p,\mu}(\{x \in \partial\Omega : f(x) > 0\}) > 0$. Then the following are true:*

(a) x_0 is regular if and only if

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) = 0;$$

(b) x_0 is semiregular if and only if

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) > 0; \tag{3.1}$$

(c) x_0 is strongly irregular if and only if the limit

$$\lim_{\Omega \ni y \rightarrow x_0} Hf(y) \tag{3.2}$$

does not exist.

Here $C_{p,\mu}$ is the Sobolev capacity defined on p. 48 in [12].

Observe that in the last case we get that $\liminf_{\Omega \ni y \rightarrow x_0} Hf(y) = 0$.

This result improves upon the characterization (d) in Theorem 4.2 in Björn–Björn [5]. It is possible to make similar generalizations of (k) in Theorem 6.1 in [5] and (A)–(C) in Remark 6.2 in [5].

See Theorems 4.2 and 6.1 in [5] for other characterizations of regular boundary points, and Theorems 5.3 and 6.1 in A. Björn [4] for other characterizations of semiregular and strongly irregular boundary points.

Proof. (a) The necessity is a direct requirement of the definition of regularity. Conversely, by the Kellogg property (Theorem 9.11 in [12]) $Hf \not\equiv 0$. Thus $Hf > 0$ in Ω , by the strong minimum principle. Hence Hf is a weak barrier, and the regularity follows from Theorem 1.1.

(b) If x_0 is semiregular, then we know that the limit $c := \lim_{\Omega \ni y \rightarrow x_0} Hf(y)$ exists, and by the assumptions on f we must have $c \geq 0$. As x_0 is not regular, (a) shows that $c > 0$.

Conversely, assume that (3.1) holds, then x_0 cannot be regular nor strongly irregular. Hence, by Theorem 2.1 in A. Björn [4], x_0 is semiregular.

(c) If x_0 is strongly irregular, then the limit (3.2) can only exist if it equals 0, but by (a) that would mean that x_0 would be regular, a contradiction, hence the limit (3.2) does not exist.

Conversely, if the limit (3.2) does not exist, then x_0 cannot be regular or semiregular. Hence, by Theorem 2.1 in A. Björn [4], x_0 is strongly irregular. $\square$

4. Metric spaces

Nonlinear potential theory concerning p -harmonic functions has been extended to quite general metric spaces in a number of papers, and the barrier characterization was obtained in Björn–Björn [5], Theorem 4.2. Perron solutions were studied in Björn–Björn–Shanmugalingam [6] (and in [5]).

In the context as in [5] and [6], the Wiener criterion has not been proved and we can therefore not generalize Theorem 1.1 to this case. (Observe that we used both the sufficiency and the necessity of the Wiener criterion when proving Theorem 1.1, and neither has been obtained in the generality of [5] and [6], even though the sufficiency has been obtained for linearly locally connected spaces, see Corollary 7.3 in [5].)

The Wiener criterion has however been extended to Cheeger p -harmonic functions on metric spaces, see Theorem 1.1 in J. Björn [8]. We follow the notation in [8] and assume that X is a complete metric space with a doubling measure supporting a weak p -Poincaré inequality. We also define Cheeger p -(super)harmonic functions and the capacities C_p and Cap_p as in [8]. We then have the following result.

Theorem 4.1. *Let $\Omega \subset X$ be a nonempty bounded open set with $C_p(X \setminus \Omega) > 0$. Then $x_0 \in \partial\Omega$ is a regular boundary point if and only if there is a weak barrier at x_0 . Here weak barriers and regularity are with respect to Cheeger p -harmonic functions.*

Proof. The proof is similar to the proof of Theorem 1.1 we only need to replace some of the references to [12] with references to results valid in metric spaces.

Assume first that x_0 is regular. By Theorem 4.2 in [5] there exists a positive barrier at x_0 which is thus a weak barrier.

The proof of the converse direction is just as in the proof of Theorem 1.1. We only need to observe that Lemma 12.11 in [12] directly generalizes to metric spaces, the only difficulty being that we have to know that the capacity is outer, i.e., that for $E \subset B(x, r)$ we have

$$\text{Cap}_p(E, B(x, 2r)) = \inf_{G \supset E \text{ open}} \text{Cap}_p(G, B(x, 2r)),$$

which was established in Björn–Björn–Shanmugalingam [7], Proposition 1.4. $\square$

After this result has been established the results corresponding to Theorem 1.2, Corollary 1.3 and Proposition 3.2 follow in the same way (always assuming that Ω is bounded and $C_p(X \setminus \Omega) > 0$). The use of Corollary 9.16 in [12] is replaced by referring to Corollary 4.4 in [5].

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