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Removable singularities for analytic functions in BMO and locally Lipschitz spaces

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Abstract In this paper we study removable singularities for holomorphic functions such that $\sup_{z \in \Omega} |f^{(n)}(z)| \operatorname{dist}(z, \partial\Omega)^s < \infty$. Spaces of this type include spaces of holomorphic functions in Campanato classes, BMO and locally Lipschitz classes. Dolzhenko (1963), Král (1976) and Nguyen (1979) characterized removable singularities for some of these spaces. However, they used a different removability concept than in this paper. They assumed the functions to belong to the function space on Ω and be holomorphic on $\Omega \setminus E$, whereas we only assume that the functions belong to the function space on $\Omega \setminus E$, and are holomorphic there. Koskela (1993) obtained some results for our type of removability, in particular he showed the usefulness of the Minkowski dimension. Kaufman (1982) obtained some results for $s = 0$.

In this paper we obtain a number of examples with certain important properties. Similar examples have earlier been obtained for Hardy H^p classes and weighted Bergman spaces, mainly by the author. Because of the similarities in these three cases, an axiomatic approach is used to obtain some results that hold in all three cases with the same proofs.

Mathematics Subject Classification (2000): Primary: 30B40; Secondary: 30D45, 30D55, 46E15.

1 Introduction

Removable singularities for analytic functions is an old subject going back to Riemann's classification of isolated singularities. Characterizations of removable singularities have been given for many different spaces. Carleson [8] characterized

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removable singularities for Hölder continuous harmonic functions in 1963. This inspired Dolzhenko [12] to give a similar characterization for Hölder continuous analytic functions. Only one direction of Dolzhenko's proof can be used to cover Lipschitz continuous analytic functions. It took until 1979 before Nguyen [37] could prove the other direction. The proof was simplified by Khrushchëv [29] the following year.

In the 1970s Král (see, e.g., [31]) developed a general theory for removable singularities enabling him to characterize the removable singularities for solutions to semielliptic partial differential operators lying in BMO, VMO, Hölder classes, little Hölder classes and Campanato spaces. His results contain the characterizations of Carleson and Dolzhenko, but not that of Nguyen, as special cases.

In Král's theory the functions are presumed to be in a function space on a domain Ω and to be distributional solutions of the operator in $\Omega \setminus E$, the question being when it can be deduced that the functions are distributional solutions of the operator in all of Ω . In this paper we presume the functions to be in the function space on $\Omega \setminus E$ only. In this case one are lead to two different notions of removability: weak removability asking when the functions are solutions in all of Ω ; and strong removability also requiring the extended solutions to lie in the function space on all of Ω . In Hölder classes there are extension theorems showing that both weak and strong removability coincide with the type of removability studied by Král. This is not the case with BMO and VMO, nor in the locLip_α classes introduced by Gehring and Martio [18], Section 2.

The main part of this paper is the development of the theory of weak and strong removable singularities for analytic functions in certain spaces including BMO, VMO, locLip_α and loclip_α . Koskela [30] showed the relevance of the Minkowski dimension for this problem.

For locLip_1 weak and strong removability coincide for all sets, and the theory bears close resemblance to the theory of removable singularities for bounded analytic functions. For instance, a Dolzhenko type theorem holds saying that countable unions of removable singularities are removable.

For BMO, VMO, locLip_α and loclip_α , $0 < \alpha < 1$, weak and strong removability coincide for compact sets but not for arbitrary sets. It turns out that this leads to a less satisfactory theory, with many counterexamples for plausible results. In particular, it is not even true that a union of two compact removable singularities needs to be removable. The results are closely resembling results for Hardy H^p spaces and weighted Bergman spaces. Moreover, many of the positive results have the same proofs. Only a few initial observations need to be proved independently for the different function spaces. For this reason, we use an axiomatic approach in this paper.

The two main axioms are requiring that a compact weakly removable set for one domain is strongly removable for any domain containing it, and that weakly removable singularities are totally disconnected. In case there are connected removable singularities the results here may be used for the subset of singularities that are totally disconnected.

The axiomatic theory is developed for analytic functions. It is straightforward to develop the same theory for solutions of other operators for which a simi-

lar uniqueness theorem holds. There are several examples of function spaces for which harmonic functions satisfy these axioms, and also examples for more general operators, but in all such cases known to the author the removable singularities have been characterized. For this reason we have chosen to develop the theory in this paper only in the context of analytic functions. It would be interesting to find an example with some other operator when it is possible to satisfy these axioms, for which a characterization of removable singularities does not yet exist.

Removable singularities have also been studied for various other spaces of analytic functions. Let us just mention: H^∞ (Tolsa [43]); the Nevanlinna class N (Rudin [39]); the Smirnov class N^+ (Khavinson [28]); the Smirnov spaces E^p (Khavinson [27]); the Dirichlet spaces AD^p (Hedberg [22]); the Zygmund class ZC (Carmona–Donaire [10]); the Lebesgue spaces L^p (Carleson [9] and Hedberg [22]); and let us finally mention the paper by Ahlfors and Beurling [1].

For those fortunate enough to be able to read Czech, the article by Král [32] is a good introduction to the results by Carleson, Dolzhenko, Král and others. Král [33] is a more extensive treatment in English, which unfortunately seems to be little known and difficult to obtain.

2 Background

Throughout this paper we make the assumption that $\Omega \subset \mathbf{C}$ is a domain, i.e. a non-empty open connected set.

Definition 2.1. Let for $n \in \mathbf{N}$, $0 \leq s < n + 1$ and $\Omega \subsetneq \mathbf{C}$,

$$\begin{aligned} \text{HL}_s^n(\Omega) &= \left\{ f \in \text{Hol}(\Omega) : \sup_{z \in \Omega} |f^{(n)}(z)| \text{dist}(z, \partial\Omega)^s < \infty \right\}, \\ \text{HL}_s^n(\mathbf{C}) &= \{ f \in \text{Hol}(\mathbf{C}) : f \in \text{HL}_s^n(\mathbf{C} \setminus \{0\}) \}. \end{aligned}$$

In this paper we study removability for HL_s^n , see Definition 3.2. These spaces include spaces of holomorphic functions in Campanato classes, BMO and locally Lipschitz classes.

Definition 2.2. A function f is in the Campanato class $\mathcal{L}_\lambda(\Omega)$, $1 \leq \lambda < 3$, if

$$\sup_{\text{discs } D \subset \Omega} \frac{1}{(\text{diam } D)^\lambda} \int_D \left| f(z) - \frac{1}{m(D)} \int_D f(w) dm(w) \right| dm(z) < \infty, \quad (2.1)$$

where the supremum is taken over all open discs $D \subset \Omega$ and m denotes the Lebesgue measure. We also let $\text{BMO}(\Omega) = \mathcal{L}_2(\Omega)$. Further $f \in \text{Lip}_\alpha(\Omega)$, $0 < \alpha \leq 1$, if

$$\sup_{z, w \in \Omega} \frac{|f(z) - f(w)|}{|z - w|^\alpha} < \infty,$$

and $f \in \text{locLip}_\alpha(\Omega)$ if

$$\sup_{\text{discs } D \subset \Omega} \sup_{z, w \in D} \frac{|f(z) - f(w)|}{|z - w|^\alpha} < \infty. \quad (2.2)$$

Note that $\text{locLip}_\alpha(\Omega)$ is not the usual local Lipschitz space, sometimes denoted $\text{Lip}_{\alpha,\text{loc}}(\Omega)$ or $C_{\text{loc}}^\alpha(\Omega)$. Our condition is local but with a global constant.

It is known that $\mathcal{L}_{2+\alpha}(\Omega) = \text{locLip}_\alpha(\Omega)$, $0 < \alpha < 1$, in the sense that every function in the former class can be modified on a set of zero Lebesgue measure so that the function belongs to the latter class, see Meyers [36], Theorem (without number).

A major reason for studying the spaces HL_s^n is the following lemma.

Lemma 2.3. *We have*

$$\text{HL}_s^1(\Omega) = \begin{cases} \text{Hol}(\Omega) \cap \mathcal{L}_{3-s}(\Omega), & 0 \leq s < 2, \\ \text{Hol}(\Omega) \cap \text{BMO}(\Omega), & s = 1, \\ \text{Hol}(\Omega) \cap \text{locLip}_{1-s}(\Omega), & 0 \leq s < 1. \end{cases}$$

Note also that $\text{HL}_0^0(\Omega) = H^\infty(\Omega)$, that the space $\text{HL}_1^1(\Omega)$ is the Bloch space for Ω , and that $\text{HL}_0^1(\Omega) = \{f \in \text{Hol}(\Omega) : f' \in L^\infty(\Omega)\}$ is a kind of Dirichlet space.

For the case $s = 1$ a proof can be found in Cima–Graham [11], pp. 693–694. This proof can easily be modified to cover the Campanato space characterization as well. The locally Lipschitz characterization was given by Hardy–Littlewood [19], Theorem 40. Gehring–Martio [18], Theorem 2.13, observed that one can replace the requirement $D \subset \Omega$ in (2.2), with the existence of a constant $M \geq 1$ such that $MD \subset \Omega$. The same modification can be made in (2.1), with only minor modifications needed in the proof of Lemma 2.3.

It should also be mentioned that Král uses squares instead of discs in the definition of Campanato spaces and BMO. However, the obtained spaces remain the same. In fact, the squares condition directly implies the restricted disc condition with the supremum in (2.1) taken over discs D with $2D \subset \Omega$. As observed above this is equivalent with the full disc condition (2.1). In the other direction, the proof of Cima–Graham that if $f \in \text{HL}_s^1(\Omega)$, then $f \in \mathcal{L}_{3-s}(\Omega)$, can easily be modified to the square case.

After these observations we can mention that the name HL may be read as “Hardy–Littlewood” or “holomorphic Lipschitz.”

The following results are relevant for us, even though the removability concept considered is not the same as in Definition 3.2. Cf. however the digression in Section 4.

Theorem 2.4. (Dolzhenko [12], Theorem 3, $0 < \alpha < 1$, Nguyen [37], Theorem 4.1, $\alpha = 1$) *Let $0 < \alpha \leq 1$. Then the set $A \subset \Omega$ is removable for $\text{Lip}_\alpha(\Omega \setminus A) \cap \text{Hol}(\Omega \setminus A) = \text{Lip}_\alpha(\mathbb{C}) \cap \text{Hol}(\Omega \setminus A)$ if and only if*

$$\Lambda_{1+\alpha}(A) = 0,$$

where Λ_d denotes d -dimensional Hausdorff measure.

For a simple proof for $\alpha = 1$, see the paper by Khrushchëv [29]. Recall also that any function in $\text{Lip}_\alpha(\Omega \setminus A)$ has an extension to a function in $\text{Lip}_\alpha(\mathbb{C})$.

Theorem 2.5. (Král [31], Theorem 3) *Let $1 < \lambda < 3$. Then the set $A \subset \Omega$ is removable for $\mathcal{L}_\lambda(\Omega) \cap \text{Hol}(\Omega \setminus A)$ if and only if*

$$A_{\lambda-1}(A) = 0.$$

See also Kaufman [26].

We get the following corollary of the results by Dolzhenko, Král and Nguyen.

Corollary 2.6. *If A is weakly removable for HL_s^1 , $0 \leq s < 2$, then $A_{2-s}(A) = 0$.*

For HL_s^1 most of the results below were reported in Björn [6], with the proofs deferred to this paper. It should be noted that in [6], $HL_\alpha(\Omega)$ was defined for $-1 < \alpha \leq 1$, and that $HL_\alpha(\Omega) = \text{HL}_{1-\alpha}^1(\Omega)$.

3 An axiomatic theory

In this paper we develop an axiomatic theory for removable singularities. This is motivated by the observation that the proofs of certain facts are very similar for Hardy H^p classes, weighted Bergman spaces, HL_s^n and hl_s^n , see Definition 2.1 and Sections 6–8 for the definitions of these spaces.

We require the following axioms to be fulfilled.

- (A1) For every domain $\Omega \subset \mathbf{C}$, $X(\Omega)$ is defined and $X(\Omega) \subset \text{Hol}(\Omega)$.
- (A2) If $\Omega_1 \subset \Omega_2 \subset \mathbf{C}$ are domains, then $X(\Omega_1) \supset X(\Omega_2)$.

Because of the uniqueness theorem we do not distinguish between restrictions and extensions of analytic functions.

We next extend the definition of $X(\cdot)$ to non-domains.

Definition 3.1. *If $A \subset \mathbf{C}$, then we define*

$$X(A) = \bigcup X(\Omega),$$

where the union is taken over all domains $\Omega \supset A$.

This definition is consistent by Axiom A2.

We are now ready to define what removable singularities are.

Definition 3.2. *Assume that $A \subset \Omega$. Then we say that A is weakly removable for $X(\Omega \setminus A)$ if $X(\Omega \setminus A) \subset \text{Hol}(\Omega)$, and that A is strongly removable for $X(\Omega \setminus A)$ if $X(\Omega \setminus A) = X(\Omega)$.*

The requirement that Ω be a domain is to avoid pathological situations such as $\Omega \setminus A$ being connected, but Ω non-connected.

An easy consequence of Axiom A2 is the following proposition, the proof of which we leave to the reader.

Proposition 3.3. *Assume that Axioms A1 and A2 are satisfied, and that $A \subset B \subset \Omega$. If B is weakly removable for $X(\Omega \setminus B)$, then A is also weakly removable for $X(\Omega \setminus A)$. Furthermore, if B is strongly removable for $X(\Omega \setminus B)$, then A is strongly removable for $X(\Omega \setminus A)$.*

We also need two axioms on removable singularities to hold for all compact sets $K \subset \mathbb{C}$ and all domains $\Omega \supset K$.

- (A3) If K is weakly removable for $X(\mathbb{C} \setminus K)$, then K is strongly removable for $X(\Omega \setminus K)$.
- (A4) If K is weakly removable for $X(\mathbb{C} \setminus K)$, then K is totally disconnected, i.e. no two different points in K can be connected by a curve in K .

Remark. In view of Axiom A3 and Proposition 3.3 we say that K is *removable* for X if there is a domain $\Omega \supset K$ such that K is weakly removable for $X(\Omega \setminus K)$, or equivalently, if K is strongly removable for $X(\Omega \setminus K)$ for all domains $\Omega \supset K$.

Further, we assume that cap_X is a non-negative set function defined for compact sets, such that if $K \subset K' \subset \mathbb{C}$ are compact sets, then

- (A5) $\text{cap}_X(K) \leq \text{cap}_X(K')$, and
- (A6) $\text{cap}_X(K) = 0$ if and only if K is removable for X .

Axiom A6 never contradicts Axiom A5, since it follows from Proposition 3.3 that a compact subset of a compact removable set is removable.

These two axioms can always be satisfied (when Axioms A1–A4 are satisfied) if we, e.g., define

$$\text{cap}_X(K) = \begin{cases} 0, & \text{if } K \text{ is removable for } X, \\ 1, & \text{if } K \text{ is not removable for } X. \end{cases}$$

In some cases it may be useful to have a differently defined capacity.

These capacities often lack several of the properties usually required of a capacity in potential theory. However, if $X = H^\infty$ one may use the analytic capacity as cap_X , and for Bergman spaces with respect to Muckenhoupt weights one may use a weighted Sobolev capacity, see Björn [3] or [4].

We extend the definition of cap_X by the following definition.

Definition 3.4. *Let $A \subset \mathbb{C}$ and define*

$$\text{cap}_X(A) = \sup\{\text{cap}_X(K) : K \subset A \text{ is compact}\}.$$

This definition is consistent by Axiom A5.

From now on we assume that Axioms A1–A6 are satisfied.

Proposition 3.5. *If $A \subset \Omega$ is weakly removable for $X(\Omega \setminus A)$, then A is totally disconnected.*

Proof. Assume that $A \subset \Omega$ is not totally disconnected. Then there is a compact connected set $K \subset A$ with more than one point. By Definition 3.1 and Axiom A4 there is a function $f \in X(\Omega \setminus K) \subset X(\Omega \setminus A)$ not analytically continuable to all of Ω , and hence A is not weakly removable for $X(\Omega \setminus A)$. $\square$

Lemma 3.6. *Assume that $E \subset \Omega$ is totally disconnected and that $\Omega \setminus E$ is a domain. Then E can be written as a countable union of well-separated compact sets K_j , where by well-separated we mean that $\text{dist}(K_k, \bigcup_{j=1, j \neq k}^\infty K_j) > 0$ for all $k = 1, 2, \dots$.*

Proof. Let $z \in E$ and let $\Omega' \Subset \Omega$ be a domain containing z . Then $E \cap \overline{\Omega}'$ is a compact totally disconnected set, and hence has topological dimension $\dim E \cap \overline{\Omega}' = 0$, see e.g. Fedorchuk [14], Theorem 5 in Section 1.3.2. By the definition of the topological dimension $\dim$ it follows that there exists an (arbitrarily small) open neighbourhood G_z of z such that $\partial G_z \cap E = \emptyset$, and hence $K_z = G_z \cap E$ is a compact totally disconnected set well-separated from $E \setminus K_z$.

The cover $\{G_z\}_{z \in E}$ of E has a countable subcover $\{G_j\}_{j=1}^\infty$. Let $G'_j = G_j \setminus \bigcup_{k=1}^{j-1} \overline{G}_k$. Since $\partial G_j \cap E = \emptyset$, $j \geq 1$, we obtain an open cover $\{\text{int } G'_j\}_{j=1}^\infty$ of E with pairwise disjoint sets. Letting $K_j = G'_j \cap E$ finishes the proof. $\square$

Proposition 3.7. *Assume that $E \subset \Omega$ is relatively closed. Then the set E is weakly removable for $X(\Omega \setminus E)$ if and only if E can be written as a countable union of well-separated compact sets removable for X .*

Proof. Assume first that $f \in X(\Omega \setminus E)$ and $E = \bigcup_{j=1}^\infty K_j$, with K_j being well-separated compact sets removable for X . Then

$$f \in X\left(\Omega \setminus \bigcup_{j=1}^\infty K_j\right) = X\left(\Omega \setminus \bigcup_{j=2}^\infty K_j\right) = \dots = X\left(\Omega \setminus \bigcup_{j=m}^\infty K_j\right).$$

Thus f can be continued to all finite unions of K_j , and by uniqueness to all of Ω , i.e. E is weakly removable for $X(\Omega \setminus E)$.

Assume, conversely, that E is weakly removable for $X(\Omega \setminus E)$. It follows from Proposition 3.5 that E is totally disconnected, and hence by Lemma 3.6, E can be written as a countable union of well-separated compact sets K_k . As $X(\Omega \setminus K_k) \subset X(\Omega \setminus E) \subset \text{Hol}(\Omega)$, the sets K_k are all removable for X . $\square$

Proposition 3.8. *Assume that $A \subset \Omega$. Then the following are equivalent:*

- (a) A is weakly removable for $X(\Omega \setminus A)$;
- (b) $\text{cap}_X(A) = 0$;
- (c) for each $z \in A$, there exists a domain $\Omega_z \ni z$ with $\text{cap}_X(A \cap \Omega_z) = 0$.

Remarks. Since parts (b) and (c) are independent of Ω , we say that A is *weakly removable* for X if there is one domain (equivalently, if for all domains) $\Omega \supset A$ such that A is weakly removable for $X(\Omega \setminus A)$.

The last condition shows that weak removability for X is a local property of A .

Proof. (b) $\Rightarrow$ (a) Let $f \in X(\Omega \setminus A)$. By definition $f \in X(\Omega')$ for some domain $\Omega' \supset \Omega \setminus A$. Without loss of generality we may assume that $\Omega' \subset \Omega$. Let $E = \Omega \setminus \Omega'$, a relatively closed subset of Ω .

If E were not totally disconnected there would exist a compact connected set $K \subset E$ with more than one point, and hence not removable for X , by Axiom A4. It would then follow that $\text{cap}_X(A) \geq \text{cap}_X(K) > 0$, a contradiction.

Thus E is totally disconnected and we can write $E = \bigcup_{j=1}^{\infty} K_j$, where K_j are well-separated compact sets, by Lemma 3.6. We have $\text{cap}_X(K_j) \leq \text{cap}_X(A) = 0$, i.e. K_j is removable for X , by Axiom A6. We conclude from Proposition 3.7 that E is weakly removable for X . It follows that $f \in X(\Omega') = X(\Omega \setminus E) \subset \text{Hol}(\Omega)$.

$\neg(b) \Rightarrow \neg(a)$ There exists a compact set $K \subset A$ with $\text{cap}_X(K) > 0$. By Axiom A6, K is not removable for X . Thus there exists $f \in X(\Omega \setminus K) \subset X(\Omega \setminus A)$ not analytically continuable to all of Ω .

(b) $\Rightarrow$ (c) This follows directly from Definition 3.4.

(c) $\Rightarrow$ (a) Let $f \in X(\mathbf{S} \setminus A) \subset X(\Omega_z \setminus A)$. Since $\text{cap}_X(A \cap \Omega_z) = 0$, $A \cap \Omega_z$ is weakly removable for X , by the already proved implication (b) $\Rightarrow$ (a), and is totally disconnected, by Proposition 3.5. Hence f can be continued analytically to $A \cap \Omega_z$. For $z, w \in A$ the continuations to the totally disconnected sets $A \cap \Omega_z$ and $A \cap \Omega_w$ must agree on their intersection. Hence f can be analytically continued to all of A , and A is weakly removable for X . $\square$

Corollary 3.9. *If $A \subset B$ and B is weakly removable for X , then A is weakly removable for X .*

Corollary 3.10. *Assume that $X(\Omega) \subset Y(\Omega)$ for all bounded domains Ω and that Axioms A1–A6 are satisfied also for Y . If $\text{cap}_Y(A) = 0$, then $\text{cap}_X(A) = 0$.*

It is actually enough if $X(\Omega) \subset Y(\Omega)$ holds for some family of domains such that for every compact set there is a domain in the family containing it, e.g. for all balls centred at the origin.

Proof. By Definition 3.4 it is enough to prove this for the case when A is a compact set. Let $\Omega \supset A$ be a bounded domain. By Proposition 3.8, $X(\Omega \setminus A) \subset Y(\Omega \setminus A) \subset \text{Hol}(\Omega)$. Hence, again using Proposition 3.8, $\text{cap}_X(A) = 0$. $\square$

Proposition 3.11. *Let $E_1, \dots, E_n \subset \Omega$ be pairwise disjoint sets such that $\Omega \setminus \bigcup_{j=1}^k E_j$ is a domain and E_k is strongly removable for $X(\Omega \setminus \bigcup_{j=1}^k E_j)$, $k = 1, \dots, n$. Then $\bigcup_{j=1}^n E_j$ is strongly removable for $X(\Omega \setminus \bigcup_{j=1}^n E_j)$.*

Proof. This is almost trivial, we have

$$X\left(\Omega \setminus \bigcup_{j=1}^n E_j\right) = X\left(\Omega \setminus \bigcup_{j=1}^{n-1} E_j\right) = \dots = X(\Omega \setminus E_1) = X(\Omega). \quad \square$$

Corollary 3.12. *Let $K_1, K_2, \dots, K_n$ be pairwise disjoint compact sets removable for X . Then $\bigcup_{j=1}^n K_j$ is removable for X .*

The following is another generalization of Corollary 3.12.

Proposition 3.13. *Let $E_k \subset \Omega$ be pairwise disjoint sets weakly removable for X and such that $\Omega \setminus E_k$ are domains, $k = 1, \dots, n$. Then $\bigcup_{k=1}^n E_k$ is weakly removable for X .*

That we cannot omit the assumption that $\Omega \setminus E_k$ are domains, follows from Proposition 5.22.

Proof. It is enough to prove the result for $n = 2$, the general result is then deduced by induction.

Since E_1 and E_2 are totally disconnected relatively closed sets, their union $E = E_1 \cup E_2$ is also totally disconnected, see, e.g., Fedorchuk [14], Proposition 9 in Section 1.3.3, and $\Omega \setminus E$ is a domain. We apply Lemma 3.6 to be able to write $E = \bigcup_{j=1}^\infty K_j$ with K_j being well-separated compact sets.

We can write $K_j = (K_j \cap E_1) \cup (K_j \cap E_2) =: K'_j \cup K''_j, j = 1, 2, \dots$. Since K_j is compact and E_1 and E_2 are relatively closed, the sets K'_j and K''_j are compact, and of course also disjoint. It follows that the collection $\{K'_j, K''_j : j = 1, 2, \dots\}$ is well-separated. Corollary 3.9 shows that K'_j and K''_j are removable for X . We conclude from Proposition 3.7 that E is weakly removable for X . $\square$

Proposition 3.14. *Let $A \subset \Omega$ be arbitrary. Then A is strongly removable for $X(\Omega \setminus A)$ if and only if E is strongly removable for $X(\Omega \setminus E)$ for all $E \subset A$ with $\Omega \setminus E$ being a domain.*

Proof. For the necessity, assume that $E \subset A$ and $\Omega \setminus E$ is a domain. Then $X(\Omega) \subset X(\Omega \setminus E) \subset X(\Omega \setminus A) = X(\Omega)$, and thus $X(\Omega) = X(\Omega \setminus E)$.

For the sufficiency, let $f \in X(\Omega \setminus A)$. Then $f \in X(\Omega \setminus E)$ for some $E \subset A$ with $\Omega \setminus E$ being a domain (see the proof of Proposition 3.8). By assumption, $f \in X(\Omega)$. $\square$

3.1 Sheaflike situations

In this section we require one more axiom to be true, the prime example here are the weighted Bergman spaces, see Section 8.

(A7) If Ω_1 and Ω_2 are domains and $\Omega_1 \cup \Omega_2$ is connected, then $X(\Omega_1) \cap X(\Omega_2) = X(\Omega_1 \cup \Omega_2)$.

Proposition 3.15. *Assume that Axiom A7 is satisfied. Let $A \subset \Omega_1 \subset \Omega_2$ be such that Ω_1 and Ω_2 are domains. If A is strongly removable for $X(\Omega_1 \setminus A)$, then A is strongly removable for $X(\Omega_2 \setminus A)$.*

Proof. Let $E \subset A$ be arbitrary such that $\Omega_2 \setminus E$ is a domain. Then,

$$X(\Omega_2 \setminus E) = X(\Omega_1 \setminus A) \cap X(\Omega_2 \setminus E) = X(\Omega_1) \cap X(\Omega_2 \setminus E) = X(\Omega_2).$$

The result follows using Proposition 3.14. $\square$

Proposition 3.16. *Assume that Axiom A7 is satisfied. Assume, furthermore, that $E_1, E_2 \subset \Omega$ are disjoint sets and such that $\Omega \setminus E_1$ and $\Omega \setminus E_2$ are domains. If E_1 and E_2 are strongly removable for $X(\Omega \setminus (E_1 \cup E_2))$, then $E_1 \cup E_2$ is strongly removable for $X(\Omega \setminus (E_1 \cup E_2))$.*

These two results are not true if we omit the assumption that Axiom A7 is satisfied, see Propositions 5.24 and 5.27.

Remark. In order for E_1 and E_2 to be strongly removable for $X(\Omega \setminus (E_1 \cup E_2))$ by Definition 3.2, it is necessary that $\Omega \setminus E_2$ and $\Omega \setminus E_1$, respectively, are domains. Thus we cannot obtain this theorem for non-relatively closed sets.

Proof. By Proposition 3.15, E_1 is strongly removable for $X(\Omega \setminus E_1)$, hence

$$X(\Omega \setminus (E_1 \cup E_2)) = X(\Omega \setminus E_1) = X(\Omega). \quad \square$$

3.2 When weak and strong removability are the same

Theorem 3.17. (Dolzhenko type theorem) *Let $K_1, K_2, \dots$, be compact sets with $\text{cap}_X(K_1) = \text{cap}_X(K_2) = \dots = 0$. Assume that weak and strong removability for X are the same for all sets. Then $\text{cap}_X(\bigcup_{j=1}^{\infty} K_j) = 0$.*

Proof. It is sufficient to prove that $\text{cap}_X(K) = 0$ for all compact sets $K \subset \bigcup_{j=1}^{\infty} K_j$. Since we can write such a set K as a similar union, it is therefore enough to prove the theorem for the case when $K = \bigcup_{j=1}^{\infty} K_j$ is compact.

Let $E_1 = K_1$, $E_2 = K_2 \setminus K_1$, $E_3 = K_3 \setminus \bigcup_{j=1}^2 K_j, \dots$. Each of these sets is a relatively closed subset of some suitable domain, and removable for X , by Axiom A6 and Proposition 3.9. Using Proposition 3.7 we can write each of them as a countable union of pairwise disjoint compact sets removable for X . Collecting all of these compact sets we see that it is enough to consider the case when the original sets K_j are pairwise disjoint.

Let $\Omega \supset K$ be a bounded domain. Since all K_j are totally disconnected so is K , see, e.g., Fedorchuk [14], Proposition 9 in Section 1.3.3. Let

$$\tilde{\Omega} = \bigcup \{ \Omega' \subset \Omega : X(\Omega \setminus K) \subset \text{Hol}(\Omega') \text{ and } \Omega' \text{ is a domain} \},$$

a domain. We have $X(\Omega \setminus K) \subset \text{Hol}(\tilde{\Omega})$, and hence, using that weak and strong removability are the same, $X(\Omega \setminus K) = X(\tilde{\Omega})$.

Let $K' = \Omega \setminus \tilde{\Omega}$. Our aim is to show that $K' = \emptyset$. The set $K' \subset K$ must be compact and we will also consider it to be a complete metric space of its own, in order to use Baire's category theorem. We can write $K' = \bigcup_{j=1}^{\infty} K'_j$, with $K'_j = K' \cap K_j$ being pairwise disjoint compact sets.

Let $E'_k = K'_k \setminus \overline{\bigcup_{j=1, j \neq k}^{\infty} K'_j}$. Since $\text{cap}_X(E'_k) \leq \text{cap}_X(K_k) = 0$ and $\tilde{\Omega} \cup E'_k$ is a domain, Proposition 3.8, together with the assumption that weak and strong removability are the same, show that $X(\tilde{\Omega}) = X(\tilde{\Omega} \cup E'_k)$. If E'_k was non-empty this would contradict the construction of $\tilde{\Omega}$. Hence $E'_k = \emptyset$. It follows that if U

is any open neighbourhood (with respect to K') of $z \in K'_k$, then U contains a point not in K'_k , i.e. K'_k is nowhere dense (in K'). Hence K' is a countable union of nowhere dense subsets, i.e. K' is a set of first category. It follows from Baire's category theorem that $K' = \emptyset$, see e.g. Rudin [40], Sections 5.5–5.7.

We have thus proved that $X(\Omega \setminus K) = X(\Omega)$, i.e. K is removable for X . By Axiom A6 we obtain $\text{cap}_X(K) = 0$. $\square$

Remarks. For H^∞ this result was proved in a similar way, by Dolzhenko [13], Lemma (without number). See also Garnett [16], Exercise I.1.6, p. 12. The result was generalized to relatively closed countable unions of relatively closed removable sets for a common domain, by O'Farrell [38], Example 3. We generalize this result further in the next corollary.

The proof above actually works under the weaker assumption that weak and strong removability for X are the same for all subsets $E \subset \bigcup_{j=1}^\infty K_j$ for which there exists a domain $\Omega \supset E$ with $\Omega \setminus E$ also being a domain, rather than for all sets. Similar modifications can also be done to Corollary 3.18 and Proposition 3.19.

Corollary 3.18. *Let $E_1, E_2, \dots$, be removable for X and assume that there exists a domain $\Omega_j \supset E_j$ with $\Omega_j \setminus E_j$ also being a domain, $j = 1, 2, \dots$. Assume also that weak and strong removability for X are the same for all sets. Then $\text{cap}_X(\bigcup_{j=1}^\infty E_j) = 0$.*

Proof. By Proposition 3.7, each E_j can be written as a countable union of compact sets removable for X . Hence $\bigcup_{j=1}^\infty E_j$ is a countable union of compact sets with zero cap_X capacity, and thus of zero cap_X capacity, by Theorem 3.17. $\square$

Remarks. A consequence is that if weak and strong removability are the same for all sets, then the well-separatedness condition in Proposition 3.7 can be removed. In general this is not possible, see Proposition 5.26.

We cannot omit the assumption that there exists Ω_j for which $\Omega_j \setminus E_j$ is a domain, see, e.g., Exercise 1.7, p. 12, in Garnett [16], or Proposition 9.7 in Björn [5].

Proposition 3.19. *Assume that weak and strong removability are the same for all sets and that all singleton sets are removable for X . Assume also that $A \subset \Omega$ is not removable for X , then $\dim X(\Omega \setminus A)/X(\Omega) = \infty$.*

Remark. It follows that either $X(\Omega) = X(\mathbb{C})$ or $\dim X(\Omega) = \infty$.

Proof. By Proposition 3.14 we can assume, without loss of generality, that $\Omega \setminus A$ is a domain, and hence that A is relatively closed in Ω . Let

$$\tilde{\Omega} = \bigcup \{ \Omega' \subset \Omega : X(\Omega \setminus A) \subset \text{Hol}(\Omega') \text{ and } \Omega' \text{ is a domain} \},$$

a domain. We have $X(\Omega \setminus A) \subset \text{Hol}(\tilde{\Omega})$, and hence, using that weak and strong removability are the same, $X(\Omega \setminus A) = X(\tilde{\Omega})$. We may therefore assume that $\tilde{\Omega} = \Omega \setminus A$.

Let $N \in \mathbf{Z}_+$ be arbitrary. If A is not totally disconnected then we can find pairwise disjoint compact connected sets $K_1, \dots, K_N \subset A$ with more than one point each, and hence not removable for X .

If A is totally disconnected, Theorem 3.17 shows that A is infinite (even uncountable, but there is no need for that here). We can therefore find distinct points $z_1, \dots, z_N \in A$. For $j = 1, \dots, N$, we can find pairwise disjoint arbitrarily small neighbourhoods $G_j \ni z_j$ such that $\partial G_j \cap A = \emptyset$ (see the proof of Lemma 3.6.) It follows that $K_j = \overline{G_j} \cap A$ is compact. If K_j were removable for X , then we would have $X(\tilde{\Omega}) \subset \text{Hol}(\tilde{\Omega} \cup K_j)$, which would contradict the construction of $\tilde{\Omega}$. Hence K_j is not removable for X .

Thus, regardless of whether A is totally disconnected or not, we have found pairwise disjoint compact sets $K_1, \dots, K_N \subset A$ which are not removable for X . We can now find $f_j \in X(\Omega \setminus K_j) \setminus X(\Omega)$, $j = 1, \dots, N$. These functions represent linearly independent equivalence classes in $X(\Omega \setminus A)/X(\Omega)$, and hence we see that $\dim X(\Omega \setminus A)/X(\Omega) \geq N$. Since N was arbitrary we are done. $\square$

3.3 Theory only for bounded domains

For certain function spaces it happens that Axioms A3 and A4 are not satisfied, but instead the following weaker axioms are satisfied for all compact sets $K \subset \mathbf{C}$ and all bounded domains $\Omega \supset \Omega' \supset K$.

- (A3') If K is weakly removable for $X(\Omega \setminus K)$, then K is strongly removable for $X(\Omega' \setminus K)$.
- (A4') If K is weakly removable for $X(\Omega \setminus K)$, then K is totally disconnected.

In this case we can develop the same theory as before, but, of course, with the results valid only for bounded domains. See Section 8 for an example where this is relevant.

4 A related type of removability

In the literature there is another type of removability for spaces of analytic functions, see, e.g., Theorems 2.4 and 2.5.

Let us assume the following axioms.

- (B1) For every domain $\Omega \subset \mathbf{C}$, $F(\Omega)$ is defined and is a set of functions.
- (B2) If $\Omega_1 \subset \Omega_2 \subset \mathbf{C}$ are domains, then $F(\Omega_1) \supset F(\Omega_2)$.

We also define

$$\text{Hol}(A) = \bigcup \text{Hol}(\Omega),$$

where the union is taken over all domains $\Omega \supset A$.

Definition 4.1. Assume that $A \subset \Omega$. Then we say that A is removable for $F(\Omega) \cap \text{Hol}(\Omega \setminus A)$ if $F(\Omega) \cap \text{Hol}(\Omega \setminus A) = F(\Omega) \cap \text{Hol}(\Omega)$.

Remark. It is equivalent to require that $F(\Omega) \cap \text{Hol}(\Omega \setminus A) \subset \text{Hol}(\Omega)$.

We further require two more axioms to be fulfilled for all domains $\Omega \subset \mathbb{C}$ and compact subsets $K \subset \Omega$.

(B3) If K is removable for $F(\mathbb{C}) \cap \text{Hol}(\mathbb{C} \setminus K)$, then K is removable for $F(\Omega) \cap \text{Hol}(\Omega \setminus K)$.

(B4) If K is removable for $F(\mathbb{C}) \cap \text{Hol}(\mathbb{C} \setminus K)$, then K is totally disconnected.

Proposition 4.2. Assume that Axioms B1–B4 are satisfied. Let $X(\cdot) = F(\mathbb{C}) \cap \text{Hol}(\cdot)$, and let $A \subset \Omega$. Then the following are equivalent:

- (a) A is removable for $F(\Omega) \cap \text{Hol}(\Omega \setminus A)$;
- (b) A is removable for $F(\mathbb{C}) \cap \text{Hol}(\mathbb{C} \setminus A)$;
- (c) A is weakly removable for $X(\Omega \setminus A)$;
- (d) A is weakly removable for $X(\mathbb{C} \setminus A)$;
- (e) A is strongly removable for $X(\Omega \setminus A)$;
- (f) A is strongly removable for $X(\mathbb{C} \setminus A)$;
- (g) $\text{cap}_X(A) = 0$.

Remarks. This shows that the theory of removability in the sense of Definition 4.1, with Axioms B1–B4, can be considered as a special case of the theory of removability for the case when weak and strong removability are the same.

If $F = \mathcal{L}_\lambda$, then Axioms B1–B4 are satisfied for $1 < \lambda \leq 2$, but Axiom B4 fails for $2 < \lambda < 3$, see Theorem 2.5. Axiom B4 also fails in the context of Theorem 2.4. When Axioms B1–B3 are satisfied, but not Axiom B4, then Proposition 4.2 holds if “removable” is replaced by “removable and totally disconnected.”

Proof. Let us first verify that Axioms A1–A4 hold for X . This is immediate apart from perhaps for Axiom A3. If, however a compact set $K \subset \Omega$ is weakly removable for $X(\mathbb{C} \setminus K)$, then it is removable for $F(\mathbb{C}) \cap \text{Hol}(\mathbb{C} \setminus K)$ and by Axiom B3,

$$\begin{aligned} X(\Omega \setminus K) &= F(\mathbb{C}) \cap (F(\Omega) \cap \text{Hol}(\Omega \setminus K)) \\ &= F(\mathbb{C}) \cap (F(\Omega) \cap \text{Hol}(\Omega)) = X(\Omega). \end{aligned}$$

(a) $\Rightarrow$ (b) This is a direct consequence of Axiom B2.

(b) $\Leftrightarrow$ (d) $\Leftrightarrow$ (f) and (c) $\Leftrightarrow$ (e) This follows from the definition of $X(\cdot)$.

(c) $\Leftrightarrow$ (d) $\Leftrightarrow$ (g) This follows from Proposition 3.8.

(b) $\Rightarrow$ (a) This is proved similarly to Proposition 3.8: We have already proved that $\text{cap}_X(A) = 0$. Let $f \in F(\Omega) \cap \text{Hol}(\Omega \setminus A)$. By definition there exists a set $E \subset \Omega$ such that $\Omega \setminus E$ is a domain and $f \in \text{Hol}(\Omega \setminus E)$. As in the proof of Proposition 3.8, E is totally disconnected and can be written as $E = \bigcup_{j=1}^{\infty} K_j$, where K_j are well-separated compact sets and $\text{cap}_X(K_j) = 0$. Hence,

by (g) $\Rightarrow$ (b), K_j is removable for $F(\mathbf{C}) \cap \text{Hol}(\mathbf{C} \setminus K_j)$. Using Axiom B3 (and B2) we are able to conclude that

$$f \in F((\Omega \setminus E) \cup K_j) \cap \text{Hol}(\Omega \setminus E) \subset \text{Hol}((\Omega \setminus E) \cup K_j)$$

for each j . And hence $f \in \text{Hol}(\Omega)$. $\square$

5 Removability for HL_s^n

In this section we study removable singularities for HL_s^n . We assume throughout this section that $n \in \mathbf{N}$ and $0 \leq s < n+1$. This makes all singleton sets removable for HL_s^n , as is easily verified. Note that if we let $s = n+1$, then singleton sets are not removable, and hence only the empty set is removable.

5.1 Axiom verification

Lemma 5.1. *If $K \subset \Omega$ is compact and weakly removable for $\text{HL}_s^n(\mathbf{C} \setminus K)$, then K is strongly removable for $\text{HL}_s^n(\Omega \setminus K)$ and K is totally disconnected.*

Remark. This shows that Axioms A1–A6 are satisfied if we, e.g., define

$$\text{cap}_{\text{HL}_s^n}(K) = \begin{cases} 0, & \text{if } K \text{ is removable for } \text{HL}_s^n, \\ 1, & \text{if } K \text{ is not removable for } \text{HL}_s^n. \end{cases}$$

Proof. Assume that $f \in \text{HL}_s^n(\Omega \setminus K)$. Let Ω_1 and Ω_2 be smooth bounded domains with $K \subset \Omega_1 \Subset \Omega_2 \Subset \Omega$. Let $K_1 \supset K_2 \supset \dots$ be compact smooth subsets of Ω_1 with $K = \bigcap_{n=1}^{\infty} K_n$ and $\partial K_n \subset \Omega_1 \setminus K$ for all $n \geq 1$. Then

$$f(z) = \frac{1}{2\pi i} \int_{\partial \Omega_2} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_{\partial K_n} \frac{f(\zeta)}{\zeta - z} d\zeta =: g(z) + h_n(z)$$

for $z \in \Omega_2 \setminus K_n$. Moreover, $h_n \in \text{Hol}(\mathbf{S} \setminus K_n)$, where $\mathbf{S} = \mathbf{C} \cup \{\infty\}$ is the Riemann sphere, and

$$h_n(z) = f(z) - g(z), \quad z \in \Omega_2 \setminus K_n.$$

Thus $\{h_n(z)\}_{n=1}^{\infty}$ is constant when defined, so if

$$h(z) = \lim_{n \rightarrow \infty} h_n(z), \quad z \in \mathbf{C} \setminus K,$$

then $h \in \text{Hol}(\mathbf{S} \setminus K)$ and, moreover, $h(\infty) = 0$.

Furthermore, $g^{(n)}$ is bounded in Ω_1 and hence $g \in \text{HL}_s^n(\Omega_1 \setminus K)$. It follows that $h = f - g \in \text{HL}_s^n(\Omega_1 \setminus K)$.

Let $G = \{z : \text{dist}(z, K) < \text{dist}(z, \partial \Omega_1)\}$. Then $|h^{(n)}(z)| \leq C \text{dist}(z, K)^{-s}$ for $z \in G \setminus K$. Moreover, $h^{(n)}$ is bounded on $\mathbf{C} \setminus G$ and $h^{(n)}(z) = \mathcal{O}(z^{-n-1})$, as $z \rightarrow \infty$, since $h(\infty) = 0$. It follows that $h \in \text{HL}_s^n(\mathbf{C} \setminus K) \subset \text{Hol}(\mathbf{C})$. Hence,

$h(z) \equiv h(\infty) = 0$, and $f = g$. We have thus proved that K is weakly removable for $\text{HL}_s^n(\Omega \setminus K)$.

There exists a constant M such that $\text{dist}(z, \partial\Omega) \leq M \text{dist}(z, K)$ for $z \in \Omega \setminus \Omega_1$. Thus $|f^{(n)}(z)| \leq C \text{dist}(z, \partial(\Omega \setminus K))^{-s} \leq C' \text{dist}(z, \partial\Omega)^{-s}$ for $z \in \Omega \setminus \Omega_1$. Since $f^{(n)}$ is bounded on Ω_1 , it follows that $f \in \text{HL}_s^n(\Omega)$ and the proof of the first part is complete.

For the second part we note that $\text{HL}_s^n(\Omega) \supset \text{HL}_0^n(\Omega)$ for bounded domains. Together with the first part we see that if K is removable for HL_s^n , then K is removable for HL_0^n .

Assume that K is not totally disconnected, then we can find a connected compact subset $\tilde{K}$ with at least two points and connected pairwise disjoint compact sets $K_0, \dots, K_n \subset K$ (with at least two points each). It is well known that K_j is not removable for H^∞ . Thus there exists a non-constant $f_j \in H^\infty(\mathbf{S} \setminus K_j)$ with $f_j(\infty) = 0$, $j = 0, \dots, n$.

We can find a non-trivial linear combination $g_n = \sum_{j=0}^n a_j f_j$ with $g_n(\infty) = g'_n(\infty) = \dots = g_n^{(n)}(\infty) = 0$. Without loss of generality assume that $a_0 \neq 0$. Then g_n has a singularity in K_0 , and is thus non-trivial.

Fix $z_0 \in \mathbf{C} \setminus K$. Let next recursively, $g_{m-1}(z) = \int_{z_0}^z g_m(\zeta) d\zeta + c_m$, where c_m is chosen so that $g_{m-1}(\infty) = 0$. Since $g_m(\infty) = g'_m(\infty) = 0$ and $\mathbf{C} \setminus \tilde{K}$ is simply connected, g_{m-1} is well-defined. It follows that $g_0^{(n)} = g_n \in H^\infty(\mathbf{C} \setminus K)$ and hence $g_0 \in \text{HL}_0^n(\mathbf{C} \setminus K)$. Since g_0 is non-constant, K is not removable for HL_0^n . $\square$

Proposition 5.2. *Weak and strong removability for HL_0^n are the same for all sets.*

Proof. Let A be weakly removable for HL_0^n , then A is totally disconnected. Take a domain Ω and a function $f \in \text{HL}_0^n(\Omega \setminus A) \subset \text{Hol}(\Omega)$. By definition, $f^{(n)}$ is bounded in $\Omega \setminus A$, and continuous on Ω . Since A has no interior and $f^{(n)}$ is continuous, $f^{(n)}$ is bounded by the same constant in Ω as in $\Omega \setminus A$, and hence $f \in \text{HL}_0^n(\Omega)$. $\square$

Proposition 5.3. *Assume that $0 \leq r \leq s < n+1$ and $n \leq m \in \mathbf{N}$. If A is weakly removable for HL_s^n , then A is weakly removable for HL_r^m .*

Proof. By Definition 3.4 and Proposition 3.8 it is enough to prove the result for $A = K$ compact. Since $\text{HL}_r^n(\Omega) \subset \text{HL}_s^n(\Omega)$ for bounded domains Ω , Corollary 3.10 shows that K is removable for HL_r^n . Thus it is enough to prove that K is removable for HL_r^{n+1} .

Let $\Omega \supset K$ be a bounded simply connected domain. Take $f \in \text{HL}_r^{n+1}(\Omega \setminus K)$. Then $f' \in \text{HL}_r^n(\Omega \setminus K) = \text{HL}_r^n(\Omega)$. Fix $z_0 \in \Omega \setminus K$ and let $g(z) = \int_{z_0}^z f'(\zeta) d\zeta + f(z_0)$, $z \in \Omega$, a well-defined function since Ω is simply connected. Now $f = g$ in $\Omega \setminus K$ and $g \in \text{HL}_r^{n+1}(\Omega)$. $\square$

5.2 Metric conditions

Theorem 5.4. *Let $n \in \mathbf{N}$ and $0 \leq s < 1$ if $n = 0$ and $0 \leq s < 2$ otherwise. Let*

$$d = \begin{cases} 1 - s, & n = 0, \\ 2 - s, & n \geq 1. \end{cases}$$

Let $K \subset \mathbf{C}$ be compact with $\dim_M K < d$. If $d > 1$ assume further that there exists two angles $0 \leq \theta_1 < \theta_2 < \pi$ such that the orthogonal projections of K onto $\{z : \arg z = \theta_j \text{ or } \arg z = \theta_j + \pi\}$, $j = 1, 2$, have one-dimensional Hausdorff measure zero. Then K is removable for HL_s^n .

(We follow Koskela [30] letting $\dim_M$ denote the upper Minkowski dimension, see also Mattila [35], Section 5.3.)

Koskela [30], Theorem A, showed this for $n = 1$ with $\theta_0 = 0$ and $\theta_1 = \frac{1}{2}\pi$. His condition is thus not rotationally invariant. Since it is easy to see that removability for HL_s^n is rotationally invariant, we have modified his condition here.

It is an interesting problem to ask if the additional condition for $d > 1$ can be replaced by the assumption that K is totally disconnected.

This result is sharp for $n = 0$ and $n = 1$, even so that the condition $\dim_M K < d$ cannot be replaced by the condition $\dim_M K \leq d$, see Corollaries 2.6 and 5.8, and Lemma 5.11. For $n \geq 2$ and $s > 0$, it is not known if this result is sharp, although it seems unlikely. Corollary 5.8 shows that the Minkowski dimension in Theorem 5.4 cannot be replaced by Hausdorff dimension, if $s \neq 0$. For $\text{HL}_0^0 = H^\infty$, Painlevé's theorem implies that we can indeed replace $\dim_M$ by $\dim_H$ in Theorem 5.4. For HL_0^n , $n \geq 1$, it is an open question if such a replacement is possible.

Proof. Koskela [30], Theorem A, showed this for $n = 1$ with $\theta_0 = 0$ and $\theta_1 = \frac{1}{2}\pi$. For the case $n = 1$ we can use Koskela's proof if we just add a linear change of variables. Using Proposition 5.3 we obtain the result for $n \geq 1$.

Consider the case $n = 0$ in the rest of the proof. We decompose $\mathbf{C} \setminus K$ into a *Whitney decomposition*, i.e. as a union of closed dyadic squares Q_j^k , where each square has sides parallel to the axes, the side length of Q_j^k is 2^{-k} , and $2^{-k}\sqrt{2} \leq \text{dist}(Q_j^k, K) \leq 4 \cdot 2^{-k}\sqrt{2}$, see Section 2 in Martio–Vuorinen [34]. Let N_k be the number of cubes in the k th generation. By Theorem 3.12 in [34], $N_k \leq 2^{d'k}$ for k large enough and some $d' < d$. Let $\gamma_k = \partial(\bigcup_{l=k-2}^{k+2} \bigcup_{j=1}^{N_l} Q_j^l)$. By Section 2 in [34], γ_k surrounds K . If k is large enough, then

$$\text{length } \gamma_k \leq \sum_{l=k-2}^{k+2} 4 \cdot 2^{-l} N_l \leq \sum_{l=k-2}^{k+2} 4 \cdot 2^{(d'-1)l} \leq 20 \cdot 2^{(d'-1)(k-2)}.$$

Moreover $\text{dist}(\gamma_k, K) \geq 2^{-k-2}$.

Let now $f \in \text{HL}_s^0(\mathbf{C} \setminus K)$. Then we can write

$$f(z) = a_0 + \sum_{m=1}^{\infty} \frac{a_m}{z^m}, \quad z \text{ large.}$$

We find that for k large enough and $m \in \mathbf{Z}_+$, with $d\sigma$ denoting arc length,

$$\begin{aligned} |a_m| &\leq \frac{1}{2\pi} \int_{\gamma_k} |z|^{m-1} |f(z)| d\sigma \leq C \int_{\gamma_k} |f(z)| d\sigma \leq C' \int_{\gamma_k} \frac{1}{\text{dist}(z, K)^s} d\sigma \\ &\leq 20C' 2^{(d'-1)(k-2)} 2^{(k+2)(1-d)} = 20C' 2^{4(1-d)} 2^{(d'-d)(k-2)}, \end{aligned}$$

where C' depends on m but not on k . Letting $k \rightarrow \infty$, we see that $a_m = 0$. Hence f is constant, and K is removable. $\square$

Definition 5.5. Let $0 < d < 2$, $r_{j,k} = 2^{-j} + k2^{-2j/(2-d)}$ and define

$$K_d = \{0\} \cup \{r_{j,k} e^{l\pi i 2^{1-jd/(2-d)}} : j, k, l \in \mathbf{Z}_+ \text{ and } k, l \leq 2^{jd/(2-d)}\}.$$

Lemma 5.6. (Koskela [30], the proof of Theorem B) Let $0 < d < 2$. Then K_d is a countable set with 0 as its only limit point, $\dim_M K_d = d$ and $1/z \in \text{HL}_{1-d/2}^0(\mathbf{C} \setminus K_d)$.

Koskela actually proved that $1/z \in \text{HL}_{2-d}^1(\mathbf{C} \setminus K_d)$, but that is equivalent to saying that $1/z \in \text{HL}_{1-d/2}^0(\mathbf{C} \setminus K_d)$.

Lemma 5.7. Let $0 < d < 2$ and $0 < s < n + 1$. Then $\text{HL}_s^n(\mathbf{C} \setminus K_d)$ consists of the functions which can be written as a sum of a polynomial in $1/z$ of degree $\leq \max\{0, 2s/(2-d) - n\}$ and of a polynomial in z of degree less than n .

Corollary 5.8. The set K_d is removable for HL_s^n if and only if $s < \frac{1}{2}(n+1)(2-d)$, or equivalently $d < 2 - 2s/(n+1)$.

This shows that Theorem 5.4 is sharp for $n = 1$, which Koskela [30] observed. The next corollary is in great contrast to Proposition 3.19.

Corollary 5.9. Let $0 < s < n + 1$ and $m \geq n$, $m \in \mathbf{Z}_+$, or $m = \infty$. Then there exists a domain Ω such that $\dim \text{HL}_s^n(\Omega) = m$.

Proof of Lemma 5.7. Let $f \in \text{HL}_s^n(\mathbf{C} \setminus K_d)$. Since $K_d \setminus \{0\}$ is a countable set of well-separated points it is weakly removable for HL_s^n by Proposition 3.7. Hence $f \in \text{Hol}(\mathbf{C} \setminus \{0\})$ and we can write

$$f(z) = \sum_{m=1}^{\infty} \frac{a'_m}{z^m} + \sum_{m=0}^{\infty} b'_m z^m =: g(z) + h(z), \quad z \neq 0.$$

Hence

$$f^{(n)}(z) = \sum_{m=n+1}^{\infty} \frac{a'_m}{z^m} + \sum_{m=0}^{\infty} b'_m z^m, \quad z \neq 0.$$

Since, by definition $f^{(n)}(z) \rightarrow 0$, as $z \rightarrow \infty$, we see that the second series must be identically zero, which is the same as saying that h is a polynomial in z of degree less than n .

By the definition we have $|f^{(n)}(z)| \leq C \operatorname{dist}(z, K_d)^{-s}$. If $|z| = r_{j,1/2}$, then $\operatorname{dist}(z, K_d) \geq 2^{-1-2j/(2-d)}$. Thus for $m \geq n+1$ we obtain, with $d\sigma$ denoting arc length,

$$\begin{aligned} |a_m| &\leq \frac{1}{2\pi} \int_{|z|=r_{j,1/2}} |z|^{m-1} |f^{(n)}(z)| d\sigma \leq C 2^{(1-j)m} (2^{-1-2j/(2-d)})^{-s} \\ &\leq C 2^{m+s} 2^{-j(m-2s/(2-d))} \rightarrow 0, \quad \text{as } j \rightarrow \infty, \end{aligned}$$

if $m > 2s/(2-d)$. So g is a polynomial in $1/z$ of degree $\leq 2s/(2-d) - n$.

Conversely, Lemma 5.6 shows that $1/z \in \operatorname{HL}_{1-d/2}^0(\mathbb{C} \setminus K_d)$. If $f(z) = z^{-m}$ and $1 \leq m \leq 2s/(2-d) - n$, then for $|z| \leq 2$,

$$|f^{(n)}(z)| = C_1 \left| \frac{1}{z} \right|^{m+n} \leq \frac{C_2}{\operatorname{dist}(z, K_d)^{(1-d/2)(m+n)}} \leq \frac{C_3}{\operatorname{dist}(z, K_d)^s}.$$

Similarly, for $|z| \geq 2$,

$$|f^{(n)}(z)| = C_1 \left| \frac{1}{z} \right|^{m+n} \leq \frac{C_1}{\operatorname{dist}(z, K_d)^s}. \quad \square$$

Definition 5.10. Let $0 < \alpha < \frac{1}{2}$ be fixed. Let $C_\alpha^{(0)} = [0, 1]$ and let $C_\alpha^{(1)}$ be the set remaining after having removed the open middle part of length $1 - 2\alpha$ from the interval $C_\alpha^{(0)}$. Continue in this way by letting $C_\alpha^{(m+1)}$ be the set remaining after having removed the open middle parts of length $(1 - 2\alpha)\alpha^m$ from each of the intervals in $C_\alpha^{(m)}$. The set $C_\alpha^{(m)}$ consists of 2^m disjoint closed intervals, each of length α^m . Finally let $C_\alpha = \bigcap_{m=0}^\infty (C_\alpha^{(m)} \times C_\alpha^{(m)})$.

Then C_α is a (planar self-similar) Cantor set.

It is well-known that $\dim_H C_\alpha = \dim_M C_\alpha = -\log 4 / \log \alpha$, see, e.g., Mätilä [35], Theorem 4.14.

For the historically interested reader it may be worth noting that Veltmann considered these planar Cantor sets in 1882 [45] (see also Veltmann [44]), before Cantor published his ternary set in 1883 [7], p. 590 (p. 407 in Acta Math.). Cantor-type sets were constructed already in 1875 by Smith [41], Sections 15–16.

Lemma 5.11. Let $0 < s < 1$, $d = 1 - s$ and let α be such that $d = \dim C_\alpha$. Then C_α is not removable for HL_s^0 .

This shows the sharpness of Theorem 5.4 for $n = 0$, $0 < s < 1$. It is well-known that there are compact sets with $\dim_H K = 1$, which are not removable for $H^\infty = \operatorname{HL}_0^0$. Thus Theorem 5.4 is sharp for $n = 0$, $0 \leq s < 1$. However, the Garnett–Ivanov set $C_{1/4}$ is known to be removable for $H^\infty = \operatorname{HL}_0^0$, see Garnett [15] and Ivanov [24], footnote on p. 346.

Proof. Note that $0 < \alpha < \frac{1}{4}$. Let μ be the standard probability measure on C_α such that the measure of a square in the m th generation, i.e. in $C_\alpha^{(m)} \times C_\alpha^{(m)}$, is

4^{-m} . Let $f(z) = \int_{C_\alpha} (z - \zeta)^{-1} d\mu(\zeta)$. Take $z \in \mathbf{C} \setminus C_\alpha$, and let $m \in \mathbf{N}$ be such that $\frac{1}{4}\alpha^m \leq \text{dist}(z, C_\alpha) < \alpha^m$. Then,

$$|f(z)| \leq 4\alpha^{-m}4^{-m} + \sum_{j=1}^m 3\alpha^{-j}4^{-j} \leq C\left(\frac{1}{4\alpha}\right)^m \leq C\left(\frac{1}{4\alpha^d\alpha^s}\right)^m.$$

Now $4\alpha^d = 1$, so

$$|f(z)| \leq C\left(\frac{1}{\alpha^m}\right)^s \leq \frac{C}{\text{dist}(z, C_\alpha)^s}. \quad \square$$

5.3 Comparison between different spaces

Let us ask when the implication

$$K \text{ removable for } \text{HL}_s^n \implies K \text{ removable for } \text{HL}_r^m \quad (5.1)$$

holds for all compact sets $K \subset \mathbf{C}$. Let us first note that, by Definition 3.4 and Proposition 3.8, the implication (5.1) is equivalent to the fact that the seemingly stronger implication

$$A \text{ weakly removable for } \text{HL}_s^n \implies A \text{ weakly removable for } \text{HL}_r^m$$

holds for all sets $A \subset \mathbf{C}$.

For the case $n = m$ we have the following complete result.

Proposition 5.12. *Let $0 \leq r < n + 1$. Then the implication*

$$K \text{ removable for } \text{HL}_s^n \implies K \text{ removable for } \text{HL}_r^n$$

holds if and only if $r \leq s$.

Proof. If $r \leq s$, then the implication holds by Proposition 5.3. Otherwise, Corollary 5.8 shows that the implication is false. $\square$

For the case $s = r$ we have the following complete result.

Proposition 5.13. *Let $0 \leq s < \min\{n, m\} + 1$. Then the implication*

$$K \text{ removable for } \text{HL}_s^n \implies K \text{ removable for } \text{HL}_s^m$$

holds if and only if $n \leq m$.

Proof. If $n \leq m$, then the implication holds by Proposition 5.3. If $n > m$ and $s > 0$, then Corollary 5.8 shows that the implication is false. Finally, if $n > m$ and $s = 0$, then the implication is false by the theorem on p. 18 of Kaufman [25]. $\square$

Proposition 5.3 gives us essentially all the positive results we know. There is only one additional result.

Proposition 5.14. *If a compact set K is removable for HL_s^1 for some $1 \leq s < 2$, then K is removable for $\text{HL}_0^0 = H^\infty$.*

Proof. By Corollary 2.6, $\Lambda_1(K) = 0$, and thus Painlevé's theorem shows that K is removable for $H^\infty = \text{HL}_0^0$. $\square$

We have the following result for when the implication fails.

Proposition 5.15. *Let $m \in \mathbf{N}$, and $0 \leq r < m + 1$. Then the implication*

$$K \text{ removable for } \text{HL}_s^n \implies K \text{ removable for } \text{HL}_r^m$$

is false if $s/(n+1) < r/(m+1)$ and also if $m = 0 < n$ and $s < 1 + r$.

Proof. Corollary 5.8 shows that the implication is false if $s/(n+1) < r/(m+1)$.

Assume next that $m = 0 < n$ and $s < 1 + r$. Let α be such that $1 - r = \dim C_\alpha$ if $r > 0$ and such that $1 < \dim C_\alpha < 2 - s$ if $r = 0$. Then C_α is not removable for HL_r^0 , by Lemma 5.11 if $r > 0$ and by well-known results for $H^\infty = \text{HL}_0^0$ if $r = 0$. On the other hand, C_α is removable for HL_s^n by Theorem 5.4. $\square$

If we want to consider the implications for strong removability we have to be more careful. Since strong removability depends on the domain (if $s \neq 0$) the implication is not even true for $m = n$, $r = s > 0$ when the domain is allowed to change. On the other hand, if we fix the domain and only vary the exponent we still have the following negative result.

Proposition 5.16. *Let $m \in \mathbf{N}$ and $0 < r < \min\{s, m+1\}$ be such that s/r is not an integer. Then the implication*

$$\begin{aligned} E \text{ strongly removable for } \text{HL}_s^n(\Omega \setminus E) \\ \implies E \text{ strongly removable for } \text{HL}_r^m(\Omega \setminus E) \end{aligned}$$

is false, i.e. there exists a domain Ω and a subset $E \subset \Omega$ with $\Omega \setminus E$ being a domain such that the implication is false.

Note that, by Proposition 3.14, the implication in Proposition 5.16 is equivalent to the fact that the seemingly stronger implication

$$\begin{aligned} A \text{ strongly removable for } \text{HL}_s^n(\Omega \setminus A) \\ \implies A \text{ strongly removable for } \text{HL}_r^m(\Omega \setminus A) \end{aligned}$$

holds for all $A \subset \Omega$ and all domains $\Omega \subset \mathbf{C}$. Similar modifications can be made in the formulations of Propositions 5.17 and 5.18.

In the case $n = m = 0$ we have the following complete result.

Proposition 5.17. *Let $0 \leq s < 1$ and $0 \leq r < 1$. Then the implication*

$$\begin{aligned} E \text{ strongly removable for } \text{HL}_s^0(\Omega \setminus E) \\ \implies E \text{ strongly removable for } \text{HL}_r^0(\Omega \setminus E) \end{aligned}$$

is true if and only if $s/r \in \mathbf{Z}_+$ or $r = 0$.

In the case $s = r$ we have the following complete result.

Proposition 5.18. *Let $m \in \mathbf{N}$ and $0 \leq s < \min\{n, m\} + 1$. Then the implication*

$$\begin{aligned} E \text{ strongly removable for } \text{HL}_s^n(\Omega \setminus E) \\ \implies E \text{ strongly removable for } \text{HL}_s^m(\Omega \setminus E) \end{aligned}$$

is true if and only if $m \geq n$.

In order to prove these results we first formulate a lemma.

Lemma 5.19. *Let $0 < s < n + 1$ and $K \supset K_d$ be such that $\text{dist}(z, K) \geq 2^{-1-2j/(2-d)}$ for $|z| = r_{j,1/2}$. Then $\text{HL}_s^n(\mathbf{C} \setminus K)$ consists of the functions that can be written as a sum of a polynomial in $1/z$ of degree $\leq \max\{0, 2s/(2-d) - n\}$ and of a polynomial in z of degree less than n .*

Proof. By Lemma 5.7 and the assumption $K \supset K_d$, it directly follows that all these functions belong to $\text{HL}_s^n(\mathbf{C} \setminus K)$. Conversely we can argue as in the proof of Lemma 5.7 to see that no other functions belong to $\text{HL}_s^n(\mathbf{C} \setminus K)$. $\square$

Proof of Proposition 5.16. Choose d_1 so that $0 < d_1 < 2$, $n < 2s/(2 - d_1) \notin \mathbf{Z}$ and $m < 2r/(2 - d_1) \in \mathbf{Z}$, this is possible since $s/r \notin \mathbf{Z}$. Choose next d_2 , $0 < d_2 < d_1$, so close to d_1 that $\lfloor 2s/(2 - d_1) \rfloor = \lfloor 2s/(2 - d_2) \rfloor$, and also $\theta \in \mathbf{R}$ so that $K_{d_1} \cap e^{i\theta} K_{d_2} = \{0\}$. (By $\lfloor x \rfloor$ we mean the largest integer $\leq x$.)

Let $\Omega = \mathbf{C} \setminus e^{i\theta} K_{d_2}$ and $E = K_{d_1} \setminus \{0\}$. By Lemma 5.19 we find that E is strongly removable for $\text{HL}_s^n(\Omega \setminus E)$, but not for $\text{HL}_r^m(\Omega \setminus E)$. $\square$

Proof of Proposition 5.17. If $r > s$ the implication is false by Proposition 5.12. If $0 < r < s$ and s/r is a non-integer, then the implication is false by Proposition 5.16.

Assume finally that $N = s/r$ is an integer or that $r = 0$. Assume that E is strongly removable for $\text{HL}_s^0(\Omega \setminus E)$. It follows that E is weakly removable for $\text{HL}_s^0(\Omega \setminus E)$ and hence for $\text{HL}_r^0(\Omega \setminus E)$, by Proposition 5.3. If $r = 0$, Proposition 5.2, shows that E is strongly removable for $\text{HL}_r^0(\Omega \setminus E)$. Otherwise, let $f \in \text{HL}_r^0(\Omega \setminus E) \subset \text{Hol}(\Omega)$ and let $g = f^N \in \text{Hol}(\Omega)$. By definition, there exists C such that for $z \in \Omega \setminus E$,

$$|g(z)| = |f(z)|^N \leq \left(\frac{C}{\text{dist}(z, \partial(\Omega \setminus E))^r} \right)^N = \frac{C^N}{\text{dist}(z, \partial(\Omega \setminus E))^s}.$$

Thus $g \in \text{HL}_s^0(\Omega \setminus E) = \text{HL}_s^0(\Omega)$. But, then it follows that there exists C_1 such that for $z \in \Omega$,

$$|f(z)| = |g(z)|^{1/N} \leq \left(\frac{C_1}{\text{dist}(z, \partial\Omega)^s} \right)^{1/N} = \frac{C_1^{1/N}}{\text{dist}(z, \partial\Omega)^r},$$

and hence $f \in \text{HL}_r^0(\Omega)$. $\square$

Proof of Proposition 5.18. If $m < n$, then the implication is false by Proposition 5.13.

For the converse we assume, without loss of generality, that $m = n + 1$. Let $f \in \text{HL}_s^m(\Omega \setminus A)$. By Proposition 5.3, A is weakly removable for HL_s^m , and hence $f \in \text{Hol}(\Omega)$. Moreover, $f' \in \text{HL}_s^n(\Omega \setminus A) = \text{HL}_s^n(\Omega)$, but this is equivalent to saying that $f \in \text{HL}_s^m(\Omega)$, since $f \in \text{Hol}(\Omega)$. $\square$

5.4 Unions of removable singularities

In this section we give a number of examples of non-removable unions of removable sets. These results contrast the positive results in Propositions 3.7 and 3.11, Corollary 3.12, Propositions 3.13 and 3.16, Theorem 3.17 and Corollary 3.18.

Proposition 5.20. *There exist compact countable sets K_1 and K_2 removable for all HL_s^n , $n \in \mathbb{N}$, $0 \leq s < n + 1$, such that $K_1 \cup K_2$ is not removable for any HL_s^n , $n \in \mathbb{N}$, $0 < s < n + 1$, and moreover such that $K_1 \cap K_2 = \{0\}$.*

It follows that $K_1 \setminus K_2$ is weakly removable for all HL_s^n , $n \in \mathbb{N}$, $0 \leq s < n + 1$, and strongly removable for all $\text{HL}_s^n(\mathbb{C} \setminus K_1)$, $n \in \mathbb{N}$, $0 \leq s < n + 1$, but not strongly removable for any $\text{HL}_s^n(\mathbb{C} \setminus (K_1 \cup K_2))$, $n \in \mathbb{N}$, $0 < s < n + 1$. Thus if $s \neq 0$, then weak and strong removability are not the same, strong removability is domain dependent, and a subset of a compact removable set does not have to be strongly removable.

Proposition 5.20 follows directly from the following lemma.

Lemma 5.21. *Let for $j \in \mathbb{Z}_+$, $E_j \subset \{z : 2^{-2j} < |z| < 2^{-2j+2}\}$ be a finite set such that $\text{dist}(z, E_j) \leq 2^{-2j^2}$ for $2^{-2j} < |z| < 2^{-2j+2}$. Let $K_1 = \bigcup_{j=1}^{\infty} E_{2j-1} \cup \{0\}$, $K_2 = \bigcup_{j=1}^{\infty} E_{2j} \cup \{0\}$ and $K = K_1 \cup K_2$.*

Then, K_1 and K_2 are removable for all HL_s^n , $n \in \mathbb{N}$, $0 \leq s < n + 1$, but $z^{-m} \in \text{HL}_s^n(\mathbb{C} \setminus K)$ for $n, m \in \mathbb{N}$, $0 < s < n + 1$.

Proof. Assume that $f \in \text{HL}_s^n(\mathbb{C} \setminus K_1)$ for some $n \in \mathbb{N}$, $0 \leq s < n + 1$. Let C be such that $|f^{(n)}(z)| \leq C \text{dist}(z, K_1)^{-s}$ for $z \notin K_1$. Since the sets E_j are finite they are removable and hence $f \in \text{Hol}(\mathbb{C} \setminus \{0\})$. Since $f^{(n)}$ remains bounded at ∞ we can write

$$f^{(n)}(z) = a_0 + \sum_{m=n+1}^{\infty} \frac{a_m}{z^m}, \quad z \neq 0.$$

For z with $|z| = 2^{-4j+1}$ we have $\text{dist}(z, K_1) \geq 2^{-4j}$. Hence, with $d\sigma$ denoting arc length, and $m \geq n + 1 > s$,

$$\begin{aligned} |a_m| &\leq \frac{1}{2\pi} \int_{|z|=2^{-4j+1}} |z|^{m-1} |f^{(n)}(z)| d\sigma \leq C 2^{(-4j+1)m} 2^{-4j(-s)} \\ &= C 2^m 2^{-4j(m-s)} \rightarrow 0, \quad \text{as } j \rightarrow \infty. \end{aligned}$$

Thus $f^{(n)}$ is a constant, f is a polynomial and K_1 is removable. The proof of the removability for K_2 is similar.

Consider next $f(z) = z^{-m}$, $m \geq 1$, and fix $0 < s < n + 1$. Let

$$c_1 = m(m+1) \dots (m+n-1) \quad \text{and} \quad c_2 = c_1 \sup_{j \in \mathbf{N}} 2^{2j(m+n-j)} < \infty.$$

For $|z| \geq 1$, we find that

$$|f^{(n)}(z)| = \frac{c_1}{|z|^{m+n}} \leq \frac{c_1}{|z|^s} \leq \frac{c_2}{\text{dist}(z, K)^s}.$$

For $2^{-2j} \leq |z| \leq 2^{-2j+2}$, $j \in \mathbf{Z}_+$, we have

$$|f^{(n)}(z)| \text{dist}(z, E_j)^s = \frac{c_1}{|z|^{m+n}} \text{dist}(z, E_j)^s \leq c_1 2^{2j(m+n-j)} \leq c_2.$$

Thus $f \in \text{HL}_s^n(\mathbf{C} \setminus K)$. $\square$

Finite unions of pairwise disjoint compact removable sets are removable, by Corollary 3.12. When generalizing to non-compact sets, one has to be careful with how the domains are handled. We have Proposition 3.13, but if we do not assume a common domain for E_k we instead obtain the following negative result.

Proposition 5.22. *There exist disjoint sets $E_1, E_2 \subset \mathbf{C}$ with $\mathbf{C} \setminus (E_1 \cup E_2)$ and $\mathbf{C} \setminus E_1$ being domains, and such that E_1 and E_2 are weakly removable for all HL_s^n , $n \in \mathbf{N}$, $0 \leq s < n + 1$, but $E_1 \cup E_2$ is a compact set not removable for any HL_s^n , $n \in \mathbf{N}$, $0 < s < n + 1$.*

Proof. This follows directly from Proposition 5.20, letting $E_1 = K_1$ and $E_2 = K_2 \setminus K_1$. $\square$

Because of the domain dependence, results for strong removability corresponding to Proposition 3.13 can be given in different forms. Here we give two such negative results.

Proposition 5.23. *There exist a domain Ω and disjoint sets $E_1, E_2 \subset \Omega$ with $\Omega \setminus E_1$ and $\Omega \setminus E_2$ being domains, and such that E_1 and E_2 are strongly removable for $\text{HL}_s^n(\Omega \setminus E_1)$ and $\text{HL}_s^n(\Omega \setminus E_2)$, respectively, for all $n \in \mathbf{N}$, $0 \leq s < n + 1$, but $E_1 \cup E_2$ is not strongly removable for any $\text{HL}_s^n(\Omega \setminus (E_1 \cup E_2))$, $n \in \mathbf{N}$, $0 < s < n + 1$.*

Remark. A consequence of the assumptions is that $\Omega \setminus (E_1 \cup E_2)$ is a domain.

Proof. This follows directly from Proposition 5.20, letting $\Omega = \mathbf{C} \setminus (K_1 \cap K_2)$, $E_1 = K_1 \setminus K_2$ and $E_2 = K_2 \setminus K_1$. $\square$

Proposition 5.24. *There exist a domain Ω and disjoint sets $E_1, E_2 \subset \Omega$ with $\Omega \setminus E_1$ and $\Omega \setminus E_2$ being domains, and such that E_1 and E_2 are strongly removable for all $\text{HL}_s^n(\Omega \setminus (E_1 \cup E_2))$, $n \in \mathbf{N}$, $0 \leq s < n + 1$, but $E_1 \cup E_2$ is not strongly removable for any $\text{HL}_s^n(\Omega \setminus (E_1 \cup E_2))$, $n \in \mathbf{N}$, $0 < s < n + 1$.*

This result shows that Axiom A7 is not satisfied for HL_s^n , $s \neq 0$, since otherwise we would have had Proposition 3.16. That Axiom A7 is not satisfied for HL_s^n , $s \neq 0$, also follows from the following lemma that we will use to prove Proposition 5.24.

Lemma 5.25. *Let for $j \in \mathbf{Z}_+$,*

$$E_{j,1}, E_{j,2} \subset \{z : 2^{-2j} < |z| < 2^{-2j+2} - 2^{-2j^2-1} - 2^{-2(j+1)^2-1}\}$$

be finite disjoint sets such that $\text{dist}(z, E_{j,k}) \leq 2^{-2(j+1)^2}$ for $2^{-2j} \leq |z| \leq 2^{-2j+2} - 2^{-2j^2-1}$ and $\text{dist}(z, E_{j,k}) \geq 2^{-2(j+1)^2-1}$ for $|z| = 2^{-2j}$. Let $K_k = \bigcup_{j=1}^{\infty} E_{j,k} \cup \{0\}$, $k = 1, 2$, and $K = K_1 \cup K_2$.

Then, $\text{HL}_s^n(\mathbf{C} \setminus K_1) = \text{HL}_s^n(\mathbf{C} \setminus K_2) = \text{HL}_s^n(\mathbf{C} \setminus K)$ for $n \in \mathbf{N}$, $0 \leq s < n + 1$.

It is not difficult to construct the sets $E_{j,1}$ and $E_{j,2}$ explicitly, but we omit such a construction here.

Proof of Proposition 5.24. Let K_1 and K_2 be given by Lemma 5.25, and let $E_1 = K_1 \setminus \{0\}$, $E_2 = K_2 \setminus \{0\}$ and $\Omega = \mathbf{C} \setminus \{0\}$.

By Lemma 5.25, $\text{HL}_s^n(\Omega \setminus E_1) = \text{HL}_s^n(\Omega \setminus E_2) = \text{HL}_s^n(\Omega \setminus (E_1 \cup E_2))$ for all $n \in \mathbf{N}$, $0 \leq s < n + 1$. Thus E_1 and E_2 are strongly removable for all $\text{HL}_s^n(\Omega \setminus (E_1 \cup E_2))$, $n \in \mathbf{N}$, $0 \leq s < n + 1$.

By Lemma 5.21 (and the above) we get that $\text{HL}_s^n(\Omega) \neq \text{HL}_s^n(\Omega \setminus E_1) = \text{HL}_s^n(\Omega \setminus (E_1 \cup E_2))$ for $n \in \mathbf{N}$, $0 < s < n + 1$. Thus $E_1 \cup E_2$ is not strongly removable for any $\text{HL}_s^n(\Omega \setminus (E_1 \cup E_2))$, $n \in \mathbf{N}$, $0 < s < n + 1$. $\square$

Proof of Lemma 5.25. Let $f \in \text{HL}_s^n(\mathbf{C} \setminus K)$. Since $K \setminus \{0\}$ is a countable set of well-separated points it is weakly removable for HL_s^n by Proposition 3.7. Hence $f \in \text{Hol}(\mathbf{C} \setminus \{0\})$. Moreover there is C such that $|f^{(n)}(z)| \leq C \text{dist}(z, K)^{-s}$. Let $C' = 2^s C$. We will show that $|f^{(n)}(z)| \leq C' \text{dist}(z, K_1)^{-s}$.

For $|z| = 2^{-2j}$, $j \in \mathbf{N}$, we have

$$|f^{(n)}(z)| \leq \frac{C}{\text{dist}(z, K)^s} \leq C' 2^{(2(j+1)^2+1)s} = C' 2^{2(j+1)^2 s}.$$

Hence for $|z| = 2^{-2j+2}$, $j \in \mathbf{Z}_+$,

$$|f^{(n)}(z)| \leq C' 2^{2j^2 s} \leq C' 2^{2(j+1)^2 s}.$$

The maximum principle shows that $|f^{(n)}(z)| \leq C' 2^{2(j+1)^2 s}$ for $2^{-2j} \leq |z| \leq 2^{-2j+2}$, $j \in \mathbf{Z}_+$. In particular,

$$|f^{(n)}(z)| \leq C' 2^{2(j+1)^2 s} \leq \frac{C'}{\text{dist}(z, K_1)^s}$$

for $2^{-2j} \leq |z| \leq 2^{-2j+2} - 2^{-2j^2-1}$, $j \in \mathbf{Z}_+$.

Further, $\text{dist}(z, K_1) \leq 2 \text{dist}(z, K)$ for $|z| \geq 1$ and for $2^{-2j+2} - 2^{-2j^2-1} \leq |z| \leq 2^{-2j+2}$, $j \in \mathbf{Z}_+$. Hence,

$$|f^{(n)}(z)| \leq \frac{C}{\text{dist}(z, K)^s} \leq \frac{C'}{\text{dist}(z, K_1)^s}$$

for $|z| \geq 1$ and for $2^{-2j+2} - 2^{-2j^2-1} \leq |z| \leq 2^{-2j+2}$, $j \in \mathbf{Z}_+$. Thus $|f^{(n)}(z)| \leq C' \text{dist}(z, K_1)^{-s}$ for all $z \in \mathbf{C} \setminus K_1$, and $f \in \text{HL}_s^n(\mathbf{C} \setminus K_1)$.

It follows that $\text{HL}_s^n(\mathbf{C} \setminus K_1) = \text{HL}_s^n(\mathbf{C} \setminus K_2) = \text{HL}_s^n(\mathbf{C} \setminus K)$. $\square$

We know that finite disjoint unions of compact removable sets are removable. That finiteness is essential is shown by the following result, which also shows that the well-separatedness in Proposition 3.7 is essential.

Proposition 5.26. *There exists a countable family of pairwise disjoint compact sets $K_1, K_2, \dots$, removable for all HL_s^n , $n \in \mathbf{N}$, $0 \leq s < n+1$, but with $\bigcup_{j=1}^\infty K_j$ being a compact set not removable for any HL_s^n , $n \in \mathbf{N}$, $0 < s < n+1$.*

Further, there exists an increasing sequence of compact sets $K'_1 \subset K'_2 \subset \dots$, removable for all HL_s^n , $n \in \mathbf{N}$, $0 \leq s < n+1$, but with $\bigcup_{j=1}^\infty K'_j$ being a compact set not removable for any HL_s^n , $n \in \mathbf{N}$, $0 < s < n+1$.

Proof. Let K be given by Lemma 5.21, and hence not removable for any HL_s^n , $n \in \mathbf{N}$, $0 < s < n+1$. The set K is countable, and can hence be written as a countable union of points all of whom are, of course, removable for all HL_s^n , $n \in \mathbf{N}$, $0 \leq s < n+1$.

The second part follows by writing K as an increasing union of finite sets. $\square$

5.5 Domain dependence for strong removability

As a consequence of Proposition 5.20 we observed that $K_1 \setminus K_2$ was strongly removable for all $\text{HL}_s^n(\mathbf{C} \setminus K_1)$, $n \in \mathbf{N}$, $0 \leq s < n+1$, but not strongly removable for any $\text{HL}_s^n(\mathbf{C} \setminus (K_1 \cup K_2))$, $n \in \mathbf{N}$, $0 < s < n+1$. Thus if $E \subset \Omega_1 \subset \Omega_2$ and E is strongly removable for $\text{HL}_s^n(\Omega_2 \setminus E)$ we cannot conclude that E is strongly removable for $\text{HL}_s^n(\Omega_1 \setminus E)$ (unless $s = 0$). One can ask if such an implication may be true if we instead assume that $\Omega_2 \subset \Omega_1$. In fact it would have been true by Proposition 3.15 if Axiom A7 were true. It turns out to be false (if $s \neq 0$). We will go one step further and prove the following result.

Proposition 5.27. *There exist $E \subset \Omega_4 \subset \dots \subset \Omega_1$ such that Ω_k and $\Omega_k \setminus E$ are domains, $k = 1, \dots, 4$, and such that E is strongly removable for all $\text{HL}_s^n(\Omega_k \setminus E)$, $n \in \mathbf{N}$, $0 \leq s < n+1$, $k = 1, 3$, but E is not strongly removable for any $\text{HL}_s^n(\Omega_k \setminus E)$, $n \in \mathbf{N}$, $0 < s < n+1$, $k = 2, 4$.*

This generalization shows also that one cannot interpolate between domains for which E is strongly removable (or for which E is not strongly removable).

From the proof it is clear that one can make arbitrarily long (finite) alternating sequences as above.

Proof. Let $E_{j,k}$ be given by Lemma 5.25, let $E' = \bigcup_{j=1}^{\infty} E_{2j,1}$ and let $E = E' \cup (3 + E')$.

Let $\Omega_1 = \mathbb{C} \setminus \{0, 3\}$. By Lemma 5.21, $\text{HL}_s^n(\Omega_1 \setminus E) = \text{HL}_s^n(\Omega_1 \setminus E') = \text{HL}_s^n(\Omega_1)$, $n \in \mathbb{N}$, $0 \leq s < n+1$. Hence E is strongly removable for all $\text{HL}_s^n(\Omega_1 \setminus E)$, $n \in \mathbb{N}$, $0 \leq s < n+1$.

Let $\Omega_2 = \Omega_1 \setminus \bigcup_{j=1}^{\infty} E_{2j-1,1}$. Then by Lemma 5.21, $\text{HL}_s^n(\Omega_2 \setminus E) = \text{HL}_s^n(\Omega_2 \setminus E') \neq \text{HL}_s^n(\Omega_2)$, $n \in \mathbb{N}$, $0 < s < n+1$. Thus E is not strongly removable for any $\text{HL}_s^n(\Omega_2 \setminus E)$, $n \in \mathbb{N}$, $0 < s < n+1$.

Let $\Omega_3 = \Omega_2 \setminus \bigcup_{j=1}^{\infty} E_{j,2}$. By Lemma 5.21, $\text{HL}_s^n(\Omega_3 \setminus E) = \text{HL}_s^n(\Omega_3 \setminus E')$, and by Lemma 5.25, $\text{HL}_s^n(\Omega_3 \setminus E') = \text{HL}_s^n(\Omega_3)$. Hence E is strongly removable for all $\text{HL}_s^n(\Omega_3 \setminus E)$, $n \in \mathbb{N}$, $0 \leq s < n+1$.

Let $\Omega_4 = \Omega_3 \setminus (3 + \bigcup_{j=1}^{\infty} E_{2j-1,1})$. Then by Lemma 5.21, $\text{HL}_s^n(\Omega_4) \subset \text{HL}_s^n(\Omega_4 \setminus E') \subsetneq \text{HL}_s^n(\Omega_4 \setminus E)$, $n \in \mathbb{N}$, $0 < s < n+1$. Thus E is not strongly removable for any $\text{HL}_s^n(\Omega_4 \setminus E)$, $n \in \mathbb{N}$, $0 < s < n+1$. $\square$

6 Removability for hl_s^n

Assume throughout this section that $n \in \mathbb{N}$ and $0 < s \leq n+1$.

Definition 6.1. For $\Omega \subsetneq \mathbb{C}$ we define

$$\begin{aligned} \text{hl}_s^n(\Omega) &= \{f \in \text{HL}_s^n(\Omega) : |f^{(n)}(z)| = o(\text{dist}(z, \partial\Omega)^s)\}, \\ \text{hl}_s^n(\mathbb{C}) &= \{f \in \text{Hol}(\mathbb{C}) : f \in \text{hl}_s^n(\mathbb{C} \setminus \{0\})\}. \end{aligned}$$

Note that for $s = 0$ we get the trivial space of polynomials of degree less than n , for all domains, making all sets removable. On the other hand, singleton sets are removable for hl_{n+1}^n , so the case $s = n+1$ is interesting in this case.

For $n = 1$, these spaces are connected with locLip_{1-s} , VMO and certain “little oh” Campanato spaces in the same way as HL_s^1 is connected with locLip_{1-s} , BMO and Campanato spaces. Theorem 3 in Král [31] gives the following corollary.

Corollary 6.2. If E is weakly removable for hl_s^1 , $0 < s < 2$, then $\Lambda_{2-s}(E) < \infty$.

The results in Section 5 can all be given for hl_s^n , $n \in \mathbb{N}$, $0 < s \leq n+1$, as well. In fact it is easy to modify all our proofs to cover this case (also for $s = n+1$, but, of course, avoiding $s = 0$). In some cases the result for hl_s^n can be given as a simple corollary of the corresponding result for HL_s^n , or with a modification of the proof.

The only results in Section 5 that have to be modified for hl_s^n are Lemmas 5.6 and 5.7, Corollary 5.8 and Lemmas 5.11 and 5.19. For hl_s^n they should have the following forms.

Lemma 6.3. Let $0 < d < 2$ and $1 - \frac{1}{2}d < s \leq 1$. Then $1/z \in \text{hl}_s^0(\mathbb{C} \setminus K_d)$.

Lemma 6.4. Let $0 < d < 2$ and $0 < s \leq n+1$. Then $\text{hl}_s^n(\mathbb{C} \setminus K_d)$ consists of the functions that can be written as a sum of a polynomial in $1/z$ of degree less than $\max\{1, 2s/(2-d) - n\}$ and of a polynomial in z of degree less than n .

Corollary 6.5. *The set K_d is removable for hl_s^n if and only if $s \leq \frac{1}{2}(n+1)(2-d)$, or equivalently $d \leq 2 - 2s/(n+1)$.*

Lemma 6.6. *Let $0 < s \leq 1$, $1-s < d < 2$ and let α be such that $d = \dim C_\alpha$. Then C_α is not removable for hl_s^0 .*

Lemma 6.7. *Let $n \in \mathbf{N}$ and $0 < s \leq n+1$ and $K \supset K_d$ be such that if $|z| = r_{j,1/2}$, then $\text{dist}(z, K) \geq 2^{-1-2j/(2-d)}$. Then $\text{hl}_s^n(\mathbf{C} \setminus K)$ consists of the functions that can be written as a sum of a polynomial in $1/z$ of degree less than $\max\{1, 2s/(2-d) - n\}$ and of a polynomial in z of degree less than n .*

From the inclusion $\text{hl}_s^n(\Omega) \subset \text{HL}_s^n(\Omega)$ it follows that if A is weakly removable for HL_s^n , then A is also weakly removable for hl_s^n . That

$$K \text{ removable for } \text{hl}_s^n \implies K \text{ removable for } \text{HL}_s^n$$

is false (for some compact set K) follows from Corollaries 5.8 and 6.5. (For $s = n+1$ we should use the observation that singleton sets are removable for hl_{n+1}^n , but not for HL_{n+1}^n .)

For strong removability the converse implication is also false.

Proposition 6.8. *Let $n \in \mathbf{N}$ and $0 < s < n+1$. Then there exist a domain Ω and a set $E \subset \Omega$ with $\Omega \setminus E$ being a domain, such that E is strongly removable for $\text{HL}_s^n(\Omega \setminus E)$, but E is not strongly removable for $\text{hl}_s^n(\Omega \setminus E)$.*

Proof. Let $d = 2 - 2s/(n+1)$ and $d' = 2 - 2s/(n + \frac{3}{2})$. Let $\Omega = \mathbf{C} \setminus K_d$ and $E = K_{d'} \setminus K_d$. By Lemmas 5.7 and 5.19, E is strongly removable for $\text{HL}_s^n(\Omega \setminus E)$. On the other hand, Lemmas 6.4 and 6.7 show that E is not strongly removable for $\text{hl}_s^n(\Omega \setminus E)$. $\square$

It is interesting to note that in all the counterexamples given in Section 5.4 the same sets can be used for all HL_s^n , $s \neq 0$, and all hl_s^n simultaneously, i.e. the counterexamples do not depend on n , s or if we use HL or hl .

7 Hardy spaces

Definition 7.1. *Let $0 < p < \infty$, and let the Hardy spaces be defined by*

$$\begin{aligned} H^p(\Omega) &= \{f \in \text{Hol}(\Omega) : |f|^p \leq u \text{ in } \Omega \text{ for some } u \text{ harmonic in } \Omega\}, \\ H^\infty(\Omega) &= \{f \in \text{Hol}(\Omega) : \sup_{z \in \Omega} |f(z)| < \infty\}. \end{aligned}$$

Axioms A1–A6 are satisfied for H^p , see Björn [2], Theorem 4.5 and the remark following Definition 4.1. The spaces H^∞ are as we have noticed equal to HL_0^0 , and in particular weak and strong removability are the same for all sets for H^∞ .

That Axiom A7 is not satisfied for H^p was observed by Hejhal [23], Theorem 2, and independently by Gauthier and Hengartner, pp. 411–412 in [17]. See also Suita [42].

The results in Section 3 have all been obtained for H^p , see Björn [5]. A number of results corresponding to those for HL_s^n given in Section 5 have also been given

for H^p , see the discussion in [5]. The big similarities are a reason behind the axiomatic approach used in this paper. However, the counterexamples for H^p are more difficult to obtain than the corresponding results for HL_s^n and not yet fully complete.

Proposition 3.19 may be a new observation for H^∞ , and stands in contrast to the following result. (This was observed in Björn [6] with the proof deferred to this paper.)

Proposition 7.2. *Let $0 < p < \infty$ and $M \in \mathbf{Z}_+$ or $M = \infty$. Then there exists a domain Ω such that $\dim H^p(\Omega) = M$.*

Proof. For $p \geq \frac{1}{4}$ this follows directly from Corollary 6.4 in Björn [5].

For general p we can use Theorem 3.4 in Hasumi [20]. Some comments may be useful. We follow the notation in [20] and let $\theta(t) = e^{pt}$, $\varphi(t) = e^{Mpt}$ and $\psi(t) = e^{(M-1)pt}$. In condition (e) on p. 219 we can replace the constant 2 in the right-hand side by $2M$. Then, the estimate (13) on p. 222 follows also if we replace φ by θ , and hence the set E constructed in Theorem 2.5 is not only belonging to the class N_φ of compact removable singularities for $H_\varphi = H^{Mp}$, but also to N_θ of compact removable singularities for $H_\theta = H^p$.

This enables us to modify the statement of Theorem 3.4 to say that the sets E_n belong to N_θ . If on line -4 on p. 224, we now assume that $f \in H_\theta(\mathbf{S} \setminus E) = H^p(\mathbf{S} \setminus E)$ and proceed in the same way as in the original proof, we obtain

$$\frac{\theta(\log |f(w)|)}{\varphi(-\log a_n)} \leq C, \quad |w| = \sigma a_n,$$

instead of (19). This implies that

$$|f(w)| \leq a_n^{-CN}, \quad |w| = \sigma a_n \text{ and } n \geq N.$$

(The exponent $-C - 1$ can be replaced by $-C$ in the original proof.) Proceeding in the same way as in the original proof we find that f must be a polynomial in $1/z$. Now, the original statement show that $1/z \in H_\psi(\mathbf{S} \setminus E) = H^{(M-1)p}(\mathbf{S} \setminus E)$, but this is equivalent to saying that $1/z^{M-1} \in H^p(\mathbf{S} \setminus E)$. Moreover, $1/z \notin H_\varphi(\mathbf{S} \setminus E) = H^{Mp}(\mathbf{S} \setminus E)$, which is equivalent to saying that $1/z^M \notin H^p(\mathbf{S} \setminus E)$. Thus $H^p(\mathbf{C} \setminus E) = H^p(\mathbf{S} \setminus E)$ is spanned by the functions $1, 1/z, \dots, 1/z^{M-1}$. $\square$

8 Bergman spaces

Let $L_\mu^p(\Omega)$ denote the usual Lebesgue space (quasi)-normed by

$$\|f\|_{L_\mu^p(\Omega)} = \left(\int_\Omega |f(z)|^p d\mu(z) \right)^{1/p},$$

where μ is a Radon measure on $\mathbf{C}$, i.e. a positive Borel measure which is finite for compact subsets of $\mathbf{C}$.

Definition 8.1. The Bergman space $\mathcal{A}_\mu^p(\Omega)$, $0 < p \leq \infty$, is defined by

$$\mathcal{A}_\mu^p(\Omega) = \{f \in \text{Hol}(\Omega) : \|f\|_{L_\mu^p(\Omega)} < \infty\}.$$

The case with infinite measure is sometimes quite different from the finite measure case. In the infinite measure case it can happen that Axioms A3 and A4 fail. However, Axioms A3' and A4' are satisfied. Instead of just studying bounded domains, one can introduce auxiliary Bergman spaces, and this gives additional information also for removability for unbounded domains.

Definition 8.2. The auxiliary Bergman space $B_\mu^p(\Omega)$, $0 < p \leq \infty$, is defined by

$$B_\mu^p(\Omega) = \bigcap \mathcal{A}_\mu^p(\Omega'),$$

where the intersection is taken over all domains $\Omega' \subset \Omega$ with $\mu(\Omega') < \infty$.

Axioms A1–A6 are satisfied for B_μ^p , see Björn [3] or [4]. In contrast to H^p , $p \neq \infty$, HL_s^n , $s \neq 0$, and hl_s^n , Axiom A7 is also satisfied for B_μ^p .

The theory for removable singularities for weighted Bergman spaces was developed in [3]. In particular, the results obtained in Section 3, with the exception of Proposition 3.19, were obtained with the restriction that in order to say that E is removable for $B_\mu^p(\Omega \setminus E)$, it was required that $\Omega \setminus E$ is a domain.

Results corresponding to many of those in Section 5 were obtained for B_μ^p in [3] and Björn [5]. Björn [4] is based on the axiomatic results of this paper, and contains the results on Bergman spaces in [3] and [5].

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