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Removable singularities for Hardy spaces

Anders Björn

Abstract

In this paper we study removable singularities for Hardy spaces of analytic functions on general domains. Two different definitions are given. For compact sets they turn out to be equal and moreover independent of the surrounding domain, as was proved by D. A. Hejhal. For non-compact sets the difference between the definitions is studied.

A survey of the present knowledge is given, except for the special cases of singularities lying on curves and singularities being self-similar Cantor sets, which the author deals with in other papers. Among the results is the non-removability for H^p of sets with dimension greater than p , $0 < p < 1$. Many counterexamples are provided and the H^p -capacities are introduced and studied.

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This paper is essentially a part of the author's thesis [3]. It contains the results, with some improvements, in Chapters 3–6 in [3]. The results in Chapter 7 in [3], about removable singularities lying on rectifiable curves, are published in [5]. The results in Chapter 8 in [3], about removability of Cantor sets, are going to be published in [6]. See also Björn [4].

1. Notation

We let $\mathbf{S} = \mathbf{C} \cup \{\infty\}$ be the Riemann sphere with the usual topology, $D(z_0, r) = \{z \in \mathbf{C} : |z - z_0| < r\}$, $\mathbf{D} = D(0, 1)$, $A(\Omega) = \{f : f \text{ is analytic in } \Omega\}$ and $f'(\infty) = \lim_{z \rightarrow \infty} z(f(z) - f(\infty))$.

By a domain we will mean a non-empty open connected set. Because of the uniqueness theorem for analytic functions we will not distinguish between restrictions and extensions of analytic functions.

We let Λ_d denote d -dimensional Hausdorff measure and $\dim$ denote Hausdorff dimension. We let $\omega(E; \Omega, z_0)$ denote the harmonic measure of the set $E \subset \partial\Omega$ for the domain Ω evaluated at z_0 .

By cap we denote the logarithmic capacity, if we are working with one complex variable, and the $(2n - 2)$ -dimensional Newtonian capacity normally used for harmonic functions in $\mathbf{R}^{2n}$, when working with $n > 1$ complex variables. The sets characterized by $\text{cap}(\cdot) = 0$ are called polar sets. We are only interested in the zero sets for the capacity and will not give the precise definition of it.

The logarithmic capacity is sometimes defined so that $\text{cap}(D(\cdot, r)) = r$ and sometimes so that $\text{cap}(D(\cdot, r)) = 1/(\log r^{-1})$. The zero sets are the same for both definitions. We choose the former alternative. For sets $E \ni \infty$ we define $\text{cap}(E) = \text{cap}(E \cap \mathbf{C})$.

There are many books treating harmonic measures, Hausdorff measures and capacities, see e.g. Carleson [7], Helms [18], Landkof [25], Nevanlinna [27] and Wermer [35].

2. Hardy spaces

2.1. The definition of Hardy spaces

Definition 2.1. For $0 < p < \infty$ and a domain Ω , $\Omega \subset \mathbf{S}$ or $\Omega \subset \mathbf{C}^n$, $n > 1$, let

$$\begin{aligned} H^p(\Omega) &= \{f \in A(\Omega) : |f|^p \text{ has a harmonic majorant in } \Omega\}, \\ H^\infty(\Omega) &= \{f \in A(\Omega) : \sup_{z \in \Omega} |f(z)| < \infty\}. \end{aligned}$$

We also want to have a norm, or for $p < 1$ a quasi-norm. If $f \in H^p(\Omega)$, $0 < p < \infty$, then $|f|^p$ has a least harmonic majorant, a harmonic majorant which is least at all points in Ω . If we choose a point $a \in \Omega$ (which we will call the norming-point) we get a norm by evaluating the least harmonic majorant at that point. Different points give different norms, but it follows from Harnack's inequalities that they are equivalent. For $p = \infty$ we use the supremum norm. For completeness we define the norm to be $+\infty$ for functions in $A(\Omega) \setminus H^p(\Omega)$.

This definition for general domains was introduced in 1950 by Parreau [29], [30], see also Rudin [31].

In the classical case, $\Omega = \mathbf{D}$, the norm, with norming-point $a = 0$, can be given by

$$\|f\|_{H^p(\mathbf{D})}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta.$$

We should point out that for the upper half plane our definition is not the usual one.

Definition 2.2. The space $H^p(\Omega)$ is *trivial* if $H^p(\Omega) = \{f : f \text{ is constant}\}$.

The Hardy spaces we are considering are conformally invariant. In the terminology of Ahlfors and Beurling, [2] p. 102, they are even analytically invariant, i.e. if $\varphi : \Omega' \rightarrow \Omega$ is a meromorphic function, not necessarily injective, and $f \in H^p(\Omega)$, then $f \circ \varphi \in H^p(\Omega')$.

The following inclusions are important:

$$H^p(\Omega) \supset H^q(\Omega) \supset H^\infty(\Omega), \quad 0 < p < q < \infty. \quad (1)$$

We also want to point out that Hardy spaces do not form sheaves.

Theorem 2.3. *Let $0 < p < \infty$. Then there exist domains $\Omega_1, \Omega_2 \subset \mathbf{C}$ and a function f , such that $f \in H^p(\Omega_1)$ and $f \in H^p(\Omega_2)$, but $f \notin H^p(\Omega_1 \cup \Omega_2)$.*

This result was proved by Hejhal [17] and independently by Gauthier and Hengartner, pp. 411–412 in [12], in 1973. See also Suita [33].

2.2. The H^p -norm expressed using harmonic measures

Lemma 2.4. *Let $\Omega \subset \mathbf{S}$ be a domain with $\text{cap}(\mathbf{S} \setminus \Omega) > 0$, or let $\Omega \subset \mathbf{C}^n$, $n > 1$, be a bounded domain. Let $f \in A(\Omega)$ be a function which is continuous on $\overline{\Omega}$. Set*

$$h(z) = \int_{\partial\Omega} |f(\zeta)|^p \omega(d\zeta; \Omega, z), \quad z \in \Omega.$$

Then h is the least harmonic majorant of $|f|^p$ in Ω and if $a \in \Omega$ is the norming-point then

$$\|f\|_{H^p(\Omega)}^p = h(a).$$

Proof: The conditions on Ω ensure the existence of the harmonic measure $\omega(\cdot; \Omega, z)$ and hence of h .

If Ω is a regular domain then h is a harmonic function with the same boundary values as the subharmonic function $|f|^p$, and hence is the least harmonic majorant.

If Ω is irregular then h and $|f|^p$ may take different boundary values at irregular boundary points, if the boundary values of h at all exist.

We first show that h is really a majorant of $|f|^p$. The function h is the generalized solution to the Dirichlet problem

$$\begin{cases} \Delta h = 0, & \text{in } \Omega, \\ h = |f|^p, & \text{on } \partial\Omega, \end{cases}$$

as solved by the Perron–Wiener–Brelot method, see e.g. Helms [18], Theorem 8.13 if $\Omega \subset \mathbf{C}^n$, $n > 1$, and Theorems 9.22 and 9.23 if $\Omega \subset \mathbf{S}$. The solution can be obtained as the so called lower solution, i.e.

$$h(z) = \sup\{g(z) : g \in \mathcal{L}\}, \quad z \in \Omega, \quad (2)$$

where

$$\mathcal{L} = \{g : g \text{ is subharmonic in } \Omega \text{ and } \limsup_{\Omega \ni z \rightarrow z_0} g(z) \leq |f(z_0)|^p \text{ for all } z_0 \in \partial\Omega\},$$

see e.g. Helms [18], p. 157. As $|f|^p \in \mathcal{L}$ it follows that h is a majorant of $|f|^p$.

Now we need to show that h is least. Let u be the least harmonic majorant of $|f|^p$ in Ω and let $g \in \mathcal{L}$. As $|f(z)|^p \leq u(z)$ for all $z \in \Omega$,

$$\liminf_{\Omega \ni z \rightarrow z_0} u(z) \geq |f(z_0)|^p \geq \limsup_{\Omega \ni z \rightarrow z_0} g(z) \quad \text{for all } z_0 \in \partial\Omega.$$

So by the maximum principle $u(z) \geq g(z)$ for all $z \in \Omega$. By (2) it follows that $u(z) \geq h(z)$ for all $z \in \Omega$, i.e. h is the least harmonic majorant of $|f|^p$ in Ω . $\square$

Lemma 2.5. *Let $\Omega_1 \subset \Omega_2 \subset \dots$ be domains with $\Omega_k \subset \mathbf{S}$ and $\text{cap}(\mathbf{S} \setminus \partial\Omega_k) > 0$ or $\Omega_k \subset \mathbf{C}^n$, $n > 1$, and Ω_k bounded, for $k = 1, 2, \dots$. Let $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$. Let $f \in A(\Omega)$ and assume that f is continuous on $\overline{\Omega}_k$, for $k = 1, 2, \dots$ (which, e.g., is true if $\overline{\Omega}_k \subset \Omega$). Let*

$$h_k(z) = \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, z), \quad z \in \overline{\Omega}_k.$$

If $a \in \Omega_1$ is the norming-point, then

$$h_k(a) = \|f\|_{H^p(\Omega_k)}^p \nearrow \|f\|_{H^p(\Omega)}^p, \quad \text{as } k \rightarrow \infty,$$

where by $b_k \nearrow b$ we mean that $b_1 \leq b_2 \leq \dots$ and $\lim_{k \rightarrow \infty} b_k = b$.

Remark: This is by no means a new result, at least not if the domains Ω_k are regular, see e.g. Theorem 1.3 in Rudin [31].

Proof: As both h_k and h_{k+1} are harmonic majorants of $|f|^p$ in Ω_k by the last lemma, and h_k is least we get that $\{h_k\}_{k=1}^{\infty}$ is a non-decreasing sequence of harmonic functions. Let $H(z) = \lim_{k \rightarrow \infty} h_k(z)$ for all $z \in \Omega$. By Harnack's theorem H is harmonic or $H \equiv \infty$. In either case it is clear that there cannot exist a smaller harmonic majorant of $|f|^p$ in Ω than H , and that H is a majorant of $|f|^p$ in Ω . $\square$

Lemma 2.6. *Let Ω be a domain and $f \in A(\Omega)$. Then the function $p \mapsto \|f\|_{H^p(\Omega)}$ is non-decreasing. Moreover, for $0 < p \leq \infty$,*

$$\lim_{q \rightarrow p^-} \|f\|_{H^q(\Omega)} = \|f\|_{H^p(\Omega)}.$$

Proof: Let the notation be given as in Lemma 2.5 and let $\omega_k = \omega(\cdot; \Omega_k, a)$. Applying Jensen's inequality to Lemma 2.5, using that $\omega(\partial\Omega_k; \Omega_k, a) = 1$, we

get for $q < p$,

$$\begin{aligned} \|f\|_{H^q(\Omega)} &= \lim_{k \rightarrow \infty} \left(\int_{\partial\Omega_k} |f(\zeta)|^q d\omega_k(\zeta) \right)^{1/q} \\ &\leq \lim_{k \rightarrow \infty} \left(\int_{\partial\Omega_k} |f(\zeta)|^p d\omega_k(\zeta) \right)^{1/p} = \|f\|_{H^p(\Omega)}. \end{aligned}$$

This proves the first part. As for the second part we first fix k . Using the monotone convergence theorem we get

$$\begin{aligned} \lim_{q \rightarrow p^-} \int_{\partial\Omega_k} |f(\zeta)|^q d\omega_k(\zeta) &= \lim_{q \rightarrow p^-} \int_{\partial\Omega_k \cap \{|f| > 1\}} |f(\zeta)|^q d\omega_k(\zeta) + \lim_{q \rightarrow p^-} \int_{\partial\Omega_k \cap \{|f| \leq 1\}} |f(\zeta)|^q d\omega_k(\zeta) \\ &= \int_{\partial\Omega_k \cap \{|f| > 1\}} |f(\zeta)|^p d\omega_k(\zeta) + \int_{\partial\Omega_k \cap \{|f| \leq 1\}} |f(\zeta)|^p d\omega_k(\zeta) \\ &= \int_{\partial\Omega_k} |f(\zeta)|^p d\omega_k(\zeta). \end{aligned}$$

Letting $k \rightarrow \infty$ we get

$$\begin{aligned} \lim_{q \rightarrow p^-} \|f\|_{H^q(\Omega)} &= \sup_{q < p} \sup_k \left(\int_{\partial\Omega_k} |f(\zeta)|^q d\omega_k(\zeta) \right)^{1/q} \\ &= \sup_k \sup_{q < p} \left(\int_{\partial\Omega_k} |f(\zeta)|^q d\omega_k(\zeta) \right)^{1/q} \\ &= \sup_k \left(\int_{\partial\Omega_k} |f(\zeta)|^p d\omega_k(\zeta) \right)^{1/p} = \|f\|_{H^p(\Omega)}. \end{aligned}$$

For $p = \infty$ we leave the details to the reader. $\square$

Remark: The norm is not continuous from the right with respect to p . See e.g. Proposition 5.9 and its proof.

3. Removable singularities

3.1. Three definitions of removable singularities

Definition 3.1. Let Ω be a domain and $E \subset \Omega$ be relatively closed such that $\Omega \setminus E$ is also a domain. Let $0 < p \leq \infty$. In the literature there are three definitions of when E is a *removable singularity* for $H^p(\Omega \setminus E)$.

- I If $H^p(\Omega \setminus E) \subset A(\Omega)$.
- II If $H^p(\Omega \setminus E) = H^p(\Omega)$ (as sets).
- III If $\|f\|_{H^p(\Omega \setminus E)} = \|f\|_{H^p(\Omega)}$ for all $f \in H^p(\Omega \setminus E)$.

In the first case we mean that every $f \in H^p(\Omega \setminus E)$ can be extended analytically to the whole of Ω . In the last case, where the same norming-point is used for $H^p(\Omega)$ and $H^p(\Omega \setminus E)$, E is called *isometrically removable*.

Remark: For H^∞ the three definitions coincide if E is compact, by the maximum principle, or if $\text{int } E = \emptyset$. In the plane case it is well-known that any set removable for H^∞ must be totally disconnected, and hence the same is true for $p < \infty$. It follows that the three definitions coincide for H^∞ in the plane case.

3.2. Isometrically removable singularities

We will first take a look at the third case, which is essentially completely solved, due to the following theorem by Conway, Dudziak and Straube [9], 1987.

Theorem 3.2. *Let $0 < p < \infty$ and $\Omega \subset \mathbf{C}^n$ be a domain. Let $E \subset \Omega$ be such that $\Omega \setminus E$ is a domain. Then E is an isometrically removable singularity for $H^p(\Omega \setminus E)$ if and only if $\text{cap}(E) = 0$ or $H^p(\Omega \setminus E)$ is trivial.*

Remarks: To get isometry one, of course, needs to use the same norming-point for $H^p(\Omega)$ and $H^p(\Omega \setminus E)$, unless $H^p(\Omega) = H^p(\Omega \setminus E) = \{f : f \text{ is constant}\}$.

Notice that the property $\text{cap}(E) = 0$ is independent of both p and Ω . The dependency lies only in the non-triviality of $H^p(\Omega \setminus E)$.

Parreau gave the first version of this result in 1952 when he proved that relatively closed polar subsets of a Riemann surface are removable for H^p , Theorem 20, p. 182, in [30]. Yamashita proved the same result as Parreau in 1968 [36] using a different method, his proof also shows that polar subsets are isometrically removable, even though he does not mention it. In 1982 Järvi [20] proved the sufficiency in the higher dimensional case. Also his proof shows isometrical removability without it being mentioned. Finally in 1987, Conway–Dudziak–Straube [9], proved the necessity.

3.3. The higher dimensional case

Theorem 3.3. *Let $\Omega \subset \mathbf{C}^n$, $n > 1$, be a domain and let $K \subset \Omega$ be compact such that $\Omega \setminus K$ is also a domain, then $A(\Omega \setminus K) = A(\Omega)$.*

This classical theorem by Hartogs can be found in almost any book on several complex variables, e.g. in Hörmander [19], Theorem 2.3.2.

This solves the problem of classifying removable singularities in the first sense for compact sets. Together with Theorem 3.5 below this also solves the problem of classifying removable singularities in the second sense for compact sets. That all compact sets are removable in the second sense was noticed by Conway–Dudziak–Straube in [9], p. 268.

Corollary 3.4. *Let $0 < p \leq \infty$. Let $\Omega \subset \mathbf{C}^n$, $n > 1$, be a domain. Let $E \subset \Omega$ be such that $\Omega \setminus E$ is a domain. Assume that E can be written as $E = \tilde{E} \cup$*

$\bigcup_{j=1}^{\infty} K_j$, where $\tilde{E}$ is a relatively closed subset of $\Omega \setminus \bigcup_{j=1}^{\infty} K_j$ with $\text{cap}(\tilde{E}) = 0$ and K_j are pairwise disjoint compact sets. Assume also that $\bigcup_{j=k}^{\infty} K_j$ is a relatively closed subset of Ω for $k = 1, 2, \dots$. Then E is removable for $H^p(\Omega \setminus E)$ in the first sense.

If there are only finitely many K_j , then E is removable for $H^p(\Omega \setminus E)$ in the second sense.

Proof: Let $f \in H^p(\Omega \setminus E)$. By Theorem 3.2, $f \in H^p(\Omega \setminus \bigcup_{j=1}^{\infty} K_j)$ (if $p = \infty$, it can be deduced from Theorem 3.2 or from other well-known results). Then

$$f \in H^p\left(\Omega \setminus \bigcup_{j=1}^{\infty} K_j\right) = H^p\left(\Omega \setminus \bigcup_{j=2}^{\infty} K_j\right) = \dots = H^p\left(\Omega \setminus \bigcup_{j=m}^{\infty} K_j\right),$$

as compact subsets are removable. Thus f can be continued to all finite unions of K_j , and by uniqueness to all of Ω , i.e. E is removable in the first sense. $\square$

3.4. Comparing the norms of $H^p(\Omega)$ and $H^p(\Omega \setminus K)$

Theorem 3.5. (Øksendal [28], 1987) *Let $0 < p \leq \infty$. Assume that $\Omega \subset \mathbf{S}$ or $\Omega \subset \mathbf{C}^n$, $n > 1$, is a domain, $K \subset \Omega$ is compact and $\Omega \setminus K$ is also a domain. Then there exists a constant $M = M(\Omega, K, a, p)$, a being the norming-point, such that*

$$\|f\|_{H^p(\Omega)} \leq M \|f\|_{H^p(\Omega \setminus K)} \quad \text{for all } f \in A(\Omega). \quad (3)$$

This version is slightly different from that of Øksendal, see Theorem 3.2 in [28], but the proof is basically the same. Notice also that, contrary to most of this paper, we do not require K to be small.

Øksendal proves this theorem using Brownian motion. We will give essentially the same proof, but without using probabilistic methods, instead we will use harmonic measures. Øksendal states this theorem for subdomains of $\mathbf{C}$, but it works in higher dimensions as well. As most of Øksendal's paper deals with $\mathbf{C}^n$ this omission is perhaps just due to a printing error. The proof in Øksendal [28] contains a minor oversight for the case when $\text{cap}(\partial\Omega) = 0$. This has been taken care of below.

Proof: For $p = \infty$ the result follows directly from the maximum principle, with $M = 1$. Assume for the rest of the proof that $0 < p < \infty$.

Assume first that $\text{cap}(\partial\Omega) = 0$ and that $\Omega \subset \mathbf{S}$. Then $\partial\Omega = \mathbf{S} \setminus \Omega$ and

$$\begin{aligned} H^p(\Omega \setminus K) \cap A(\Omega) &= H^p((\mathbf{S} \setminus K) \setminus \partial\Omega) \cap A(\Omega) \\ &= H^p(\mathbf{S} \setminus K) \cap A(\Omega) \subset A(\mathbf{S}) = \{f : f \text{ is constant}\}, \end{aligned}$$

where the second equality follows from Theorem 3.2. This means that the two norms in (3) are equal, either to $+\infty$ or to the value of the constant function f . We can take $M = 1$.

Assume next that $\text{cap}(\partial\Omega) = 0$ and $\Omega \subset \mathbf{C}^n$, $n > 1$. Let $f \in H^p(\Omega \setminus K) \cap A(\Omega)$. As in the plane case we have

$$H^p(\Omega \setminus K) \cap A(\Omega) = H^p((\mathbf{C}^n \setminus K) \setminus \partial\Omega) \cap A(\Omega) = H^p(\mathbf{C}^n \setminus K) \cap A(\Omega).$$

Thus $|f|^p$ has a harmonic majorant in $\mathbf{C}^n \setminus K$ which by Corollary 1 following Theorem 18.2 in Wermer [35] has a limit at infinity and is therefore bounded near ∞ . Therefore f is bounded on $\mathbf{C}^n$ and by Liouville's theorem it is constant. We can take $M = 1$.

Assume now instead that $\text{cap}(\partial\Omega) > 0$. In the plane case we can assume that $\infty \notin \Omega$. This is not really needed in the proof but it makes the exposition more uniform for the plane and the higher dimensional cases.

Choose bounded domains Q and $\{\Omega_k\}_{k=1}^\infty$, such that $a \in \partial Q$, $K \subset Q \subset \bar{Q} \subset \Omega_1 \subset \Omega_2 \subset \dots$, $\Omega = \bigcup_{k=1}^\infty \Omega_k$ and $\bar{\Omega}_k \subset \Omega$.

Using Lemma 2.5 we see that it is enough to show that there exists B independent of k , p and f such that

$$\int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, a) \leq B \int_{\partial(\Omega_k \setminus K)} |f(\zeta)|^p \omega(d\zeta; \Omega_k \setminus K, a)$$

for all $f \in A(\Omega)$, all k and all p , $0 < p < \infty$. Notice that B corresponds to M^p , and thus that M^p will be proved to be independent of p .

We have, for all $z \in \Omega_k \setminus K$,

$$\begin{aligned} \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, z) &= \int_{\partial(\Omega_k \setminus K)} \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, \eta) \omega(d\eta; \Omega_k \setminus K, z) \\ &= \int_{\partial K} \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, \eta) \omega(d\eta; \Omega_k \setminus K, z) \\ &\quad + \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k \setminus K, z) \\ &\leq \omega(\partial K; \Omega_k \setminus K, z) \sup_{\xi \in \partial K} \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, \xi) \\ &\quad + \int_{\partial(\Omega_k \setminus K)} |f(\zeta)|^p \omega(d\zeta; \Omega_k \setminus K, z). \end{aligned} \tag{4}$$

The first equality follows from the fact that both sides are harmonic in $\Omega_k \setminus K$ and equal at all regular boundary points of $\partial(\Omega_k \setminus K)$. The second equality is due to the fact that $\omega(\cdot; \Omega_k, \eta)$ is the Dirac measure at the point η , when $\eta \in \partial\Omega_k$.

Let, for $z \in \Omega_k$,

$$\begin{aligned} u_k(z) &= \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, z), & U_k &= \sup_{z \in \partial Q} u_k(z), \\ v_k(z) &= \int_{\partial(\Omega_k \setminus K)} |f(\zeta)|^p \omega(d\zeta; \Omega_k \setminus K, z), & V_k &= \sup_{z \in \partial Q} v_k(z) \end{aligned}$$

and ρ be given by

$$\sup_{z \in \partial Q} \omega(\partial K; \Omega_k \setminus K, z) \leq \sup_{z \in \partial Q} \omega(\partial K; \Omega \setminus K, z) = \rho < 1.$$

The first inequality follows from Carleman's principle, see e.g. Chapter 4.2 in Nevanlinna [27]. The second inequality follows from the fact that $\text{cap}(\partial\Omega) > 0$, as then $\omega(\partial\Omega; \Omega \setminus K, z) > 0$.

The maximum principle, applied to u_k , gives

$$u_k(z) = \int_{\partial\Omega_k} |f(\zeta)|^p \omega(d\zeta; \Omega_k, z) \leq U_k \quad \text{for all } z \in \partial K$$

and from (4) we get $U_k \leq \rho U_k + V_k$, which gives $U_k \leq V_k/(1 - \rho)$.

Harnack's inequalities say that there exist $0 < c < C$ such that $ch(a) \leq h(z) \leq Ch(a)$ for all positive harmonic functions h on $\Omega_1 \setminus K$ and all $z \in \partial Q$. If we assume that $V_k = v_k(z_0)$, $z_0 \in \partial Q$, we get

$$u_k(a) \leq \frac{1}{c} u_k(z_0) \leq \frac{1}{c(1 - \rho)} v_k(z_0) \leq \frac{C}{c(1 - \rho)} v_k(a).$$

Thus we can choose $B = C/c(1 - \rho)$, independently of f , k and p . $\square$

Remark: This theorem also shows that the set $H^p(\Omega)$ is a closed subspace of $H^p(\Omega \setminus K)$ in the topology of the latter space.

4. Removable singularities in the plane

4.1. The classes $\mathcal{O}_p$ and $\mathcal{N}_p$

Definition 4.1.

$$\begin{aligned} \mathcal{O}_p &= \{\Omega \subset \mathbf{S} : \Omega \text{ is a domain and } H^p(\Omega) \text{ is trivial}\}, \quad 0 < p \leq \infty, \\ \mathcal{O}_{\text{cap}} &= \{\Omega \subset \mathbf{S} : \Omega \text{ is a domain and } \text{cap}(\mathbf{S} \setminus \Omega) = 0\}. \end{aligned}$$

The set $\mathcal{O}_{\text{cap}}$ is also the set of trivial domains for the Nevanlinna class, defined by $N(\Omega) = \{f \in A(\Omega) : \log^+ |f| \text{ has a harmonic majorant in } \Omega\}$, see Rudin [31], and the Smirnov class, $N^+(\Omega)$, see Khavinson [21].

We get from (1) and Theorem 3.2 that

$$\mathcal{O}_{\text{cap}} \subset \mathcal{O}_p \subset \mathcal{O}_q \subset \mathcal{O}_\infty, \quad 0 < p < q < \infty.$$

In the case when Ω is ranging over all arbitrary Riemann surfaces Heins [15] showed that these inclusions are strict. His examples were Myrberg type surfaces of infinite genus. In the plane case partial results were obtained by Hejhal [16],

[17], and Kobayashi [22], [23]. In 1978 Hasumi [13] showed that all these inclusions are strict also in the plane case. The classification scheme obtained by Hasumi is (in terms of Hardy–Orlicz classes he obtained even more)

$$\mathcal{O}_{\text{cap}} \subsetneq \bigcap_{q>0} \mathcal{O}_q \subsetneq \bigcup_{q<p} \mathcal{O}_q \subsetneq \mathcal{O}_p \subsetneq \bigcap_{q>p} \mathcal{O}_q \subsetneq \bigcup_{q<\infty} \mathcal{O}_q \subsetneq \mathcal{O}_{\infty}, \quad 0 < p < \infty.$$

All the sets constructed by Hasumi are zero-dimensional, as a consequence we obtain the following result (which is not explicitly mentioned in Hasumi’s paper).

Proposition 4.2. *There exists a zero dimensional set $K \subset \mathbf{S}$ such that K is not removable for $H^p(\mathbf{S} \setminus K)$, for any $p < \infty$.*

Definition 4.3. Let $0 < p \leq \infty$. We say that $K \in \mathcal{N}_p$ if $K \subset \mathbf{S}$ is compact, $\mathbf{S} \setminus K$ is a domain and for all domains $\Omega \subset \mathbf{S}$ with $K \subset \Omega$ we have $H^p(\Omega \setminus K) = H^p(\Omega)$ (as sets).

Notice that the classes $\mathcal{O}_p$ contain very large sets, whereas the classes $\mathcal{N}_p$ contain very small sets.

4.2. Theorems on removable singularities

For $p = \infty$ we have the following classical theorem.

Theorem 4.4. (Painlevé’s theorem) *If $\Lambda_1(K) = 0$ and $K \subset \mathbf{S}$ is compact then*

$$K \in \mathcal{N}_{\infty} \quad \text{and} \quad \mathbf{S} \setminus K \in \mathcal{O}_{\infty}.$$

The sets in $\mathcal{N}_{\infty}$ are precisely those with zero analytic capacity, see Definition 5.2, in fact removability is one of the main motivations for introducing the analytic capacity. For $K \in \mathcal{N}_{\infty}$ it is well-known that $\text{cap}_1(K) = 0$, and therefore $\dim K \leq 1$, see Section 4.4.

There are several examples of sets with positive Λ_1 -measure and zero analytic capacity. The first example was given by Vitushkin [34] in 1959. For discussions of this see, e.g., Christ [8], Garnett [10], [11] and Murai [26].

It is an open question, or at least not known to the author, if there exist any sets of positive Λ_1 -measure removable for any $p < \infty$.

Hejhal [16], [17] gave an important theorem connecting the classes $\mathcal{O}_p$ and $\mathcal{N}_p$.

Theorem 4.5. *Assume that $0 < p < \infty$, $K \subset \mathbf{S}$ is compact and $\mathbf{S} \setminus K$ is a domain. Then the following are equivalent:*

- (i) $K \in \mathcal{N}_p$,
- (ii) $\mathbf{S} \setminus K \in \mathcal{O}_p$,
- (iii) *there exists $\Omega \subset \mathbf{S}$ such that $K \subset \Omega$ and $H^p(\Omega \setminus K) = H^p(\Omega)$,*
- (iv) *there exists $\Omega \subset \mathbf{S}$ such that $K \subset \Omega$ and $H^p(\Omega \setminus K) \subset A(\Omega)$.*

Proof: That (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) is trivial. That (iv) $\Rightarrow$ (ii) follows from

$$\begin{aligned} H^p(\mathbf{S} \setminus K) &= H^p(\mathbf{S} \setminus K) \cap H^p(\Omega \setminus K) \subset H^p(\mathbf{S} \setminus K) \cap A(\Omega) \\ &\subset A(\mathbf{S}) = \{f : f \text{ is constant}\}. \end{aligned}$$

That (ii) $\Rightarrow$ (i) is omitted here, the proof is in Hejhal [17]. $\square$

Part (iv) was not in Hejhal's theorem, as Hejhal only considered removability in the second sense, however, it gives the following corollary without making the proof more complicated.

Corollary 4.6. *For compact subsets of $\mathbf{S}$ the first and second definitions of removability are equivalent.*

Compare this result with Proposition 4.12. The corollary also follows from Theorem 3.5.

Theorem 4.7. *Let $\Gamma \subset \mathbf{C}$ be a $C^{1+\varepsilon}$ Jordan curve, $\varepsilon > 0$, and $K \subset \Gamma$ be compact. Then $K \in \mathcal{N}_1$ if and only if $\Lambda_1(K) = 0$.*

This is only a special case of the results proved in Section 7 in Björn [3] and also in Björn [5], but is enough for us here. We refer to [3] or [5] for a detailed discussion of singularities lying on simple and intersecting rectifiable Jordan curves of various types.

4.3. Unions of removable singularities

First we have an almost trivial theorem.

Proposition 4.8. *Let $0 < p \leq \infty$. Assume that $K_j \in \mathcal{N}_p$, $1 \leq j \leq m$, are pairwise disjoint, then $\bigcup_{j=1}^m K_j \in \mathcal{N}_p$.*

Proof: It is enough to prove the theorem for $m = 2$. As $K_2 \subset \mathbf{S} \setminus K_1$ we have

$$H^p(\mathbf{S} \setminus (K_1 \cup K_2)) = H^p((\mathbf{S} \setminus K_1) \setminus K_2) = H^p(\mathbf{S} \setminus K_1) = H^p(\mathbf{S}). \quad \square$$

Remark: This together with the total disconnectedness of removable sets shows that removability in the plane case is a local property.

Lemma 4.9. *Let $\Omega \subset \mathbf{S}$ be a domain and $E \subset \Omega$ be a totally disconnected relatively closed set. Then E can be written as a countable union of pairwise disjoint compact sets, K_j , with the sets $\bigcup_{j=k}^{\infty} K_j$ being relatively closed subsets of Ω for $k = 1, 2, \dots$.*

Sketch of proof: Using the total disconnectedness, we can draw closed curves in $\Omega \setminus E$ splitting E into several pieces. Those pieces that are separated from $\partial\Omega$ are compact. By letting the curves tend to the boundary the result is obtained.

Theorem 4.10. *Let $0 < p \leq \infty$. Let $\Omega \subset \mathbf{S}$ be a domain and $E \subset \Omega$ be such that $\Omega \setminus E$ is also a domain. Then E is removable in the first sense for $H^p(\Omega \setminus E)$ if and only if E can be written as a countable union of pairwise disjoint compact sets $K_j \in \mathcal{N}_p$, where $\bigcup_{j=k}^{\infty} K_j$ is a relatively closed subset of Ω for $k = 1, 2, \dots$.*

Proof: Assume first that $f \in H^p(\Omega \setminus E)$ and $E = \bigcup_{j=1}^{\infty} K_j$, $K_j \in \mathcal{N}_p$, are as in the statement, then

$$f \in H^p\left(\Omega \setminus \bigcup_{j=1}^{\infty} K_j\right) = H^p\left(\Omega \setminus \bigcup_{j=2}^{\infty} K_j\right) = \dots = H^p\left(\Omega \setminus \bigcup_{j=m}^{\infty} K_j\right).$$

Thus f can be continued to all finite unions of K_j , and by uniqueness to all of Ω , i.e. E is removable in the first sense.

Assume now instead that E is removable in the first sense. It follows that E is totally disconnected and hence E can be written as a union of pairwise disjoint compact sets K_j , as in Lemma 4.9. As $H^p(\Omega \setminus K_j) \subset H^p(\Omega \setminus E) \subset A(\Omega)$ it follows from Theorem 4.5 that $K_j \in \mathcal{N}_p$. $\square$

Remark: It follows from this that removability in the first sense is independent of the domain, even in the non-compact case.

Proposition 4.11. *Let $1 \leq p < \infty$. Then there exist $K_1, K_2 \in \mathcal{N}_p$ such that $K_1 \cup K_2 \notin \mathcal{N}_p$.*

Remarks: A consequence is that the pairwise disjointness condition in Proposition 4.8 is significant, at least for $1 \leq p < \infty$.

Hejhal gave this result for $p = 1$, Example 1, p. 19, in [17]. In Example 2, p. 20 [17], Hejhal dealt with a different problem, but the construction he gave there will be used by us to prove Proposition 4.11 for $1 < p < \infty$.

Proof: Take an arbitrary compact totally disconnected set $A \subset \mathbf{T} = \partial \mathbf{D}$ such that $\Lambda_1(A) = 0 < \text{cap}(A)$, $1 \notin A$ and A is symmetric with respect to $\mathbf{R}$, e.g. A can be a curvilinear Cantor set. Let

$$B = \log A = \{z \in \mathbf{C} : e^z \in A\} \subset \{z \in \mathbf{C} : \text{Re } z = 0\}.$$

The set B is totally disconnected, closed and unbounded, as a subset of $\mathbf{C}$, and does not contain the point 0.

Let $N \geq p$ be an integer and let

$$\tilde{K}_1 = (\log A)^{-1/N} \cup \{0\} = \{z \in \mathbf{C} : z^{-N} \in B\} \cup \{0\} = \{z \in \mathbf{C} : e^{z^{-N}} \in A\} \cup \{0\}.$$

The set $\tilde{K}_1$ is a compact totally disconnected set, symmetric with respect to the origin, and $\Lambda_1(\tilde{K}_1) = 0 < \text{cap}(\tilde{K}_1)$.

Let $\tilde{K}_2$ be $\tilde{K}_1$ rotated $\pi/2N$ radians around 0. (By the symmetry it does not matter in which direction we rotate.)

Let $K = \tilde{K}_1 \cup \tilde{K}_2$ and $L_j = \{z \in K : \arg z = j\pi/2N\} \cup \{0\}$ for $0 \leq j < 4N$. Then $K = \bigcup_{j=0}^{4N-1} L_j$, and by Theorem 4.7, $L_j \in \mathcal{N}_1 \subset \mathcal{N}_p$.

In Example 2, p. 20, in [17] Hejhal proves that the function $z \mapsto 1/z$ belongs to $H^N(\mathbf{S} \setminus K)$. Thus $K \notin \mathcal{N}_N \supset \mathcal{N}_p$. The essential ingredient in Hejhal's proof is a generalized Nevanlinna–Frostman theorem. Proposition 4.11 follows by reductio ad absurdum. $\square$

Proposition 4.12. *Let $1 \leq p < \infty$. Then there exists a domain $\Omega \subset \mathbf{C}$ and a set $E \subset \Omega$, with $\Omega \setminus E$ being a domain, such that $H^p(\Omega \setminus E) \subset A(\Omega)$ and $H^p(\Omega \setminus E) \neq H^p(\Omega)$ (as sets), i.e. E is removable in the first sense, but not in the second sense, for $H^p(\Omega \setminus E)$.*

Proof: Let K be the set constructed in the proof of Proposition 4.11. Let $\Omega = \mathbf{C} \setminus \{0\}$ and let $E = K \setminus \{0\}$. Write $E = \bigcup_{j=1}^{\infty} K_j$, with K_j being pairwise disjoint compact sets, as in Lemma 4.9. As each K_j is a finite pairwise disjoint union of subsets of lines and $\Lambda_1(K_j) = 0$, Theorem 4.7 together with Proposition 4.8 show that $K_j \in \mathcal{N}_1 \subset \mathcal{N}_p$. Thus Theorem 4.10 shows that $H^p(\Omega \setminus E) \subset A(\Omega)$. The space $H^p(\Omega \setminus E) = H^p(\mathbf{C} \setminus K) = H^p(\mathbf{S} \setminus K)$ is non-trivial by the proof of Proposition 4.11, whereas $H^p(\Omega)$ is trivial. $\square$

Proposition 4.13. *Let $0 < p \leq \infty$. Let $\Omega_1, \Omega_2 \subset \mathbf{S}$ be domains and $E_j \subset \Omega_j$ be such that $\Omega_j \setminus E_j$ is a domain ($j = 1, 2$). Assume that $E_1 \subset E_2$ and that E_2 is removable for $H^p(\Omega_2 \setminus E_2)$ in the first sense. Then E_1 is removable for $H^p(\Omega_1 \setminus E_1)$ in the first sense.*

Remark: If E_1 is compact then it follows from Corollary 4.6 that E_1 is also removable in the second sense.

Proof: Assume first that $K \subset E_2$ is compact. As $H^p(\Omega_2 \setminus K) \subset H^p(\Omega_2 \setminus E_2) \subset A(\Omega_2)$ it follows from Theorem 4.5 that $K \in \mathcal{N}_p$.

Write E_1 as a countable union of pairwise disjoint compact sets K_j , as in Lemma 4.9. For each j , $K_j \subset E_1 \subset E_2$ and by the observation above, $K_j \in \mathcal{N}_p$. It follows from Theorem 4.10 that E_1 is removable for $H^p(\Omega_1 \setminus E_1)$ in the first sense. $\square$

Proposition 4.14. *Let $1 \leq p < \infty$. Then there exist a set $K \in \mathcal{N}_p$, a domain $\Omega \subset \mathbf{S}$ and a set $E \subset \Omega$, with $\Omega \setminus E$ being a domain, such that $E \subset K$, but E is not removable for $H^p(\Omega \setminus E)$ in the second sense.*

Proof: Let K_1 and K_2 be as in Proposition 4.11. Let $K = K_1$, $\Omega = \mathbf{S} \setminus K_2$ and $E = K_1 \setminus K_2$. By the assumption in Proposition 4.11, $K \in \mathcal{N}_p$.

Assume that E is removable in the second sense for $H^p(\Omega \setminus E)$. Then $H^p(\Omega \setminus E) = H^p(\Omega) = H^p(\mathbf{S})$, as $K_2 \in \mathcal{N}_p$, and $K_1 \cup K_2 = E \cup K_2 \in \mathcal{N}_p$, but this is a contradiction. $\square$

Remark: Since E is removable for $H^p((\mathbf{S} \setminus (K_1 \cap K_2)) \setminus E) = H^p(\mathbf{S} \setminus K) = \{f : f \text{ is constant}\}$ it follows that removability in the second sense depends on the domain.

4.4. Non-removability for $p < \dim K$

We will need some classical potential theory.

Definition 4.15. Let $K \subset \mathbf{C}(=\mathbf{R}^2)$ be a compact set and $\mu \in \mathcal{M}^+(K)$, the set of positive Borel measures on K . We define, for $0 < \alpha < 2$, the *Riesz potential*

$$U_\alpha^\mu(z) = \int_K \frac{d\mu(w)}{|z - w|^{2-\alpha}}, \quad z \in \mathbf{C},$$

and the *Riesz capacity*

$$\text{cap}_\alpha(K) = \sup\{\mu(K) : \mu \in \mathcal{M}^+(K) \text{ and } U_\alpha^\mu(z) \leq 1 \text{ for all } z \in \mathbf{C}\}.$$

Let $E \subset \mathbf{C}$ be a Borel set, then we define the *Riesz capacity* as

$$\text{cap}_\alpha(E) = \sup\{\text{cap}_\alpha(K) : K \subset E \text{ is compact}\}.$$

This definition of capacity is one of several equivalent definitions. There are many books about classical (linear) potential theory, see e.g. Carleson [7], Helms [18] or Landkof [25].

The following relates capacity with Hausdorff measures.

Theorem 4.16. *Let $E \subset \mathbf{C}$ be a Borel set and $0 < \alpha < 2$. Then*

$$\begin{aligned} \dim E > 2 - \alpha &\implies \text{cap}_\alpha(E) > 0, \\ \Lambda_{2-\alpha}(E) < \infty &\implies \text{cap}_\alpha(E) = 0. \end{aligned}$$

See e.g. Theorem 1 in Section 4 in Carleson [7].

Now we can formulate our result.

Theorem 4.17. *Let $\Omega \subset \mathbf{S}$ be a domain and $E \subset \Omega$ be such that $\Omega \setminus E$ is also a domain. Let $0 < p \leq \infty$, $\alpha = 2 - \min(1, p)$ and assume that $\text{cap}_\alpha(E) > 0$. Then E is not removable for $H^p(\Omega \setminus E)$ in the first sense, and hence not in the second sense either.*

Proof: By the definition of Riesz capacity there exists a compact set $K \subset E$ with $\text{cap}_\alpha(K) > 0$. If E is removable in the first sense for $H^p(\Omega \setminus E)$ then $K \in \mathcal{N}_p$ by Proposition 4.13, hence it is enough to show the theorem for $K = E$ compact.

We start with the case $p \geq 1$. As $\text{cap}_1(K) > 0$ there exists $\mu \in \mathcal{M}^+(K)$, $\mu \neq 0$ with $U_1^\mu(z) \leq 1$ for all $z \in \mathbf{C}$. Therefore the analytic function

$$f(z) = \int_K \frac{d\mu(w)}{z - w}$$

is bounded, by 1, on $\mathbf{S} \setminus K$ and $H^p(\mathbf{S} \setminus K)$ is non-trivial.

Assume now instead that $p < 1$. Let $\Omega_1 \subset \Omega_2 \subset \dots$ be domains such that $\overline{\Omega}_k \subset \mathbf{S} \setminus K = \bigcup_{j=1}^{\infty} \Omega_j$, for $k = 1, 2, \dots$. Let $\omega_k = \omega(\cdot; \Omega_k, a)$, where $a \in \Omega_1$ is the norming-point, and $g_w(z) = 1/(z - w)$. As $\text{cap}_\alpha(K) > 0$ there exists $\mu \in \mathcal{M}^+(K)$, $\mu \neq 0$ with $U_\alpha^\mu(z) \leq 1$ for all $z \in \mathbf{C}$. Then, using Lemma 2.5, the monotone convergence theorem and Fubini's theorem,

$$\begin{aligned} \int_K \|g_w\|_{H^p(\mathbf{S} \setminus K)}^p d\mu(w) &= \int_K \lim_{k \rightarrow \infty} \int_{\partial\Omega_k} \frac{1}{|z - w|^p} d\omega_k(z) d\mu(w) \\ &= \lim_{k \rightarrow \infty} \int_{\partial\Omega_k} \int_K \frac{1}{|z - w|^p} d\mu(w) d\omega_k(z) \\ &\leq \lim_{k \rightarrow \infty} \int_{\partial\Omega_k} d\omega_k(z) = 1. \end{aligned}$$

Hence, $g_w \in H^p(\mathbf{S} \setminus K)$, for μ -almost all w , and $H^p(\mathbf{S} \setminus K)$ is non-trivial. $\square$

Remark: For $p \geq 1$ this is a well-known result, see e.g. Ahlfors [1].

5. H^p -capacities

In this chapter we introduce the H^p -capacities. The H^p -capacities have some important properties, but also lack some other. The name capacity is perhaps not so appropriate as the H^p -capacities lack several of the properties often required of capacities. The close resemblance with analytic capacity, at least in some aspects, is the main reason for using the name capacity.

5.1. The fundamental functional $f \mapsto f'(\infty)$

Proposition 5.1. *Let $K \subset \mathbf{C}$ be compact with $\mathbf{S} \setminus K$ connected. Then $K \notin \mathcal{N}_p$ if and only if there is a function $f \in H^p(\mathbf{S} \setminus K)$ with $f(\infty) = 0$ and $f'(\infty) \neq 0$.*

Proof: The sufficiency is trivial. For the necessity let $f \in H^p(\mathbf{S} \setminus K)$ be non-constant. Without loss of generality we can assume that $f(\infty) = 0$. Expand f in a Laurent series,

$$f(z) = \sum_{k=1}^{\infty} c_k z^{-k} \quad \text{for } |z| \text{ large.}$$

As f is non-constant there exists $k \geq 1$ with $c_k \neq 0$. Let k_0 be the least such k . Then $g(z) = z^{k_0-1} f(z)$, $z \in \mathbf{S} \setminus K$, is a well-defined holomorphic function with $g(\infty) = 0$ and $g'(\infty) = c_{k_0} \neq 0$.

It follows that there exists M such that $|g(z)| \leq 1$ for all z with $|z| \geq M$. For $|z| \leq M$ we have $|g(z)| \leq M^{k_0-1} |f(z)|$. Let u be a harmonic majorant of $|f|^p$ in $\mathbf{S} \setminus K$, then

$$|g(z)|^p \leq 1 + M^{p(k_0-1)} |f(z)|^p \leq 1 + M^{p(k_0-1)} u(z), \quad z \in \mathbf{S} \setminus K.$$

Hence $g \in H^p(\mathbf{S} \setminus K)$. $\square$

Remarks: This theorem is, of course, also true if we evaluate the derivative at any other point in $\mathbf{S} \setminus K$.

Rudin showed, Theorem 2.9 in [31], that the functional $f \mapsto f'(\infty)$ is bounded on $H^p(\mathbf{S} \setminus K)$, for every compact set K .

5.2. The definition of H^p -capacities

Definition 5.2. Let $K \subset \mathbf{C}$ be a compact set with connected complement and let $0 < p \leq \infty$. Define

$$\begin{aligned}\gamma_p(K) &= \sup\{|f'(\infty)| : f \in H^p(\mathbf{S} \setminus K), \|f\|_{H^p(\mathbf{S} \setminus K)} \leq 1\}, \\ \dot{\gamma}_p(K) &= \sup\{|f'(\infty)| : f(\infty) = 0, f \in H^p(\mathbf{S} \setminus K), \|f\|_{H^p(\mathbf{S} \setminus K)} \leq 1\},\end{aligned}$$

where we use the norming-point $a = \infty$.

Remarks: If $K \subset \mathbf{C}$ is compact but without connected complement we have not defined the capacity. It is possible, if one wants, to put $\gamma_p(K) = \gamma_p(\tilde{K})$ and $\dot{\gamma}_p(K) = \dot{\gamma}_p(\tilde{K})$, where $\tilde{K}$ is the complement of the component of $\mathbf{S} \setminus K$ containing ∞ .

For $p = \infty$ we obtain the usual analytic capacity, often denoted by γ . It is not difficult to prove that $\gamma_\infty = \dot{\gamma}_\infty$, see e.g. the proof of Theorem 1.1 in Garnett [11].

The main reason for defining H^p -capacities is the following immediate corollary of Proposition 5.1.

Corollary 5.3. *Let $K \subset \mathbf{C}$ be compact with $\mathbf{S} \setminus K$ connected. Then*

$$K \in \mathcal{N}_p \iff \gamma_p(K) = 0 \iff \dot{\gamma}_p(K) = 0.$$

Let us also give the following definition.

Definition 5.4. Let Ω be a domain and $a \in \Omega$ be the norming-point for $H^p(\Omega)$. Define

$$H_0^p(\Omega) = \{f \in H^p(\Omega) : f(a) = 0\},$$

normed by the same norm as $H^p(\Omega)$.

5.3. Properties of the H^p -capacities

Proposition 5.5. *Let $K, K_1, K_2 \subset \mathbf{C}$ be compact sets with connected complements and $K_1 \subset K_2$. Let $0 < p < q \leq \infty$ and let $c \in \mathbf{C}$. Then the following are*

true:

- (i) $\gamma_p(K_1) \leq \gamma_p(K_2), \quad \dot{\gamma}_p(K_1) \leq \dot{\gamma}_p(K_2),$
- (ii) $\gamma_p(K) \geq \gamma_q(K), \quad \dot{\gamma}_p(K) \geq \dot{\gamma}_q(K),$
- (iii) $\gamma_p(cK) = |c|\gamma_p(K), \quad \dot{\gamma}_p(cK) = |c|\dot{\gamma}_p(K),$
- (iv) $\gamma_p(K+c) = \gamma_p(K), \quad \dot{\gamma}_p(K+c) = \dot{\gamma}_p(K),$
- (v) $\dot{\gamma}_p(\overline{\mathbf{D}}) = 1,$
- (vi) $\gamma_p(K) \leq \gamma_p(\overline{\mathbf{D}}) \text{diam}(K), \quad \dot{\gamma}_p(K) \leq \text{diam}(K),$
- (vii) $\gamma_p(K) \geq \dot{\gamma}_p(K).$

Proof: Parts (i), (iii), (iv) and (vii) are trivial. Part (ii) follows directly from Lemma 2.6. For (vi) notice that if $z \in K$, then $K \subset \overline{D(z, \text{diam}(K))}$. Thus (vi) follows directly from parts (i), (iv) and (v). To prove (v) we do as follows, using well-known boundary properties, see e.g. Theorem 17.11 in Rudin [32]. Let $L^p(\mathbf{T}) = L^p(\mathbf{T}, d\theta/2\pi)$, where $\mathbf{T} = \partial\mathbf{D}$. Then

$$\begin{aligned}
\dot{\gamma}_p(\overline{\mathbf{D}}) &= \sup\{|f'(\infty)| : f(\infty) = 0, \|f\|_{H^p(\mathbf{S} \setminus \overline{\mathbf{D}})} \leq 1\} \\
&= \sup\{|f'(0)| : f(0) = 0, \|f\|_{H^p(\mathbf{D})} \leq 1\} \\
&= \sup\{|g(0)| : \|zg(z)\|_{H^p(\mathbf{D})} \leq 1\} \\
&= \sup\{|g(0)| : g \in A(\mathbf{D}), \lim_{r \rightarrow 1-} \|rzg(rz)\|_{L^p(\mathbf{T})} \leq 1\} \\
&= \sup\{|g(0)| : g \in A(\mathbf{D}), \lim_{r \rightarrow 1-} \|g(rz)\|_{L^p(\mathbf{T})} \leq 1\} \\
&= \sup\{|g(0)| : \|g\|_{H^p(\mathbf{D})} \leq 1\}.
\end{aligned}$$

Let $g \in H^p(\mathbf{D})$, $p < \infty$, with $\|g\|_{H^p(\mathbf{D})} \leq 1$, and let u be the least harmonic majorant of $|g|^p$. Then $|g(0)| \leq u(0)^{1/p} \leq 1$ and hence $\dot{\gamma}_p(\overline{\mathbf{D}}) \leq 1$, which also holds for $p = \infty$. But for $g \equiv 1$ we have equality. $\square$

Proposition 5.6. *Let $K \subset \mathbf{C}$ be a compact set with more than one point and such that $\mathbf{S} \setminus K$ is simply connected. Let $\varphi : \mathbf{S} \setminus \overline{\mathbf{D}} \rightarrow \mathbf{S} \setminus K$ be a conformal mapping with $\varphi(\infty) = \infty$ having the power series expansion $\varphi(z) = cz + \mathcal{O}(1)$ near ∞ . Then $\gamma_p(K) = |c|\gamma_p(\overline{\mathbf{D}})$ and $\dot{\gamma}_p(K) = |c|$.*

Proof: Let $f \in H^p(\mathbf{S} \setminus K)$ with $f(z) = b_0 + b_1 z^{-1} + \mathcal{O}(z^{-2})$ near ∞ . Then $f \circ \varphi \in H^p(\mathbf{S} \setminus \overline{\mathbf{D}})$ and $f \circ \varphi(z) = b_0 + b_1 c^{-1} z^{-1} + \mathcal{O}(z^{-2})$, as $c \neq 0$. By conformal invariance we have

$$\gamma_p(K) = |c| \sup\{|(f \circ \varphi)'(\infty)| : \|f \circ \varphi\|_{H^p(\mathbf{S} \setminus \overline{\mathbf{D}})} \leq 1\} = |c|\gamma_p(\overline{\mathbf{D}}),$$

and similarly for $\dot{\gamma}_p$, using Proposition 5.5(v). $\square$

Proposition 5.7. *Let $K_1 \supset K_2 \supset \dots$ be compact subsets of $\mathbf{C}$ with connected complements, and let $K = \bigcap_{k=1}^{\infty} K_k$. Let $0 < q_1 \leq q_2 \leq \dots$ and let $p =$*

$\lim_{k \rightarrow \infty} q_k$. Then

$$\begin{aligned} \text{(a)} \quad \gamma_p(K) &= \lim_{k \rightarrow \infty} \gamma_{q_k}(K_k), \\ \text{(b)} \quad \mathring{\gamma}_p(K) &= \lim_{k \rightarrow \infty} \mathring{\gamma}_{q_k}(K_k). \end{aligned}$$

Remark: For the case $q_k = \infty$, $k = 1, 2, \dots$, this result is well-known, see e.g. Ahlfors [1], Theorem 1, and its proof. For the case $q_k < p = \infty$, $k = 1, 2, \dots$, however, this theorem gives new identities involving the analytic capacity $\gamma = \gamma_\infty = \mathring{\gamma}_\infty$.

Proposition 5.7 follows from the following more general result.

Lemma 5.8. *Let $\Omega_1 \subset \Omega_2 \subset \dots$ be domains and let $\Omega = \bigcup_{k=1}^\infty \Omega_k \subset \mathbf{S}$. Assume that $\overline{\Omega}_1 \subset \Omega$. Let $0 < q_1 \leq q_2 \leq \dots$ and let $p = \lim_{k \rightarrow \infty} q_k$. Let $a \in \Omega_1$ be the common norming-point and let $\Lambda \in H^{q_1}(\Omega_1)^*$, where $*$ denotes the dual space. Then*

$$\begin{aligned} \text{(a)} \quad \|\Lambda\|_{H^p(\Omega)^*} &= \lim_{k \rightarrow \infty} \|\Lambda\|_{H^{q_k}(\Omega_k)^*}, \\ \text{(b)} \quad \|\Lambda\|_{H_0^p(\Omega)^*} &= \lim_{k \rightarrow \infty} \|\Lambda\|_{H_0^{q_k}(\Omega_k)^*}. \end{aligned}$$

Proof: We will prove (a), the proof of (b) is similar.

It follows from Lemmas 2.5 and 2.6 that $\{\|f\|_{H^{q_k}(\Omega_k)}\}_{k=1}^\infty$ is non-decreasing for every f . Hence $\{\|\Lambda\|_{H^{q_k}(\Omega_k)^*}\}_{k=1}^\infty$ is non-increasing and

$$\|\Lambda\|_{H^p(\Omega)^*} \leq \lim_{k \rightarrow \infty} \|\Lambda\|_{H^{q_k}(\Omega_k)^*}.$$

We proceed by assuming that $\|\Lambda\|_{H^p(\Omega)^*} < \lim_{k \rightarrow \infty} \|\Lambda\|_{H^{q_k}(\Omega_k)^*}$. Let

$$\widetilde{M} = \|\Lambda\|_{H^p(\Omega)^*} < \lim_{k \rightarrow \infty} \|\Lambda\|_{H^{q_k}(\Omega_k)^*} = M.$$

We can thus, for $k = 1, 2, \dots$, find $f_k \in H^{q_k}(\Omega_k)$ with

$$\|f_k\|_{H^{q_k}(\Omega_k)} = 1 \quad \text{and} \quad \Lambda f_k \geq \frac{1}{2}(M + \widetilde{M}).$$

Let u_k be the least harmonic majorant of $|f_k|^{q_k}$ in Ω_k , if $q_k < \infty$, and let $u_k \equiv 1$ if $q_k = \infty$. It follows that $u_k(a) = 1$.

Let $K \subset \Omega_1$ be an arbitrary compact subset. By Harnack's inequalities all u_k are bounded on K by a constant independent of k . As

$$|f_k| \leq u_k^{1/q_k} \leq u_k^{1/q_1} + 1,$$

all f_k are also bounded on K by a constant independent of k .

A normal families argument shows that there is a subsequence of $\{f_k\}_{k=1}^\infty$ which converges uniformly on all compact subsets of Ω_1 . We can take a subsequence of this sequence to ensure that also $\{u_k\}_{k=1}^\infty$ converges uniformly on all

compact subsets of Ω_1 . We denote the new subsequences with the same names as the original sequences. If we repeat this process for each Ω_k , $k = 1, 2, \dots$, leaving the first $k - 1$ elements intact in the k th step (a diagonal argument) we obtain sequences $\{f_k\}_{k=1}^\infty$ and $\{u_k\}_{k=1}^\infty$ ($\{q_k\}_{k=1}^\infty$ and $\{\Omega_k\}_{k=1}^\infty$) such that

$$f_k \in H^{q_k}(\Omega_k), \quad \|f_k\|_{H^{q_k}(\Omega_k)} \leq 1, \quad |f_k| \leq u_k^{1/q_k}, \quad \Lambda f_k \geq \frac{1}{2}(M + \widetilde{M})$$

and such that the sequences $\{f_k\}_{k=1}^\infty$ and $\{u_k\}_{k=1}^\infty$ converge uniformly on all compact subsets of Ω .

Then there is an analytic (pointwise) limit function f of $\{f_k\}_{k=1}^\infty$ and a harmonic (pointwise) limit function u of $\{u_k\}_{k=1}^\infty$. As

$$|f| = \lim_{k \rightarrow \infty} |f_k| \leq \lim_{k \rightarrow \infty} u_k^{1/q_k} = u^{1/p}$$

and $u(a) = 1$ it follows that $f \in H^p(\Omega)$ and $\|f\|_{H^p(\Omega)} \leq 1$. Since $\overline{\Omega}_1 \subset \Omega$, $\{f_k\}_{k=1}^\infty$ converges uniformly on Ω_1 and

$$\Lambda f = \lim_{k \rightarrow \infty} \Lambda f_k \geq \frac{1}{2}(M + \widetilde{M}),$$

which contradicts $\|\Lambda\|_{H^p(\Omega)^*} = \widetilde{M}$. Our assumption must have been wrong. $\square$

5.4. The failure of some properties for the H^p -capacities

Unfortunately, the H^p -capacities lack many of the properties one would like them to have. We list some of these properties. All these properties are derived from the properties of the sets removable for H^p . The same results would hold for any other capacity satisfying Corollary 5.3.

Proposition 5.9. *Let $0 < p < \infty$. Then there exists a compact set $K \subset \mathbf{C}$, with connected complement, such that*

$$\begin{aligned} \text{(a)} \quad & \lim_{q \rightarrow p^+} \gamma_q(K) < \gamma_p(K), \\ \text{(b)} \quad & \lim_{q \rightarrow p^+} \dot{\gamma}_q(K) < \dot{\gamma}_p(K). \end{aligned}$$

Proof: Let $K \in (\bigcap_{q>p} \mathcal{N}_q) \setminus \mathcal{N}_p$, i.e. K is removable for all $q > p$, but not for p . Such a set exists by Hasumi's classification, see Section 4.1. Then

$$\lim_{q \rightarrow p^+} \dot{\gamma}_q(K) = \lim_{q \rightarrow p^+} \gamma_q(K) = 0 < \dot{\gamma}_p(K) \leq \gamma_p(K). \quad \square$$

Proposition 5.10. *Let $1 \leq p < \infty$. Then there exist two compact sets $K_1, K_2 \subset \mathbf{C}$, with connected complements, such that $\dot{\gamma}_p(K_1) = \dot{\gamma}_p(K_2) = 0 < \dot{\gamma}_p(K_1 \cup K_2)$ and hence also $\gamma_p(K_1) = \gamma_p(K_2) = 0 < \gamma_p(K_1 \cup K_2)$. Thus the H^p -capacities are not sub-additive.*

This is a direct consequence of Proposition 4.11.

Remark: For the analytic capacity, γ_∞ , one of the most important open questions is whether it is sub-additive or not.

Proposition 5.11. *Let $1 \leq p < \infty$. Then there exist sets $K_1 \subset K_2 \subset \dots$, $K = \bigcup_{k=1}^\infty K_k \subset \mathbf{C}$, all compact with connected complements, such that*

$$\begin{aligned} \text{(a)} \quad & \lim_{k \rightarrow \infty} \gamma_p(K_k) < \gamma_p(K), \\ \text{(b)} \quad & \lim_{k \rightarrow \infty} \mathring{\gamma}_p(K_k) < \mathring{\gamma}_p(K). \end{aligned}$$

Proof: Let K be the set constructed in the proof of Proposition 4.11. The set K is not removable for H^p and hence $\gamma_p(K) \geq \mathring{\gamma}_p(K) > 0$. Let $K_k = (K \setminus D(0, 1/k)) \cup \{0\}$. Thus $K = \bigcup_{k=1}^\infty K_k$. Each K_k is a pairwise disjoint union of subsets of lines, and $\Lambda_1(K_k) = 0$. It follows from Theorem 4.7 and Proposition 4.8 that K_k is removable for H^p and thus that $\gamma_p(K_k) = \mathring{\gamma}_p(K_k) = 0$. $\square$

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