Examensarbete

Future Upgrades of the LHC Beam Screen Cooling System

Björn Backman

LITH-IFM-EX--06/1639--SE
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Supervisor: Rob van Weelderen  CERN

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Linköping, October 2006
Sammanfattning
Abstract

The topic of this thesis concerns the LHC, the next large particle accelerator at CERN which will start operating in 2007. Being based on superconductivity, the LHC needs to operate at very low temperatures, which makes great demands on the cryogenic system of the accelerator. To cope with the heat loads induced by the particle beam, a beam screen cooled with forced flow of supercritical helium is used.

There is an interest in upgrading the energy and luminosity of the LHC in the future and this would require a higher heat load to be extracted by the beam screen cooling system. The objective of this thesis is to quantify different ways to upgrade this system by mainly studying the effects of different pressure and temperatures levels as well as a different cooling medium, neon.

For a numerical program which simulates one-dimensional pipe flow was constructed. The frictional forces were accounted for by the empirical concept of friction factor. For the fluid properties, software using empirically made correlations was used. To validate the numerical program, a comparison with previous experimental work was done. The agreement with experimental data was good for certain flow configurations, worse for others. From this it was concluded that further comparisons with experimental data must be made in order to tell the accuracy of the mathematical model and the correlations for fluid properties used.

When using supercritical helium, thermo-hydraulic instabilities may arise in the cooling loop. It was of special interest to see how well a numerical program could simulate and predict this phenomenon. It was found that the numerical program did not function for such unstable conditions; in fact it was much more sensitive than what reality is.

For the beam screen cooling system we conclude that to cope with the increased heat loads of future upgrades, an increase in pressure level is needed regardless if the coolant remains helium, or is changed to neon. Increasing the pressure level also makes that the problems with thermo-hydraulic instabilities can be avoided. Of the two coolants, helium gave the best heat extraction capacity. Unlike neon, it is also possible to keep the present temperature level when using helium.

Nyckelord
Keywords
LHC, Cryogenics, Beam Screen, Future Upgrades, Supercritical Helium, Computational Fluid Dynamics.
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Keywords: LHC, Cryogenics, Beam Screen, Future Upgrades, Supercritical Helium, Computational Fluid Dynamics.
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Last, I would like to thank all my friends that I got to know during my year in Geneva, for making this a great time. Hope to see you soon!

Linköping in June 2006

Björn Backman
Nomenclature

Here follow the most common symbols, abbreviations and acronyms used in this thesis. For each symbol the corresponding SI unit is given (for dimensionless quantities “-” is given).

Symbols

\[\begin{align*}
A & \quad \text{Area} \ [\text{m}^2] \\
c_p & \quad \text{Specific heat capacity at constant pressure} \ [\text{J} \cdot \text{K}^{-1} \cdot \text{kg}^{-1}] \\
d & \quad \text{Diameter} \ [\text{m}] \\
D_h & \quad \text{Hydraulic diameter} \ [\text{m}] \\
e & \quad \text{Material roughness} \ [\text{m}] \\
E & \quad \text{Relativistic energy} \ [\text{J}] \\
f & \quad \text{Friction factor} \ [-] \\
F & \quad \text{Force} \ [\text{N}] \\
g & \quad \text{Acceleration of gravity} \ [\text{m} \cdot \text{s}^{-2}] \\
i & \quad \text{Specific internal energy} \ [\text{J} \cdot \text{kg}^{-1}] \\
b_i & \quad \text{Bunch intensity} \ [\text{A}] \\
k & \quad \text{Thermal conductivity} \ [\text{W} \cdot \text{m}^{-1} \cdot \text{K}^{-1}] \\
L & \quad \text{Luminosity} \ [\text{m}^{-2} \cdot \text{s}^{-1}] \\
n_b & \quad \text{Bunch number} \ [-] \\
p & \quad \text{Pressure} \ [\text{N} \cdot \text{m}^{-2}] \\
P & \quad \text{Duct perimeter} \ [\text{m}] \\
q & \quad \text{Heat rate per unit length} \ [\text{W} \cdot \text{m}^{-1}] \\
Q & \quad \text{Total heat rate} \ [\text{W}] \\
r & \quad \text{Position vector} \ [\text{m}] \\
R & \quad \text{Reynolds number} \ [-] \\
t & \quad \text{Time} \ [\text{s}] \\
T & \quad \text{Temperature} \ [\text{K}] \\
u & \quad \text{Velocity} \ [\text{m} \cdot \text{s}^{-1}] \\
x, y, z & \quad \text{Cartesian coordinates} \ [\text{m}] \\
\mu & \quad \text{Viscosity} \ [\text{kg} \cdot \text{m}^{-1} \cdot \text{s}^{-1}] \\
\rho & \quad \text{Density} \ [\text{kg} \cdot \text{m}^{-3}] \\
\sigma_b & \quad \text{Bunch length} \ [\text{m}] \\
\tau & \quad \text{Shear stress} \ [\text{N} \cdot \text{m}^{-2}] \\
\phi & \quad \text{Inclination angle} \ [-] \\
\end{align*}\]

Abbreviations & Acronyms

\begin{align*}
\text{Eq} & \quad \text{Equation} \\
\text{Fig} & \quad \text{Figure} \\
\text{CV} & \quad \text{Control volume}
\end{align*}
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1 Introduction

This chapter gives a short introduction to the topic of the thesis and also a brief overview of the contents of each chapter.

1.1 Background

This diploma work was carried out at the Cryogenics for Accelerators group at CERN\(^1\), the European Laboratory for Particle Physics. The topic of the study concerns the LHC\(^2\), the next large particle accelerator of CERN which will start operating in 2007. The magnets in the LHC are based on superconductivity and need to operate at a temperature of 1.9 K. This makes great demands on the cryogenic system of the accelerator.

When the accelerator is running, heat will be induced in the magnets by the particle beam. To conduct this heat away there is a beam screen surrounding the beam, which is cooled by forced flow of supercritical helium.

There is an interest to upgrade the LHC in the future and this would require a higher heat extraction capacity of the beam screen cooling system. The objective of this study is to quantify different ways to upgrade this system, by constructing a numerical program which simulates the flow of helium through the cooling loop.

1.2 Outline of Thesis

Here follows a brief overview of the contents of the following chapters of this thesis.

**Chapter 2: Background**

This chapter contains a background concerning the LHC, the beam screen cooling system, future upgrade scenarios and also some properties of supercritical helium.

**Chapter 3: Problem definition**

In this chapter we define the problem of the thesis in more detail.

**Chapter 4: Mathematical model**

Here we account for the one-dimensional mathematical model used to simulate pipe flow. Empirical correlations for the frictional forces are introduced by the concept of friction factor.

**Chapter 5: Design of numerical program**

This chapter contains the numerical methods used to construct a computer program which simulates one-dimensional pipe flow.

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\(^1\)Conseil Européen pour la Recherche Nucléaire.

\(^2\)Large Hadron Collider.
Chapter 6: Implementation and validation
Here we first look at matters concerning the implementation of the numerical algorithm in a computer program. Secondly, we validate the functionality of the program and compare with experimental results.

Chapter 7: Upgrade of the beam screen cooling loop
In this chapter we use the numerical program to compare different ways to upgrade the LHC beam screen cooling loop.

Chapter 8: Conclusion and future work
In the last chapter we summarize the outcome of the study and also look at possible future work.
2 Background

In this chapter we will give a background necessary to better understand the problem of the thesis.

2.1 The LHC Project

The Large Hadron Collider (LHC) is a particle accelerator currently being installed at the Swiss/French border outside Geneva. Due to switch on in 2007, it will be the world’s most powerful accelerator, able to dig deeper into the fundamental particle properties than ever before. This will be done by colliding beams of particles with each other at very high energies. Two types of particles will be used for this; either one collides two proton beams with each other or two beams containing heavy ions (typically lead ions). Being constructed in the existing tunnel of LEP, the previous large accelerator at CERN, LHC will measure 27 km in circumference and will be located at an average of 100 m underground, see fig 2.1.

There will be four large detectors, each being the place for the main experiments using the LHC. The biggest one is ATLAS, which in itself is the biggest collaborative effort ever attempted in the physical sciences. ATLAS and CMS, being general purpose detectors, will be used for further tests of the Standard model (especially the search for the Higgs boson) and

Figure 2.1 Overall view of the LHC and the different experiments.
also to look for theories beyond this, such as Supersymmetry and String theories. At ALICE the primary goal is to study the nucleon – nucleon interactions that take place when heavy ions are smashed into each other. A new phase, the quark – gluon plasma, is expected to be formed at these events. Finally, at LHCb one will study CP-violation and try to find answers to why there are not equal amounts of matter and antimatter in the universe.

2.2 The Superconducting Magnets

Generally in a circular particle accelerator two types of magnets are needed [1]. First there is the dipole which is used to bend the particle beam so that it follows the right trajectory. Secondly we have the quadrupole which task is to focus the particle beam. This is done by repeatedly focusing and de-focusing, pretty much like with ordinary lenses. In the LHC one also uses other types of magnets to correct for field errors. They are all put together in a cell like in fig 2.2. Since in the LHC one collides equally charged protons with each other, one must have two separate beam lines (since the beams go in opposite directions), each made up of magnets like in fig 2.2.

To accelerate the particles to high energies one uses not magnetic but electric fields [1]. In the LHC such fields are created in so called radiofrequency chambers, which are not relevant for this thesis.

![Figure 2.2 A schematic layout of the different magnets in a LHC cell.](image)

2.2.1 Superconductivity

The vast majority of the magnets used in the LHC are based on a superconducting technology. For this to work they will need to operate at extremely low temperatures. This is due to that a material is only superconducting in a region limited by [2]:

1) The electrical current density
2) The magnetic field
3) The temperature

This can be visualized in the 3D-space made up of the three mentioned quantities above. The superconducting region is that below the critical surface drawn in fig 2.3.

The peak value for the magnetic field in the LHC magnets will be 8.33 T, which is at the limit of what can be achieved when it comes to mass production. With the required current density taken into account, this gives a critical temperature of 2.3 K for the superconducting material, Niobium-Titanium (Nb-Ti), used in the LHC magnets. If the temperature of the magnets goes above this value, the superconductivity will vanish and we will have a so called *quench* which
can lead to the breakdown of the magnet.

The process of quenching can be described as follows [3]: the part where superconductivity disappears will now have a resistivity, leading to Joule heating, capable of melting the cable if nothing was done. To handle this, one deliberately heats up the entire magnet so that the heat is dissipated over as large superconductor volume as possible. At the same time, the current is being ramped down at a controlled rate in order not to induce high voltages and release the enormous amount of energy stored in the magnetic field.

Thus, keeping the temperature below the critical value is important when running the accelerator. The operational temperature of the LHC magnets is 1.9 K, leaving a margin temperature of 0.4 K to the critical value.

2.2.2 Cooling of the Magnets

To get a rough idea of how the cooling of the magnets is done, a cross section of a LHC dipole magnet is shown in fig 2.4, where the essential parts for our understanding are specified (the cooling of the other types of magnets is done in a similar way). The core of the cryodipole is the dipole cold mass. It is made up of all parts that are cooled to 1.9 K and can in fig 2.4 be seen as the area inside the shrinking cylinder (7) (the beam pipes (4) excluded). The magnetic field is generated by the superconducting coils (3).

The cold mass is in fact not totally solid, but contains thin layers of empty space (not visible in the figure, since they run in parallel with the cross section) where superfluid helium (He II), the medium used to cool the cold mass, is contained. The empty space makes up about 2 % of the magnet. The reasons for using superfluid helium are several; it has a very high thermal conductivity, making it ideal as a cooling medium. Moreover, its extremely low viscosity leads to that it fills up the very thin layers of empty space to 100 %.

Outside the cold mass there is vacuum in the space between the shrinking cylinder and the vacuum vessel (5). To insulate the cold mass from the room temperature outside the cryodipole, there are two layers of superinsulation (2) and also a thermal shield (8). Any heat that reaches the cold mass is conducted away through the helium bath and extracted by the heat exchanger pipe (1). The object of interest in this thesis, the beam screen (6), will be described separately in section 2.4.
Figure 2.4 Cross section of a cryodipole.

2.3 Beam Induced Heat Loads

When the LHC is running, heat will be induced into the magnets from a number of sources. This heat must be extracted by either the cold mass or the beam screen (as described in section 2.4) in order to maintain superconductivity in the magnets and avoid quenches. The heat loads of main interest in this thesis will be those extracted by the beam screen. They have in common that they are induced by the particle beam in different ways and they are predominantly dynamic, i.e. they vary in time.

2.3.1 Sources of Heat

Here follows a brief description of the mechanism behind each of the heat loads induced by the particle beam [4].

Synchrotron radiation

When the particle beam is bent by the dipole magnets, the particles radiate high energy photons due to the laws of electrodynamics. This radiation will hit the beam pipe and induce heat in the magnets.
**Image current**

When charged particles pass by an electric conductor (metal) an image current is induced on the surface of the conductor (so that zero electric field is maintained in the interior of the conductor). This leads to resistive heating. As the particle beam is surrounded by metal in the magnets, we will have this effect in the LHC.

**Electron cloud**

When the high energy photons from the synchrotron radiation strike the walls of the beam pipe, they can free electrons from the surface of the wall. These so called photo-electrons will then be accelerated by the electric potential generated by the positively charged protons in the particle beam. When the accelerated photo-electrons hit the wall of the beam pipe they can in turn free more electrons and so on. This will lead to a build up of an electron cloud accompanying the beam. Apart from having effects on the beam dynamics, it will also induce heat into the system when the electrons hit the beam pipe.

**Inelastic beam – gas scattering**

Since the vacuum is not perfect, there are still some gas molecules there that the beam will collide with. This will lead to inelastic scattering of some of the protons in the beam and in this process their kinetic energy will be lost and transformed into heat. The gas molecules will tend to condensate on the walls of the beam pipe and in that way induce the heat in the cold mass. This condensate gas will however not remain on the surface forever, but can be knocked off again by various particles (energetic photons, electrons and beam particles). They will then collide with more beam particles and so on.

### 2.3.2 Dependency on Beam Parameters

To see how these heat loads depend on the particle beam we must introduce some key parameters of beam physics [5].

The particle beam is not a continuous stream of particles, instead the particles are grouped together in so called bunches. This is due to that the electrical fields used to accelerate the particles vary in time in a periodic way. In the LHC, the bunches are distanced at about 25 ns, and each bunch contains about $1.1 \cdot 10^{11}$ particles.

The beam parameters of interest to us are:

**Beam energy** - This is the total relativistic energy of each particle (fully accelerated), given by the formula $E = \frac{m_0c^2}{\sqrt{1 - \frac{u^2}{c^2}}}$.

**Bunch intensity** - This is a measure of how intense each bunch is (in terms of particles). It is defined as the electrical current made up of the passing charged particles in a bunch.

**Bunch number** - This is simply the number of bunches in one of the LHC beam pipes.

---

*These figures are for protons, not ions. This applies to every figure from now on.*
**Bunch length** - This is the physical length of each bunch.

We can now look at the principle dependence of each of the heat loads on the different parameters of the beam. This is shown in table 2.1.

<table>
<thead>
<tr>
<th>Heat load</th>
<th>Beam energy</th>
<th>Bunch intensity</th>
<th>Bunch number</th>
<th>Bunch length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Synchrotron radiation</td>
<td>$E^4$</td>
<td>$I_b$</td>
<td>$n_b$</td>
<td>-</td>
</tr>
<tr>
<td>Image current</td>
<td>-</td>
<td>$I_b^2$</td>
<td>$n_b$</td>
<td>$\sigma_b^{3/2}$</td>
</tr>
<tr>
<td>Electron cloud</td>
<td>-</td>
<td>$I_b^3$</td>
<td>$n_b$</td>
<td>-</td>
</tr>
<tr>
<td>Beam – gas scattering</td>
<td>-</td>
<td>$I_b$</td>
<td>$n_b$</td>
<td>-</td>
</tr>
</tbody>
</table>

Table 2.1 The different heat loads dependencies on the beam parameters [6].

2.4 The Beam Screen

Because of the high thermodynamic cost of refrigeration at a temperature of 1.9 K, the cold mass should not extract all heat induced in the magnets. This leads to the necessity of a beam screen, maintained at a higher temperature, which shields the walls of the beam pipe from the particle beam and thus intercepts most of the beam induced heat loads.

2.4.1 Design and Functionality

How the beam screen is integrated in the beam pipes is shown in fig 2.5, where also the dimensions are specified. A photo of the beam screen can be seen in fig 2.6. The beam screen is cooled with forced flow of supercritical helium through two capillaries, situated in the empty spaces at the top and bottom. This is described in more detail in the next section.

As can be seen in fig 2.5 and 2.6, the beam screen contain small pumping slots on the upper and lower sides. As mentioned in the part on inelastic beam – gas scattering above, the gas molecules hit by the particle beam will tend to condensate on the surrounding surface and they may be knocked out again by various particles. The purpose of the pumping slots is to bring some of the gas molecules out of the beam screen so that they condensate on the beam pipe instead of on the beam screen. Shielded by the beam screen, this condensate gas will remain

![Figure 2.5](image-url) Cross section of beam pipe with beam screen (the two diameters refer to inner respectively outer diameter).
on the beam pipe and in this way we will get a better vacuum and thereby less heat due to inelastic beam – gas scattering. In this way the heat load due to beam – gas scattering will mostly be extracted by the cold mass.

The pumping slots have but one drawback, namely that the electron cloud can penetrate through the slots and induce a significant amount of heat into the cold mass. Therefore there is a pumping slot shield that stops most of the electrons from passing, but still lets the gas molecules by.

To better extract the heat loads, the inside surface of the beam screen is prepared in a special way. The beam screen itself is made by stainless steel, but this has an unacceptable high resistivity which will lead to a too high heat load due to image currents. Therefore, there is a 75 μm thin layer of copper on the inside of the beam screen. In this way the image currents will only be conducted in the copper, which has a much lower resistivity.

2.4.2 Cooling Scheme

We now look at how the overall cooling scheme of the beam screen is designed. As mentioned before, the beam screen is cooled by forced flow of supercritical helium through two capillaries. They have an inner diameter of 3.7 mm and are made of stainless steel. The reason that one uses supercritical helium is to avoid two phase flow, which is more difficult both in practical handling and theoretical modelling (more about this in section 2.6).

The helium goes in a cooling loop as outlined in fig 2.7. It goes over half a LHC cell (see fig 2.2) and contains four cooling capillaries in total (two for each beam pipe). The supply and return headers are shared by many LHC cells. As can be seen the two pairs of capillaries go in a crossed pattern, changing to the other beam pipe after each magnet. The reason for this is to make the heat load on each pair of capillaries equally high.

The temperature and pressure for the supply header vary a bit depending on what sector of the LHC you look at. Typically the supply header has a pressure of 3 bar and a temperature of 4.6 K. The helium from the supply header (point 1) first passes a part with extra heating. This heat
is not related to the beam screen, but has to do with the insulation of the mechanical supports that the magnets rest on. This causes the temperature and pressure at point 2 to be slightly different than at point 1. Then follows 53 m of heating of each capillary from the beam screen.

The return header has a fixed pressure of 1.3 bar and its temperature is the same as the maximum temperature of the beam screen (at point 3). This temperature is maintained fixed at the maximum allowed value, about 20 K, by the use of a control valve between point 3 and the return header, point 4. The valve is controlled by a control unit which changes the mass flow through the valve so that the temperature at point 3 is fixed even though the heat loads on the beam screen vary in time.

There is of course an upper limit to the amount of heat that can be extracted by this cooling system if a maximum temperature of 20 K should be maintained. This corresponds to the valve being opened to 100 %, thus maximizing the mass flow through the capillaries. In practice however, the maximum value is a bit lower, since typically one third of the total pressure difference between the supply and return header must be over the control valve; this to make it possible to control the maximum temperature. A typical working line for the cooling loop in a pressure – enthalpy diagram can be found in fig 2.8.

**Figure 2.7** Cooling scheme for the beam screens.

**Figure 2.8** Typical working line for the cooling loop in a pressure – enthalpy diagram. The points 1 – 4 are defined in fig 2.7.
2.4.3 Thermo-Hydraulic Instabilities

We have previously mentioned that working with supercritical helium in the beam screen cooling loop avoids many of the difficulties of two phase flow. There is however a drawback; due to the strongly varying properties of helium close to the critical point (see section 2.6) there is a risk that instabilities in the form of big pressure-density wave oscillations can arise. This must not happen since it will stop the beam screen cooling loop to function as intended.

These instabilities have been studied in theory and verified in experiments [4]. The pressure – density oscillations have their origin in fluctuations in the properties of the fluid. When these fluctuations occur close to the critical point, they are magnified because of the high variation in fluid properties (see fig 2.11 in section 2.6.2 below) close to this point and will, with appropriate phase, be self-sustained and lead to the build up of pressure – density waves. In table 2.2 the factors that have been found to affect the stability of the flow are summarized.

There have been investigations made into the possibility of handling any instabilities that might arise in the cooling loop. For example a control unit with a heater could be installed between the supports and the start of the beam screens. This control unit would control the inlet temperature to the beam screens so that any instabilities are suppressed.

<table>
<thead>
<tr>
<th>Stabilizing</th>
<th>Destabilizing</th>
</tr>
</thead>
<tbody>
<tr>
<td>- Increase of mass flow</td>
<td>- Increase of heat load</td>
</tr>
<tr>
<td>- Increase of pressure level</td>
<td>- Increase of capillary length</td>
</tr>
<tr>
<td>- Increase of capillary diameter</td>
<td></td>
</tr>
</tbody>
</table>

*Table 2.2 Factors affecting the stability of the flow [7].*

2.5 The Upgrade of the LHC

There will be an interest in upgrading the LHC after about 8 years of operation. This will require a higher heat extraction capacity of the beam screen cooling system [6].

2.5.1 The Needs for an Upgrade

In particle accelerator experiments there are two quantities that one wants to maximize. The first is the beam energy $E$ (as defined in 2.3.2); at higher energies new particles can be discovered and new phenomena can be encountered. The other quantity is the luminosity, $L$. The luminosity is a measure of the intensity of the beams at the point of collision. More precisely, it can be found that the number of events (collisions) $N$ per unit time is proportional to the interaction cross section $\sigma$ (a measure of how likely it is that two particles will interact at head on collision), with the luminosity $L$ as the proportionality factor [8]:

$$\frac{dN}{dt} = L \cdot \sigma$$

Since the outcome of a collision is only governed by the probabilities given by quantum mechanics, a high luminosity is needed in order to discover events that have a low probability of happening.
With the current design of the LHC, a beam energy of 7 TeV and a luminosity of \(1 \cdot 10^{34} \text{ cm}^{-2} \text{s}^{-1}\) will be reached. To fully exploit the potential of the machine and of the detectors, possibilities to double the energy and to increase the luminosity by an order of magnitude have been studied [6]. We will now have a look at the two main scenarios to achieve such a Super-LHC and how each of them affects the heat loads.

### 2.5.2 Different Upgrade Scenarios

To see how each of the two upgrade scenarios affects the heat loads we will relate back to the beam parameters defined in section 2.3.2. The beam energy is the same for the two; it is 14 TeV (i.e. double the energy of the LHC without upgrades). What differs are the different ways to raise the luminosity by an order of magnitude. The beam parameters for each scenario are summarized in table 2.3 where also the parameters for the current LHC without upgrades are given.

**Scenario 1: Bunched beam**

In the first scenario one increases the number of bunches, \(n_b\), and for this to be possible one must also make them shorter, i.e. decrease the bunch length, \(\sigma_b\). The bunch intensity is also increased a bit.

**Scenario 2: Super-bunch**

The other way is totally different from the current configuration; here one only uses one single bunch per beam. Such a super-bunch would be much longer than the current bunches and also have much greater bunch intensity.

| Scenario          | \(E\) [TeV] | \(I_b\) [mA] | \(n_b\) [-] | \(\sigma_b\) [mm] | \(L\) [\text{cm}^{-2} \text{s}^{-1}]| |
|-------------------|-------------|---------------|-------------|------------------|----------------------------------|
| No upgrades       | 7           | 0.2           | 2808        | 77               | \(1.0 \cdot 10^{14}\)           |
| Bunched beam      | 14          | 0.23          | 5616        | 54.4             | \(1.0 \cdot 10^{15}\)           |
| Super-bunch       | 14          | 720           | 1           | 75000            | \(1.0 \cdot 10^{15}\)           |

**Table 2.3** Beam parameters for the LHC, without any upgrades and in the two different upgrade scenarios [6].

The main interest for us lies in how the different upgrade scenarios affect the heat loads that need to be extracted by the beam screen. This is given in table 2.4 together with the heat loads of a non-upgraded LHC. Apart from the heat loads induced by the beam, there is also a static heat load due to insulation issues (this is the same with or without upgrades). As can be seen, an upgrade scenario with a single long super-bunch is more favourable from a cryogenic point of view.
### Table 2.4 Heat loads on the LHC beam screen, without any upgrades and in the two different upgrade scenarios. The values given are the time averaged heat loads for one beam screen (not two) [6].

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Synchrotron radiation [W/m]</th>
<th>Image currents [W/m]</th>
<th>Electron cloud [W/m]</th>
<th>Static [W/m]</th>
<th>Total [W/m]</th>
</tr>
</thead>
<tbody>
<tr>
<td>No upgrades</td>
<td>0.33</td>
<td>0.36</td>
<td>0.89</td>
<td>0.13</td>
<td>1.71</td>
</tr>
<tr>
<td>Bunched beam</td>
<td>12.49</td>
<td>1.70</td>
<td>2.94</td>
<td>0.13</td>
<td>17.3</td>
</tr>
<tr>
<td>Super-bunch</td>
<td>6.83</td>
<td>0.06</td>
<td>0.15</td>
<td>0.13</td>
<td>7.16</td>
</tr>
</tbody>
</table>

#### 2.6 Supercritical Helium

Supercritical fluids have many interesting properties that make them useful for a wide range of applications. In this section we will first look at the general concept of supercritical region and critical point for a fluid. Then we will study the special case of helium in more detail.

##### 2.6.1 Supercritical Fluids

In 1822 G. de la Tour discovered that the two phases liquid and gas can only coexist up to a certain temperature [9]. This temperature is named the critical temperature of the material. There exists likewise a critical pressure, above which gas and liquid can not coexist. This can be seen in a typical phase diagram such as the one in fig 2.9 (for carbon dioxide). The temperature – pressure points where gas and liquid can coexist are found on the saturation line. However, when we increase the temperature and pressure along this line we reach a point, the critical point, where the distinction between gas and liquid can no longer be made. Above this point there is only one phase, called the supercritical phase. This is neither a gas nor a liquid, but is simply called a fluid.

As can be seen in the phase diagram there is no sharp distinction between the supercritical region and the liquid and gas regions respectively; this change is gradual. When we are close

![Figure 2.9 Phase diagram of carbon dioxide][9]
to the liquid region, the fluid behaves more like a liquid, when we are close to the gas region it is more gas like. However, when we are somewhere in between both regions we have a unique fluid which has properties of both. A supercritical fluid has a density typical for liquids, but unlike liquids it is compressible. Just like gases it has a low viscosity and surface tension, but it also has a high diffusivity like liquids, making it ideal as a solvent for various solids. Indeed, most applications of supercritical fluids are found in chemistry; extractions, dry cleaning and chemical waste disposal are some examples. The supercritical fluids most frequently used are water and carbon dioxide. In table 2.5 the critical temperature and pressure for a number of materials are shown, of which neon will be of special interest to us later on (see chapter 7).

<table>
<thead>
<tr>
<th>Material</th>
<th>Critical temperature $T_c$ [K]</th>
<th>Critical pressure $p_c$ [bar]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Carbon dioxide</td>
<td>304.1</td>
<td>73.8</td>
</tr>
<tr>
<td>Water</td>
<td>647.3</td>
<td>221.2</td>
</tr>
<tr>
<td>Helium</td>
<td>5.195</td>
<td>2.275</td>
</tr>
<tr>
<td>Neon</td>
<td>44.44</td>
<td>26.53</td>
</tr>
</tbody>
</table>

Table 2.5 Critical temperature and pressure for some materials [4].

2.6.2 Supercritical Helium

The fluid of main interest to us is of course helium. Therefore we will now in some more detail study the properties of helium in the supercritical region and in the vicinity of the critical point.

In fig 2.10 the phase diagram of helium is shown. It differs from those of most materials in a number of ways; lack of a triple point, existence of two different liquid phases, He I (normal helium) and He II (superfluid helium), and that it does not solidify for pressures below a certain value. These unique properties are however not of interest to us, we are interested in the region above and around the critical point.

Figure 2.10 Phase diagram of helium with working lines for two phase flow (a) and supercritical flow (b) [4].
First we can relate back to section 2.4.2, where the cooling scheme of the beam screen was described. By looking at the phase diagram we can now see why we must choose to either have two phase flow or to pass through the supercritical region. The reason for this is that the point in the phase diagram corresponding to the helium at the inlet of the cooling system is in the He I liquid phase. When the helium goes through the cooling loop the temperature will increase and the pressure will drop and we can in the phase diagram see that this can be done in only two ways; either one crosses the saturation line and gets two phase flow (a) or one goes above the critical point and through the supercritical region (b).

As we have mentioned before, passing by the supercritical region, especially close to the critical point, can lead to pressure – density instabilities due to strongly varying properties of helium in this region. We will now have a closer look at what is happening to various properties close to the critical point.

The critical point is a singular point in the equation of state of a material where the first and second derivatives of pressure with respect to density are zero and the derivative of pressure with respect to temperature goes to infinity [10]. The consequences include infinite heat capacities and thermal conductivity. Moreover most properties vary greatly in the vicinity of the critical point. Examples of this can be seen in fig 2.11 where the density and specific heat are plotted with respect to temperature for a number of pressures. Note that when the pressure is lower than the critical pressure we cross the saturation line and thus have a discontinuity when going from one phase to another.
Figure 2.11 Variation in density (a) and specific heat capacity (b) at constant pressure close to the critical point (calculated by Hepak, see section 6.1.2).
3 Problem Definition

Given the background presented in the previous chapter we will now in detail define the problem of the thesis.

3.1 The Problem

As said in the previous chapter, there is an interest to upgrade the energy and luminosity of the LHC in the future. This would lead to higher beam induced heat loads and thus require a higher heat extraction capacity of the beam screen cooling system which also must be upgraded. The objective of this thesis is to study different ways of doing this.

In section 2.5.2 we looked at two different upgrade scenarios to achieve a Super-LHC which has a beam energy twice that of the current LHC and a luminosity which is an order of magnitude higher. As can be seen in table 2.4 the scenario with a bunched beam gives the highest heat load, which is about one order of magnitude higher than the one we have today; this thus gives an upper limit for the required heat extraction capacity.

There has been previous work in theoretical modelling and experimental studies of the flow of helium through the cooling capillaries [4]. The outcome of this work has led to the need of comparing the theoretical model used with numerical simulations. Thus, a second objective of this thesis is to construct a numerical program that simulates compressible pipe flow. The program must be versatile enough to make it possible to simulate different kinds of fluids; this would make it a useful tool for various future applications.

3.2 Limitations

We now look at how the upgrade of the beam screen cooling system may be done. The different parts that may be changed are given below together with limitations for how they may be changed. They are also outlined in fig 3.1.

Capillaries

For the purpose of this study, there is no limitation to the number of capillaries and they may have any form, not just cylindrical. However, the size of the beam screen will not be changed, so the capillaries must fit into the current available space, see fig 2.5 for dimensions. Moreover they must not cover the pumping slots in the beam screen. The preferred scenario is to keep the capillaries the way they are; thus, that is what we will mainly consider in this study.

Supply header

The temperature and pressure of the supply header is limited by what the cryoplant supplying the helium can achieve; the minimum temperature is 4.5 K (the present temperature is 4.6 K i.e. in practice already at the limit) and the maximum pressure is about 19 bar.
Figure 3.1 Beam screen cooling loop with parts that may be changed outlined.

Return header
The pressure of the return header must not be decreased below its present level (1.3 bar) since this would lead to sub-atmospheric pressure and the risk of air entering the system if there is a leak.

Maximum temperature
The maximum temperature of the beam screen is limited by the effects it has on the beam pipe vacuum; a too high temperature leads to a vacuum of lower quality. For the LHC the maximum temperature of the beam screen has been estimated to be about 30 K [11].

Cooling medium
The fluid in the capillaries does not necessarily have to be helium; different cooling mediums may be used, for example neon.

Thermo-hydraulic instabilities
Pressure-density wave instabilities as described in section 2.4.3 must not occur. It is however uncertain how well a numerical program can simulate such unstable conditions. This study should give some insight in this matter.
4 Mathematical Model

In this chapter we will present the mathematical model used to describe the beam screen cooling system. A complete set of equations for the system will be derived and they will then be used to construct a numerical program that simulates the cooling loop in the next chapter. The material in this chapter is, where nothing else is stated, derived from [12].

4.1 One-Dimensional Pipe Flow

To simulate the beam screen cooling system we will use a very simple model; we consider each capillary as a 53 m straight pipe subject to a uniform heat load from the beam screen, see fig 4.1. Even though, as can be seen in fig 2.7, the capillaries have a number of bends, these are rather few and have a negligible effect for our purposes. We will now see how we can make a one-dimensional model for pipe flow.

![Figure 4.1 Straight pipe model for each of the capillaries.](image)

In fig 4.2a the velocity profile, or flow field, for a typical pipe flow is drawn. The flow field is symmetric around the mid axis of the pipe and only depends on the distance from the axis and on the coordinate (x in the figure) parallel with the axis. Thus it is two-dimensional. Note that for viscous fluids (defined below) we have a no-slip condition at the wall of the pipe, i.e. the velocity goes to zero close to the wall. However, when we get just a bit off the wall the velocity is more or less constant (for high enough flow rates). This makes it often possible to assume a uniform flow field over the whole cross section of the pipe as in fig 4.2b, and still get good correspondence with experimental results. With such an assumption the flow only depends on the x-coordinate and is thus one-dimensional. In the same way one can assume a one-dimensional profile for the other variables of interest, such as pressure, temperature, density etc.

One often also assumes a uniform flow profile at each section of the duct even if it is not

![Figure 4.2 Velocity profile for a typical pipe flow (a) and uniform velocity approximation of the same (b).](image)
cylindrical (for example ellipsoidal or rectangular). This because a one-dimensional flow is much simpler than a two- or three-dimensional and such an approximation is often sufficient for engineering accuracy.

4.2 Classification of Flows

We will now have a brief look at what different types of flows there are and the characteristics of each of them.

**Viscous - Inviscid flow**

The viscosity of a fluid is defined by looking at how the fluid deforms under the application of a shear stress. This can be done by placing the fluid between two plates, and move the upper one, see fig 4.3. The rate of deformation is given by how much the velocity, \( u \), changes in the direction normal to the stress. A fluid is defined as Newtonian if the rate of deformation is proportional to the shear stress, \( \tau \), which is the force applied to the upper plate, \( F \), divided by the area of the upper plate, \( A \). The constant of proportionality in such a relation is defined as one over the viscosity, \( \mu \), of the fluid:

\[
\frac{du}{dy} = \frac{1}{\mu} \tau ; \quad \tau = \frac{F}{A}
\]

We may think of the viscosity as a measure of the internal friction of the fluid subject to a shear stress. More vaguely it can be thought of the “thickness” or resistance to pouring; water is an example of a fluid with low viscosity, as opposed to vegetable oil which has a high viscosity.

One consequence of viscous pipe flow is, as already mentioned, that the velocity will be zero at the wall of the pipe. Moreover we will have an energy loss due to friction. Truly inviscid fluids (having zero viscosity) do not exist (except for superfluids). However, in many applications fluids with a low viscosity may be approximated as being inviscid. An inviscid fluid will flow without friction with a uniform flow field as in fig 4.2b.

**Laminar - Turbulent flow**

There are two different types of viscous flow, namely laminar and turbulent flow. In the former, the flow field is characterized by motion in laminae or layers, see fig 4.4a. A thin filament of dye injected into laminar flow appears as a single line, there is no dispersion of dye throughout the flow (except the slow dispersion due to molecular motion). As can be seen in fig 4.4b, the turbulent flow has a time averaged flow field similar to the laminar flow, but

![Figure 4.3 Definition of viscosity for a Newtonian fluid by looking at the rate of deformation at the application of a shear stress.](image-url)
there are small random fluctuations of the field at every instant of time. The instantaneous time dependent flow field is not ordered in the same way as the laminar one and the filament of dye would quickly disperse throughout the fluid, and not appear as a single line. But even though the flow field is clearly three-dimensional, turbulent flow can still be approximated with a one-dimensional model as in section 4.1, since the time averaged flow field has a similar character to that of laminar flow.

What determines if we get laminar or turbulent flow is the so called Reynolds number, see section 4.4.2.

**Compressible - Incompressible flow**

An incompressible fluid has constant density, as opposed to a compressible one. In most cases where the fluid is a liquid, the flow is considered incompressible, whereas flows of gases are compressible. Modelling incompressible flow is considerably easier than compressible flow.

**Steady - Unsteady flow**

The definition of steady flow is that every flow property (such as velocity, density, temperature, etc.) in every point along the pipe does not vary in time. If this does not hold, then we have unsteady flow. Note that, as mentioned above, turbulent flow has, in the same way as laminar, a structured flow field when averaged over time; thus turbulent flow may be regarded as steady if this time averaged field does not vary in time (even though the instantaneous flow field vary in a random way in time).

When modelling the beam screen cooling system, we will be mainly interested in viscous (mostly turbulent), compressible flow (since Helium is highly compressible) which may be both steady and unsteady.

### 4.3 Derivation of the General Differential Equation

We will now look at the general method (derived from [13]) used to derive each of the equations we need to fully describe the flow through the pipe. In fluid mechanics, unlike regular mechanics, it is not convenient to work with systems of constant mass; rather it is better to consider a system made up by a fixed volume of space, called a control volume (CV).

Consider the infinitesimal control volume in fig 4.5. To derive a differential equation we look at the balance (or conservation) of some property (for example energy) in the CV. Stated in words, such a balance can be put as:
Change of property in CV in time =

\[ \text{net flux of property into CV + net creation of property inside CV} \]

To go from words to a mathematical formula, let \( \Phi(r) \) denote the property of interest, expressed per unit mass, i.e. \( \Phi \) is a specific property. The flux field is called \( J(r) \) and the density \( \rho(r) \). The source field, which at each point in space gives the net creation (per unit volume) of the property, is called \( S(r) \).

Since the CV is small we can make a first order Taylor approximation of the flux field in each direction as shown in fig 4.5. For the \( x \)-direction this gives a net flux of

\[
\text{Flux in} - \text{flux out} = J_x \cdot dydz - (J_x + \frac{\partial J_x}{\partial x} dx) \cdot dydz = - \frac{\partial J_x}{\partial x} dx dydz .
\]

Doing the same thing in the \( y \)- and \( z \)-direction finally gives the total net flux into the CV as

\[
- \left( \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} \right) dx dydz = - \text{div} J dxdydz = - \text{div} J dV .
\]

Thus, the net flux per unit volume is minus the divergence of the flux field. We can now turn the balance statement above into a differential equation:

\[
\frac{\partial}{\partial t} \left[ \Phi \cdot \rho dV \right] = - \text{div} J dV + \rho \cdot S dV \quad \Rightarrow
\]

\[
\frac{\partial}{\partial t} [\rho \Phi] + \text{div} J = S
\]  

(4.1)

So, we have now found the general form of each of the equations that we will use to describe the pipe flow. Below we will look at each of them in detail.

**Figure 4.5** Infinitesimal control volume with flux field in \( x \)-direction.

### 4.4 Complete System of Equations

To fully describe viscous, compressible flow a total of four equations are needed. Those are:

1. Conservation of energy (First law of thermodynamics)
2. Conservation of momentum (Newton’s second law)
3. Conservation of mass (Continuity equation)
4. Equation of state
Figure 4.6 Control volume in the one-dimensional approximation of the pipe flow.

We will now in detail derive each of the equations 1 – 3, starting from the general form (4.1) and assume that we can use the one-dimensional approximation accounted for in section 4.1. Our only coordinate is thus the x-coordinate parallel to the pipe and our CV will look like the one in fig 4.6.

4.4.1 Energy Equation

First off, we look at equation number 1; conservation of energy. The total energy is made up of three parts; internal, kinetic and potential (due to gravity). The last of these is included since we might be interested in an inclined pipe. If \(i\) is the specific internal energy, \(u\) the velocity of the fluid, \(g\) the acceleration of gravity and \(z\) the height compared to some reference level, then the total energy in a CV such as the one in fig 4.6 is

\[
E_{\text{tot}} = E_{\text{internal}} + E_{\text{kinetic}} + E_{\text{potential}} = i\rho dV + \frac{1}{2} \rho dVu^2 + g\rho dVz = \rho dV \left(i + \frac{1}{2} u^2 + gz\right).
\]

Hence, we see that the specific property \(\Phi\) in the general equation (4.1) corresponds here to

\[
\Phi_{\text{energy}} = i + \frac{1}{2} u^2 + gz.
\]

What remains is to find expressions for the flux \(J\) and the source field \(S\). The energy flux is made up of three parts. First we have the flux due to the flow of mass (this mass carries with it energy). Then there is the net work made on the control volume by the pressure of the fluid. Last we have the heat flux by thermal conduction (given by Fourier’s law [13]) due to a difference in temperature:

\[
J_{\text{tot}} = J_{\text{massflow}} + J_{\text{work}} + J_{\text{heat}} = u\rho \cdot \Phi_{\text{energy}} + pu - k \frac{\partial T}{\partial x},
\]

where \(p\) is the pressure of the fluid, \(k\) is the thermal conductivity and \(T\) is the temperature.

The source field \(S\) is only given by the heat from the beam screen, \(q\), which is assumed to be constant along the pipe. If \(q\) is given per unit length then we must divide it by the cross section area of the pipe, \(A\), since the source field should be energy per unit volume. The source field can thus be written as

\[
S = \frac{q}{A}.
\]
Putting everything together in the form of the general differential equation (4.1) we finally get the differential equation stating the conservation of energy to be

\[
\frac{\partial}{\partial t} \left[ \rho \left( i + \frac{1}{2} u^2 + gz \right) \right] + \frac{\partial}{\partial x} \left[ u \rho \left( i + \frac{1}{2} u^2 + gz \right) + p u - k \frac{\partial T}{\partial x} \right] = \frac{q}{A}.
\]

4.4.2 Momentum Equation

Equation number 2 is nothing else than Newton’s second law, stating that for a system of constant mass, the change in linear momentum with time is given by the sum of the different forces acting on the system. Since we are not working with a system of constant mass, but one of constant volume, our equation will look a bit different than Newton’s second law.

Relating back to the derivation of the general differential equation; our conserved property is linear momentum which means that the specific property is linear momentum per unit mass, i.e. the velocity of the fluid:

\[ \Phi_{\text{momentum}} = u \]

The flux is given only by that due to the flow of mass (this mass carries with it momentum):

\[ J_{\text{tot}} = J_{\text{massflow}} = u \rho \cdot \Phi_{\text{momentum}} \]

The sum of the forces will appear as the source term \( S \). In view of Newton’s second law we may think of this as that the forces create momentum; some of this created momentum goes to raising the level of momentum in the control volume (first term on the left hand side of eq (4.1)), the rest is the net flux of momentum out of the control volume (second term on the left hand side of eq (4.1)). In our case we have three different forces; one due to the difference in pressure of the fluid, then there is the gravitational force, and last we have a force due to friction:

\[ S = F_{\text{pressure}} + F_{\text{gravity}} + F_{\text{friction}} \]

The force due to the pressure difference is simply

\[ F_{\text{pressure}} = -\frac{\partial p}{\partial x} \]

The gravitational force can be expressed in the angle of the slope of the pipe, \( \varphi \):

\[ F_{\text{gravity}} = -g \rho \sin \varphi , \]

where the sign of \( \varphi \) is such that a positive angle means upwards slope. We wait for a moment to look at the form of the frictional force, for now we just call it \( F_{\text{friction}} \). We can then put together the momentum equation according to eq (4.1):

\[
\frac{\partial}{\partial t} [\rho u] + \frac{\partial}{\partial x} [\rho u^2] = -\frac{\partial p}{\partial x} - g \rho \sin \varphi + F_{\text{friction}} \quad (4.2)
\]
Now, to find the frictional force we have to use the concept of friction factor. The friction factor is an empirical quantity defined for steady, incompressible, fully developed flow (with zero gravity). Fully developed flow means that the velocity is constant along the pipe. In such a flow the momentum equation (4.2) would reduce to

\[ \frac{\partial p}{\partial x} = F_{\text{friction}}. \]

Assuming that the frictional force is constant (which is reasonable since the velocity is), we can integrate along the pipe and thereby find the pressure drop, \( \Delta p \) (which is positive if the pressure decreases along the pipe), over the whole pipe as

\[ F_{\text{friction}} = -\frac{\Delta p}{L}. \]

It is at this point that one introduces the dimensionless friction factor, \( f \). It is empirically defined by the relation

\[ \frac{\Delta p}{L} \equiv \frac{f \rho u |u|}{2D_h}, \]

where \( D_h \) is the so called hydraulic diameter (as defined below) of the pipe. This leads to the frictional force being

\[ F_{\text{friction}} = -\frac{f \rho u |u|}{2D_h}. \] (4.3)

The reason for taking the absolute value of the velocity is that in case we are interested in backward flow (i.e. a negative \( u \)), the pressure drop must have the right sign. By using the hydraulic diameter, other duct shapes than circular may be included. It is defined in terms of the cross section area and the duct perimeter, \( P \):

\[ D_h \equiv \frac{4A}{P}. \]

In the case of a circular pipe it is reduced to the regular diameter, \( d \). Using the hydraulic diameter gives good results for duct shapes that are not too exaggerated; typically the ratio of height to width should be less then 3 – 4.

The friction factor depends on the dimensionless Reynolds number, \( R \), defined as

\[ R \equiv \frac{\rho u D_h}{\mu}, \]

where \( \mu \) is the viscosity of the fluid. As mentioned in section 4.2 it is also the Reynolds number that determines if we have laminar or turbulent flow; a Reynolds number less then about 2300 gives laminar flow. Above this value we have first a critical transition region up to
about 4000 in which the flow is neither laminar nor turbulent, and the fluid behaviour is
difficult to predict. Above 4000 we have turbulent flow.

We may think of the Reynolds number as the ratio between inertial forces, \( \rho u \), and the viscous
forces, \( \mu \), a low Reynolds number means that the viscous forces dominate and gives the
flow an ordered structure; for high Reynolds numbers the mass flow is so high that it can not
be controlled by the viscous forces and we will have random motion and turbulence.

For laminar flow, the friction factor can be calculated analytically, which yields

\[
f = \frac{64}{R}.
\]  

(4.4)

For turbulent flow \( f \) has to be determined experimentally, which was done by L.F. Moody in
1944. It was found that for turbulent flow the friction factor also depends on the relative
roughness of the pipe \( e/D_h \) (where \( e \) is the roughness of the pipe material, given in units of
length). The results of the experiments can be put together in a so called Moody chart, as
shown in fig 4.7, where also the laminar region and the critical region is included. There exist
several formulae made to fit to these experimental data, more about this in section 6.1.1.

As mentioned above, the friction factor only gives the frictional force when one has steady,
incompressible, fully developed flow. Since we are interested in compressible, not fully
developed flow which may be unsteady these conditions obviously do not hold for us. What
we do is to assume that the changes in flow variables (such as velocity) are gradual along the
pipe so that locally we can use the formula (4.3) for the frictional force. This assumption

![Moody chart](image)

**Figure 4.7** Moody chart for the experimentally determined friction factor as a function of Reynolds number and relative roughness of the pipe.
gives, when compared to experimental data, sufficiently good agreement for engineering accuracy.

We have thus found the momentum equation to be
\[
\frac{\partial}{\partial t} [\rho u] + \frac{\partial}{\partial x} [\rho u^2] = -\frac{\partial p}{\partial x} - g\rho \sin \phi - \frac{f|u|}{2D_h},
\]
where \(f\) is given by the Moody chart in fig 4.7.

### 4.4.3 Continuity Equation

The third equation, the continuity equation, states that mass can not be created or destroyed. Thus, in the view of the derivation of our general differential equation (4.1), the property conserved must be mass. This means that the specific property is mass per unit mass, i.e. simply 1:
\[
\Phi_{\text{mass}} = 1
\]

The flux field is, of course, only given by that due to flow of mass, i.e.
\[
J_{\text{tot}} = J_{\text{massflow}} = u\rho \cdot \Phi_{\text{mass}}.
\]

Last, since mass must not be created or destroyed, the source term must be zero:
\[
S = 0
\]

In the form of the general equation (4.1) the continuity equation thus becomes
\[
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} (\rho u) = 0.
\]

### 4.4.4 Equation of State

Since we are interested in compressible flow, the density must be calculated using the equation of state for the fluid, which will be the fourth and last equation. This is the only one of the equations that is not of the general form (4.1). The equation of state for a material is a relation between three state variables; the most common example is temperature, pressure and density. In our case we will substitute the temperature by the internal energy, \(i\). Hence, our equation of state will look like
\[
f_{\text{state}} (i, p, \rho) = 0.
\]

The function \(f_{\text{state}}\) can in some rare cases have an analytical form (an example is an ideal gas), but most of the time it can only be approximated by empirically made correlations.
4.5 Boundary Conditions

For a system of partial differential equations, such as ours, to have a unique solution, a number of boundary conditions are needed. They can be chosen in different ways, but not in any way. We will here not make a thorough account on this matter but simply see what boundary conditions we will use and give a short motivation.

Typically in compressible pipe flow, one specifies:

- Mass flow at inlet: \( \dot{m}_{in} = (\rho u A)_{in} = \text{fixed} \)
- Temperature at inlet: \( T_{in} = \text{fixed} \)
- Pressure at outlet: \( p_{out} = \text{fixed} \)

The reasons to choose these boundary conditions are several. It is natural to specify the mass flow and temperature at the inlet since the flow is mainly one-way, i.e. each property at a fixed point depends mainly on the properties upstream of that point, not downstream. A strong one-way character of the flow not only makes it natural to specify these properties at the inlet; it may also be necessary for a computer program to work.

When it comes to the pressure, it can not be specified at the inlet if the mass flow already is. This is due to computational issues discussed in section 5.4.3. It must thus be specified at the outlet. While the interpretation of a fixed mass flow and temperature at the inlet is quite natural, a fixed pressure at the outlet may be thought of as the pipe going out into a large reservoir which has a constant pressure.

The boundary conditions above are not sufficient to give a unique solution to our equations; we need further specifications at the outlet. What one often does when solving problems like this numerically is to assume that flow properties (such as temperature and velocity for example) are more or less constant at the outlet, i.e. that the gradient of these properties is zero \([13]\). We leave this matter for now and discuss it further in the next chapter.

If we consider steady state, the boundary conditions given above are sufficient to give a unique solution, but when we consider unsteady flow we have time as a second independent variable. For the system of equations to have a unique solution we must then, apart from specifying the three spatial boundary conditions, also give a “boundary condition” with respect to time; this is done by specifying every property of the flow along the pipe at the initial time.
5 Design of Numerical Program

In this chapter we will account for the numerical methods leading to the construction of a computer program which will solve the system of partial differential equations given by the mathematical model for the cooling loop derived in the previous chapter. The material in this chapter is derived from [13].

5.1 The Idea

The task of our numerical program will be to solve the following system of partial differential equations:

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ \rho \left( i + \frac{1}{2} u^2 + gz \right) \right] + \frac{\partial}{\partial x} \left[ u \rho \left( i + \frac{1}{2} u^2 + gz \right) \right] + pu - k \frac{\partial T}{\partial x} &= \frac{q}{A} \quad (5.1a) \\
\frac{\partial}{\partial t} [pu] + \frac{\partial}{\partial x} [pu^2] &= -\frac{\partial p}{\partial x} - g \rho \sin \varphi - \frac{f \rho u |u|}{2D_h} \quad (5.1b) \\
\frac{\partial p}{\partial t} + \frac{\partial}{\partial x} (pu) &= 0 \quad (5.1c) \\
f_{\text{state}}(i, p, \rho) &= 0, \quad (5.1d)
\end{align*}
\]

with the following boundary conditions:

- Mass flow at inlet: \( \dot{m}_{\text{in}} = \text{fixed} \) \quad (5.1e)
- Temperature at inlet: \( T_{\text{in}} = \text{fixed} \) \quad (5.1f)
- Pressure at outlet: \( p_{\text{out}} = \text{fixed} \) \quad (5.1g)

The idea is to construct a numerical algorithm which as an end result gives the properties of the fluid in a finite number of points along the pipe; we will thus discretize the different property fields. When considering unsteady, i.e. time dependent, flow the program will likewise give flow properties at discrete points in time. The idea of the numerical scheme (known as the Finite Volume method) will be to turn the differential equations above into simple linear algebraic equations for the discrete fields. To see exactly how this can be done we will look at a general example and afterwards apply the technique to each of our differential equations.

5.2 Discretization of a Differential Equation

We start by looking at the case of steady state, i.e. the flow is time independent. The generalization to unsteady flow is easily done which we will see later on.

5.2.1 Steady Flow

Consider the following general differential equation, derived from a conservation principle as the one in section 4.3:
\[
\frac{d}{dx} [\rho u \Phi] = \frac{d}{dx} \left[ \Gamma \frac{d\Phi}{dx} \right] + S(\Phi), \tag{5.2}
\]

where \( \Gamma \) describes diffusive processes and the source term \( S(\Phi) \) represents any additional terms. The variable \( \Phi(x) \) denotes any specific property of the fluid.

Our task is this: given every other property of the flow (such as flow field, density field etc.) except \( \Phi \): find \( \Phi \) in a finite number of points along the pipe. These so called grid points are assumed to be uniformly distributed along the \( x \)-axis, separated by a distance \( \Delta x \), as in fig 5.1.

As mentioned before we will solve the task by turning the differential equation (5.2) into a system of linear algebraic equations for the values of \( \Phi \) at the grid points. This is done by integrating eq (5.2) over a control volume (CV) of length \( \Delta x \) set symmetrically around each grid point as in fig 5.1. The boundaries of such a CV are located midway between two grid points at so called mid points. The grid point around which the CV is set is called \( P \) (for point), the neighbouring grid points \( W \) (for west) and \( E \) (for east), and the midpoints in between are for the same reasons called \( w \) and \( e \). The integration of eq (5.2) over the CV should result in a linear relation for \( \Phi \) at the grid points \( P, E \) and \( W \):

\[
\int_{w}^{e} (eq(5.2)) \Rightarrow a_p \Phi_P = a_e \Phi_E + a_w \Phi_W + b
\]

We now look at how each of the terms in eq (5.2) is integrated.

**Convection – Diffusion terms**

It is easiest to handle the two terms in eq (5.2) containing derivatives together. They describe flux due to convection (left hand derivative of eq (5.2)) respectively diffusion (right hand derivative of eq (5.2)) and are named thereafter. The integration is straight forward:

\[
\int_{w}^{e} \left( \frac{d}{dx} [\rho u \Phi] - \frac{d}{dx} \left[ \Gamma \frac{d\Phi}{dx} \right] \right) dx = \left( \rho u \Phi \right)_e - \left( \rho u \Phi \right)_w - \left( \Gamma \frac{d\Phi}{dx} \right)_e + \left( \Gamma \frac{d\Phi}{dx} \right)_w \tag{5.3}
\]

To proceed we must express the right hand side of eq (5.3) in \( \Phi \) taken at the grid points\(^4\): \( \Phi_W, \Phi_P \) and \( \Phi_E \). There are several ways of doing this; the most natural is to calculate \( \Phi \) and the derivatives at the mid points by assuming a linear profile for \( \Phi \):

![Figure 5.1 Grid points, mid points and control volume for a part of the pipe.](image)

\(^4\)Even though the properties assumed to be known, as for example \( \rho \), are also only defined at the grid points (except the velocity which is defined at mid points; see section 5.3.2), they can easily be approximated at the mid points; unlike as is the case with the property \( \Phi \), for which we intend to solve the discretized equation, it does not matter how this approximation is done; a linear profile may always be used.
For the diffusion terms (b) this works fine, but not always for the convection terms (a); it can give non-physical solutions in the case of very high mass flows. In these cases one has to explicitly take the movement of the fluid into account. One way of doing this is by the so called upwind approximation:

\[
\Phi_e = \Phi_p \quad \text{if } u_e > 0 \quad ; \quad \Phi_e = \Phi_E \quad \text{if } u_e < 0
\]  

(5.4)

A more accurate way is to use the so called power law scheme, which approximates the profile for \( \Phi \) in a way that is very close to the true solution. It is what we will use. We will not derive it here, but simply give the results (a derivation can be found in [13]):

\[
(\rho u \Phi)_e - (\rho u \Phi)_w - \left( \int \frac{d\Phi}{dx} \right)_e + \left( \int \frac{d\Phi}{dx} \right)_w = \lambda_p \Phi_p - \lambda_e \Phi_E - \lambda_w \Phi_w,
\]

where

\[
\lambda_e \equiv D_e \max(0, (1 - \frac{0.1|F_e|}{D_e})^5) + \max(0, -F_e)
\]

\[
\lambda_w \equiv D_w \max(0, (1 - \frac{0.1|F_w|}{D_w})^5) + \max(0, F_w)
\]

\[
\lambda_p \equiv \lambda_e + \lambda_w + F_e - F_w,
\]

and two new parameters are defined:

\[
F = \rho u \quad ; \quad D = \frac{\Gamma}{\Delta x}
\]

Source term

The source term can have any form, so in general we must linearize it. Such linearizations are necessary at several points in the numerical process. They will however not affect the end result, see section 5.2.2 below. The source term can thus be written as

\[
S(\Phi) = S_p \Phi + S_C.
\]

When we integrate the source term we assume that the value of \( S \) at the grid point \( P \) prevails over the whole control volume, i.e. we assume that \( S \) has this constant value over the whole control volume. Since we will be using this method many times it is from now on referred to as the principle of prevailing. We thus get

\[
\int (S_p \Phi + S_C) dx = \int (S_p \Phi_p + S_C) dx = (S_p \Phi_p + S_C) \Delta x.
\]

Putting everything together we see that integration of eq (5.2) over the control volume yields
the following algebraic equation for each grid point $P$:

$$a_P \Phi_P = a_E \Phi_E + a_W \Phi_W + b,$$

where

$$a_P \equiv \hat{\lambda}_P - S_P \Delta x$$
$$a_E \equiv \hat{\lambda}_E$$
$$a_W \equiv \hat{\lambda}_W$$
$$b \equiv S_C \Delta x.$$

Assuming that we do this over a total of $N$ grid points, this gives us $N$ linear equations with $N + 2$ unknown variables, labelled $\Phi_0$, $\ldots$, $\Phi_{N+1}$, as in fig 5.2. In order to get a solvable system of equations with only $N$ unknown variables we must either know the value of $\Phi$ at the boundaries or express it in the $N$ inner grid point values. Typically one only knows $\Phi$ at one of the boundaries (from a specified boundary condition), most often the inlet:

$$\Phi_0 = \Phi_{in}$$

At the other, most often the outlet, one usually assumes (as mentioned in section 4.5) that the change in $\Phi$ is small and that we can approximately put the gradient of $\Phi$ to zero:

$$\left( \frac{d\Phi}{dx} \right)_{outlet} \approx 0$$

With such an assumption we can calculate $\Phi$ at the boundary as

$$\Phi_{N+1} = \Phi_N.$$

In our case, the change in properties such as internal energy and velocity is not always small at the outlet. This is however not a problem; due to the strong one-way character of the flow (see section 4.5), the coefficient $a_E$ will be negligible at the outlet and it does not matter how we approximate $\Phi_{N+1}$. This applies to the outlet boundary conditions in both the energy and the momentum equation, see section 5.4.3. The reason that our flow is mainly one-way is that the effects of thermal conduction are small.

So, we now have a system of $N$ linear equations with $N$ unknown variables, $\Phi_1$, $\ldots$, $\Phi_N$. Solving this gives us the property $\Phi$ at every grid point along the section of interest.
5.2.2 Non-Linearity

Most often, the differential equation one tries to solve numerically by discretization is non-linear. As we have already mentioned, the source term may be non-linear and that is handled by linearizing it, although we did not say how to do this. Moreover, non-linearity in the other terms will lead to that the coefficients \( \alpha_p, \alpha_E \) and \( \alpha_W \) depend on the property \( \Phi \).

All of this leads to that the algebraic system of equations must be solved on an iterative basis, where we first guess the value of \( \Phi \) at each grid point. This guessed \( \Phi \) field is then used to calculate the coefficients \( \alpha_p, \alpha_E \) and \( \alpha_W \). Moreover it is used to linearize the source term by making a Taylor expansion around the guessed value, \( \Phi^* \):

\[
S(\Phi) \approx S(\Phi^*) + \left( \frac{dS}{d\Phi} \right)_{\Phi=\Phi^*} (\Phi - \Phi^*)
\]  

(5.5)

Sometimes one can not differentiate \( S(\Phi) \); then one has to resort to

\[
S(\Phi) \approx S(\Phi^*).
\]  

(5.6)

After having solved the system of linear equations obtained in this way, we use the solution as a new guess for \( \Phi \) and solve the linearized equations once more and so on. If this iteration process converges it will always yield the true solution to the non-linear equation; how we for example linearize the source term only affects the rate of convergence, not the converged solution itself.

5.2.3 Unsteady Flow

When we consider the most general case, i.e. unsteady flow, the differential equation of interest will typically look like

\[
\frac{\partial}{\partial t} [\rho \Phi] + \frac{\partial}{\partial x} [\rho u \Phi] = \frac{\partial}{\partial x} \left[ \Gamma \frac{\partial \Phi}{\partial x} \right] + S(\Phi).
\]  

(5.7)

We will now assume that we know all properties, including \( \Phi \), of the flow at an initial time \( t_0 \) and we want to calculate \( \Phi \) after a step in time, i.e. at \( t_1 = t_0 + \Delta t \). Moreover, every other property of the flow except \( \Phi \) is also assumed to be known at the new time \( t_1 \). To distinguish properties at the “old” time \( t_0 \) and from those at the “new” time \( t_1 \) we denote the former with a superscript 0 and the latter with a superscript 1.

To find a system of algebraic equations for \( \Phi \) at the grid points at the new time \( t_1 \) we go about in the same way as for steady flow, but in addition to doing a spatial integration of eq (5.7) over each control volume, we now also integrate it over one step in time, from \( t_0 \) to \( t_1 \). Referring to fig 5.1 this is what we will do:

\[
\int_{t_0}^{t_1} \int_{V_p} (eq(5.7)) \Rightarrow a_p \Phi^i_p = a_E \Phi^i_E + a_W \Phi^i_W + b
\]

The spatial integration is done with exactly the same methods used for steady flow. We will now see how the time integration is done for each of the terms in eq (5.7).
Unsteady term

First out is the time derivative term, which is the only new term compared to the differential equation for steady flow (5.2). The integration over time is straightforward and the spatial integration is done by using the principle of prevailing defined in section 5.2.1. We thus get

\[
\int_{w}^{e} \int_{t}^{t+\Delta t} \frac{\partial \Phi}{\partial t} dt \, dx = \int_{w}^{e} \left( (\rho \Phi)^{i} - (\rho \Phi)^{0} \right) dx = \left( (\rho \Phi)^{p} - (\rho \Phi)^{0} \right) \Delta x .
\]

The other terms

All the other terms of eq (5.7) do not contain a time derivative and are handled in the same way, namely by the so-called fully implicit method. In this method one assumes that the value of the integrand at the new time, \( t_{1} \), prevails over the time step, i.e. that the integrand has this constant value over the whole time step. This might seem like a crude approximation but most often it is accurate enough. Moreover it avoids problems with non-physical results that can occur with other methods. Using the same method for spatial integration as we did for steady flow (including linearization of the source term) we thus get

\[
\int_{t_{0}}^{t_{0}+\Delta t} \int_{w}^{e} \left( \frac{d}{dx} [\rho \Phi] - \frac{d}{dx} \left( \Gamma \frac{d \Phi}{dx} \right) - (S_{p} \Phi + S_{c}) \right) dx \, dt = \\
\int_{t_{0}}^{t_{0}+\Delta t} \left( \lambda_{p} \Phi^{p} - \lambda_{E} \Phi^{E} - \lambda_{w} \Phi^{W} - (S_{p} \Phi^{p} + S_{c}) \Delta x \right) dt = \\
\int_{t_{0}}^{t_{0}+\Delta t} \left( \lambda_{p} \Phi^{p} - \lambda_{E} \Phi^{E} - \lambda_{w} \Phi^{W} - (S_{p} \Phi^{p} + S_{c}) \Delta x \right) dt = \\
\left( \lambda_{p} \Phi^{p} - \lambda_{E} \Phi^{E} - \lambda_{w} \Phi^{W} - (S_{p} \Phi^{p} + S_{c}) \Delta x \right) \Delta t .
\]

Putting all parts together we see that integration of eq (5.7) over the control volume and over one time step from \( t_{0} \) to \( t_{1} \) gives the following algebraic equation for \( \Phi \) at the new time \( t_{1} \):

\[
a_{p} \Phi^{i}_{p} = a_{E} \Phi^{i}_{E} + a_{w} \Phi^{i}_{w} + b ,
\]

where, after dividing by \( \Delta t \), we get the coefficients to be

\[
a_{p} \equiv \rho_{p} \frac{\Delta x}{\Delta t} + \lambda_{p} - S_{p} \Delta x \\
a_{E} \equiv \lambda_{E} \\
a_{w} \equiv \lambda_{w} \\
b \equiv \left( \rho \Phi \right)^{0}_{p} \frac{\Delta x}{\Delta t} + S_{c} \Delta x .
\]

By comparing with the results in section 5.2.1 we see that we get the case of steady flow by letting \( \Delta t \to \infty \). The matter of boundary conditions is of course handled in exactly the same way as for steady state, described above.
5.3 Discretization of Our Differential Equations

We will now in detail see how we can discretize each of our equations; the energy, momentum and continuity equation. This will be done in the same manner as in section 5.2. We will consider the most general case, namely unsteady flow; as we have already noticed, the case of steady flow is found by letting $\Delta t \to \infty$.

5.3.1 Discretization of the Energy Equation

The most natural choice of a specific property for which we will set up algebraic equations is the internal energy, $i$. Doing this we will assume that the other properties, i.e. pressure, density and velocity, are known.

To write the energy equation (5.1a) in a form similar to that of the general unsteady differential equation (5.7) we must first express the temperature gradient in the internal energy $i$:

$$\frac{\partial T}{\partial x} = \left( \frac{\partial T}{\partial t} \right)_p \frac{\partial i}{\partial x} + \left( \frac{\partial T}{\partial p} \right)_i \frac{\partial p}{\partial x} = \frac{1}{c_p} \frac{\partial i}{\partial x} + \left( \frac{\partial T}{\partial p} \right)_i \frac{\partial p}{\partial x},$$

where $c_p$ is the specific heat capacity at constant pressure. The energy equation (5.1a) can then be written as

$$\frac{\partial}{\partial t} \left[ \rho \frac{1}{2} u^2 \right] + \frac{\partial}{\partial x} \left[ u \rho \frac{1}{2} u^2 + gz \right] = \frac{k}{c_p} \frac{\partial i}{\partial x} - \frac{\partial}{\partial t} \left[ \rho \left( \frac{1}{2} u^2 + gz \right) \right] - \frac{\partial}{\partial x} \left[ \frac{\partial T}{\partial p} \right] \frac{\partial p}{\partial x} + q \frac{A}{I} \tag{5.8}$$

We will now, with a short reference to the methods used in section 5.2, account for how each term of eq (5.8) is integrated over a CV and a time step.

The terms 1, 2 and 3 are identical to the corresponding terms in eq (5.7) and thus handled in exactly the same way. Spatial and time integration of term 4 is done in exactly the same way as for term 1. Spatial integration of term 5 is straightforward since it is a derivative with respect to $x$ (it is done in the same way as for the convection term, see eq (5.3)) and the time integration is done by the fully implicit method. This also goes for term 6 but since it depends on $i$ in a non-linear way we must linearize it. This is done according to eq (5.6). Finally, term 7 is constant which makes both spatial and time integration trivial.

So, the integration of eq (5.8) over the CV and a step in time yields

$$a_p i_p = a_E i_E + a_w i_w + b, \quad \tag{5.9}$$

where
\[ a_p \equiv \frac{\Delta x}{\Delta t} \rho_p + \lambda_p \]
\[ a_E \equiv \lambda_E \]
\[ a_w \equiv \lambda_w \]
\[ b \equiv (i\rho)^0 \frac{\Delta x}{\Delta t} + \left( \rho_p^0 \left( \frac{1}{2} (u_p^0)^2 + g z_p \right) - \rho_p \left( \frac{1}{2} u_p^2 + g z_p \right) \right) \frac{\Delta x}{\Delta t} + u_w P_w \left( \frac{u_w^2}{2} + g z_w \right) \]
\[-u_e P_e \left( \frac{u_e^2}{2} + g z_e \right) + u_e P_e - u_e P_e + \left( \frac{\partial T}{\partial p} \right)_{i, e} - \left( \frac{\partial T}{\partial p} \right)_{i, s} \right] + \frac{q}{S} \Delta x.\]

The coefficients \( \lambda_i \) are defined in section 5.2.1 (with \( \Gamma = k / c_p \) in this case). Note that we have now dropped the superscript 1 for properties at the new time \( t_1 \). This applies from now on wherever there is no risk of confusion. As mentioned before, the case of steady flow is obtained by letting \( \Delta t \to \infty \).

This discretized energy equation will have to be solved on an iterative basis as described in section 5.2.2. This, because it is non-linear; as already mentioned, term 6 in eq (5.8) is non-linear, so is term 3, since the thermal conductivity \( k \) and the specific heat \( c_p \) depend on the internal energy \( i \).

### 5.3.2 Discretization of the Momentum Equation

The momentum equation will be solved for the velocity \( u \) as the unknown property. We will thus assume that the other properties, i.e. internal energy, pressure and density, are known. Unlike the other properties which are defined at the grid points, one finds it best to calculate the velocity at the mid points. The reasons for using such a displaced, or staggered grid, are several; it helps avoiding certain non-realistic solutions and it also makes the algorithm more efficient. We will still refer to the points where the velocity is calculated as mid points, even though they correspond to the term grid points in section 5.2.

The momentum equation is

\[ \frac{1}{\partial t} [\rho u] + \frac{2}{\partial x} [\rho u \cdot u] = \frac{3}{\partial x} \frac{\partial p}{\partial x} - g \rho_0 \sin \phi - \frac{f p u |u|}{2D_h} \quad (5.10) \]

As we can see, it has the same form as the general differential equation (5.7) that we have already discretized. The only difference (apart from the form of the source term) is that we don’t have a diffusion term, but this is just equivalent to the diffusion coefficient \( \Gamma \) being zero. We can also note some obvious non-linearities; the convection coefficient \( \rho u \) on the left hand side depends on \( u \) and the friction term on the right hand side is non-linear. This is not a problem; it will be handled with the methods described in section 5.2.2.

We will integrate eq (5.10) over the CV in fig 5.3 and over a step in time from \( t_0 \) to \( t_0 + \Delta t \). Here is how each term is integrated:

Terms 1 and 2 are identical to the corresponding terms in eq (5.7) and thus handled in exactly the same way. Term 3 is handled in the same way as term 5 in the energy equation. For term 4 the prevailing principle and fully implicit scheme is used. Term 5 is linearized by a Taylor
expansion according to eq (5.5); however, the friction factor \( f \) depends on \( u \) in a complicated way and must be directly calculated from the guessed velocity field before making the Taylor expansion.

Putting all parts together we find that integration of eq (5.10) over the CV and time step gives

\[
a_p u_p = a_e u_e + a_u u_w + b + p_w - p_E, \tag{5.11}
\]

where

\[
a_p \equiv \rho_p \frac{\Delta x}{\Delta t} + \lambda_p + \left( \frac{f \rho |u|}{D_h} \right)_p \Delta x
\]

\[
a_e \equiv \lambda_e
\]

\[
a_u \equiv \lambda_u
\]

\[
b \equiv \rho_p^0 u_p^0 \frac{\Delta x}{\Delta t} + \left( \frac{f \rho |u|^*}{2D_h} \right)_p \Delta x - g \rho_p \sin \phi \Delta x.
\]

As before, the coefficients \( \lambda_i \) are defined in section 5.2.1 (with \( \Gamma \equiv 0 \) in this case). The guessed velocity field used in the iterative procedure to handle the non-linearities is denoted by an asterisk. Note that we have chosen to not include the pressure difference in the term \( b \). Why we have done this way will be clear in section 5.4.1.

### 5.3.3 Discretization of the Continuity Equation

Last we have the continuity equation (5.1c). As we will see in section 5.4.1 it will not be used to directly yield a property, as the energy and momentum equation are, but it will indirectly be used to find the pressure field. For now, we will only integrate it, without setting up any algebraic equations for a particular property.

The continuity equation is

\[
\frac{1}{\rho} \frac{\partial p}{\partial t} + \frac{1}{\rho} \frac{\partial}{\partial x} (\rho u) = 0.
\]

We will integrate this equation over a CV around a grid point, as in fig 5.1, and over a step in time from \( t_0 \) to \( t_0 + \Delta t \). The integration is straight forward; term 1 is integrated in the same way as the unsteady term in section 5.2.3 and term 2 is handled in the same way as the convection term in the same section. This yields
\[
(\rho_p - \rho_p^0) \frac{\Delta x}{\Delta t} + \rho_p u_e - \rho_p u_w = 0. \tag{5.12}
\]

We will leave it in this form for now.

## 5.4 Assembly of Program

We have now derived the discrete form of our equations. It is however not clear exactly how they will be used to solve the total flow problem. We know from the discretized energy equation how to find the internal energy field given the pressure, density and velocity field. In the same way we know from the momentum equation how to find the velocity field when the internal energy, pressure and density are given. Finally, we can use the equation of state (5.1d) to find the density given the pressure and internal energy. It seems that we must use the continuity equation to find the pressure. We will start by looking at how this can be done.

### 5.4.1 The Pressure Correction Equation

As indicated in section 5.3.3 the continuity equation will not give the pressure field directly. Instead it will work on an iterative basis and correct a guessed pressure, density and velocity field so that, in the end, the continuity equation is satisfied.

Assume that we have a guessed pressure and density field, denoted \( p^* \) and \( \rho^* \). These fields can be used to solve the discretized momentum equation (5.11) (assuming that the internal energy is known); denote the velocity field obtained in this way, \( u^* \). We now introduce the pressure, velocity and density corrections, \( p', u' \) and \( \rho' \) by the definition

\[
p = p^* + p' \quad ; \quad u = u^* + u' \quad ; \quad \rho = \rho^* + \rho', \tag{5.13}
\]

where \( p, u \) and \( \rho \) are the exact solutions to our system of discretized equations. The idea is to transform the continuity equation into an algebraic equation for the pressure corrections. To do this we must express both the velocity and density corrections in the pressure corrections. We will start with the velocity corrections. From the momentum equation (5.11) we have

\[
a_p u_p = a_e u_e + a_w u_w + b + p_w - p_E \tag{5.14a}
\]

\[
a_p' u_p' = a_e' u_e' + a_w' u_w' + b' + p_w' - p_E'. \tag{5.14b}
\]

The reason that the coefficients in eq (5.14b) are denoted by an asterisk is that they depend on the guessed density. We will now however make the first of two quite crude approximations; we will exchange the coefficients in eq (5.14b), \( a_i^* \) and \( b^* \), by the corresponding coefficients in eq (5.14a), \( a_i \) and \( b \). Both approximations are justified at the end of section 5.4.2. We can now subtract eq (5.14b) from eq (5.14a) and get

\[
a_p u_p' = a_e u_e' + a_w u_w' + p_w' - p_E'. \tag{5.15}
\]

The second approximation we will make is to now drop the terms \( a_e u_e' + a_w u_w' \) from eq (5.15). In this way we can express the velocity corrections in the pressure corrections:
\[ u'_p = \zeta_p (p'_w - p'_E), \]  
(5.16)

where

\[ \zeta_p = \frac{1}{a_p}. \]

Keep in mind that this is the \( a_p \) from the momentum equation, so don’t get confused when we later on use a similar notation for the coefficients in the pressure correction equation.

The density corrections are expressed as a linear function of the pressure corrections:

\[ \rho' = K \rho', \]
(5.17)

where the parameter \( K \) can be calculated from the equation of state as

\[ K = \left( \frac{\partial \rho}{\partial p} \right) \]

We now turn to the discretized continuity equation (5.12) to find an equation for the pressure correction \( p' \). This is done by inserting the formulae (5.13) into the continuity equation (5.12) and to use the formulae for the different corrections, (5.16) and (5.17). In order to only get linear terms we use the following approximation:

\[ \rho u = (\rho^* + \rho')(u^* + u') \approx \rho^* u^* + \rho' u^* + \rho^* u' \]

As in the case of the discretization of the general differential equation in section 5.2.1 we must think about how to express the pressure corrections at the mid points in the pressure corrections at the grid points. This will be done according to the upwind scheme (5.4):

\[ p'_e = p'_p \quad \text{if } u_e > 0 \quad ; \quad p'_e = p'_E \quad \text{if } u_e < 0 \quad \text{(analogously for } p'_w) \]

After some algebraic work we then get the so called pressure correction equation, which is an algebraic equation for the pressure corrections at each grid point:

\[ a_p p'_p = a_E p'_E + a_w p'_w + b, \]
(5.18)

where

\[ a_p = \frac{\Delta x}{\Delta t} K_p + \rho_e^* \zeta_e + \rho_w^* \zeta_w + K_e \max(0, u_e^*) + K_w \max(0, -u_w^*) \]
\[ a_E = K_e \max(0, -u_e^*) + \rho_e^* \zeta_e \]
\[ a_w = K_w \max(0, u_w^*) + \rho_w^* \zeta_w \]
\[ b = \frac{\Delta x}{\Delta t} (-\rho_e^* + \rho_p^0) - \rho_w^* u_e^* + \rho_w^* u_w^*. \]
We can notice that, unlike the discretized energy and momentum equations, this equation is linear (i.e. none of the coefficients depend on the pressure corrections), so no iterations are necessary when solving it.

5.4.2 Main Algorithm
After having obtained discretized equations to calculate each property of the flow we will now use them together in a numerical algorithm that will solve the total flow problem. This algorithm is called SIMPLE (Semi-Implicit Method for Pressure-Linked Equations) and can be described in the following steps:

1. Guess internal energy, pressure and density fields, \( i^* \), \( p^* \) and \( \rho^* \).

2. Solve the momentum equation (5.11) using the guessed fields from step 1; call the velocity solution \( u^* \).

3. Solve the pressure correction equation (5.18) using the guessed fields; \( i^* \), \( p^* \), \( \rho^* \) and \( u^* \).

4. Add the corrections obtained this way to the guessed fields using eq (5.16) and (5.17) to obtain new pressure, density and velocity fields; \( p \), \( \rho \) and \( u \):

\[
p = p^* + p' ; \quad u = u^* + u' ; \quad \rho = \rho^* + \rho'
\]

5. Solve the energy equation (5.9) using the corrected pressure, density and velocity fields; \( p \), \( \rho \) and \( u \).

6. Use the obtained internal energy field to calculate a new density from the equation of state (5.1d) (using the corrected pressure \( p \)). Then return to step 1 using the now obtained internal energy, pressure and density as a new guess for these fields.

The steps 1 – 6 are repeated until a converged solution is obtained. For unsteady flow this procedure is done for each step in time.

As promised, we will now justify the approximations made when deriving the pressure correction equation in section 5.4.1. The reason that these approximations will not affect the final solution lies in the iterative nature of the procedure. In this view, the pressure correction equation should only be regarded as an intermediate step on the iterative way to the converged solution; the continuity equation, from which the pressure equation is derived, is not required to be perfectly satisfied until we have found the converged solution. It is no different than the linearization of the source terms in section 5.2.2. Of course, any approximations made when deriving the pressure correction equation will affect the rate of convergence of the overall SIMPLE algorithm. But the final converged solution will still be the same.

5.4.3 Boundary Conditions
To make the SIMPLE algorithm complete we must look at what boundary conditions to use when solving the different discretized equations.
Energy equation

The boundary conditions used for the internal energy are of the same type as the ones in section 5.2.1; at the inlet the internal energy is given, since we can, given the pressure field, calculate it from the specified inlet temperature (5.1f) using an equation of state. At the outlet we use the assumption of zero gradient (as said in section 5.2.1 this may be used even when the variations in the internal energy are high at the outlet since the flow will have a strong one-way character).

Momentum equation

Also here the boundary conditions used are of the same type as the ones in section 5.2.1; at the inlet the velocity can be calculated from the specified mass flow (5.1e) given the density field. At the outlet we use the assumption of zero gradient.

Pressure correction equation

Here the boundary conditions are not as straight forward as for the other two equations. We start with the easiest, namely the outlet. Here the pressure is specified according to (5.1g) so the boundary condition for the pressure corrections at the outlet, \( p'_{\text{out}} \), is simply the difference between this specified value, \( p_{\text{out}} \), and the current (guessed) pressure at the outlet, \( p^*_{\text{out}} \):

\[
p'_{\text{out}} = p_{\text{out}} - p^*_{\text{out}}
\]

At the inlet we must look back at the discretized continuity equation for the left most grid point:

\[
(\rho_p - \rho_p^0) \frac{\Delta x}{\Delta t} + \rho_e u_e - \rho_w u_w = 0
\]

Since the mass flow is specified at the inlet, eq (5.1e), we know the factor \( \rho u \) at the western mid point, see eq (5.19). As can be easily seen, this will lead to that when deriving the pressure correction equation for this grid point, the correction at the western neighbouring grid point, \( p'_{W} \), will never appear. There is thus no need to specify \( p'_{W} \). Relating back to section 4.5, we now see the reason why it would have been impossible to specify a fixed pressure and a fixed mass flow at the same boundary.

5.5 Relaxation

Whenever some sort of iterative process is used there might be a need to either slow down or speed up the convergence. In our case we will only be interested in the former, which is called under relaxation. The reason for us wanting to slow down the convergence is to avoid divergence.

In our program, there will be two sorts of iterations. First, the main algorithm works by doing iterations, secondly we need to use iterations when solving the non-linear momentum and energy equation.
5.5.1 Relaxation when Solving a Discretized Equation

We will first look at the relaxation used when solving a discretized equation. Consider a general discretized equation of the form

$$a_D \Phi_p = a_E \Phi_E + a_W \Phi_W + b.$$  \hfill (5.20)

Let $\widetilde{\Phi}_p$ be the value of $\Phi_p$ from the previous iteration round. Eq (5.20) can be written as

$$\Phi_p = \widetilde{\Phi}_p + \left( \frac{a_E \Phi_E + a_W \Phi_W + b}{a_D} - \widetilde{\Phi}_p \right).$$

In order to slow down the convergence we introduce a relaxation factor $\gamma$ according to

$$\Phi_p = \widetilde{\Phi}_p + \gamma \left( \frac{a_E \Phi_E + a_W \Phi_W + b}{a_D} - \widetilde{\Phi}_p \right),$$  \hfill (5.21)

where $\gamma$ is a number between 0 and 1. A $\gamma$ close to 0 will lead to slow convergence as opposed to a $\gamma$ close to 1 which will make the convergence rate close to what we would have if no under relaxation was used. Eq (5.21) can further be written as

$$\gamma \frac{a_D}{\gamma} \Phi_p = a_E \Phi_E + a_W \Phi_W + b + (1 - \gamma) \frac{a_D}{\gamma} \widetilde{\Phi}_p.$$

We have thus derived a new linear equation that has exactly the same form as the original equation (5.20), but will give us a slower convergence, and hence less chance of divergence.

This relaxation method will be used when solving the momentum and energy equation. In both cases it will be used twice with respect to two different iterations; first we have the iterations between the different steps of the main algorithm. Secondly, we have the iterations used to handle the non-linearity of these equations.

5.5.2 Relaxation when Correcting Pressure, Density and Velocity

When correcting the pressure, density and velocity one should only add a fraction of the correction term, i.e.

$$p = p^* + \gamma p',$$

for some $\gamma$ between 0 and 1 (analogously for density and velocity).

5.5.3 Relaxation when Calculating Density from Equation of State

We also need to use relaxation when calculating the density from the equation of state. If $\rho'$ denotes the density calculated in step 4 of the SIMPLE algorithm and $\rho''$ the density given by the equation of state we should in step 6 calculate the density as

$$\rho = (1 - \gamma) \rho' + \gamma \rho''.$$
for some $\gamma$ between 0 and 1.

### 5.6 Solving the Algebraic Equations

Consider a discretized differential equation of the following form:

$$a_i \Phi_i = b_i \Phi_{i+1} + c_i \Phi_{i-1} + d_i,$$  \hspace{1cm}  (5.22)

where the index $i$ spans over all grid points; $i = 1, \ldots, N$.

We further assume that using our boundary conditions we get $c_1 = 0$ (left boundary) and $b_N = 0$ (right boundary). As can be seen above, this is true for all of our discretized differential equations. With such boundary conditions there is a very simple and fast way to solve the linear equation system called the TriDiagonal-Matrix Algorithm (TDMA). It is based on the following facts:

Eq (5.22) is a relation between only 3 neighbours at a time. But since we at the left boundary have a relation between $\Phi_1$ and $\Phi_2$ only, we can substitute this relation into the relation for $\Phi_1$, $\Phi_2$ and $\Phi_3$, hence obtaining a relation between $\Phi_2$ and $\Phi_3$, which in turn can be substituted into the next relation and so on.

Doing this on and on we end up with a relation between $\Phi_{N-1}$ and $\Phi_N$. But since we also have the last equation ($i = N$) left to use, which is another relation between $\Phi_{N-1}$ and $\Phi_N$, we can thus find the numerical value of $\Phi_N$. Then we can, by using our 2 neighbour relations just derived, go backwards and find $\Phi_{N-1}$, $\Phi_{N-2}$, ... $\Phi_1$.

After now having presented the main idea of the TDMA, we give a step by step description of it together with a brief derivation.

**TDMA algorithm**

Let the 2-neighbour relations mentioned above be of the form

$$\Phi_i = P_i \Phi_{i+1} + Q_i.$$  \hspace{1cm}  (5.23)

Using eq (5.22) one can find the following recurrence relation for the coefficients $P_i$ and $Q_i$:

$$P_i = \frac{b_i}{a_i - c_i P_{i-1}},$$  \hspace{1cm}  (5.24a)

$$Q_i = \frac{d_i + c_i Q_{i-1}}{a_i - c_i P_{i-1}}.$$  \hspace{1cm}  (5.24b)

$P_1$ and $Q_1$ are found from the first two neighbour relation between $\Phi_1$ and $\Phi_2$ to be

$$P_1 = \frac{b_1}{a_1} ; \quad Q_1 = \frac{d_1}{a_1}.$$  \hspace{1cm}  (5.25)
At the right boundary we have $b_N = 0$ and this gives in eq (5.24a) $P_N = 0$ and hence we get in eq (5.23):

$$\Phi_N = Q_N$$  \hspace{1cm} (5.26)

The TDMA algorithm is thus as follows:

1. Calculate $P_1$ and $Q_1$ from eq (5.25).

2. Use the recurrence relations (5.24a) and (5.24b) to calculate $P_i$ and $Q_i$ for $i = 2, \ldots, N$.

3. Set $\Phi_N = Q_N$ according to eq (5.26).

4. Use eq (5.23) to find $\Phi_i$ for $i = N-1, \ldots, 1$.

We have thus accounted for how each of our discretized equations can be solved in a fast and efficient way.
6 Implementation and Validation

In this chapter we will look at how the numerical algorithm derived in the previous chapter is implemented in a computer program. We will also validate the functionality of the program and discuss the accuracy of the mathematical model by comparing with experimental results.

6.1 Implementation

The SIMPLE algorithm derived in chapter 5 was implemented using C++. We will in this section address some matters of special interest concerning the implementation.

6.1.1 Friction Factor

As said in section 4.4.2, the friction factor can only be calculated analytically for laminar flow; see eq (4.4). For turbulent flow, which is what we in general will have, one has to resort to the experimental data given by the Moody chart in fig 4.7. As mentioned, there have been many suggestions of formulae made to fit these experimental data. The one we will use is

\[
 f \left( R_e, \frac{e}{D_h} \right) = 4 \cdot 1.375 \cdot 10^{-3} \left( 1 + 21.544 \cdot \frac{2e}{D_h} + \frac{100}{R} \right)^3.
\]

6.1.2 Fluid Properties

The matter of fluid properties includes calculation of density from the equation of state, but also calculation of thermal conductivity \( k \), specific heat \( c_p \) and viscosity \( \mu \). Various derivatives of the equation of state are also needed, such as

\[
 \left( \frac{\partial T}{\partial p} \right)_i \quad \text{and} \quad \left( \frac{\partial \rho}{\partial p} \right)_i. \tag{6.1}
\]

Moreover we must be able to calculate the internal energy as a function of temperature and pressure in order to get the inlet boundary condition for the internal energy.

Helium

In the normal case of the fluid being helium, we use the software Hepak, developed by Cryodata Inc, a company of former NIST\(^5\) employees located in Colorado. It calculates the thermophysical properties of helium from fundamental state equations which are valid for temperatures in between 0.8 K – 1500 K and for pressures up to 20000 bar [10].

The accuracy of the equation of state between temperature, density and pressure in the region we are interested in is about 0.2 % [10]. When calculating properties derived from this equation using differentiation and integration the accuracy is typically lowered by an order of magnitude. These properties include internal energy, which means that when calculating

\(^5\)National Institute of Standards and Technology.
density as a function of internal energy and pressure in eq (5.1d) we will have an accuracy of about 2 % which also goes for the specific heat. This leads to that for the properties in eq (6.1) (which will be estimated by simple linear interpolation of the values from Hepak) we will have a rather poor accuracy of about 20 %. This is however not a problem; as we will see, thermal conduction plays a negligible part in the energy equation and as for the parameter $K$, defined in eq (5.17), it is only needed to be tentatively known (for the same reason that we could make crude approximations when deriving the pressure correction equation). As for transport properties such as viscosity and thermal conductivity, an accuracy of about 10 % can be expected.

In the so called critical region close to the critical point, the accuracy is much less then what is given above; for some properties it can not be estimated. Thus simulations in this region should be avoided. The critical region is defined [10] as the state points within 15 % of the critical density, $\rho_c = 69.64$ kg/m$^3$, and within 2 % of the critical temperature, $T_c = 5.1953$ K.

**Neon**

As we will see in chapter 7, we will be interested in another fluid besides helium, namely neon. The properties of neon can be calculated using the software Gaspak, also developed by Cryodata. It calculates the properties of neon for temperatures and pressures from the triple point (24.55 K, 0.4335 bar) and melting line up to temperatures of 1300 K and pressures of 1000 bar [14].

The accuracy in the region we will be interested in is about 5 % when calculating the equation of state from the internal energy [14]. As for the specific heat an accuracy of 5 – 10 % can be expected. For transport properties such as thermal conductivity and viscosity the accuracy is somewhat less than for thermodynamical properties.

### 6.2 Validation

We will now validate the functionality of the program and make a comparison with experimental results.

#### 6.2.1 Functionality

To validate the functionality of the program is to see how well it completes its task; to solve eq (5.1a) – (5.1d) and fulfil the specified boundary conditions (5.1e) – (5.1g). The case of steady state is what we are mainly interested in and it is also easiest to validate, so it is what we now consider. One can then integrate eq (5.1a) – (5.1c) and get:

\[
\int_0^x (eq.5.1a) \Rightarrow \rho u \left( \frac{u^2}{2} + i + \frac{p}{\rho} + gz \right) - k \frac{\partial T}{\partial x} - \frac{q}{A} \chi = \text{constant} \tag{6.2a}
\]

\[
\int_0^x (eq.5.1b) \Rightarrow \rho u^2 + p + \int_0^x \left( \frac{fp|u|}{2D_h} + g\rho \sin \phi \right) dx = \text{constant} \tag{6.2b}
\]

\[
\int_0^x (eq.5.1c) \Rightarrow \rho u = \text{constant} \tag{6.2c}
\]
How well eq (6.2a) – (6.2c) are satisfied is easy to check (the integrals can be calculated using simple step functions and the temperature gradient can be approximated in a linear way). The last equation we need to check is eq (5.1d); the equation of state. This is done by simply comparing the density at a given point \( x \) with the density one gets when calculating Hepak’s density as a function of the internal energy and pressure at that point, i.e. how well the following equation is satisfied:

\[
\rho(x) = \rho^{Hepak}(i(x), \rho(x))
\] (6.3)

Several simulations have been made to check eq (6.2a) – (6.2c), (6.3) and that the boundary conditions are fulfilled. The relative errors in these equations depend on the simulation; how much the different fields vary, what the resolution of the grid points is etc. What we found was that when the resolution of the grid was increased, the relative errors went to zero. This shows that the program succeeds in finding a numerical solution to the equations of our mathematical model; eq (5.1a) – (5.1d) with boundary conditions (5.1e) – (5.1g).

6.2.2 Steady State Comparison with Experimental Results

As mentioned before, there has been previous work made in theoretical modelling and experiments for the beam screen cooling system [4]. We will use the experimental data from that work to make a comparison with our numerical program. The setup of the experiments is shown in fig 6.1. As can be seen the pipe is not straight but goes in coils which leads to a higher pressure drop over the pipe compared to a straight pipe. This can be taken into account by modifying the friction factor \( f \) according to [15]

\[
f_c = f_s + 0.03 \sqrt{\frac{d}{D}},
\] (6.4)

where \( f_c \) denotes the friction factor for the coiled pipe and \( f_s \) the one for a straight pipe. Moreover, \( d \) is the diameter of the pipe and \( D \) is the diameter of the coils. The total length of the pipe is 53 m, the diameter of the pipe is 3.7 mm and the diameter of the coils is 0.45 m. The slope of the pipe corresponds to an angle of 0.80 degrees. The pipe is made of the same type of stainless steel used in the beam screen cooling loop which has a relative roughness of 2.1·10^{-4}.

We will now compare the results given by our numerical program with the experimental data for a number of experiments with various inlet temperatures and pressures, various mass

\[\text{Figure 6.1 Setup of the experiments used for a comparison with our numerical program.}\]
As can be seen, the results given by the numerical program agree quite well with experimental data in the case of an inlet temperature of 6.5 K, although the deviation seems to increase for higher mass flows. For the lower inlet temperature of 5 K, we have a worse agreement with experimental data. This may be due to several reasons, of which inaccuracy of the data given by Hepak is perhaps the explanation closest at hand, since an inlet temperature of 5 K brings us nearer to the critical region. However, this does not seem to be the case, as when studying the results in detail one sees that we in fact are outside of the critical region defined in section 6.1.2. Other explanations may be inaccuracy in the formula for the modification of the friction factor due to the coils (6.4) and uncertainty in experimental parameters; the pressure drop is for example very sensitive to variations in the diameter of the pipe (for example; for the numerical program to yield the same pressure drop as experiments for a mass flow of 1.0 g/s in fig 6.2a, the pipe diameter needs to be changed from 3.7 mm to about 3.9 mm). To tell how
accurate our one-dimensional model is we would need to make comparisons with more experimental data, preferably for straight pipes.

6.2.3 Prediction of Thermo-Hydraulic Instabilities

As said when defining the problem of the thesis in chapter 3, there is an interest to see how well a numerical program can detect thermo-hydraulic instabilities (as described in section 2.4.3). In fig 6.4 the experimental results for when instabilities occur are given [7]. These were simply found by varying the mass flow and total heat load on the pipe (recall that increased heat load and/or decreased mass flow lead to instabilities).

It was uncertain how the numerical program would respond when carrying out simulations under conditions for which one expects instabilities. Our first approach was simply to solve the steady state equations. Doing this we found that for certain configurations of the mass flow/heat load the program was unable to find a steady solution; it simply could not converge. This thus gave a way to tell stable configurations (the program could find a steady state solution) from unstable ones (they program could not find a steady state solution) and make a comparison with the experimental results. This is also shown in fig 6.4.

As can be seen the region in which the program functions is just a subset of the region which in experiments gives a stable flow. Even though, of course, a more thorough investigation of this matter is necessary to make any general conclusions, it seems that if the program can find a steady state solution, then we get stable flow in experiments as well.

Another way to tell when instabilities occur is to make an unsteady simulation and see if some sort of pressure – density oscillations could be simulated in time. We hoped that this would give a better agreement with the experimental results in fig 6.4. However, we found that the same conditions that made it impossible to find a steady solution, also led to that the program did not function in unsteady simulations; as with the steady simulation, the program would simply not converge. Thus, unsteady simulations do not improve the ability of the program to predict instabilities.

![Figure 6.4](image)  
**Figure 6.4** Diagram of what values for mass flow and total heat $Q$ that lead to instabilities (both experimental data and values given by control theoretical modelling) [7]. The region in which the numerical program functions is to the left of the limit in the figure.
In this chapter we will use the numerical program to compare different options for the upgrade of the LHC beam screen cooling loop.

7.1 Upgrade Options

In section 3.2 the limitations for how the beam screen cooling loop may be upgraded were given. We shall now see what different changes in the cooling loop we will study in this chapter.

**Cooling medium**

The two cooling mediums we will consider is helium, which is used at present, and neon. Why the latter is of interest is discussed in section 7.2 below.

**Capillary shape**

At present the cooling capillaries are circular tubes with a diameter of 3.7 mm. To see how a change in shape affects the heat extraction capacity we will also consider an ellipsoidal shape of the capillaries, where the vertical diameter remains 3.7 mm and the horizontal diameter is doubled, i.e. 7.4 mm. This is shown in fig 7.1. As said in chapter 3 one prefers to keep the capillaries the way they are at present, but there is still an interest to see what effect a change of shape would have on the heat extraction capacity.

**Pressure of supply header**

A larger difference in pressure between the supply and return header will lead to a higher mass flow and thereby a higher heat extraction capacity. Since the pressure of the return header is already at its lowest, an increase in pressure difference can only be achieved by increasing the pressure of the supply header. As said in section 3.2 the upper limit for this is 19 bar.

**Maximum temperature of beam screen**

At present the maximum temperature of the beam screen is 20 K. We will consider an

![Figure 7.1](image.png)  
**Figure 7.1** Cooling capillaries with a circular (a) and ellipsoidal (b) cross section.
increase in this temperature up to 30 K.

7.2 Neon as Cooling Medium

Liquid neon is nowadays finding important applications as an economical cryogenic refrigerant. Due to its high density it has a refrigerating capacity per unit volume which is 40 times that of liquid helium \([16]\). There is therefore an interest to see if it is possible to exchange the helium in the beam screen cooling loop for neon.

However, using neon as the cooling medium implies certain limitations in the pressure and most of all temperature range. This can be seen by looking at the phase diagram of neon in fig 7.2.

First of all we can note that, unlike helium, we will not be interested in supercritical neon; the critical point of neon is way out of the pressure and temperature range that we may use for the cooling loop. Instead we will look at liquid neon. The specific region we are interested in is given in fig 7.2 and is limited by the following facts: since we must stay above the melting line, the minimum temperature we may use is about 25 K. Furthermore, to avoid two phase flow, we must not cross the saturation line, which implies the maximum temperature to be a bit below 31 K; this is however not a limitation since we are anyway only interested in a maximum temperature of 30 K. Moreover, in order to not cross the saturation line, the pressure must not be lower than about 2.5 bar.

Thus when using neon, the temperature range is from 25 to 30 K and the pressure range is from 2.5 to 19 bar.

![Figure 7.2 Phase diagram of neon (calculated by Gaspak, see section 6.1.2) with region of interest for this application, including a typical working line for the beam screen cooling loop.](image-url)
7.3 Results

We start by recalling the heat loads on the beam screen in the two different upgrade scenarios accounted for in section 2.5.2 compared to the heat loads of the present LHC. They are summarized in table 7.1 and tell us how high heat loads we are interested in, i.e. up to 18 W/m.

The simulations were carried out according to the outline in section 7.1: for each of the two cooling mediums, two different shapes (given in fig 7.1) and two different maximum temperatures of the beam screen were considered. The results are shown in fig 7.3, where for each specific heat load that one wants to extract from the beam screen, the required pressure for the supply header is given.

For these simulations we assumed one third of the pressure drop to be over the control valve after the cooling capillaries, the reason for this was given in section 2.4.2. Relating back to fig 2.7, we further assumed the temperature and pressure at the inlet of the cooling capillaries, i.e. at point 2, to be the same as for the supply header, i.e. point 1. As said in section 2.4.2, this is not true since we have heating from the supports between point 1 and 2, but in the cases we are interested in here, the mass flows are high enough to make this difference negligible. Last, we assume zero gravity; even though the LHC tunnel is inclined, the gravitational effects are rather small and may be neglected for our purposes.

After now having presented the simulation results, we can make a quantitative comparison of the different upgrade options of the beam screen cooling loop. As previously said, we are mainly interested in upgrades where the capillary shape is preserved, i.e. what we have in fig 7.3a and c. Thus this is what we consider to begin with.

If the cooling medium remains helium, it is, as we see in fig 7.3a, possible to keep the shape of the capillaries and also the maximum temperature of 20 K and still extract the heat loads required for both upgrade scenarios in table 7.1. This would require the supply header to have a pressure of up to 11 bar. Increasing the maximum temperature to 30 K only make the required pressure about 2 bar less, i.e. 9 bar; a quite small change for a big change in the maximum temperature.

As we can see in fig 7.3c, changing the cooling medium to neon would mean a significant decrease in the heat extraction capacity. It also requires a high maximum temperature of 29 – 30 K, where the higher one is required if one wants to extract heat loads higher than 9 W/m. Even then, the maximum heat load extractable is only about 11 W/m. For heat loads above this, a change in the capillary shape is necessary (a significant increase in maximum temperature can not be made, since we then would get two phase flow\(^6\), see section 7.2).

<table>
<thead>
<tr>
<th>Scenario:</th>
<th>No upgrades</th>
<th>Bunched beam</th>
<th>Super-bunch</th>
</tr>
</thead>
<tbody>
<tr>
<td>Heat load [W/m]:</td>
<td>1.71</td>
<td>17.3</td>
<td>7.16</td>
</tr>
</tbody>
</table>

Table 7.1 Summary of the total heat loads on the beam screen with the present setup and for the two upgrade scenarios.

\(^6\)There have however been investigations (not related to CERN) \(17\) \(18\) made into the possibility to work on the saturation line when using pipe flow cooling. This might increase the heat extraction capacity when using neon.
Figure 7.3 Required pressure of supply header as a function of heat load to be extracted by the beam screen for four different upgrade options: helium as cooling medium with circular capillaries (a) and ellipsoidal (b); neon as cooling medium with circular capillaries (c) and ellipsoidal (d).
Fig 7.3b and fig 7.3d make it clear that changing the shape to an ellipsoidal one with twice the horizontal diameter makes a great change in the heat extraction capacity. Using neon, it would for such shape be possible to extract heat loads up to 18 W/m, i.e. sufficient for the most demanding upgrade scenario. This would require the supply header to have a pressure of about 10 – 14 bar, depending on the maximum temperature. As for helium, we have already seen in fig 7.3a that a change in capillary shape is not necessary, even though we in fig 7.3b see that it would make the required pressure much lower.

7.4 Conclusion

Our first conclusion is that in order to extract heat loads as high as in both upgrade scenarios, the inlet pressure must go up if no change in capillary shape is made. This is the case for both coolants considered. Further we can conclude that keeping helium as the cooling medium seems to be the best option; it makes it possible to extract high enough heat loads for the most demanding upgrade scenarios of the LHC and still keep the shape of the capillaries. If a maximum temperature of 20 K is kept, it is only necessary to increase the pressure level to about 5 – 11 bar, where the lower value corresponds to the super-bunch scenario and the higher to the scenario with a bunched beam. If the maximum temperature is raised to 30 K the required pressures would be about 4 – 9 bar.

Concerning pressure wave instabilities, it is necessary to do further investigations to see if they would occur for these upgrades (when keeping helium). The fact that the numerical program encountered no problem when carrying out the simulations points at that no instabilities will occur. Another argument is that raising the pressure level of the supply header will bring us further away from the critical point, thus making the flow more stable.
8 Conclusion and Future Work

In this last chapter we will summarize the outcome of the study. A brief look at possible future work will also be made.

8.1 Conclusion

The primary objective of this thesis was to quantify different ways to upgrade the LHC beam screen cooling system, so that higher heat loads can be handled in the future. The secondary objective was to do this by constructing a numerical program for compressible pipe flow.

The approach was to use a one-dimensional mathematical model for pipe flow, where empirical correlations were used to calculate frictional forces. The partial differential equations of the model were numerically solved by using the SIMPLE algorithm. To calculate fluid properties the software packages Hepak and Gaspak were used.

The functionality of the numerical program was validated by making sure that it indeed gave an accurate solution to the system of partial differential equations. When comparing with experiments, the results showed good agreement when we are far away from the critical region and the mass flow is low. The deviations are larger for higher mass flows, especially when we are close to the critical region. This suggests that further comparisons with experiments need to be done to validate the mathematical model used and the accuracy of the software used to calculate fluid properties.

When working with supercritical pipe flow close to the critical point of the fluid, thermo-hydraulic instabilities may arise. Of special interest for this thesis was to see how well a numerical program would function under such conditions and if it could be used to predict the occurrence of such phenomena. It was found that the program could not converge to a solution when simulating under conditions for which instabilities could be expected. By comparing with experimental results it was found that the conditions for which the program functioned made up a subset of the set of conditions which gave stable flow in experiments, i.e. the numerical program was more sensitive to unstable conditions than what reality is.

When using the numerical program to compare different upgrade options of the beam screen cooling loop, it was found that regardless of what coolant that is used, an increase in pressure is necessary, if no change in capillary shape is made. At the same time, such an increase in pressure makes that the problems with thermo-hydraulic instabilities will be avoided. Of the two coolants, helium gave the best heat extraction capacity. An increase in the temperature range made the required pressure lower, although the difference was marginal, compared to the change in temperature range. The main reason that neon is not as good an option as helium is due to limitations of the region of the phase diagram in which one can work in single phase.

8.2 Future Work

Concerning the upgrade of the LHC beam screen cooling system, possible further work includes experimental testing of different fluids, capillary shapes etc. It could also be
interesting to see if it is possible to improve the use of neon by working on the saturation line, as has been investigated in [17] [18].

As mentioned in section 8.1 above, there is a need for further comparison of the numerical program constructed in this work with experimental data. Preferably such an experimental setup would contain straight pipes and the possibility to test many different flow conditions.

To study the dynamic behaviour of the cooling loop, it would be interesting to have the numerical program interacting with a control unit for the control valve at the end of the loop.
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