Examensarbete

On Poincaré’s Uniformization Theorem

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A compact Riemann surface can be realized as a quotient space $U/\Gamma$, where $U$ is the sphere $\Sigma$, the euclidian plane $\mathbb{C}$ or the hyperbolic plane $\mathbb{H}$ and $\Gamma$ is a discrete group of automorphisms. This induces a covering $p : U \to U/\Gamma$.

For each $\Gamma$ acting on $\mathbb{H}$ we have a polygon $P$ such that $\mathbb{H}$ is tesselated by $P$ under the actions of the elements of $\Gamma$. On the other hand if $P$ is a hyperbolic polygon with a side pairing satisfying certain conditions, then the group $\Gamma$ generated by the side pairing is discrete and $P$ tesselates $\mathbb{H}$ under $\Gamma$. 

**Nyckelord**

Hyperbolic plane, automorphism, Fuchsian group, Riemann surface, covering, branched covering, orbifold, uniformization, fundamental domain, Poincaré's theorem
Abstract

A compact Riemann surface can be realized as a quotient space $\mathcal{U}/\Gamma$, where $\mathcal{U}$ is the sphere $\Sigma$, the euclidian plane $\mathbb{C}$ or the hyperbolic plane $\mathcal{H}$ and $\Gamma$ is a discrete group of automorphisms. This induces a covering $p: \mathcal{U} \rightarrow \mathcal{U}/\Gamma$.

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Chapter 1

Introduction

1.1 Background

The study of Riemann surfaces begun in the 19th century when Riemann introduced them in his doctoral dissertation *Foundations for a general theory of functions of a complex variable*.

An important perspective in the study of Riemann surfaces is the concept of uniformization, which was developed by Poincaré, Klein and others. The theory states that every closed orientable Riemann surface admits a Riemann metric of constant curvature. The main geometries are the sphere, the Euclidian plane and the hyperbolic plane with curvature 1, 0 and $-1$ respectively.

Poincaré discovered that groups of Möbius transformations preserved the structures of the non-Euclidian geometries. The Riemann surfaces can be realized as quotient spaces $U/\Gamma$ where $U$ is either the sphere, the Euclidian plane or the hyperbolic plane and $\Gamma$ is a discrete group of structure preserving Möbius transformations.

Poincaré’s classical theorem states that a hyperbolic polygon satisfying certain conditions is a fundamental domain for a Fuchsian group acting on the hyperbolic plane. Poincaré generalized the theorem to polyhedra in 3-dimensional hyperbolic space where the groups are Kleinian rather that Fuchsian. The 3-dimensional manifolds are much more complicated than surfaces, but Thurston showed a connection between the 3-dimensional case and the 2-dimensional. There are still open questions in the 3-dimensional case.

Here we will look at the 2-dimensional case. Several proofs of Poincaré’s theorem have been published, but their validity has been questioned. We will limit ourselves to the case where the polygon is a triangle. The discrete groups associated with triangles, known as triangle groups, where introduced

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by Schwarz.

1.2 Chapter outline

Chapter 2 Here we introduce the concepts of the non-euclidian geometries, in particular the hyperbolic plane as well as the automorphism groups of them. We also construct Riemann surfaces and the functions defined on them.

Chapter 3 In our study of surfaces we need some basic topological tools. We look at coverings and homotopy and how homotopy classes of paths are related to coverings.

Chapter 4 In chapter 3 we only used the topological structures of surfaces. In order to include the complex structures of Riemann surfaces we need to introduce the concept of orbifolds and its relation to Riemann surfaces. We will also look at the concept of uniformization for Riemann surfaces.

Chapter 5 Here we construct the fundamental domains for Fuchsian groups and look at the quotient space $\mathcal{H}/\Gamma$. Lastly we will look at Poincaré's classical theorem for polygons and make a complete proof for hyperbolic triangles.

The theory of Riemann surfaces, Möbius transformations and Fuchsian is found in [1], [2] and [8]. The coverings are found in [3] and branched coverings and orbifolds are based on [3] and [6]. Fundamental domains are found in [1], [2] and the proof of Poincaré's theorem is based on the ideas used in [1], [5] and [7].
First we need to construct the non-euclidian geometries we need for our later work. We will also look at Möbius transformations which will play an important role.

2.1 The Extended Complex Plane

We begin with the extended complex plane which will be our model for the spherical geometry. To better understand the extended complex plane and Möbius transformations we will start with constructing what is called The Riemann Sphere. We do this by stereographic projection.

2.1.1 The Riemann Sphere

Consider the unit sphere in \( \mathbb{R}^3 \) given by

\[
x_1^2 + x_2^2 + x_3^2 = 1.
\] (2.1)

We want to associate a point on the sphere with a point in the complex plane \( \mathbb{C} \). Think of \( \mathbb{C} \) as the plane given by \( x_3 = 0 \). Now take the line that passes through \( N = (0, 0, 1) \) and a point \( z \in \mathbb{C} \). This line intersects the sphere at a single point \( \zhat \), which is the stereographic projection of \( z \).

Now if \( z = x + iy \) and \( \zhat = (x_1, x_2, x_3) \) then we have

\[
x_1 = \frac{2x}{|z|^2 + 1}, \quad x_2 = \frac{2y}{|z|^2 + 1}, \quad x_3 = \frac{|z|^2 - 1}{|z|^2 + 1} \tag{2.2}
\]

and

\[
x = \frac{x_1}{1 - x_3}, \quad y = \frac{x_2}{1 - x_3}. \tag{2.3}
\]
Chapter 2. General Setting

Now we look at a nice property of the Riemann sphere. Lines and circles in \( \mathbb{C} \) are given by an equation of the form

\[
a(x^2 + y^2) + bx + cy + d = 0. \tag{2.4}
\]

With the formulas 2.1 and 2.3 we can rewrite this as

\[
bx_1 + cx_2 + (a - d)x_3 + a + d = 0. \tag{2.5}
\]

This is the equation of a plane in \( \mathbb{R}^3 \) and the intersection of a plane and a sphere is a circle. This means that all lines and circles in \( \mathbb{C} \) are projected to circles on the sphere. Note that the lines are projected to circles through \( N \).

With this in mind we let \( \Sigma = \mathbb{C} \cup \{ \infty \} \), known as the extended complex plane and we project \( \infty \) on \( N \).

With stereographic projection and the Riemann sphere we will think of \( \Sigma \) as a sphere and lines as circles through \( \infty \), or \( N \). This will make some of the properties of Möbius Transformations easier to understand.

### 2.1.2 Möbius Transformations

A Möbius Transformation on \( \Sigma \) is a function of the form

\[
T(z) = \frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0 \tag{2.6}
\]

where that last requirement is to make sure \( T \) is not constant. We also set \( T(\infty) = a/c \) and \( T(-d/c) = \infty \) if \( c \neq 0 \) else \( T(\infty) = \infty \).

All Möbius Transformations can be written as a composition of

(i) translations \( z \mapsto z + c \ \ c \in \mathbb{C} \)

(ii) rotations \( z \mapsto e^{i\phi}z \ \ 0 \leq \phi \leq 2\pi \)
(iii) magnifications \( z \mapsto \lambda z \quad \lambda \in \mathbb{R}^+ \)

(iv) inversion \( z \mapsto \frac{1}{z} \)

**Example 1**
Let \( T(z) = 2 - z + i \). Let \( T_0(z) = e^{i\pi}z = -z \) then \( T(z) = \frac{2}{T_0(z) + i} \).
Similarly let \( T_1(z) = z + i \), \( T_2(z) = \frac{1}{z} \) and \( T_3(z) = 2z \) then we see that \( T(z) = T_3(T_2(T_1(T_0(z)))) = T_3T_2T_1T_0(z) \).

It’s easy to see that all translations, rotations, magnifications and inversion map \( \Sigma \) bijectively to itself. Remembering that lines are circles on \( \Sigma \) we also see that all those transform circles to circles.

**Example 2**
Let \( L = \{ z = x + iy \in \Sigma \mid ax + by = c \neq 0 \} \), that is a line not passing through origo, then
\[
\begin{align*}
   w = \frac{1}{z} & \iff z = x + iy = \frac{1}{w} = \frac{1}{u + iv} \\
   c = ax + by & = \frac{au}{u^2 + v^2} + \frac{-bv}{u^2 + v^2} \iff u^2 + v^2 - \frac{a}{c}u + \frac{b}{c}v = 0,
\end{align*}
\]
where the last equation gives a circle.

As we know from geometry three noncollinear points uniquely determines a circle. If they’re collinear they uniquely determines a line. We can map any circle on \( \Sigma \) to any other circle. To see this we choose three distinct points \( z_1, z_2, z_3 \in \Sigma \). If all points are finite, let
\[
T(z) = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} \quad (2.7)
\]
which satisfies \( T(z_1) = 0 \), \( T(z_2) = 1 \), \( T(z_3) = \infty \). If any of the \( z_i \) is \( \infty \) then we get the same result by removing the factors containing that point.
If \( w_1, w_2, w_3 \in \Sigma \) and and \( S \) is a Möbius transformation such that \( S(w_1) = 0 \), \( S(w_2) = 1 \), \( S(w_3) = \infty \) we see that \( S^{-1}T \) maps \( z_i \) to \( w_i \) for each \( i \).
Assume that \( U \) also maps \( z_1, z_2, z_3 \) to \( 0, 1, \infty \) respectively then \( UT^{-1} \) fixes \( 0, 1, \infty \). Now let \( UT^{-1}(z) = \frac{az + b}{cz + d} \). Solving \( UT^{-1}(z_i) = z_i \) for \( i = 1, 2, 3 \) gives \( a = d \neq 0, b = c = 0 \), that is \( UT^{-1} = Id \) and \( U = T \). So \( T \) is unique. It follows that there’s a unique transformation mapping \( z_i \) to \( w_i \) \( i = 1, 2, 3 \).

**Example 3**
The Möbius Transformation mapping \( \mathbb{R} \cup \{ \infty \} \) to the unit circle with \( T(0) = -1, T(1) = -i, T(\infty) = 1 \) is given by
\[
T(z) = \frac{z - i}{z + i}. \quad (2.8)
\]
We also note that the upper half plane is mapped to the interior of the circle.
We want to work with the Möbius transformations as a group. We do this by representing a transformation $T$ with a matrix $A$. The general linear group $GL(2, \mathbb{C})$ is the group of invertible 2x2-matrices. Now if $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$, we let $T_A(z) = \frac{az + b}{cz + d}$. We see that for $k \neq 0$, $T_{kA} = T_A$. So we can always choose $a, b, c, d \in \mathbb{C}$ such that $ad - bc = 1$. This restriction gives us the special linear group $SL(2, \mathbb{C})$. We also note that if $A \in SL(2, \mathbb{C})$ then $-A \in SL(2, \mathbb{C})$ and $T_{-A} = T_A$ as seen before. Identifying $A$ and $-A$ gives us the special projective group $PSL(2, \mathbb{C})$. Unless other noted, we will think of Möbius transformations as elements of $PSL(2, \mathbb{C})$.

Two elements $g_1, g_2$ of a group $G$ are called conjugate if there exists $h \in G$ such that $g_2 = hg_1h^{-1}$. Conjugacy is an equivalence relation and the equivalence classes are called conjugacy classes. It will be useful to divide $PSL(2, \mathbb{C})$ into conjugacy classes but first we need to look at a particular property of elements is $PSL(2, \mathbb{C})$. A fixed point of an element $T$ is a point $z_0$ such that $T(z_0) = z_0$. We see that $S(z_0)$ is a fixed point of $STS^{-1}$, this means that conjugate elements of $PSL(2, \mathbb{C})$ have the same number of fixed points. The conjugacy classes are distinguished by their fixed points. We have seen that only the identity has three (or more) fixed points. Now let $T = (az + b)/(cz + d) \in PSL(2, \mathbb{C})$. If $c \neq 0$ then $z \in \mathbb{C}$ is a fixed point of $T$ if and only if

$$cz^2 + (d - a)z - b = 0$$

which has two solutions unless $(a + d)^2 = 4$. If $c = 0$ then $\infty$ is a fixed point of $T$ and $ad = 1$ which gives $T(z) = az + ab$. There is a second fixed point if and only if $a^2 \neq 1$, that is $(a + d)^2 \neq 4$. Let $\pm A \in SL(2, \mathbb{C})$ be the pair of matrices associated with $T$. The trace $tr(A) = a + d$ is invariant under conjugation and $tr(-A) = -tr(A)$ so $tr^2(\pm A) = (a + d)^2$ depends only on the conjugacy class of $T$.

**Theorem 4** Two Möbius transformations $T_1, T_2 \neq Id$ are conjugate if and only if $tr^2(T_1) = tr^2(T_2)$.

With those results we divide $PSL(2, \mathbb{C})$ in classes. Let $T \in PSL(2, \mathbb{C})$, then $T$ is

(i) parabolic if $(a + d)^2 = 4$. $T$ has one fixed point and is conjugate to $z + 1$

(ii) hyperbolic if $(a + d)^2 > 4$. $T$ has two fixed points and is conjugate to $\lambda z$ ($|\lambda| \neq 1, \lambda \in \mathbb{R}$)

(iii) elliptic $0 \leq (a + d)^2 < 4$. $T$ has two fixed points and is conjugate to $\lambda z$ ($\lambda = e^{i\theta}$)
(iv) loxodromic if \((a + d^2 < 0)\) or \((a + d)^2 \notin \mathbb{R}\). \(T\) has two fixed points and is conjugate to \(\lambda z (|\lambda| \neq 1, \lambda \notin \mathbb{R})\)

## 2.2 Surfaces

Before we look at some additional properties of Möbius transformations, we have to look at surfaces and functions on surfaces. First we need some types of functions on \(\mathbb{C}\).

### 2.2.1 Riemann Surfaces

Let \(\gamma_1, \gamma_2\) be two paths \([0, 1] \rightarrow \mathbb{C}\) passing through a point \(p \in \mathbb{C}\). Let \(\theta\) be the angle between the tangents of \(\gamma_1, \gamma_2\) at \(p\). We say that \(\gamma_1, \gamma_2\) intersect with angle \(\theta\) at \(p\). A function \(f\) is called conformal at \(p\) if \(f(\gamma_1), f(\gamma_2)\) intersects with angle \(\theta\) at \(f(p)\).

We call a function \(f : U \rightarrow \mathbb{C}\), where \(U \subset \mathbb{C}\), analytic on \(U\) if \(f'(z)\) exists for all \(z \in U\).

When working with Riemann surfaces we will encounter singularities. There are three types of singularities. Let \(f : U \rightarrow \mathbb{C}\) be analytic on \(U \setminus \{p\}\) where \(U \subset \mathbb{C}\).

(i) \(p\) is called a removable singularity if \(f\) is bounded in a neighbourhood of \(p\). The Laurent series \(f(z) = \sum_n c_n(z - p)^n\) has no terms with negative exponent.

(ii) \(p\) is called a pole if \(f \rightarrow \infty\) as \(z \rightarrow p\). The Laurent series \(f(z) = \sum_n c_n(z - p)^n\) has a finite number of terms with negative exponents.

(iii) Lastly \(p\) is called an essential singularity if \(f\) has no limit as \(z \rightarrow p\). The Laurent series \(f(z) = \sum_n c_n(z - p)^n\) has infinite number of terms with negative exponents.

Now we look at the concept of a surface. Let \(S\) be a Hausdorff space. \(S\) is called a surface if each \(s \in S\) has an open neighbourhood \(U_i\) such that there exists a homeomorphism \(\Phi_i : U_i \rightarrow V_i\), where \(V_i \subset \mathbb{C}\) is open. We call \((\Phi_i, U_i)\) a chart. An atlas is a set of charts \(A\) such that for each \(s \in S\) there exists \((\Phi_i, U_i)\) where \(s \in U_i\). Now let \((\Phi_i, U_i), (\Phi_j, U_j)\) be two charts at \(s \in S\) then the functions

\[
\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \rightarrow \Phi_i(U_i \cap U_j)
\]
are called \textit{transition functions}. We call an atlas analytic where all transition functions are analytic. We call two analytic atlases $A, B$ \textit{compatible} if all the transition functions of charts $(\Phi, U) \in A, (\Psi, V) \in B$ are analytic. Such atlases form an equivalence class called a \textit{complex structure}. A surface with a complex structure is called a \textit{Riemann surface}.

**Example 5** $\mathbb{C}$ with the single chart $(\text{Id} : \mathbb{C} \to \mathbb{C}, \mathbb{C})$.

**Example 6** $\Sigma$, with the two charts $(\text{Id}, \mathbb{C})$, $(1/z : \Sigma \setminus \{0\} \to \mathbb{C}, \mathbb{C})$. We see that transition function $f : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}, z \mapsto 1/z$ is analytic.

**Example 7** $\Sigma$, with the two charts $(w_1, \Sigma \setminus \{\infty\}), (w_2, \Sigma \setminus \{0\})$ where $w_1 = z^n$ and $w_2 = (1/z)^n$. The transition function $f : z \mapsto 1/z$ is analytic on $\mathbb{C} \setminus \{0\}$. Further we see that those charts are not compatible with the charts in example 6 thus provides $\Sigma$ with a different complex structure.

Now we will define some function types on Riemann surfaces. Let $S$ be a Riemann surface. A function $f : S \to \mathbb{C}$ is called analytic if the function $f \circ \Phi^{-1} : \Phi(U) \to \mathbb{C}$ is analytic for all charts $\Phi$ on $S$. Now let $S_1$ and $S_2$ be Riemann surfaces and $\Phi_1$ and $\Phi_2$ be charts of respective surface. Then a function $f : S_1 \to S_2$ is called \textit{holomorphic} if the function

$$\Phi_2 \circ f \circ \Phi_1^{-1} : \Phi_1(U_1 \cap f^{-1}(U_2)) \to \mathbb{C} \quad (2.9)$$

is analytic whenever $U_1 \cap f^{-1}(U_2) \neq \emptyset$. In a similar way we call a function $f : S_1 \to S_2$ conformal if the function in equation 2.9 is conformal.

**Example 8** The Möbius transformations are holomorphic.

**Example 9** When we work with $\Sigma$ we will study functions at $\infty$. A function $f : \Sigma \to \Sigma$ is conformal at $z = \infty$ if $f(\infty) \neq \infty$ and $f(1/z)$ is conformal at $z = 0$. If $f(\infty) = \infty$ then $f$ is conformal at $z = \infty$ if $1/f(1/z)$ is conformal at $z = 0$.

Let $f : S \to \mathbb{C}$ be a holomorphic function, where $S$ is a Riemann surface. A singular point is a point $p \in S$ such that the point $\Phi(p)$ is a singularity to the function $f \circ \Phi^{-1} : \Phi(U) \to \mathbb{C}$. We say that $p$ is a singularity of the same type as $\Phi(p)$ is for $f \circ \Phi^{-1}$. Similarly $f : S_1 \to S_2$ has a singularity at $p$ if the function in 2.9 has a singularity at $\Phi_1(p)$. If $f$ only has non-essential singularities we can replace $\mathbb{C}$ with $\Sigma$. In this case we call $f$ meromorphic.

**Example 10** The Möbius transformations are meromorphic.

More generally, a structure preserving map $f : X \to Y$ is called a morphism.
2.2. Surfaces

2.2.2 Automorphisms

Let $S_1, S_2$ be two surfaces. A function $f : S_1 \to S_2$ is called a homeomorphism if $f$ is bijective, continuous and $f^{-1}$ is continuous. $f$ is called a local homeomorphism if it is open and continuous. Let $S$ be a surface, then a conformal homeomorphism $f : S \to S$ is called an automorphism. We shall denote the group of automorphisms of a surface $S$ as $Aut S$.

**Example 11** The automorphisms on $\mathbb{C}$ are the functions $az + b$ where $a, b \in \mathbb{C}, a \neq 0$.

Now we want to show that the Möbius transformations are the automorphisms of $\Sigma$. We show that they are conformal by using the following theorem.

**Theorem 12** An holomorphic function $f$ is conformal at every point $z_0$ for which $f'(z_0) \neq 0$.

**Theorem 13** All Möbius transformations are conformal on $\Sigma$.

**Proof.** Let $T$ be a Möbius transformation, then $T$ is conformal since

$$T'(z) = \frac{ad - bc}{cz + d} 
eq 0 \quad (2.10)$$

for $z \neq -d/c, \infty$. The remain cases are:

(i) $z = \infty$, $T(z) \neq \infty$. In this case we have $c \neq 0$

$$S(z) = T(1/z) = \frac{a + bz}{c + dz}$$

$$S'(0) = -\frac{1}{c^2} \neq 0$$

$S$ is conformal at $z = 0$ so $T$ is conformal at $z = \infty$

(ii) $z = \infty$, $T(z) = \infty$. In this case we have $c = 0$ and thus $a \neq 0$

$$U(z) = 1/T(1/z) = \frac{c + dz}{a + bz}$$

$$U'(0) = \frac{1}{a^2} \neq 0$$
$U$ is conformal at $z = \infty$ so $T$ is conformal at $z = \infty$

(iii) $z = -d/c \ (\neq \infty)$. In this case we have $c \neq 0$

$$V(z) = 1/T(z) = \frac{cz + d}{az + b}$$

$$V'(-d/c) = -c^2 \neq 0$$

$V$ is conformal at $z = -d/c$ so $T$ is conformal at $z = -d/c$. $\blacksquare$

**Theorem 14** Each conformal map $f : \Sigma \to \Sigma$ is an automorphism.

Note that we do not require $f$ to be bijective in theorem 14, it follows as a consequence.

As automorphisms are conformal we see that on $\Sigma$ they map circles to circles. Thus we conclude that $\text{Aut } \Sigma$ is the group of Möbius transformation. This leads to the following identification

**Theorem 15** $\text{Aut } \Sigma \cong \text{PSL}(2, \mathbb{C})$.

The functions

$$T(z) = \frac{a\bar{z} + b}{c\bar{z} + d}, \ a, b, c, d \in \mathbb{C}, \ ad - bc \neq 0 \quad (2.11)$$

are called anti-automorphisms of $\Sigma$. As the complex conjugation, $z \mapsto \bar{z}$, is anti-conformal and anti-automorphisms are a composition of an automorphism and complex conjugation we see that anti-automorphisms are anti-conformal. The composition of two anti-automorphisms is an automorphism and the composition of an anti-automorphism and an automorphism is an anti-automorphism. So the anti-automorphisms and automorphisms forms a group denoted by $\overline{\text{Aut}(\Sigma)}$ which has $\text{Aut}(\Sigma)$ as a subgroup of index 2. $\overline{\text{Aut}(\Sigma)}$ is also known as the extended Möbius group.

### 2.3 The Hyperbolic Plane

In this section we will study the hyperbolic plane and the automorphisms on it. We start with constructing a model of the hyperbolic plane called the Poincaré half-plane model.

Let $\mathcal{H} = \{ z \in \Sigma; \text{Im}(z) > 0 \}$. We see that an automorphism of $\mathcal{H}$ is an automorphism of $\Sigma$ such that $\mathbb{R} \cup \{ \infty \}$ is preserved. So if $T \in \text{Aut } \mathcal{H}$ then

$$T(0) = \frac{b}{d} \in \mathbb{R} \cup \{ \infty \} \quad T(\infty) = \frac{a}{c} \in \mathbb{R} \cup \{ \infty \} \quad (2.12)$$

That means $a, c, b, d \in \mathbb{R}$ as we want $\mathcal{H}$ mapped to itself. Thus we conclude that
2.3. The Hyperbolic Plane

**Theorem 16** \( \text{Aut } \mathcal{H} \cong PSL(2, \mathbb{R}) \).

We define the *hyperbolic length* by the Poincaré metric

\[
d s^2 = \frac{d x^2 + d y^2}{y^2} = |d z|^2
\]

(2.13)

Let \( \gamma \) be a piecewise differentiable path \( \gamma : [0, 1] \rightarrow \mathcal{H}, \gamma(t) = x(t) + y(t) = z(t) \) then the hyperbolic length of \( \gamma, h(\gamma) \), is given by

\[
h(\gamma) = \int_0^1 \sqrt{\frac{(d x)^2 + (d y)^2}{y}} \, dt = \int_0^1 \frac{|d z|}{y} \, dt
\]

(2.14)

**Theorem 17** Let \( \gamma \) be defined as above, then \( T \in PSL(2, \mathbb{R}) \Rightarrow h(T(\gamma)) = h(\gamma) \)

**Proof.** Let

\[
T(z) = \frac{a z + b}{c z + d} \in PSL(2, \mathbb{R})
\]

then

\[
T'(z) = \frac{a d - b c}{(c z + d)^2} = \frac{1}{(c z + d)^2}
\]

Also if \( z = x + iy \) and \( T(z) = u + iv \) then \( v = \frac{y}{|c z + d|^2} \) so \( \frac{d T}{d z} = v/y \). So

\[
h(T(\gamma)) = \int_0^1 \frac{|d T|}{v} \, dt = \int_0^1 \frac{|d T| |d z|}{v} \, dt = \int_0^1 \frac{|d T| |d z|}{y} \, dt = \int_0^1 \frac{|d z|}{y} \, dt = h(\gamma)
\]

**Hyperbolic lines**, or \( H \)-lines, in \( \mathcal{H} \) are Euclidean lines perpendicular to the real axis and semicircles centered at a point on the real axis. The following theorem motivates this definition.

**Theorem 18** Let \( z_0, z_1 \in \mathcal{H} \) then the path of shortest hyperbolic length from \( z_0 \) to \( z_1 \) is a segment of the unique \( H \)-line joining \( z_0 \) and \( z_1 \).

With this result we define the *hyperbolic metric* \( \rho \) as \( \rho(z_0, z_1) = h(\gamma) \), where \( \gamma \) is the \( H \)-line segment joining \( z_0 \) and \( z_1 \). \( \rho(z_0, z_1) \) is given by

\[
\rho(z_0, z_1) = \ln \left( \frac{|z_0 - \bar{z}_1| + |z_0 - z_1|}{|z_0 - \bar{z}_1| - |z_0 - z_1|} \right)
\]

(2.15)

**Example 19** Let \( a < b \in \mathbb{R} \) then by formula 2.15

\[
\rho(ia, ib) = \ln \left( \frac{|ia + ib| + |ia - ib|}{|ia + ib| - |ia - ib|} \right) = \ln \left( \frac{2b}{2a} \right) = \ln \left( \frac{b}{a} \right).
\]
Chapter 2. General Setting

Figure 2.2: Hyperbolic lines in $\mathcal{H}$ and $\mathcal{D}$.

$\mathcal{H}$ with the hyperbolic metric gives us a model for the hyperbolic plane, the Poincaré halfplane model. Sometimes we will use another model, the Poincaré disc model $\mathcal{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. $\mathcal{H}$ can be mapped biholomorphically to $\mathcal{D}$ by $z \mapsto \frac{z-i}{z+i}$ as we have seen before in example 3. This induces a metric on $\mathcal{D}$ given by

$$
\begin{align*}
    w &= \frac{z-i}{z+i} \iff z = \frac{iw+i}{-w+1} \Rightarrow \frac{dz}{dw} = \frac{2i}{(1-w)^2} \Rightarrow |dz| = \frac{2|dw|}{|1-w|^2} \\
    y &= \frac{1-|w|^2}{|1-w|^2}
\end{align*}
$$

$$
    ds = \frac{|dz|}{y} \frac{2|dw|}{|1-w|^2} = \frac{2|dw|}{1-|w|^2}.
$$

From now on we will call the hyperbolic plane $\mathcal{H}$ and the hyperbolic metric $\rho$ regardless which of the models we use.

Let $E \subseteq \mathcal{H}$, then the hyperbolic area of $E$ is defined as

$$
\mu(E) = \int \int_E \frac{dxdy}{y^2} \quad (2.16)
$$

Theorem 20 Let $E \subseteq \mathcal{H}$ and $T \in \text{PSL}(2, \mathbb{R})$ then $\mu(T(E)) = \mu(E)$

A hyperbolic $n$-sided polygon is a closed set in the closure of $\mathcal{H}$ in $\Sigma$ bounded by $n$ hyperbolic line segments. A point where two line segments intersect is called a vertex of the polygon. A vertex may be on $\mathbb{R} \cup \{\infty\}$. A subset $C$ of $\mathcal{H}$ is hyperbolically starlike if there is a point $p$ in the interior of $C$ such that $\forall q \in C$, the H-line joining $p$ and $q$ is in $C$. If this is true for all $p \in C$, $C$ is called hyperbolically convex.
2.3. The Hyperbolic Plane

We define the *angle* of two intersecting H-lines by the angle between their tangents at the point of intersection. If they intersect on $\mathbb{R} \cup \{\infty\}$ the angle is zero. The angles of a hyperbolic triangle are related to its area in a fascinating way.

**Theorem 21 Gauss-Bonnet.** Let $\Delta$ be a hyperbolic triangle with angles $\alpha, \beta, \gamma$. then

$$\mu(\Delta) = \pi - \alpha - \beta - \gamma.$$  \hfill (2.17)

**Proof.** We assume that $\Delta$ has two sides that are vertical H-lines then the base is a segment of a semi-circle. As previously stated the area is invariant under transformations in $PSL(2, \mathbb{R})$, thus we can assume that the semi-circle has center in 0 and radius 1. Assume that the vertical H-lines intersects the real axis at $a, b$ where $a < b$. Let $\alpha, \beta$ be the angles at the vertices of $\Delta$ were $x = a$ and $x = b$ respectively. Then $\gamma = 0$ since the last vertex is $\infty$. Now

$$\mu(\Delta) = \int \int_E \frac{dx dy}{y^2} = \int_a^b dx \int_0^\infty \frac{dy}{\sqrt{1-x^2}} y^2 = \int_a^b dx \frac{dx}{\sqrt{1-x^2}} = [x = \cos \theta] =$$

$$= \int_{\pi - \alpha}^\beta \frac{-\sin \theta d\theta}{\sin \theta} = \pi - \alpha - \beta$$

If $\Delta$ has one vertex $c$ in $\mathbb{R}$ then there exists a $T \in PSL(2, \mathbb{R})$ such that $T(c) = \infty$. As this transformation does not change the area we see that $\mu(\Delta) = \pi - \alpha - \beta$. Lastly if $\Delta$ has no vertices in $\mathbb{R} \cup \{\infty\}$ we can construct $\Delta$ as the difference of two hyperbolic triangles $\Delta_1, \Delta_2$ which has one common vertex on $\mathbb{R} \cup \{\infty\}$. We can easily see that $\mu(\Delta) = \mu(\Delta_1) - \mu(\Delta_2) = \pi - \alpha - \beta - \gamma$  .

The Gauss-Bonnet formula can be generalized to an arbitrary hyperbolically starlike polygon.

![Hyperbolic triangle](image-url)
Theorem 22 Let $\Pi$ be a $n$-sided hyperbolically starlike polygon with angles $\alpha_1, \alpha_2 \ldots \alpha_n$. Then

$$\mu(\Pi) = (n - 2)\pi - \alpha_1 - \alpha_2 \cdots - \alpha_n$$

(2.18)

2.4 Fuchsian groups

We have seen how $\text{PSL}(2,\mathbb{C})$ can be divided in conjugacy classes. $\text{PSL}(2,\mathbb{R})$ can be divided in conjugacy classes in a similar way, though some elements may be conjugate when regarded as elements of $\text{PSL}(2,\mathbb{C})$ but not when regarded as elements of $\text{PSL}(2,\mathbb{R})$. Let $T \in \text{PSL}(2,\mathbb{R})$, as in 2.6 with $\mathbb{R}$ instead of $\mathbb{C}$, then $T$ is

(i) parabolic if $(|a + d| = 2)$. $T$ has one fixed point $\alpha \in \mathbb{R} \cup \{\infty\}$ and is conjugate to $z + 1$ or $z - 1$

(ii) hyperbolic if $(|a + d| > 2)$. $T$ has fixed points $\alpha, \beta \in \mathbb{R} \cup \{\infty\}$ and is conjugate to $\lambda z$ ($\lambda > 1$)

(iii) elliptic if $(|a + d| < 2)$. $T$ has fixed point $\xi \in \mathcal{H}$ conjugate to a rotation of $\mathcal{H}$

Before looking at Fuchsian groups we need to define topological groups. A topological group is a topological space $G$ which also is a group where the maps

$$m : G \times G \to G \quad m(g, h) = gh,$$

$$i : G \times G \quad i(g) = g^{-1}$$

are continuous.

Example 23 $\text{PSL}(2,\mathbb{C})$ and $\text{PSL}(2,\mathbb{R})$ are topological groups.

A Fuchsian group $\Gamma$ is a subgroup of $\text{PSL}(2,\mathbb{R})$ such that, when regarded as a subspace of $\text{PSL}(2,\mathbb{R})$, it has the discrete topology. This means the set $\{g \in \Gamma; \|g\| \leq k\}$ is finite for every $k > 0$, where $\|g\|$ is the matrix norm.

It is difficult to decide when a subgroup of $\text{PSL}(2,\mathbb{R})$ is discrete. We shall look at a few examples.

Example 24 The simplest Fuchsian groups are the following cyclic groups:

Hyperbolic cyclic groups are generated by a hyperbolic element, such as $z \mapsto \lambda z$, ($\lambda > 0$). Parabolic cyclic groups are generated by a parabolic element, such as $z \mapsto z + 1$. Finite elliptic cyclic groups are generated by an elliptic element.
Example 25 A more complicated group is the modular group $\text{PSL}(2, \mathbb{Z})$ consisting of the following transformations

$$T(z) = \frac{az + b}{cz + d} \quad a, b, c, d, \in \mathbb{Z} \quad ad - bc = 1.$$ 

Now let $Q$ be an H-line. An H-reflection is an isometry of $\mathcal{H}$, other than identity, that fixes $Q$.

Example 26 The H-reflection in the imaginary axis $Q_0$ is given by

$$R_0 : z \mapsto -\bar{z}.$$ 

As for every H-line $Q$ there exists a $T$ such that $T(Q) = Q_0$. So the H-reflection in $Q$ is given by $T^{-1}R_0T$. H-reflections are anti-conformal homeomorphisms, that is they preserve angles but reverse orientation. We see that all H-reflections and other anti-conformal homeomorphisms of $\mathcal{H}$ are given by

$$z \mapsto \frac{a\bar{z} + b}{c\bar{z} + d} \quad ad - bc = -1. \quad (2.19)$$

Example 27 As a last example let $\Delta$ be a hyperbolic triangle with vertices $v_1, v_2, v_3$ with angles $\pi/m_1, \pi/m_2, \pi/m_3$ respectively and with opposing sides $M_1, M_2, M_3$. We call the reflection in the line containing $M_i, R_i$. Let $\Gamma^*$ be the group generated by $R_1, R_2, R_3$. If $\Gamma = \Gamma^* \cap \text{PSL}(2, \mathbb{R})$, then $\Gamma^* = \Gamma \cup \Gamma R_1$. $\Gamma$ is a Fuchsian group called a triangle group and is generated by $R_1R_2, R_2R_3, R_3R_1$ where

$$(R_1R_2)^{m_3} = (R_2R_3)^{m_1} = (R_3R_1)^{m_2} = \text{Id}.$$ 

With presentation we have

$$\Gamma = \langle x, y | x^{m_3} = y^{m_1} = (xy)^{m_2} = \text{Id} \rangle = \langle x, y, z | x^{m_3} = y^{m_1} = z^{m_2} = xyz = \text{Id} \rangle.$$ 

We will later see that a group $\Gamma$ acting on $\mathcal{H}$ is discrete if and only if $\Gamma$ acts properly discontinuous. Before we look at this property we will look at the concept of covering maps.
Figure 2.5: Triangle with reflections
Chapter 3
Covering Maps

Let $X, Y$ be topological spaces. A map $p : Y \to X$ is called a covering map (or covering) when each point $x \in X$ has a neighbourhood $V$ such that

$$p^{-1}(V) = \bigcup_{\alpha} U_{\alpha}$$

is a union of pairwise disjoint sets $U_{\alpha}$, each mapped homeomorphically onto $V$. $V$ is called a distinguished neighbourhood. $Y$ is called the covering space of $X$ and for each $x \in X$ $p^{-1}(x)$ is called a fiber over $x$.

Theorem 28: If $X$ is connected and $p : Y \to X$ is a covering then each fiber $p^{-1}(x)$, $x \in X$ has the same cardinal number, which is called the number of sheets (or leaves) of the covering. If $|p^{-1}| = n < \infty$ we say that the covering is an $n$-sheeted covering.

Example 29: Let $Y$ be a discrete space with $|Y| = n$. Then $p : X \times Y \to X$ defined as $p(x, y) = x$ is a covering. It’s easy to see that $|p^{-1}(x)| = n$, that is the covering is an $n$-sheeted covering.

Example 30: Let $p = e^{it}$. Then $p : \mathbb{R} \to S^1$ is a covering map. We also see that each fiber $p^{-1}(x)$ is infinite.

Example 31: Let $h : X \to Y$ be a homeomorphism. Then $h$ is a 1-sheeted covering map.

3.1 Fundamental Groups

Let $X, Y$ be topological spaces. Two functions $f, g : X \to Y$ are called homotopic if there exists a continuous map $H : X \times [0, 1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$. $H$ is called a homotopy between $f$ and $g$. 

Bartolini, 2006.
Example 32 Let \( f, g \) be two constant maps \( f, g : X \to Y, \) \( f(x) = p, \) \( g(x) = q. \) \( f \) and \( g \) are homotopic if and only if \( p \) and \( q \) are in the same pathwise connected part of \( Y. \) If there exists a path \( \gamma \) such that \( \gamma(0) = p \) and \( \gamma(1) = q \) we can define a homotopy \( H \) between \( f \) and \( g \) by \( H(x, t) = \gamma(t) \) for all \( (x, t) \in X \times [0,1]. \)

A particular case of homotopy is path homotopy. A path on a topological space \( X \) is a continuous function \( a : [s_0, s_1] \to X. \) We call the initial point \( a(s_0) \) and the final point \( a(s_1) \) endpoints. A path \( a \) is closed if \( a(s_0) = a(s_1). \)

Now we are interested in homotopies of paths with their endpoints fixed. Let \( a, b : [s_0, s_1] \to X \) be paths with the same endpoints, we say that \( a, b \) are homotopic paths if there exists a homotopy \( H \) between \( a \) and \( b \) such that

\[
\begin{align*}
H(s, 0) &= a(s), \quad H(s, 1) = b(s) \\
H(s_0, t) &= a(s_0) = b(s_0) \\
H(s_1, t) &= a(s_1) = b(s_1) \quad \forall s \in [s_0, s_1], \forall t \in [0,1]
\end{align*}
\]

(3.2)

Path homotopy is a equivalence relation which we will denote \( a \cong b. \) We call the equivalence class \( [a] \) of a path \( a \) a homotopy class.

Let \( a, b : [s_0, s_1] \to X \) be two paths such that \( a(s_1) = b(s_0). \) We then define the product \( ab : [s_0, s_1] \to X \) by

\[
ab(s) = \begin{cases} 
    a(2s - s_0) & \text{if } s \in [s_0, \frac{s_1 - s_0}{2}] \\
    b(2s - s_1) & \text{if } s \in \left[\frac{s_1 - s_0}{2}, s_1\right]
\end{cases}
\]

(3.3)

It’s easy to see that \( ab \) is a path with endpoints \( a(s_0), b(s_1). \) we also need the inverse of a path. We define the inverse of a path \( a \) by \( a^{-1}(s) = a(s_0 + s_1 - s), \) \( s \in [s_0, s_1]. \)

Theorem 33 Let \( a, b \) be two paths with \( a(s_1) = b(s_0). \) If \( a', b' \) are paths such that \( a \cong a', \) \( b \cong b' \) then \( ab \cong a'b' \) and \( a^{-1} \cong a'^{-1}. \)

We now define \( [a][b] = [ab] \) and \( [a]^{-1} = [a^{-1}] \). By theorem 33 we see that neither depend on the choices of \( a, b. \) We are most interested in closed paths, or loops, that is paths \( a \) such that \( a(s_0) = a(s_1) = x. \) We call \( x \) the base point of \( a. \) The following theorem shows that the homotopy classes of loops based at a point \( x \) forms a group.

Theorem 34 Let \( a, b, c \) be loops based at \( x \in X. \) Let \( e_x \) be that constant path at \( x. \) Then

(i) \([a][a^{-1}] = [a^{-1}][a] = [e_x]\)
(ii) \([e_x][a] = [a][e_x] = [a]\)

(iii) \([a][(b)[c]] = ([a][b])[c]\).

By theorem 34 we see that the homotopy classes of loops based at a point \(x_0 \in X\) forms a group. We call this group the fundamental group of \(X\) and denote it by \(\pi_1(X, x_0)\).

**Theorem 35** If \(x_0, x_1\) belong to the same pathwise connected component of \(X\) then \(\pi_1(X, x_0)\) and \(\pi_1(X, x_1)\) are isomorphic.

It follows that if \(X\) is pathwise connected then \(\pi_1(X, x_0) = \pi_1(X, x_1)\) for all \(x_0, x_1 \in X\). In this case we simply write \(\pi_1(X)\).

**Example 36** \(\pi_1(\Sigma) = \pi_1(C) = \pi_1(H) = \{Id\}\), that is the trivial group. In fact if \(X\) is a simply-connected space then \(\pi_1(X) = \{Id\}\) and as we know \(\Sigma, C, H\) are all simply-connected.

**Example 37** Let \(T\) be a torus. The torus is pathwise connected so we don’t need to choose a base point. Let \(a, b\) be the classes of loops one ”lap” around the meridian and longitude respectively. We note that each class \(c\) of loop on \(T\) can be expressed in terms of \(a, b\), \(c = a^{m_1} + b^{m_2}\). \(\pi_1(T) = \langle a, b | [a, b] = Id \rangle\) where \([a, b] = aba^{-1}b^{-1}\) is the commutator. Similarly for a surface \(S\) of genus \(g\) we have \(\pi_1(S) = \langle a_1, b_1, \ldots, a_g, b_g | \Pi[a_i, b_i] = Id \rangle\).

**Example 38** Let \(S\) be \(\Sigma\) with three points removed. Let \(a, b, c\) be classes of loops one ”lap” around each point respectively. As \(ab = c\) we can conclude that \(\pi_1(S) = \langle a, b | - \rangle\).
3.2 Group Actions on Surfaces

Let \( G \) be a group of homeomorphisms of a topological space \( X \). Then \( G \) acts \( \text{freely} \) on \( X \) if each point \( x \in X \) has a neighbourhood \( V \) such that \( g(V) \cap V = \emptyset \) for all \( g \in G \). All groups we will look at do not satisfy this, sometimes we need a weaker condition. \( G \) acts \( \text{properly discontinuously} \) on \( X \) if each point \( p \in X \) has a neighbourhood \( V \) such that \( g(V) \cap V \neq \emptyset \) for a finite number of \( g \in G \). If \( G \) acts freely on \( X \) then it acts properly discontinuous.

The quotient space of the orbits \( Gx, x \in X \) is denoted by \( X/G \). The canonical projection \( p : X \to X/G \) associates each \( x \in X \) with its orbit \( Gx \). An open set \( A \subset X/G \) is a set such that \( p(A)^{-1} \) is open in \( X \).

Example 39 Let \( w_1, w_2 \in \mathbb{C}, w_1/w_2 \notin \mathbb{R} \) and

\[
\Lambda = \{ z \mapsto z + mw_1 + nw_2; m, n \in \mathbb{Z} \}.
\]
The two paths \( a, b : [0, 1] \to \mathbb{C}/\Lambda \) defined by \( a(s) = sw_1 \) and \( b(s) = sw_2 \) are closed as \( a(0) = a(1) = 0 = b(0) = b(1) \). It's easy to see that \( a, b \) are not homotopic to each other nor to a constant path. Now we see that \( \pi_1(\mathbb{C}/\Lambda) = \langle [a], [b]; [[a], [b]] = 1 \rangle \). So \( \mathbb{C}/\Lambda \) is a torus.

Now recall that a group \( G \) acts freely on a space \( X \) if each point \( x \in X \) has a neighbourhood \( U \) such that \( g(U) \cap U = \emptyset \) for all \( g \in G \).

**Theorem 40** Let \( G \) be a group of homeomorphisms acting on a space \( X \) then the following statements are equivalent

(i) \( G \) acts freely on \( X \)

(ii) The canonical projection \( p : X \to X/G \) is a covering

(iii) \( p : X \to X/G \) is locally injective

**Proof.** (i)⇒(ii): As \( G \) acts freely there exists a neighbourhood \( V \) for each \( x \in X \) such that \( g(V) \cap V = \emptyset \) for all \( g \in G \). This means that \( g(V) \cap h(V) = \emptyset \) for all \( g, h \in G \) and \( W = p(V) \) is isomorphic to \( V \). Now let \( U_g = g(V) \) then

\[
p^{-1}(W) = \bigcup_{g \in G} U_g.
\]

That means \( p \) is a covering.

(ii)⇒(i): As \( p : X \to X/G \) is a covering for each open subset \( W \in X/G \) there exists disjoint open subsets \( U_i \in X \) such that \( p(U_i) = W \) which means no points \( x_1, x_2 \in U_i \) are in the same orbit. Thus \( g(U_i) \cap U_i = \emptyset \) and we conclude that \( G \) acts freely on \( X \). ■

**Example 41** From theorem 40 we see that \( p : \mathbb{C} \to \mathbb{C}/\Lambda \) is a covering where \( \Lambda \) is the lattice in example 39.

**Example 42** Let \( \Gamma \) be a Fuchsian group without elliptic elements. As no \( g \in \Gamma \) has any fixed point in \( \mathcal{H} \) we can for each \( z \in \mathcal{H} \) choose a neighbourhood \( V \) such that \( g(V) \cap V = \emptyset \), which means that \( \Gamma \) acts freely on \( \mathcal{H} \). From theorem 40 it follows that the canonical projection \( p : \mathcal{H} \to \mathcal{H}/\Gamma \) is a covering.

**Example 43** Fuchsian groups with elliptic elements acts properly discontinuous on \( \mathcal{H} \).

In fact we have the following relation between discreteness and properly discontinuity.
Theorem 44 A subgroup $\Gamma \subset PSL(2, \mathbb{R})$ is discrete if and only if it acts properly discontinuous on $\mathcal{H}$.

Proof. Let $\Gamma$ be properly discontinuous. Assume that $\Gamma$ is not discrete. Then there exists elements $T_n \in \Gamma$ such that $T_n \rightarrow Id$ as $n \rightarrow \infty$. This means $T_n(z) \rightarrow z$ for all $z \in \mathcal{H}$. Thus for all neighbourhoods $V$ of $z$ there exists an $N$ such that $T_n(z) \in V, n > N$. This violates the definition of properly discontinuous and we conclude that $\Gamma$ is discrete. 

Let $p: X \rightarrow Y$ be a covering. A homeomorphism $f: X \rightarrow X$ such that $p \circ f = p$ is called a deck transformation. The set of deck transformations of a covering $p$ forms a group under composition.

Example 45 In example 30 we saw that $p = e^t$ is a covering map $p: \mathbb{R} \rightarrow S^1$. Let $f_k(t) = t + 2\pi k$, then $p \circ f_k = f_k$. That is, $f_k$ is a deck transformation. It’s easy to see that the group of deck transformations of $p$ is isomorphic to $\mathbb{Z}$.

Example 46 Now let $p: \mathbb{C} \rightarrow \mathbb{C}/\Lambda$ be the covering seen in example 41. The lattice $\Lambda = \{ z \mapsto z + mw_1 + nw_2; m, n \in \mathbb{Z}\}$ is the deck transformation group and is isomorphic to $\mathbb{Z} \times \mathbb{Z} = \{(m, n) | m, n \in \mathbb{Z}\}$.

Let $f: Y \rightarrow X$ be a continuous surjective map. If there for any path $a: [s_0, s_1] \rightarrow X$ and any point $y \in Y$ such that $f(y) = a(s_0)$ there exists a path $\tilde{a}: [s_0, s_1] \rightarrow Y$ such that $a(s_0) = y$ and $f \circ \tilde{a} = a$ we say that $f$ has the path lifting property. If the path $\tilde{a}$ is unique we say that $f$ has the unique path lifting property.

Let $p: Y \rightarrow X$ and $y \in Y$ such that $p(y) = x$. By $H(y)$ we denote the homomorphism $p# : \pi_1(Y, y) \rightarrow \pi_1(X, x)$.

Theorem 47 Let $p: Y \rightarrow X$ be a covering and $a, b: I \rightarrow X$ be paths with the same endpoints and $\tilde{a}, \tilde{b}: I \rightarrow Y$ be their liftings at a point $y \in Y$. Then $\tilde{a}(1) = \tilde{b}(1)$ if and only if $[ab^{-1}] \in H(y)$.

3.3 Universal Coverings

Let $p: \mathcal{U} \rightarrow X$ be a covering. We call $p$ a universal covering if $\mathcal{U}$ is simply connected, that is the fundamental group is trivial. We call $\mathcal{U}$ the universal cover of $X$. If $X$ is simply connected it is its own universal cover. If $p: \mathcal{U} \rightarrow X$ is a universal covering and $p': Y \rightarrow X$ is another covering then there exists a covering $p'': \mathcal{U} \rightarrow Y$ such that $p' \circ p''$ is a covering. For Riemann surfaces we have the following theorem.
Theorem 48 Uniformization Let $p : \mathcal{U} \to S$ be a universal covering of a surface $S$. Then $\mathcal{U}$ is one of the following spaces:

(i) $\mathbb{C}$

(ii) $\Sigma$

(iii) $\mathcal{H}$

Example 49 As we have seen before there exists a covering $p : \mathbb{C} \to T$ where $T$ is a torus. That means $\mathbb{C}$ is the universal cover of $T$.

Example 50 Let $T_g$ be a surface of genus $g > 1$. Then there exists a universal covering $p : \mathcal{H} \to T_g$.

Example 51 In fact, $\mathcal{H}$ is the universal cover of all surfaces except the sphere $\Sigma$, which is its own universal cover, the torus $T$ and the plane $\mathbb{C}$ which both have $\mathbb{C}$ as their universal cover.
Chapter 4

Riemann Surfaces as Orbifolds

In our work with surfaces in chapter 3 we have only looked at surfaces as topological spaces, but Riemann surfaces have a differential structure which we want to include. When we have worked with the hyperbolic plane $\mathcal{H}$ we have so far excluded the elliptic elements of $\text{PSL}(2, \mathbb{R})$. In order to include them and the differential structure of Riemann surfaces we need some additional concepts namely orbifolds and branched coverings.

4.1 2-Orbifolds

A 2-orbifold $O$ is a 2-dimensional space such that there exists an atlas of folding charts $(V_i, G_i, \Phi_i, U_i)$. Each chart consists of an open set $V_i \in \mathbb{C}$, a finite cyclic or dihedral group $G_i$ acting on $V$, an open set $U_i \in O$ and a folding map $\Phi_i : U_i \rightarrow V_i$ which induces a homeomorphism $U_i \rightarrow V_i/G_i$. The charts in an atlas satisfy

$$\bigcup_i U_i = O \quad (4.1)$$

and if $\Phi_i^{-1}(x) = \Phi_j^{-1}(y)$ then there exists neighbourhoods $x \in V_x \subset V_i$ and $y \in V_y \subset V_j$ and the transition function $\Phi_j \circ \Phi_i^{-1} : V_x \rightarrow V_y$ is a diffeomorphism.

Example 52 A dihedral group is a group

$$D_n = \langle a, b | a^2 = b^2 = (ab)^n = Id \rangle$$

A cyclic group is a group

$$C_n = \langle a | a^n = Id \rangle$$

Bartolini, 2006. 25
Let \( O \) be an orbifold. A \( \text{singular point} \ p \) is a point such that there exists a chart with a group \( G_i \neq \{Id\} \) stabilizing \( p \). We call the set of such points \( \text{the singular set} \) of \( O \). The singular set of \( O \) together with the stabilizers of the points in it are called \( \text{the stratification} \) of \( O \). Here the only singular points we will consider are called \( \text{cone points} \). A cone point is a singular point such that \( G_i \) is a cyclic group, in our case generated by an elliptic element in \( \text{PSL}(2, \mathbb{C}) \). Let \( m_i \) be the order of the cyclic groups \( G_i \) stabilizing the cone points \( p_i \), then \( O \) has Euler characteristic

\[
\chi(O) = 2 - 2g - \sum_i \left( 1 - \frac{1}{m_i} \right)
\]  

(4.2)

**Example 53** Let all \( G_i = \{Id\} \). Then \( O \) is a Riemann surface without singularities.

**Example 54** In general, observe that the complex structure of a Riemann surface in page 7 provides the surface with an orbifold structure.

**Example 55** The charts in example 7 provides \( \Sigma \) with an orbifold structure with folding charts which induces homeomorphisms \( w_1 = z \in \mathbb{C}/\mathbb{C}_n \) and \( w_2 = 1/z \in \mathbb{C}/\mathbb{C}_n \). We see that \( 0 \) and \( \infty \) are cone points.

### 4.2 Branched Coverings

Here we will construct branched coverings with coverings in mind. Let \( X \) and \( Y \) be topological spaces. A map \( p : Y \to X \) is called a \( \text{branched covering} \) when each point \( x \in X \) has a neighbourhood \( V \) such that

\[
p^{-1}(V) = \bigcup_\alpha U_\alpha
\]

(4.3)

is a union of pairwise disjoint sets \( U_\alpha \), where \( U_\alpha/G_\alpha \to V \) is a homeomorphism, where \( G_\alpha \) is a finite cyclic, possibly trivial, group. A point \( x \in X \) such that \( G \neq \{Id\} \) is called a \( \text{branch point} \). Let \( x \) be a branch point, a point \( y \) such that \( p(y) = x \) is called a \( \text{ramification point} \). Let \( B = \{x_1, \ldots, x_r\} \), the set \( p^{-1}(B) \subset Y \) is called the \( \text{ramification} \) of the covering. Note that \( p : Y \setminus p^{-1}(B) \to X \setminus B \) is a covering.

**Example 56** The coverings constructed in chapter 3 are branched coverings such that all \( G_\alpha \) are trivial.

**Example 57** Let \( S_1, S_2 \) be two Riemann surfaces. A surjective continuous function \( p : S_1 \to S_2 \) is a branched covering.
In theorem 40 we saw a relation between groups of free homeomorphisms and coverings. Now we state a similar relation for properly discontinuous groups and branched coverings.

**Theorem 58** Let $X$ be a topological space and $G$ be a group of homeomorphisms acting on $X$ then the following statements are equivalent:

(i) $G$ acts properly discontinuous on $X$.

(ii) The canonical projection $p : X \to X/G$ is a branched covering

Note that we don’t have a statement about injectivity. Assume that $G = C_n$, that is a cyclic group of order $n$. If $x \in X$ is fixed by $G$ then $p : X \to X/G$ is $n$-to-one near $x$.

A diffeomorphism $f : Y \to Y$ such that $p \circ f = p$, where $p : Y \to X$ is a branched covering, is called a deck transformation. Let $G$ be the deck transformation group of a covering $p$, we call $p$ a regular covering if $Y/G = X$.

### 4.2.1 Universal Branched Coverings

Let $p : U \to X$ be a branched covering of $X$. We call $p$ a universal branched covering if $U$ is simply connected. Now we have a theorem similar to theorem 48 for Riemann surfaces. This theorem is known as Poincaré’s first theorem.

**Theorem 59** Uniformization Let $p : U \to S$ be a universal covering of a Riemann surface $S$. Then $U$ is one of the following spaces:

(i) $\mathbb{C}$

(ii) $\Sigma$

(iii) $\mathcal{H}$

A 2-orbifold covered by $U$ is called a good orbifold, otherwise it’s called bad.

**Example 60** The only bad Riemann surfaces not covered by $U$ are:

(i) The "Teardrop", which is a sphere with one cone point.

(ii) The "Football", which is a sphere with two cone points with different order.
Example 61 With the orbifold structure of Riemann surfaces in mind we get a similar result as in example 51. $H$ is the universal branched covering of all good Riemann surfaces except the sphere which has $\Sigma$ as universal cover and the tori that has $\mathbb{C}$ as universal cover.

Now we define the fundamental group $\pi_1(S)$ of a Riemann surface $S$ as the deck transformation group of its universal covering. In this case we have $U/\pi_1(S) = S$. If $S$ has no singular points then $\pi_1(S) = \pi_1(S)$ as defined in chapter 3.

![Figure 4.1: The pillow case](image)

Example 62 The orbifold with fundamental group

$$\pi_1(S) = \langle x_1, x_2, x_3, x_4 | x_1^2 = x_2^2 = x_3^2 = x_4^2 = Id \rangle.$$  

is given in the figure above. $S$ is the sphere with four cone points of order 2.

Example 63 Let $S$ be a Riemann surface with genus 2 and with singular points $p_i$ with orders $m_i$, $i = 1 \ldots 3$. Then the fundamental group is given by

$$\pi_1(S) = \langle a, b, c, d, x_1, x_2, x_3 | x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = x_1x_2x_3[a, b][c, d] = Id \rangle.$$
Example 64 In a similar way the fundamental group of a Riemann surface $S$ with genus $g$ and cone points $p_i$ with orders $m_i, 1 \ldots r$ is given by

$$\bar{\pi}_1(S) = \langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r \mid x_1^{m_1} = \cdots = x_r^{m_r} = \Pi_i x_i \Pi_j [a_j, b_j] = Id \rangle. $$

A branched covering $p : Y \to X$ is regular if and only if $\bar{\pi}_1(Y) \leq \bar{\pi}_1(X)$, that is the fundamental group of $Y$ is a normal subgroup of the fundamental group of $X$. The deck transformation group $G$ is given by $\bar{\pi}_1(X)/\bar{\pi}_1(Y)$. Let $p : Y \to X$ be a branched covering and $\gamma : [0, 1] \to X$ be a closed path. Then the liftings $\tilde{\gamma} : [0, 1] \to Y$ are all closed or all open. The deck transformation group is then given by $\bar{\pi}_1(X)/\bar{\pi}_1(Y)$.
Chapter 5

Poincarés Theorem

Now we will work with Fuchsian groups acting the hyperbolic plane $\mathcal{H}$. Remember that a Fuchsian group is a discrete subgroup of the automorphisms, $PSL(2, \mathbb{R})$, of $\mathcal{H}$.

5.1 Fundamental Domains

Let $\Gamma$ be a Fuchsian group. $F \subset \mathcal{H}$ is a fundamental domain to $\Gamma$ if $F$ is a closed set such that

(i) $\bigcup_{T \in \Gamma} T(F) = \mathcal{H}$

(ii) $\overset{\circ}{F} \cap T(\overset{\circ}{F}) = \emptyset \forall T \in \Gamma \setminus \{Id\}$ where $\overset{\circ}{F}$ is the interior of $F$

(iii) $\mu(\delta F) = 0$ where $\delta F$ is the boundary of $F$

A special kind of fundamental domain is constructed the following way. Let $\Gamma$ be a Fuchsian group and $p \in \mathcal{H}$ be a point not fixed by any element in $\Gamma$. Then the Dirichlet domain of $\Gamma$ centered at $p$ is

$$D_p(\Gamma) = \{z \in \mathcal{H} | \rho(z, p) \leq \rho(z, T(p)) \forall T \in \Gamma\} \quad (5.1)$$

**Theorem 65** If $p$ isn’t fixed by any element of $\Gamma \setminus \{Id\}$ then $D_p(\Gamma)$ is connected.

A fundamental domain $F$ of a Fuchsian group $\Gamma$ is called locally finite if $\forall p \in F \exists$ neighbourhood $V(p)$ such that $V(p) \cap T(F) \neq \emptyset$ for finitely many $T \in \Gamma$.

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Example 66 Let $\Gamma$ be a triangle group as seen in example 27. Let $p$ be a point in the middle of the side $M_i$ of the triangle $\Delta$. Then $D_p(\Gamma) = \Delta \cup R_i(\Delta)$ where $R_i$ is the reflection in $M_i$.

Theorem 67 Let $F_1, F_2$ be fundamental domains for a Fuchsian group $\Gamma$. Then $\mu(F_1) = \mu(F_2)$.

Proof.

$$
\mu(F_1) = \mu(f_1 \cap (\bigcup_{g \in \Gamma} gF_2)) = \sum_{g \in \Gamma} \mu(F_1 \cap gF_2)
$$

$$
= \sum_{g \in \Gamma} \mu(g^{-1}F_1 \cap F_2) = \mu(F_2)
$$

\[\hfill\]

5.2 The Quotient Space $\mathcal{H}/\Gamma$

Let $F$ be a fundamental domain for the Fuchsian group $\Gamma$. Then we say that $\mathcal{H}/\Gamma$ is compact if $F$ is a compact subset of $\mathcal{H}$.

Theorem 68 If $\mathcal{H}/\Gamma$ is compact then $\Gamma$ contains no parabolic elements.

Example 69 A Dirichlet domain $F$ can be non-compact in several ways. First if it has an infinite number of sides it is non-compact. Otherwise it is non-compact if it has a vertex on $\mathbb{R} \cup \{\infty\}$, called a parabolic vertex, or is bounded by a part of the real line.
A Dirichlet domain is bounded by H-lines and possibly by sections of the real line. A point \( p \in \mathcal{H} \) where such H-lines intersect is called a vertex of the Dirichlet domain. Let \( F \) be a Dirichlet domain for \( \Gamma \) and let \( u, v \) be vertices of \( F \). We say \( u \) and \( v \) are congruent if there exists \( T \in \Gamma \) such that \( T(u) = v \). Congruent vertices form an equivalence class, and those are called cycles. We are interested in vertices fixed by elliptic element. If a vertex of a cycle is fixed by an elliptic element so are all vertices in that cycle. Such a cycle is called an elliptic cycle and its vertices elliptic vertices.

Let \( s \) be a side of a Dirichlet domain for a Fuchsian group \( \Gamma \). If \( T \in \Gamma \setminus \{Id\} \) and \( T(s) \) is a side of \( F \) then \( s \) and \( T(s) \) are called congruent sides. \( T(s) \) is a side of \( T(F) \) and \( T(s) = F \cap T(F) \). The sides of \( F \) fall into congruent pairs.

Let \( \Gamma \) be a Fuchsian group with \( \mathcal{H}/\Gamma \) compact. Then a Dirichlet domain \( F \) of \( \Gamma \) is compact and so has a finite number of sides. Therefore \( F \) has finitely vertices and elliptic cycles. So \( \Gamma \) has a finite number of elliptic generators with periods \( m_1, m_2, \ldots, m_r \). If \( \mathcal{H}/\Gamma \) is a Riemann surface with genus \( g \) and singular points \( p_1, \ldots, p_r \) with stabilizers \( G_i = C_{m_i} \), the when say that \( \Gamma \) has the signature \( (g; m_1, m_2, \ldots, m_r) \). With the signature we get a generalization of Gauss-Bonnets theorem.

**Theorem 70 (Gauss-Bonnet)** Let \( \Gamma \) have signature \( (g; m_1, m_2, \ldots, m_r) \). If \( F \) is a fundamental domain for \( \Gamma \) then

\[
\mu(F) = 2\pi \left( (2g - 2) + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right).
\]

(5.2)

**Theorem 71** If \( F \) is a locally finite fundamental domain for a Fuchsian group \( \Gamma \) then \( F/\Gamma \) is isomorphic to \( \mathcal{H}/\Gamma \).

**Proof.** Let \( p_1 : \mathcal{H} \to \mathcal{H}/\Gamma \) and \( p_2 : F \to F/\Gamma \) be the canonical projections and \( i : F \to \mathcal{H} \) be the inclusion map. We define \( \theta : F/\Gamma \to \mathcal{H}/\Gamma \) as \( \theta(p_2(z)) = p_1(z), z \in F \). \( \theta \) is bijective since if \( p_2(z_1) = p_2(z_2) \) then then there is a side pairing \( T \in \Gamma \) and \( p_1(z_1) = p_1(z_2) \). Now let \( V_1 \subset \mathcal{H}/\Gamma \) be open, then

\[
p_2^{-1}(\theta^{-1}(V_1)) = F \cap p_1^{-1}(V_1)
\]

which is an open subset of \( F \) as \( p_1 \) is continuous. This means \( \theta^{-1}(V_1) \) is open and therefore \( \theta \) is continuous. Now let \( V_2 \subset F/\Gamma \) be open. There exists \( U \subset \mathcal{H} \) such that

\[
p_2^{-1}(V_2) = F \cap U, \quad p_2(F \cap U) = V_2.
\]
Now let 
\[ V = \bigcup_{g \in \Gamma} g(F \cap U). \]
Then 
\[ p_1(V) = p_1(F \cap U) = p_1 \circ i(F \cap U) = \theta \circ p_2(F \cap U) = \theta(V_2). \]
We want to show that $\theta(V_2)$ is open. To do this it is sufficient to show that $V$ is open since $p_1$ is an open map. Let $z \in V$, as $V$ is $\Gamma$-invariant we may assume that 
\[ z \in F \cap U \]
As $F$ is locally finite there exists a neighbourhood $N$ of $z$ such 
\[ N \cap T_i(F) \neq \emptyset, i = 1 \ldots n. \]
We suppose that $z \in T_i(F), i = 1 \ldots n$, then $T_i^{-1}(z) \in F$ and 
\[ p_2(T_i^{-1}(z)) = p_2(z) \in V_2. \]
Thus $z \in T_i(F \cap U)$ and for sufficiently small radius of $N$ we have 
\[ N \subset \bigcap_i T_i(F \cap U) \]
and $N \subset V$. 

5.3 Poincaré's Theorem

**Theorem 72 (Poincaré)** Let $P \in \mathcal{U}$ be a polygon with a side pairing generating a group $\Gamma$ satisfying

(i) for each vertex $x_0$ of $P$ there are vertices $x_0, x_1, \ldots, x_n$ of $P$ and elements $g_0 (= \text{Id}), g_1, \ldots, g_n$ of $\Gamma$ such that $g_i(N_i)$ are non-overlapping and $\bigcup g_i(N_i) = B(x_0, \varepsilon)$ where $N_i = \{ y \in P | d(y, x_0) < \varepsilon \}$.

(ii) each $g_{i+1} = g_i g_s$ where $g_s$ is a side pairing and $g_{n+1} = \text{Id}$.

Further there exists $\varepsilon$ such that for each point $p \in P$ $B(p, \varepsilon)$ is in a union of images of $P$. Then $\Gamma$ is a Fuchsian group $\Gamma$ and $P$ is a fundamental domain for $\Gamma$. 

However, this is difficult to prove as there are many cases to consider. We will look at the case where $P$ is a hyperbolic triangle or a pair of hyperbolic triangles.

![Hyperbolic triangle](image)

**Figure 5.2: Hyperbolic triangle**

**Theorem 73** Let $\Delta$ be a triangle with vertices $v_1, v_2, v_3$ with angles $\pi/m_1$, $\pi/m_2$, $\pi/m_3$ respectively and with opposing sides $M_1, M_2, M_3$. Let $R_i$ be the reflection in $M_i$, $i = 1, 2, 3$. The group $\Gamma^*$ generated by the reflections $R_i$ is discrete and $\Delta$ is a fundamental domain for $\Gamma^*$. Further let $\Gamma \subset \Gamma^*$ be the conformal subgroup. then $\Delta \cup R_i(\Delta)$ is a fundamental domain for $\Gamma$.

Before proving the theorem we will outline the steps of the proof. First we create a space $X$ with pairs $(g, z) \in \Gamma^* \times \Delta$. Then we show that $X$ is homeomorphic to $\mathcal{H}$ and we get a covering $\mathcal{H} \rightarrow \mathcal{H}/\Gamma^*$. So by theorem 58 we get that $\Gamma^*$ is properly discontinuous and by theorem 44 $\Gamma$ is discrete.
Proof. We will start with the pairs \((g, z) \in \Gamma^* \times \Delta\). We can think of this as disjoint copies of \(\Delta\), \((g, \Delta)\). Let \(X\) be the space with elements \(\langle g, z \rangle\) given by

(i) \(\{ (g, z) \} \) if \(z \in \overset{0}{\Delta}\)

(ii) \(\{ (g, z), (gR_i, z) \} \) if \(z \in \overset{i}{\Delta}\)

(iii) \(\{ (g, z), (gR_i, z), (gR_iR_{i+1}, z) \ldots, (g(R_iR_{i+1})^{m_i+2-1}, z) \} \) if \(z = v_{i+2}\).

That is, we identify pairs \((g_1, z), (g_2, z)\) if \(g_1(z) = g_2(z)\). Let \(\tilde{g} \colon X \to X\) defined by \(\tilde{g} : \langle h, z \rangle \to \langle gh, z \rangle\). The group \(G\) of \(\tilde{g}\) is isomorphic to \(\Gamma^*\) since if \(\tilde{g} = \tilde{f}\) then if \(z \in \overset{0}{\Delta}\)

\[ \langle g, z \rangle = \tilde{g} \langle Id, z \rangle = \tilde{f} \langle Id, z \rangle = \langle f, z \rangle \]
and so \(g = f\). Now it's easy to see that

\[ \bigcup_{\tilde{g} \in G} \tilde{g} \langle Id, \Delta \rangle = X \quad (5.3) \]

and if \(\tilde{g} \neq \tilde{f}\) then

\[ \tilde{g} \langle Id, \Delta \rangle \cap \tilde{f} \langle Id, \Delta \rangle = \emptyset. \quad (5.4) \]

Let

\[ N_i = \{ z \in \Delta | \rho(z, z_i) < \epsilon \} \quad (5.5) \]
for some $\varepsilon$. Suppose that $z_i = v_1$. Then for sufficiently small $\varepsilon$

$$N_1 \cup R_3N_1 \cup R_2R_3N_1 \cup \cdots \cup (R_2R_3)^{m_1-1}N_1 = B(v_1, \varepsilon).$$

Similarly if $z_i \in M_j$ then $N_i \cup R_jN_i = B(z_i, \varepsilon)$ and if $z_i \in \Delta$ then $N_i = B(z_i, \varepsilon)$. We note that $N_i \subset B(z_i, \varepsilon)$. Now let $h : X \to \mathcal{H}$ be defined by $h : \langle g, z \rangle \mapsto g(z)$. Let $A \subset \mathcal{H}$ be open. Then

$$h^{-1}(A) = \{ \langle g, z \rangle | g(z) \in A \}$$

which is open since for each $z_i$ there exists $N_i$, $g(N_i) \subset B(g(z_i), \varepsilon) \subset A$ and $\{ \langle g, z \rangle | z \in N_i \}$ is open. That means $h$ is continuous.

Let $B \subset X$ be open. Then

$$h(B) = \{ g(z) | \langle g, z \rangle \in B \}$$

which is open since each $\{ \langle g, z \rangle | z \in N_i \}$ maps to $B(z_i, \varepsilon)$ where $N_i$ is as above. The last two results mean that $h$ is a local homeomorphism. Now we will look at the bijectivity. Assume that $h \langle g_1, z_1 \rangle = h \langle g_2, z_2 \rangle$. Then

$$g_1(z_1) = g_2(z_2) \Rightarrow z_1 = g_1^{-1}g_2(z_2).$$

It’s easy to see that $z_1 = z_2$. If $z_1 \in \overset{\circ}{\Delta}$ then $g_1^{-1}g_2 = Id$ and $g_1 = g_2$. If $z_1 \in M_i$ then $g_1 = g_2$ or $g_1 = g_2R_i$ and similarly when $z_1 = v_i$. Thus we conclude that $\langle g_1, z_1 \rangle = \langle g_2, z_2 \rangle$, that is $h$ is injective.

Now let $w \in \mathcal{H}$. Then there exists a path $\gamma$ such that $\gamma(0) = z \in \Delta$ and $\gamma(1) = w$. We note that we can lift $\gamma(t) \in \Delta$ to a path $\tilde{\gamma}(t) \in X$ by

$$\tilde{\gamma}(t) = (Id, \gamma(t)).$$

As $(Id, \gamma(t)) = (R_i, \gamma(t))$ if $\gamma(t) \in M_i$ we see that we can continue this lifting as

$$\tilde{\gamma}(t) = (R_i, R_i(\gamma(t)))$$

if $\gamma(t) \in R_i(\Delta)$. Continuing this way we see that there exist $g \in \Gamma^*$ such that $\tilde{\gamma}(1) = \langle g, z \rangle$ and $g(z) = \gamma(t) = w$. Thus we can conclude that $h$ is surjective.

We have shown that $h$ is a homeomorphism and with equations 5.3 and 5.4 we see that

$$\bigcup_{g \in \Gamma^*} g(\Delta) = \mathcal{H} \quad \text{(5.6)}$$

$$\overset{\circ}{g(\Delta)} \cap f(\overset{\circ}{\Delta}) = \emptyset. \quad \text{(5.7)}$$
Now let $p : \mathcal{H} \to \mathcal{H}/\Gamma^*$ be the canonical projection. For each point $z \in \mathcal{H}/\Gamma^*$ there is a neighbourhood $N$ constructed as above such that

$$h^{-1}(N) = \bigcup_i B(z_i, \varepsilon)$$

where $z_i$ are the points in the orbit of $z$. For sufficiently small $\varepsilon$ the open sets $B(z_i, \varepsilon)$ are pairwise disjoint. Each $B(z_i, \varepsilon)/\Lambda$ is homeomorphic to $N$ where $\Lambda$ is the stabilizer of $z$ in $\Gamma^*$. Thus $p$ is a branched covering and by theorem 58 $\Gamma^*$ is properly discontinuous and by theorem 44 discrete.

Now we finally look at $\Gamma \subset \Gamma^*$, the conformal subgroup. $\Gamma$ is discrete as $\Gamma^*$ is discrete. Thus we only have to show that $\Delta \cup R_i(\Delta)$ is a fundamental domain for $\Gamma$. Let $g \in \Gamma^*$ be anti-conformal. Then $gR_i$ is conformal and thus $gR_i = T \in \Gamma$ which means $g = TR_i$. It follows that

$$\mathcal{H} = \bigcup_{g \in \Gamma^*} g(\Delta) =$$

$$\left( \bigcup_{T \in \Gamma} T(\Delta) \right) \cup \left( \bigcup_{T \in \Gamma} T(R_i(\Delta)) \right) =$$

$$\bigcup_{T \in \Gamma} T(\Delta \cup R_i(\Delta)).$$

Further it is easy to see that if $T \neq S$ then $T(\hat{F}) \cap S(\hat{F}) = \emptyset$, where $F = \Delta \cup R_i(\Delta)$. We conclude that $\Gamma$ is a Fuchsian group and $\Delta \cup R_i(\Delta)$ is a fundamental domain for $\Gamma$. $\blacksquare$

With Poincaré’s theorem we get the following theorem.
Theorem 74 If $g \geq 0$ and $m_i \geq 2$ are integers and if

$$2g - 2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) > 0 \quad (5.8)$$

then there exists a Fuchsian group $\Gamma$ with signature $(g; m_1, m_2, \ldots, m_r)$.

We will end this chapter with a few examples.

Example 75 Let $P$ be the polygon with side pairing we get with the union of triangles and with the side pairing in figure 5.5. We see that all the vertices which are endpoints to the sides $a_i, b_i$ are congruent and with angle sum $2\pi$. We can then scale the polygon such that

$$\mu(P) = 2\pi \left( 2g - 2 + \sum_{i=1}^{r} \left( 1 - \frac{1}{m_i} \right) \right).$$

We note that the cycle conditions are satisfied and by theorem 72 $P$ is a fundamental domain and the group $\Gamma$ generated by the side pairing is a Fuchsian group. With presentation we have

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, x_1, \ldots, x_r | x_1^{m_1} = \cdots = x_r^{m_r} = \Pi_i x_i \Pi_j [a_j, b_j] = Id \rangle.$$
Figure 5.5: Fundamental polygon
Figure 5.6: The poincaré disc tesselated by the modular group [4]
Figure 5.7: The poincaré disc tesselated by the \((2, 3, 7)\) triangle group [4]
Conclusions and Further work

We have seen how Riemann surfaces can be expressed as quotient spaces with the use of discrete groups. We have also seen how hyperbolic triangles with side reflections generate discrete groups. The procedure we have used can easily be used for spherical or euclidian triangles as well. It is also possible to extend it to arbitrary starlike polygons.

Further the theorem has many applications. It can be used as a tool to find automorphisms of a surface, in the work with Teichmüller spaces and in many other applications.
Bibliography


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