Linköping Studies in Science and Technology. Dissertations
No. 1044

Asymptotic analysis of solutions to elliptic and parabolic problems

Peter Rand

INSTITUTE OF TECHNOLOGY
LINKÖPING UNIVERSITY

Matematiska institutionen
Linköpings universitet, SE-581 83 Linköping, Sweden

Linköping 2006

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Matematiska institutionen
Linköpings universitet
SE-581 83 Linköping, Sweden
peran@mai.liu.se

ISBN 91-85523-04-6
ISSN 0345-7524

Printed by UniTryck, Linköping 2006

Abstract

In the thesis we consider two types of problems. In Paper 1, we study small solutions to a time-independent nonlinear elliptic partial differential equation of Emden-Fowler type in a semi-infinite cylinder. The asymptotic behaviour of these solutions at infinity is determined. First, the equation under the Neumann boundary condition is studied. We show that any solution small enough either vanishes at infinity or tends to a nonzero periodic solution to a nonlinear ordinary differential equation. Thereafter, the same equation under the Dirichlet boundary condition is studied, the nonlinear term and right-hand side now being slightly more general than in the Neumann problem. Here, an estimate of the solution in terms of the right-hand side of the equation is given. If the equation is homogeneous, then every solution small enough tends to zero. Moreover, if the cross-section is star-shaped and the nonlinear term in the equation is subject to some additional constraints, then every bounded solution to the homogeneous Dirichlet problem vanishes at infinity.

In Paper 2, we study asymptotics as $t \to \infty$ of solutions to a linear, parabolic system of equations with time-dependent coefficients in $\Omega \times (0, \infty)$, where Ω is a bounded domain. On $\partial\Omega \times (0, \infty)$ we prescribe the homogeneous Dirichlet boundary condition. For large values of t, the coefficients in the elliptic part are close to time-independent coefficients in an integral sense which is described by a certain function $\kappa(t)$. This includes in particular situations when the coefficients may take different values on different parts of Ω and the boundaries between them can move with t but stabilize as $t \to \infty$. The main result is an asymptotic representation of solutions for large t. As a corollary, it is proved that if $\kappa \in L^1(0, \infty)$, then the solution behaves asymptotically as the solution to a parabolic system with time-independent coefficients.

Acknowledgements

I would like to thank my supervisors Vladimir Kozlov and Mikael Langer for all their support and invaluable hints during this work and Anders Björn and Jonna Gill for helping me with LaTeX. Thanks also to everyone else who has helped me in some way.

Contents

Introduction

Most differential equations and systems are not possible to solve exactly. Hence, it is important to develop other methods of analyzing the properties of solutions. One of these methods is based on asymptotic analysis. Although the solutions are unknown, it may be possible to find information about their behaviour as some variables tend to some finite value or to infinity.

Asymptotic analysis is used to study time-dependent evolution problems as well as time-independent stationary problems. Frequently, one is interested in behaviour of solutions as time tends to infinity, for example in questions concerning stability, periodicity, rate of growth etc. A survey of evolution problems and a general theory of analyzing them, including asymptotic analysis, can be found in Dautray, Lions [1], [2] or Lions, Magenes [8], [9], [10]. An important class of evolution problems are reaction-diffusion problems. Such occur frequently in biology and chemistry, see for example Fife [3] or Murray [11]. Important contributions to asymptotic methods for evolution problems can be found in Friedman [4], Pazy [12] and Vishik [13].

In this thesis, we use an approach developed in Kozlov, Maz'ya, [6], [7]. Starting with a linear or nonlinear equation or system of equations, the problem is reduced to first order ordinary differential equations with operator coefficients. Then, by use of a spectral splitting, a finite dimensional system of first order ordinary differential equations perturbed by a small integro-differential term is obtained for the leading term. The main difficulty is to perform the above reduction and the study of the system of ordinary differential equations for the leading term. The main result is that the asymptotic behaviour of solutions of the initial system of equations is described by solutions of the above finite dimensional system.

We use and extend this approach. In paper 1 we consider a nonlinear Emden-Fowler type time-independent partial differential equation in a semi-infinite cylinder and study the asymptotics of solutions when the unbounded coordinate tends to infinity. The equation is complemented by the Neumann or Dirichlet boundary condition. We analyze the asymptotic behaviour of a given, small solution of the problem. In the Neumann case, we obtain a nonlinear ordinary differential equation for the leading term in the asymptotics of solution. We also find an estimate for the remainder term. From this asymptotic formula it follows that the solution behaves asymptotically like a periodic solution. In the Dirichlet case we show that small solutions decrease exponentially. We also consider the case of a star-shaped cross section and show that if the nonlinear term in the equation is subject to some additional constraints, then every bounded solution of the homogeneous Dirichlet problem vanishes at infinity. The use of Pohožaev's identity is essential in the proof.

Paper 2 is devoted to the study of a linear parabolic system of equations in a bounded domain under Dirichlet boundary conditions and with prescribed initial values. We consider the asymptotic behaviour of solutions as time tends to infinity. The elliptic part of the system is here considered as a perturbation of time-independent coefficients. We consider a larger class of perturbations than Kozlov, Maz'ya [6]. Smallness of the perturbations is assumed only in integral sense. In particular, we include such situations when the leading coefficients may take different values on different parts of Ω and the boundaries between them can move with t but stabilize as $t \to \infty$. Here we use another reduction than Kozlov, Maz'ya [6] to obtain the first order system of ordinary differential equations perturbed by an integro-differential term for the leading terms. Then an approach from Kozlov [5] is used to study the asymptotic behaviour of solutions to this system.

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Paper 1

Asymptotic analysis of a nonlinear partial differential equation in a semicylinder

Peter Rand

Abstract

We study small solutions of a nonlinear partial differential equation in a semi-infinite cylinder. The asymptotic behaviour of these solutions at infinity is determined. First, the equation under the Neumann boundary condition is studied. We show that any solution small enough either vanishes at infinity or tends to a nonzero periodic solution of a nonlinear ordinary differential equation. Thereafter, the same equation under the Dirichlet boundary condition is studied, but now the nonlinear term and right-hand side are slightly more general than in the Neumann problem. Here, an estimate of the solution in terms of the right-hand side of the equation is given. If the equation is homogeneous, then every solution small enough tends to zero. Moreover, if the cross-section is star-shaped and the nonlinear term in the equation is subject to some additional constraints, then every bounded solution of the homogeneous Dirichlet problem vanishes at infinity. An estimate for the solution is given.

1 Introduction

Let Ω be a bounded domain in $\mathbf{R}^{n-1}$ with C^2-boundary. We define the semi-infinite cylinder $\mathcal{C}_+ = \{x = (x', x_n) : x' \in \Omega, x_n > 0\}$. In Section 2 we study bounded solutions of the equation

$$\Delta U + q(U)U = H \quad \text{in } \mathcal{C}_+ \tag{1.1}$$

under the boundary condition

$$\frac{\partial U}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty). \tag{1.2}$$

Our aim is to describe the asymptotic behaviour as $x_n \to \infty$ of solutions U of problem (1.1), (1.2) subject to

$$|U(x)| \le \Lambda \quad \text{for } x \in \mathcal{C}_+, \tag{1.3}$$

where Λ is a positive constant.

We assume that $q(u) > 0$ if $u \neq 0$. Moreover, q is continuous and

$$|s|, |t| \leq \Lambda \Rightarrow |q(s)s - q(t)t| \leq C_\Lambda |s-t| \tag{1.4}$$

with $C_\Lambda < \lambda_1$. Here, λ_1 is the first positive eigenvalue of the Neumann problem for the operator

$$-\Delta' = -\sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2}$$

in Ω.

We set $\mathcal{C}_t = \Omega \times (t, t+1)$ and define $L^r_{\text{loc}}(\mathcal{C}_+)$, $1 \leq r \leq \infty$, as the space of functions which belong to $L^r(\mathcal{C}_t)$ for every $t \geq 0$. We also suppose $H \in L^p_{\text{loc}}(\mathcal{C}_+)$ and

$$\int_0^\infty (1+s) \|H\|_{L^p(\mathcal{C}_s)} \, ds < \infty, \tag{1.5}$$

where

$$\begin{cases} p > n/2 & \text{if } n \geq 4 \\ p = 2 & \text{if } n = 2, 3. \end{cases} \tag{1.6}$$

The main result of Section 2 is Theorem 2.1, which states that one of two alternatives is valid:

1. U admits the asymptotic representation

   $$U(x) = u_h(x_n) + w(x) \qquad \text{as } x_n \to +\infty,$$

   where u_h is a nonzero periodic solution of

   $$u_h'' + q(u_h)u_h = 0$$

   and $w \to 0$ as $x_n \to \infty$. An estimate for the remainder term w is given.

2. $U \to 0$ as $x_n \to \infty$. An estimate for U is given in the theorem.

If, for example, $H = 0$ and $C_\Lambda \to 0$ as $\Lambda \to 0$, then Corollary 2.3 gives the following estimate for U in the second case:

$$|U(x', x_n)| \leq C_\epsilon e^{-\sqrt{\lambda_1 - \epsilon}\, x_n},$$

where ϵ is an arbitrary small positive number and C_ϵ is a constant depending on ϵ.

In Sections 3.1-3.4 we study solutions U of (1.1) subject to (1.3) under the Dirichlet boundary condition

$$U = 0 \qquad \text{on } \partial\Omega \times (0, \infty). \tag{1.7}$$

Now we suppose that q is continuous and that

$$|q(v)| \leq C_\Lambda \quad \text{if } |v| \leq \Lambda,$$

where $C_\Lambda < \lambda_D$. Here, λ_D is the first eigenvalue of the Dirichlet problem for $-\Delta'$ in Ω. We assume also that $\|H\|_{L^p(\mathcal{C}_t)}$, with p as in (1.6), is a bounded function of t, $t \geq 0$. The main result is Theorem 3.1 which gives an explicit bound for $\|U\|_{L^\infty(\mathcal{C}_t)}$ in terms of the function $\|H\|_{L^p(\mathcal{C}_t)}$. This implies in particular that $\|U\|_{L^\infty(\mathcal{C}_t)} \to 0$ as $t \to \infty$ if the same is valid for $\|H\|_{L^p(\mathcal{C}_t)}$. If $H = 0$ and q(0)=0, then the estimate from Theorem 3.1 implies that

$$|U(x', x_n)| \leq C_\epsilon e^{-\sqrt{\lambda_D - \epsilon}\, x_n}, \tag{1.8}$$

where $\epsilon > 0$ is arbitrary.

In Section 3.5 we study all bounded solutions of (1.1), (1.7). Here we suppose additionally that $n \geq 4$, that the domain Ω is star-shaped with respect to the origin, that the function q is continuous with $q(0) = 0$ and that $q(u)$ is positive for $u \neq 0$. We also assume that

$$\frac{n-3}{2} q(u)u^2 - (n-1) \int_0^u q(v)v \, dv \geq \epsilon \, q(u)u^2 \tag{1.9}$$

for some $\epsilon > 0$. Then Theorem 3.4 states that every bounded solution of (1.1), (1.7) with $H = 0$ satisfies (1.8). Some examples of functions satisfying (1.9) are

- $q(u) = |u|^p, \quad p > \frac{4}{n-3},$
- $q(u) = |u|^p e^{|u|}, \quad p > \frac{4}{n-3},$
- $q(u) = |u|^p (e^{|u|} - 1), \quad p > \frac{7-n}{n-3},$
- linear combinations with positive coefficients of the functions above.

A natural question: under which conditions on q is it possible to remove ϵ in the relation (1.8)? The aim of Section 3.6 is to study a similar question for the ordinary differential equation

$$u'' - \lambda u + q(u)u = 0, \tag{1.10}$$

where $\lambda > 0$, q is continuous with $q(0) = 0$ but $q(u) > 0$ for $u \neq 0$ and

$$\int_{-1}^{1} \frac{q(u)}{|u|} \, du < \infty.$$

Theorem 3.5 states that every solution u of (1.10) subject to $u(t) \to 0$ as $t \to \infty$ satisfies

$$|u(t)| + |u'(t)| = O\left(e^{-\sqrt{\lambda} t}\right)$$

for large positive t.

The problem (1.1) under the boundary conditions (1.2) or (1.7) with $q(U) = |U|^{p-1}$, $p > 1$ has been studied in Kozlov [14]. There it is shown that the restriction (1.3) is essential for Theorems 2.1 and 3.1. One of the goals of this thesis is to extend some results from [14] to the equation (1.1).

The equation

$$\Delta u - a|u|^{q-1}u = 0 \quad \text{in } \mathcal{C}_+, \tag{1.11}$$

where $q > 1, a > 0$ and with the boundary condition (1.2) is considered in Kondratiev [11]. Furthermore, the problem

$$\begin{cases} Lu = 0 & \text{in } \mathcal{C}_+ \\ \dfrac{\partial u}{\partial \nu} + a|u|^{q-1}u = 0 & \text{on } \partial\Omega \times (0, \infty), \end{cases}$$

where L is an elliptic partial differential operator, $a > 0$ and $q > 1$ are constants is studied in Kondratiev [12]. In both these cases it is proved that the solutions of these problems have asymptotics of the form $u(x', x_n) = Cx_n^{-\sigma}$ with $\sigma > 0$. This shows that the minus sign in (1.11) essentially changes the asymptotic behaviour of solutions at infinity.

There is a lot of research on positive solutions of nonlinear problems in an infinite cylinder and other unbounded domains. We direct the reader to Bandle and Essén [3], Berestycki [4], Berestycki, Caffarelli and Nirenberg [5], Berestycki, Larrouturou and Roquejoffre [6], Berestycki and Nirenberg [7] and Kondratiev [13] where also further references can be found.

Small global solutions of the equation

$$\Delta u + \lambda u + f(u, u_x, u_y) = 0$$

in a two-dimensional strip with homogeneous Dirichlet boundary conditions are studied in Amick, Toland [2] and Kirchgässner, Scheurle [10].

2 The Neumann problem

2.1 Notation

The Laplace operator and the gradient in $\mathbf{R}^n$ are denoted by Δ and ∇, respectively. For the corresponding operators in $\mathbf{R}^{n-1}$ we introduce

$$\Delta' = \sum_{k=1}^{n-1} \frac{\partial^2}{\partial x_k^2}$$

and

$$\nabla' = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_{n-1}} \right).$$

By Ω we denote a bounded domain in $\mathbf{R}^{n-1}$ with C^2-boundary and $(n-1)$-dimensional Lebesgue measure $|\Omega|$. We introduce the cylinder

$$\mathcal{C} = \{(x', x_n) : x' \in \Omega \text{ and } x_n \in \mathbf{R}\}$$

and the semicylinder

$$\mathcal{C}_+ = \{(x', x_n) : x' \in \Omega \text{ and } x_n > 0\}.$$

We let ν and ν' denote the outward unit normals to $\partial\mathcal{C}$ and $\partial\Omega$, respectively. Thus $\nu \in \mathbf{R}^n$ while $\nu' \in \mathbf{R}^{n-1}$ and $\nu = (\nu', 0)$. After introducing $\mathcal{C}_t = \Omega \times (t, t+1)$, we say that a function $u : \mathbf{R}^n \to \mathbf{R}$ belongs to $L^r_{\text{loc}}(\mathcal{C})$ or $W^{k,r}_{\text{loc}}(\mathcal{C})$, $1 \le r \le \infty$ and $k = 0, 1, \ldots$, if it belongs to $L^r(\mathcal{C}_t)$ or $W^{k,r}(\mathcal{C}_t)$ for every $t \in \mathbf{R}$.

2.2 Problem formulation and assumptions

Assume that p is subject to (1.6). We study the asymptotic behaviour as $x_n \to \infty$ of solutions $U \in W^{2,p}_{\text{loc}}(\mathcal{C}_+)$ of the problem

$$\begin{cases} \Delta U + q(U)U = H & \text{in } \mathcal{C}_+ \\ \dfrac{\partial U}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty) \end{cases} \tag{2.1}$$

satisfying (1.3).

We assume that q is continuous and positive for $u \neq 0$ and satisfies (1.4). We suppose further that $H \in L^p_{\text{loc}}(\mathcal{C}_+)$ is subject to (1.5)

In order to motivate (1.6), let us consider a bounded solution $U \in W^{1,2}_{\text{loc}}(\mathcal{C}_+)$ of (2.1). By Lemma A.16 in Section A.3 we get that $U \in W^{2,p}_{\text{loc}}(\mathcal{C}_+)$. Furthermore, it follows from well-known Sobolev inequalities, see for example Theorem 5.6 in Evans [8], that, since $p > n/2$, there exists a positive γ such that either $U \in C^{0,\gamma}(\mathcal{C}_t)$ or $U \in C^{1,\gamma}(\mathcal{C}_t)$ for every $t > 0$. Hence it is meaningful to assume that the studied solution belongs to $W^{2,p}_{\text{loc}}(\mathcal{C}_+)$ and is bounded.

2.3 The main asymptotic result

The aim of Section 2 is to prove the following theorem concerning the asymptotic behaviour of solutions of (2.1) subject to (1.3):

Theorem 2.1 *Suppose that $U \in W^{2,p}_{\text{loc}}(\mathcal{C}_+)$, where p satisfies* (1.6)*, is a solution of* (2.1) *subject to* (1.3)*. Suppose also that q is continuous, $q(u) > 0$ if $u \neq 0$ and that the Lipschitz condition* (1.4) *is fulfilled. Finally, assume that $H \in L^p_{\text{loc}}(\mathcal{C}_+)$ satisfies* (1.5)*. Then one of the following alternatives is valid:*

1. *$U(x) = u_h(x_n) + w(x)$, where u_h is a nonzero periodic solution of*

$$u_h'' + q(u_h)u_h = 0$$

and

$$\|w\|_{L^\infty(\mathcal{C}_t)} \le C\left(\int_t^\infty s\|H\|_{L^p(\mathcal{C}_s)}\,ds + t\int_0^t e^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)}\|H\|_{L^p(\mathcal{C}_s)}\,ds + te^{-\sqrt{\lambda_1 - C_\Lambda}t}\right)$$

for $t \ge 1$. The right-hand side tends to 0 as $t \to \infty$.

2. *$\|U(\cdot, x_n)\|_{L^\infty(\Omega)} \to 0$ as $x_n \to \infty$. Furthermore, $U(x) = u_0(x_n) + w(x)$, where*

$$\frac{(u_0'(t))^2}{2} + \int_0^{u_0(t)} q(v)v\,dv \le C\left(\int_t^\infty \|H\|_{L^p(\mathcal{C}_s)}\,ds + \int_0^t e^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)}\|H\|_{L^p(\mathcal{C}_s)}\,ds + e^{-\sqrt{\lambda_1 - C_\Lambda}t}\right) \tag{2.2}$$

and

$$\|w\|_{L^\infty(\mathcal{C}_t)} \le C\left(\int_0^\infty e^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|}\|H\|_{L^p(\mathcal{C}_s)}\,ds + e^{-\sqrt{\lambda_1 - C_\Lambda}t}\right) \tag{2.3}$$

for $t \ge 1$.

The proof of this theorem is contained in Sections 2.5 - 2.9.

Theorem 2.1 is a generalization of Theorem 3 in Kozlov [14], where the case $q(U) = |U|^{p-1}$, $p > 1$ is studied. We use the same approach in this thesis. Since most of the proofs in [14] are brief or absent, we present here complete proofs of all assertions. Our restriction (1.5) is different from the corresponding restriction in Theorem 3 [14]. This rigorous analysis of the proofs indicates that possibly (1.5) is the right assumption also in [14].

In the next section we give some corollaries of Theorem 2.1.

2.4 Corollaries of Theorem 2.1

Corollary 2.2 *Suppose, in addition to the conditions in Theorem 2.1, that the nonlinear term q has the property that the constant C_Λ in (1.4) tends to 0 as Λ tends to 0. Then the estimates (2.2) and (2.3) can be improved, namely, the constant $\sqrt{\lambda_1 - C_\Lambda}$ can be replaced by $\sqrt{\lambda_1 - \epsilon}$ where $\epsilon > 0$ is arbitrary. In this case, the constant C appearing in (2.2) and (2.3) is dependent of ϵ.*

Proof. Since $\|U(\cdot, x_n)\|_{L^\infty(\Omega)} \to 0$ as $x_n \to \infty$ we can apply the theorem for the semicylinder $\Omega \times (T, \infty)$ where T is sufficiently large and Λ small enough. □

Corollary 2.3 *Suppose, in addition to the conditions in Corollary 2.2, that $H = 0$. If case 2 in Theorem 2.1 occurs, then, for every $\epsilon \in (0, \lambda_1)$, there exists a constant C_ϵ such that*

$$|U(x', x_n)| \le C_\epsilon e^{-\sqrt{\lambda_1 - \epsilon}\, x_n} \quad \text{for } x_n \ge 1. \tag{2.4}$$

Proof. We begin with proving that there exists a constant A_ϵ such that

$$|U(x', x_n)| \le A_\epsilon e^{-\frac{1}{2}\sqrt{\lambda_1 - \epsilon}\, x_n} \quad \text{for } x_n \ge 1. \tag{2.5}$$

From (2.2) it follows that

$$|u_0'(t)| \le C e^{-\frac{1}{2}\sqrt{\lambda_1 - C_\Lambda}\, t}$$

and

$$u_0(t) \to 0 \quad \text{as } t \to \infty.$$

Since

$$u_0(t) = -\int_t^\infty u_0'(s)\, ds$$

we get

$$|u_0(t)| \le C e^{-\frac{1}{2}\sqrt{\lambda_1 - C_\Lambda}\, t}. \tag{2.6}$$

Furthermore, (2.3) gives

$$\|w\|_{L^\infty(\mathcal{C}_t)} \le C e^{-\sqrt{\lambda_1 - C_\Lambda} t}, \tag{2.7}$$

where C does not depend on t. Since

$$U(x) = u_0(x_n) + w(x)$$

we get from (2.6) and (2.7) that

$$|U(x', x_n)| \le C e^{-\frac{1}{2}\sqrt{\lambda_1 - C_\Lambda}\, x_n}. \tag{2.8}$$

Using that $C_\Lambda \to 0$ as $\Lambda \to 0$ and considering problem (2.1) in a semicylinder $\Omega \times (t_0, \infty)$, where t_0 is sufficiently large, we can suppose that $C_\Lambda < \epsilon$. Then the estimate (2.5) follows from (2.8).

We set

$$\mathcal{C}_t' = \Omega \times (t + 1/4, t + 3/4).$$

Lemma A.16 in Section A.3 implies that

$$\|U\|_{W^{2,2}(\mathcal{C}_t')} \le C(\|U\|_{L^2(\mathcal{C}_t)} + \|q(U)U\|_{L^2(\mathcal{C}_t)})$$

so from (1.4) it follows that

$$\|U\|_{W^{2,2}(\mathcal{C}_t')} \le C\|U\|_{L^2(\mathcal{C}_t)}$$

and by using (2.5) we get the estimate

$$\|U\|_{W^{2,2}(\mathcal{C}'_t)} \le Ce^{-\frac{1}{2}\sqrt{\lambda_1-\epsilon}\,t}.$$

Corollary 6.2.5 in Kozlov and Maz'ya [15], with the parameters $k_- = 0$, $k_+ = \sqrt{\lambda_1}$, $m_- = 2$ and $m_+ = 1$ together with the fact that R can be made suitably small, implies that

$$\|U\|_{W^{2,2}(\mathcal{C}'_t)} \le Ce^{-\sqrt{\lambda_1-\epsilon}\,t}. \tag{2.9}$$

We can now use local estimates and an iteration procedure as in the proof of Lemma A.16 in Section A.3 to obtain

$$\|U\|_{L^\infty(\mathcal{C}''_t)} \le C\|U\|_{W^{2,2}(\mathcal{C}'_t)},$$

where $\mathcal{C}''_t = \Omega \times (t+3/8, t+5/8)$. Combination of the last estimate and (2.9) gives

$$\|U\|_{L^\infty(\mathcal{C}_t)} \le Ce^{-\sqrt{\lambda_1-\epsilon}\,t}$$

if $t \ge 3/8$. This implies (2.4). □

2.5 The corresponding problem in $\mathcal{C}$

We now begin proving Theorem 2.1. Before turning to equation (2.1), we study a solution $u \in W^{2,p}_{\text{loc}}(\mathcal{C})$ of the problem

$$\begin{cases} \Delta u + q(u)u = h & \text{in } \mathcal{C} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{C} \end{cases} \tag{2.10}$$

satisfying

$$\sup_{x\in\mathcal{C}} |u(x)| \le \Lambda, \tag{2.11}$$

where Λ is the same constant as in (1.3) and p is subject to (1.6). By the Sobolev embedding theorem the solution is continuous. Thus we do not need to use essential supremum.

As before, we assume that q is continuous and positive for $u \neq 0$ and satisfies (1.4). We also suppose that $h \in L^p_{\text{loc}}(\mathcal{C})$ and that

$$\int_{-\infty}^{\infty} (1+|s|)\|h\|_{L^p(\mathcal{C}_s)}\, ds < \infty. \tag{2.12}$$

Theorem A.8 in Section A.1 states that there exists an ON-basis of $L^2(\Omega)$ consisting of eigenfunctions of the operator $-\Delta'$ for the Neumann problem in Ω. Let $\phi_0(\cdot, x_n)$ denote the eigenfunction with $L^2(\Omega)$-norm equal to 1 corresponding to the eigenvalue $\lambda_0 = 0$, i.e. $\phi_0 = |\Omega|^{-1/2}$. Set $\overline{u}$

to the orthogonal projection of u onto the subspace of $L^2(\Omega)$ spanned by ϕ_0, that is

$$\overline{u}(x_n) = \frac{1}{|\Omega|}\int_\Omega u(x', x_n)\, dx',$$

and define $v(x)$ by the equality

$$u(x) = \overline{u}(x_n) + v(x). \tag{2.13}$$

Inserting (2.13) in (2.10) and integrating over Ω, we obtain

$$\overline{u}''(x_n) + \frac{1}{|\Omega|}\int_\Omega \Delta v(x', x_n)\, dx' + \frac{1}{|\Omega|}\int_\Omega f(u(x', x_n))\, dx' = \overline{h}(x_n), \tag{2.14}$$

where

$$\overline{h}(x_n) = \frac{1}{|\Omega|}\int_\Omega h(x', x_n)\, dx'$$

and

$$f(t) = q(t)t. \tag{2.15}$$

Due to the homogeneous boundary condition in (2.10), Greens formula gives

$$\int_\Omega \Delta' u\, dx' = 0.$$

Therefore

$$\begin{aligned}\frac{1}{|\Omega|}\int_\Omega \Delta v\, dx' &= \frac{1}{|\Omega|}\int_\Omega \Delta' v\, dx' + \frac{d^2}{dx_n^2}\left(\frac{1}{|\Omega|}\int_\Omega v\, dx'\right)\\ &= \frac{1}{|\Omega|}\int_\Omega \Delta' u\, dx' = 0.\end{aligned}$$

Equation (2.14) can now be written as

$$\overline{u}''(x_n) + \frac{1}{|\Omega|}\int_\Omega f(u(x', x_n))\, dx' = \overline{h}(x_n) \tag{2.16}$$

and by defining

$$K(\overline{u}, v)(x_n) = f(\overline{u}(x_n)) - \overline{f(\overline{u} + v)}(x_n) \tag{2.17}$$

we get

$$\overline{u}''(x_n) + f(\overline{u}(x_n)) = \overline{h}(x_n) + K(\overline{u}, v)(x_n). \tag{2.18}$$

We will often write $K(x_n)$ or K instead of $K(\overline{u}, v)(x_n)$.

Using (2.10), the equation $\Delta v = h - f(\overline{u} + v) - \overline{u}''$ is obtained which, together with (2.16), implies that

$$\begin{cases}\Delta v = -M(\overline{u}, v) + h_1 & \text{in } \mathcal{C}\\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\mathcal{C},\end{cases} \tag{2.19}$$

where

$$M(\overline{u}, v)(x) = f(\overline{u}(x_n) + v(x)) - \overline{f(\overline{u} + v)}(x_n) \tag{2.20}$$

and

$$h_1 = h - \overline{h}. \tag{2.21}$$

The equations (2.18) and (2.19) will play a central role in the sequel.

2.6 An auxiliary ordinary differential equation

In this section we study the equation

$$\xi''(t) + q(\xi(t))\xi(t) = g(t), \quad t \geq t_0, \tag{2.22}$$

where $t_0 \geq 1$ is given. We assume that $q(v)$ is positive for every $v \in \mathbf{R}$, possibly except for $v = 0$, and continuous. We also assume that $q(v)v$ is Lipschitz continuous on every finite interval and that

$$\int_{t_0}^{\infty} |sg(s)|\, ds < \infty. \tag{2.23}$$

Since solutions of (2.22) with $g = 0$ will play an important role in the asymptotic representation of ξ as $t \to \infty$, we will now describe them. We have thus the equation

$$\xi''(t) + q(\xi(t))\xi(t) = 0, \quad t \geq t_0. \tag{2.24}$$

Multiplying (2.24) by ξ' and integrating, we obtain

$$\frac{1}{2}(\xi'(t))^2 + \int_{t_0}^{t} q(\xi)\xi\xi'\, ds = \frac{1}{2}(\xi'(t_0))^2.$$

Using that

$$\frac{dG(\xi(t))}{dt} = q(\xi)\xi\xi', \tag{2.25}$$

where

$$G(v) = \int_0^v q(w)w\, dw,$$

we get

$$\frac{1}{2}(\xi'(t))^2 + G(\xi(t)) = c_0,$$

where c_0 is constant. If

$$c_0 < \min\left\{\int_0^{\infty} q(s)s\, ds, \int_0^{-\infty} q(s)s\, ds\right\}, \tag{2.26}$$

then ξ and ξ' are bounded and ξ is periodic. Conversely, if we know in advance that ξ is a bounded solution, then c_0 satisfies (2.26). This fact and an even more general result will be deduced in the proof of Theorem 2.4.

We also note that if the right-hand side of (2.26) is infinite, then every solution of (2.24) is periodic.

In the following theorem we describe the asymptotic behaviour of bounded solutions of (2.22).

Theorem 2.4 *Let ξ be a bounded solution of* (2.22)*. Then one of the two following alternatives occurs:*

1. $\xi(t) = \xi_h(t) + w(t)$*, where $\xi_h(t)$ is a nonzero periodic solution of* (2.24) *and*
$$|w(t)| + |w'(t)| = O\left(\int_t^\infty |sg(s)|\, ds\right) \tag{2.27}$$
as $t \to \infty$.

2. *Both $\xi(t)$ and $\xi'(t)$ tend to* 0 *as $t \to \infty$ and*
$$\frac{(\xi'(t))^2}{2} + \int_0^{\xi(t)} q(v)v\, dv = O\left(\int_t^\infty |g(s)|\, ds\right). \tag{2.28}$$

The remaining part of this section is devoted to the proof of this theorem. We start with the following lemma:

Lemma 2.5 *Let ξ be a bounded solution of* (2.22)*. Then*
$$\frac{1}{2}(\xi'(t))^2 + G(\xi(t)) = c_0 + O\left(\int_t^\infty |g(s)|\, ds\right) \tag{2.29}$$
as $t \to \infty$, where c_0 is a nonnegative constant depending on t_0 and ξ.

Proof. We begin with proving that ξ' is bounded. Multiplying (2.22) by ξ' and integrating, we obtain
$$\frac{1}{2}(\xi'(t))^2 - \int_{t_0}^t g(s)\xi'(s)\, ds = \frac{1}{2}(\xi'(t_0))^2 - \int_{t_0}^t q(\xi(s))\xi(s)\xi'(s)\, ds. \tag{2.30}$$
Now (2.25) implies that
$$\int_{t_0}^t q(\xi(s))\xi(s)\xi'(s)\, ds = G(\xi(t)) - G(\xi(t_0)),$$
where $G(\xi(t))$ is uniformly bounded in t due to the boundedness of ξ. This means that also the left hand side of (2.30) is bounded in t.

Set
$$C_T = \sup_{t_0 \le t \le T} |\xi'(t)|.$$
We have for $t_0 \le t \le T$
$$\int_{t_0}^t g\xi'\, ds \le C_T \int_{t_0}^\infty |g|\, ds,$$

which is finite because of (2.23). Therefore, from (2.30) it follows that

$$\frac{1}{2}(\xi'(t))^2 - C_T \int_{t_0}^{\infty} |g|\,ds \le C$$

with a constant C independent of t and T. Taking supremum over $t_0 \le t \le T$ we get

$$\frac{1}{2}C_T^2 - C_T \int_{t_0}^{\infty} |g|\,ds \le C$$

which gives an upper bound for C_T independent of T. Thus $\xi'(t)$ is bounded for all $t \ge t_0$.

The equation (2.30) is equivalent to

$$\frac{1}{2}(\xi'(t))^2 + G(\xi(t)) = \frac{1}{2}(\xi'(t_0))^2 + G(\xi(t_0)) + \int_{t_0}^{t} g\xi'\,ds. \tag{2.31}$$

From (2.23) and the boundedness of ξ', we get

$$\int_{t_0}^{t} g\xi'\,ds = C_1 - \int_{t}^{\infty} g\xi'\,ds = C_1 + O\left(\int_{t}^{\infty} |g|\,ds\right)$$

for some constant C_1. This equality applied to (2.31) finally implies (2.29). By letting $t \to \infty$ it follows that $c_0 \ge 0$. □

Proof of Theorem 2.4: Since $\xi(t)$ is bounded, there exists a number L such that

$$|\xi(t)| \le L, \quad t \ge t_0. \tag{2.32}$$

We rewrite (2.22) as the system of first order equations

$$\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = g(t) - q(y_1(t))y_1(t), \end{cases}$$

where $y_1 = \xi$ and $y_2 = \xi'$. In polar coordinates, $y_1 = r\cos\phi$, $y_2 = r\sin\phi$, the above system takes the form

$$\begin{cases} r'\cos\phi - r\phi'\sin\phi = r\sin\phi \\ r'\sin\phi + r\phi'\cos\phi = g - q(r\cos\phi)r\cos\phi. \end{cases}$$

This implies that

$$\begin{cases} r' = g\sin\phi + (1 - q(r\cos\phi))r\sin\phi\cos\phi \\ \phi' = \dfrac{g\cos\phi}{r} - q(r\cos\phi)\cos^2\phi - \sin^2\phi. \end{cases} \tag{2.33}$$

Define the function ρ as

$$\rho = \frac{r^2\sin^2\phi}{2} + \int_0^{r\cos\phi} q(v)v\,dv \tag{2.34}$$

and observe that $\rho \geq 0$. Since

$$\rho(t) = \frac{1}{2}(\xi'(t))^2 + \int_0^{\xi(t)} q(v)v\,dv,$$

it follows from Lemma 2.5 that

$$\rho(t) = c_0 + O\left(\int_t^\infty |g(s)|\,ds\right) \tag{2.35}$$

with $c_0 \geq 0$. If $c_0 = 0$, then the second alternative in the theorem is valid. Suppose now that $c_0 > 0$.

It is more convenient to use the variables (ρ, ϕ) instead of (r, ϕ). The second equation of (2.33) becomes

$$\phi' = \frac{g\cos\phi}{r(\rho,\phi)} - F(\rho,\phi), \tag{2.36}$$

where

$$F(\rho,\phi) = q(r(\rho,\phi)\cos\phi)\cos^2\phi + \sin^2\phi. \tag{2.37}$$

For readability we will make an abuse of notation and sometimes consider r as a function of t and sometimes of ρ and ϕ.

We now show that c_0 satisfies (2.26). If both integrals in (2.26) are infinite, this is obvious. Suppose that one of them is finite. First we conclude that there exists a positive number r_0 and a $t_1 \geq t_0$ such that

$$r(t) \geq r_0 \tag{2.38}$$

for $t \geq t_1$. Indeed, if $r(t_j) \to 0$ for some sequence $\{t_j\}_{j=2}^\infty$ then also $\rho(t_j) \to 0$ which contradicts (2.35). Thus (2.38) follows.

Integrating (2.36) from t_1 to t, where $t \geq t_1$, we obtain

$$\phi(t) = \phi(t_1) + \int_{t_1}^t gr^{-1}\cos\phi\,ds - \int_{t_1}^t q\cos^2\phi\,ds - \int_{t_1}^t \sin^2\phi\,ds. \tag{2.39}$$

From (2.23) and (2.38) it follows that the first integral in (2.39) has a finite limit as $t \to \infty$. Let us show that one of the last two integrals tends to infinity as $t \to \infty$. Suppose that

$$\int_{t_1}^\infty \sin^2\phi\,ds < \infty.$$

Then $l(\{t \geq t_1 : \sin^2\phi(t) > 1/2\}) < \infty$, where $l(D)$ denotes the Lebesgue measure of D. Thus $l(E) = \infty$ for $E = \{t \geq t_1 : \cos^2\phi(t) \geq 1/2\}$. From (2.38) and (2.32) it then follows that

$$r_0/\sqrt{2} \leq |r\cos\phi| \leq L$$

on E. Therefore, there exists a positive constant q_0 such that $q(r\cos\phi) \geq q_0$ on E, which implies that

$$\int_{t_1}^{\infty} q\cos^2\phi\, ds \geq \frac{q_0}{2} l(E) = \infty.$$

This proves that

$$\phi(t) \to -\infty \tag{2.40}$$

as $t \to \infty$.

Because of (2.38) and (2.40), we can find sequences $\{\tau_j\}, \{s_j\}$ such that $\tau_j, s_j \to \infty$ as $j \to \infty$ and $r(\tau_j)\sin\phi(\tau_j) = r(s_j)\sin\phi(s_j) = 0$, $r(\tau_j)\cos\phi(\tau_j) < 0$ but $r(s_j)\cos\phi(s_j) > 0$. Equations (2.34) and (2.32) then imply that

$$\rho(s_j) = \int_0^{r(s_j)\cos\phi(s_j)} q(v)v\, dv \leq \int_0^L q(v)v\, dv < \int_0^\infty q(v)v\, dv.$$

Analogously,

$$\rho(\tau_j) \leq \int_0^{-L} q(v)v\, dv < \int_0^{-\infty} q(v)v\, dv.$$

Since $\rho(t) \to c_0$ as $t \to \infty$, we have

$$\begin{aligned} c_0 &\leq \min\left\{\int_0^L q(v)v\, dv, \int_0^{-L} q(v)v\, dv\right\} \\ &< \min\left\{\int_0^\infty q(v)v\, dv, \int_0^{-\infty} q(v)v\, dv\right\} \end{aligned}$$

and (2.26) is proved.

By (2.26) there exists an ϵ such that

$$0 < \epsilon < \min\left\{\int_0^\infty q(v)v\, dv, \int_0^{-\infty} q(v)v\, dv\right\} - c_0.$$

Next, we prove that there exists positive constants A_1, A_2, B_1 and B_2 such that

$$A_1 \leq r(\rho, \phi) \leq A_2 \tag{2.41}$$

and

$$B_1 \leq F(\rho, \phi) \leq B_2 \tag{2.42}$$

if $c_0 - \epsilon \leq \rho \leq c_0 + \epsilon$. By the same argument as in the proof of the existence of r_0 in (2.38), we can show that there exists a positive A_1 such that $r \geq A_1$. To prove the right inequality in (2.41) we introduce $r_1(\phi)$ as the unique solution of the equation

$$\rho(r, \phi) = c_0 + \epsilon.$$

That this equation has one solution $r = r_1(\phi)$ for every ϕ is a consequence from these three facts which all follow from (2.34):

1. $\rho(\cdot,\phi)$ is strictly increasing.

2. $\rho(r,\phi) \to 0$ as $r \to 0$ for every ϕ.

3. We have for $\phi \neq n\pi$

$$\rho \to \infty \qquad \text{as } r \to \infty$$

and for $\phi = n\pi$

$$\rho \to \int_0^{\pm\infty} q(v)v\,dv,$$

where both integrals are larger than $c_0 + \epsilon$.

Since ρ is continuous, so is r_1. We can thus define

$$A_2 = \max_{\phi\in[0,2\pi]} r_1(\phi)$$

and conclude that the constant A_2 satisfies the right inequality in (2.41).

The existence of B_2 follows from (2.37) together with the continuity of q and the right inequality in (2.41). To show the existence of B_1 we proceed as follows. Suppose that there exists sequences $\{\rho_j\}_{j=1}^{\infty}$ and $\{\phi_j\}_{j=1}^{\infty}$ with

$$\rho_j \in [c_0 - \epsilon, c_0 + \epsilon], \quad j = 1, 2, \ldots$$

such that

$$F(\rho_j,\phi_j) = \frac{q(\rho_j,\phi_j)r(\rho_j,\phi_j)^2\cos^2\phi_j + r(\rho_j,\phi_j)^2\sin^2\phi_j}{r(\rho_j,\phi_j)^2} \to 0$$

as $j \to \infty$. Here we use the abbreviation $q(\rho,\phi)$ for $q(r(\rho,\phi)\cos\phi)$. Due to (2.41), this implies that

$$\begin{cases} q(\rho_j,\phi_j)\cos^2\phi_j \to 0 \\ \sin^2\phi_j \to 0. \end{cases}$$

This can happen only if $\cos\phi_j \to 0$ and $\sin\phi_j \to 0$ which is impossible so the left inequality in (2.42) is proved.

Let us show that

$$F(\rho,\phi) = F(c_0,\phi) + O(|\rho - c_0|) \tag{2.43}$$

for $|\rho - c_0| < \epsilon$. In order to make the computations somewhat visually clearer, we set

$$r = r(\rho,\phi) \qquad r_0 = r(c_0,\phi)$$
$$q = q(r\cos\phi) \qquad q_0 = q(r_0\cos\phi)$$

and obtain

$$\begin{aligned}|F(\rho,\phi)-F(c_0,\phi)| &= \left|q\cos^2\phi - q_0\cos^2\phi\right| \\ &\le \left|\frac{qr_0r\cos\phi - q_0r_0r\cos\phi}{rr_0}\right| \\ &\le \frac{r_0|qr\cos\phi - q_0r_0\cos\phi| + |r_0-r||q_0r_0\cos\phi|}{rr_0}.\end{aligned} \tag{2.44}$$

From (2.34) we get that

$$\frac{\partial r}{\partial \rho} = (rF)^{-1}$$

and by (2.41) and (2.42), the right-hand side is bounded uniformly in ϕ. Therefore

$$|r-r_0| \le C|\rho - c_0|, \tag{2.45}$$

where C is a constant independent of r, ρ and ϕ. Using the Lipschitz continuity of the function $q(v)v$ and (2.45), we derive from (2.44) the relation (2.43).

If ρ and ϕ are considered as functions of t, the representation (2.35) shows that

$$F(\rho,\phi) = F(c_0,\phi) - h(t),$$

where

$$h(t) = O\left(\int_t^\infty |g|\,ds\right). \tag{2.46}$$

Now choose $t_2 \ge t_0$ such that

$$|\rho(t) - c_0| \le \epsilon$$

if $t \ge t_2$ and set

$$g_1 = r^{-1}g\cos\phi. \tag{2.47}$$

The equation (2.36) can now be rewritten as

$$\phi' = -F(c_0,\phi) + g_1(t) + h(t).$$

This implies that

$$\begin{aligned}\int_{\phi(t_2)}^{\phi(t)} \frac{d\psi}{F(c_0,\psi)} &= \int_{t_2}^{t}\left(-1 + \frac{g_1(s)}{F(c_0,\phi(s))} + \frac{h(s)}{F(c_0,\phi(s))}\right) ds \\ &= -t + t_2 + \int_{t_2}^{t} \frac{g_1(s)}{F(c_0,\phi(s))}\,ds + \int_{t_2}^{t}\frac{h(s)}{F(c_0,\phi(s))}\,ds.\end{aligned} \tag{2.48}$$

Using (2.41), (2.42) and (2.47), we get

$$\int_{t_2}^{t} \frac{g_1(s)}{F(c_0,\phi(s))}\,ds = C_1 - \int_t^\infty \frac{g\cos\phi}{rF(c_0,\phi)}\,ds = C_1 - O\left(\int_t^\infty |g(s)|\,ds\right). \tag{2.49}$$

Furthermore, the relations (2.42) and (2.46) imply that

$$\int_{t_2}^{t} \frac{h(s)}{F(c_0,\phi(s))}\,ds = C_2 + O\left(\int_t^\infty |h(s)|\,ds\right) = C_2 + O\left(\int_t^\infty |sg(s)|\,ds\right). \tag{2.50}$$

By using (2.49) and (2.50), we derive from (2.48) the relation

$$\int_0^{\phi(t)} \frac{d\psi}{F(c_0,\psi)} = c_1 - t + O\left(\int_t^\infty |sg(s)|\,ds\right), \quad t \ge t_2. \tag{2.51}$$

Let us rewrite the equation (2.24) in the variables ρ and ϕ. The above calculations can be used with $g = 0$. Equations (2.35) and (2.51) become

$$\begin{cases} \rho_h(t) = a_0 \\ \displaystyle\int_0^{\phi_h(t)} \frac{d\psi}{F(c_0,\psi)} = a_1 - t, \quad t \ge t_0. \end{cases} \tag{2.52}$$

We show that the homogeneous equation (2.24) has a solution which satisfies (2.52) with $a_0 = c_0$ and $a_1 = c_1$. First choose $\rho_h(t_0)$ and $\phi_h(t_0)$ such that

$$\begin{cases} \rho_h(t_0) = c_0 \\ t_0 + \displaystyle\int_0^{\phi_h(t_0)} \frac{d\psi}{F(c_0,\psi)} = c_1. \end{cases}$$

The last equation is solvable because of (2.42). We can now reconstruct $\xi_h(t_0)$ and $\xi_h'(t_0)$ and take these as the Cauchy data for (2.24).

The equality (2.35) and the first equality in (2.52) (with $a_0 = c_0$) imply that

$$\rho(t) = \rho_h(t) + O\left(\int_t^\infty |g(s)|\,ds\right) \tag{2.53}$$

and by using (2.42) we obtain

$$|\phi(t) - \phi_h(t)| \le B_2 \left| \int_{\phi_h(t)}^{\phi(t)} \frac{d\psi}{F(c_0,\psi)} \right|.$$

Furthermore, (2.51) together with the second equation in (2.52) (with $a_1 = c_1$) gives

$$\int_{\phi_h(t)}^{\phi(t)} \frac{d\psi}{F(c_0,\psi)} = O\left(\int_t^\infty |sg(s)|\,ds\right).$$

Therefore we have

$$\phi(t) = \phi_h(t) + O\left(\int_t^\infty |sg(s)|\, ds\right). \tag{2.54}$$

The next step is to study the relation between r and $r_h = r(\rho_h, \phi_h)$. Expanding $r(\rho, \phi)$ near (ρ_h, ϕ_h) and using (2.53), (2.54), we obtain

$$r(t) = r_h(t) + O\left(\int_t^\infty |sg(s)|\, ds\right).$$

This, together with the fact that $\xi = r\cos\phi$, $\xi' = r\sin\phi$, finally gives

$$\begin{cases} \xi(t) = \xi_h(t) + O\left(\int_t^\infty |sg(s)|\, ds\right) \\ \xi'(t) = \xi_h'(t) + O\left(\int_t^\infty |sg(s)|\, ds\right) \end{cases}$$

and the theorem follows. □

2.7 The equation for v

We now study solutions v of (2.19). In Section 2.5 we introduced M and h_1 by (2.20) and (2.21). It is straightforward to check that $h_1 \in L^p_{\text{loc}}(\mathcal{C})$,

$$\int_\Omega h_1(x', x_n)\, dx' = 0, \tag{2.55}$$

$$\int_\Omega M\, dx' = 0, \tag{2.56}$$

$$\int_\Omega v(x', x_n)\, dx' = 0$$

and

$$M = w - \overline{w}, \tag{2.57}$$

where

$$w = f(u) - f(\overline{u}) \tag{2.58}$$

and f is given by (2.15). Furthermore, by noting that $\|h_1\|_{L^p(\mathcal{C}_t)} \leq 2\|h\|_{L^p(\mathcal{C}_t)}$, we get from (2.12) that

$$\int_{-\infty}^\infty (1 + |s|)\|h_1\|_{L^p(\mathcal{C}_s)}\, ds < \infty. \tag{2.59}$$

Clearly, the function v belongs to $L^\infty(\mathcal{C})$ and we have the following result:

Lemma 2.6 *The function v in* (2.19) *satisfies the estimate*

$$\|v\|_{L^\infty(\mathcal{C}_t)} \leq C \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|} \|h\|_{L^p(\mathcal{C}_s)}\, ds, \tag{2.60}$$

where C depends on p, n, Ω, Λ and C_Λ.

Proof. We start by proving the inequality

$$\|v(\cdot, x_n)\|_{L^2(\Omega)} \leq \frac{1}{2\sqrt{\lambda_1 - C_\Lambda}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1 - C_\Lambda}|x_n - s|} \|h_1(\cdot, s)\|_{L^2(\Omega)}\, ds. \tag{2.61}$$

Defining w as in (2.58), inequality (1.4) implies that $|w| \leq C_\Lambda |v| \leq 2\Lambda C_\Lambda$. It follows from (2.57) that $|M| \leq 4\Lambda C_\Lambda$, which together with (2.59) gives that

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \| - M + h_1\|_{L^2(\mathcal{C}_s)}\, ds < \infty.$$

Also, the equations (2.55) and (2.56) imply that

$$\int_\Omega (-M(\overline{u}, v)(x', x_n) + h_1(x', x_n))\, dx' = 0.$$

Because of the orthogonality between $w - \overline{w}$ and $\overline{w}$ in $L^2(\Omega)$, we get from (2.57) and the Pythagorean theorem that $\|M(\overline{u}, v)(\cdot, x_n)\|_{L^2(\Omega)} \leq \|w(\cdot, x_n)\|_{L^2(\Omega)}$. The condition (1.4) then gives

$$\|M(\overline{u}, v)(\cdot, x_n)\|_{L^2(\Omega)} \leq C_\Lambda \|v(\cdot, x_n)\|_{L^2(\Omega)}.$$

This and Lemma A.12 in Section A.2 show that (2.19) has a solution v fulfilling

$$\begin{aligned} &\|v(\cdot, x_n)\|_{L^2(\Omega)} \\ &\leq \frac{1}{2\sqrt{\lambda_1}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|x_n - s|} \big(\|h_1(\cdot, s)\|_{L^2(\Omega)} + C_\Lambda \|v(\cdot, s)\|_{L^2(\Omega)}\big)\, ds. \end{aligned} \tag{2.62}$$

Inserting the right-hand side of this expression in the last occurrence of $\|v(\cdot, s)\|_{L^2(\Omega)}$ and iterating, we obtain $\|v(\cdot, x_n)\|_{L^2(\Omega)} \leq \sum_{k=0}^\infty T_k$, where

$$\begin{aligned} T_k = {} & \frac{C_\Lambda^k}{(2\sqrt{\lambda_1})^{k+1}} \\ & \times \int_{\mathbf{R}^{k+1}} e^{-\sqrt{\lambda_1}(|x_n - t_0| + \sum_{j=0}^{k-1} |t_j - t_{j+1}|)} \|h_1(\cdot, t_k)\|_{L^2(\Omega)}\, dt_0\, dt_1 \cdots dt_k. \end{aligned}$$

The variable transformation $t_j = s + s_{j+1}$ for $j = 0, \ldots, k-1$, $t_k = s$ applied to (2.62), gives

$$\|v(\cdot, x_n)\|_{L^2(\Omega)} \leq \int_{-\infty}^{\infty} G(x_n - s) \|h_1(\cdot, s)\|_{L^2(\Omega)}\, ds, \tag{2.63}$$

where

$$G(t) = \frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|t|} + \sum_{k=1}^{\infty} \frac{C_\Lambda^k}{(2\sqrt{\lambda_1})^{k+1}} \times \int_{\mathbf{R}^k} e^{-\sqrt{\lambda_1}(|t-s_1|+|s_1-s_2|+\ldots+|s_{k-1}-s_k|+|s_k|)}\, ds_1\, ds_2 \ldots ds_k. \quad (2.64)$$

In order to calculate the function $G(t)$ we consider the two differential operators $-\frac{d^2}{dt^2}+\lambda_1-C_\Lambda$ and $-\frac{d^2}{dt^2}+\lambda_1$. Clearly, their fundamental solutions are

$$g(t) = \frac{1}{2\sqrt{\lambda_1 - C_\Lambda}} e^{-\sqrt{\lambda_1 - C_\Lambda}|t|}$$

and

$$h(t) = \frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|t|}.$$

Therefore

$$-\frac{dg^2}{dt^2} + \lambda_1 g(t) = \delta(t) + C_\Lambda g(t),$$

which implies that $g = h * (\delta + C_\Lambda g)$, i.e.

$$g(t) = h(t) + C_\Lambda \int_{-\infty}^{\infty} h(t-s) g(s)\, ds.$$

Inserting this expression for $g(t)$ into the right-hand side and repeating this procedure, we obtain

$$g(t) = \frac{1}{2\sqrt{\lambda_1}} e^{-\sqrt{\lambda_1}|t|} + \sum_{k=1}^{\infty} \frac{C_\Lambda^k}{(2\sqrt{\lambda_1})^{k+1}} \times \int_{\mathbf{R}^k} e^{-\sqrt{\lambda_1}(|t-s_1|+|s_1-s_2|+\ldots+|s_{k-1}-s_k|+|s_k|)}\, ds_1\, ds_2 \ldots ds_k,$$

where the right-hand side coincides with the right-hand side of (2.64). Hence,

$$G(t) = \frac{1}{2\sqrt{\lambda_1 - C_\Lambda}} e^{-\sqrt{\lambda_1 - C_\Lambda}|t|}$$

and (2.61) now follows from (2.63).

In the remaining part of this section, let C denote a generic constant depending only on p, n, Ω, Λ and C_Λ. We define $\mathcal{C}_t' = \Omega \times (t+1/4, t+3/4)$ and $\mathcal{C}_t'' = \Omega \times (t+1/8, t+7/8)$. From Corollary A.18 in Section A.3 it follows that

$$\|v\|_{L^\infty(\mathcal{C}_t')} \le C(\|v\|_{L^p(\mathcal{C}_t'')} + \|M\|_{L^p(\mathcal{C}_t'')} + \|h\|_{L^p(\mathcal{C}_t'')}).$$

Furthermore, since $|w| \le C_\Lambda |v|$ and $\|\overline{w}\|_{L^p(\mathcal{C}_t'')} \le |\Omega|^{-1/p} \|w\|_{L^p(\mathcal{C}_t'')}$, it follows from (2.57) that

$$\|M\|_{L^p(\mathcal{C}_t'')} \le C_\Lambda (1 + |\Omega|^{-1/p}) \|v\|_{L^p(\mathcal{C}_t'')}.$$

Thus,

$$\|v\|_{L^\infty(\mathcal{C}_t')} \le C(\|v\|_{L^p(\mathcal{C}_t'')} + \|h\|_{L^p(\mathcal{C}_t'')})$$

and, after iterating as in the proof of Lemma A.16 in Section A.3, we get

$$\|v\|_{L^\infty(\mathcal{C}_t')} \le C(\|v\|_{L^2(\mathcal{C}_t)} + \|h\|_{L^p(\mathcal{C}_t)}). \tag{2.65}$$

We are now in position to perform the last step of the proof of (2.60). Begin by looking at $\|h\|_{L^p(\mathcal{C}_t)}$ and set $\mathcal{C}_{t,1} = \Omega \times (t, t+1/2)$. For $\tau \in (t-1/2, t)$ we have $\|h\|_{L^p(\mathcal{C}_{t,1})} \le \|h\|_{L^p(\mathcal{C}_\tau)}$ and integrating from $t-1/2$ to t we get

$$\|h\|_{L^p(\mathcal{C}_{t,1})} \le 2e^{\sqrt{\lambda_1 - C_\Lambda}/2} \int_{t-1/2}^{t} e^{-\sqrt{\lambda_1 - C_\Lambda}|t-\tau|} \|h\|_{L^p(\mathcal{C}_\tau)}\, d\tau.$$

With a similar estimate of $\|h\|_{L^p(\mathcal{C}_{t,2})}$, where $\mathcal{C}_{t,2} = \Omega \times (t+1/2, t+1)$, we get

$$\|h\|_{L^p(\mathcal{C}_t)} \le C \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1 - C_\Lambda}|t-\tau|} \|h\|_{L^p(\mathcal{C}_\tau)}\, d\tau. \tag{2.66}$$

To find an estimate for $\|v\|_{L^2(\mathcal{C}_t)}$, we use (2.61) together with Minkowski's inequality yielding

$$\begin{aligned}\|v\|_{L^2(\mathcal{C}_t)} &= \left(\int_t^{t+1} \|v(\cdot,\tau)\|_{L^2(\Omega)}^2\, d\tau \right)^{1/2} \\ &\le C \int_{-\infty}^{\infty} \left(\int_t^{t+1} e^{-2\sqrt{\lambda_1 - C_\Lambda}|\tau - s|}\, d\tau \right)^{1/2} \|h(\cdot, s)\|_{L^2(\Omega)}\, ds.\end{aligned}$$

Using the inequality

$$\left(\int_t^{t+1} e^{-2\sqrt{\lambda_1 - C_\Lambda}|\tau - s|}\, d\tau \right)^{1/2} \le Ce^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|}$$

and making the substitution $s \mapsto s + \tau$, we arrive at

$$\|v\|_{L^2(\mathcal{C}_t)} \le C \int_{-\infty}^{\infty} \int_0^1 e^{-\sqrt{\lambda_1 - C_\Lambda}|t-s-\tau|} \|h(\cdot, s+\tau)\|_{L^2(\Omega)}\, d\tau ds.$$

From the inequality

$$\int_0^1 \|h(\cdot, s+\tau)\|_{L^2(\Omega)}\, d\tau \le \|h\|_{L^2(\mathcal{C}_s)},$$

we now obtain

$$\|v\|_{L^2(\mathcal{C}_t)} \le C \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|} \|h\|_{L^2(\mathcal{C}_s)}\, ds.$$

This, together with (2.65) and (2.66), gives the estimate (2.60) with $\|v\|_{L^\infty(\mathcal{C}_t)}$ replaced by $\|v\|_{L^\infty(\mathcal{C}_t')}$. Since

$$\|v\|_{L^\infty(\mathcal{C}_t)} \le \|v\|_{L^\infty(\mathcal{C}_{t-1/2}')} + \|v\|_{L^\infty(\mathcal{C}_t')} + \|v\|_{L^\infty(\mathcal{C}_{t+1/2}')},$$

the inequality for $\|v\|_{L^\infty(\mathcal{C}_t')}$ implies (2.60) and the proof is complete. □

2.8 Asymptotics of small solutions of problem (2.10)

In this section we continue to study solutions of problem (2.10). Our goal is to find asymptotics of u as $x_n \to \infty$.

Lemma 2.7 *Let $u \in W^{2,p}_{\text{loc}}(\mathcal{C})$ be a solution of* (2.10) *subject to* (2.11). *Then either*

1. $u(x) = u_h(x_n) + w(x)$, *where u_h is a nonzero periodic solution of*

$$u_h'' + q(u_h)u_h = 0 \tag{2.67}$$

and

$$\|w\|_{L^\infty(\mathcal{C}_t)} \le C\left(\int_t^\infty s\|h\|_{L^p(\mathcal{C}_s)}\,ds + t\int_{-\infty}^t e^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)}\|h\|_{L^p(\mathcal{C}_s)}\,ds\right) \tag{2.68}$$

for $t \ge 1$

or

2. $\|u(\cdot, x_n)\|_{L^\infty(\Omega)} \to 0$ *as $x_n \to \infty$. If $u = \overline{u} + v$ as before, then*

$$\frac{(\overline{u}'(t))^2}{2} + \int_0^{\overline{u}(t)} q(v)v\,dv \le C\left(\int_t^\infty \|h\|_{L^p(\mathcal{C}_s)}\,ds + \int_{-\infty}^t e^{-\sqrt{\lambda_1 - C_\Lambda}(t-s)}\|h\|_{L^p(\mathcal{C}_s)}\,ds\right) \tag{2.69}$$

and

$$\|v\|_{L^\infty(\mathcal{C}_t)} \le C\int_{-\infty}^\infty e^{-\sqrt{\lambda_1 - C_\Lambda}|t-s|}\|h\|_{L^p(\mathcal{C}_s)}\,ds. \tag{2.70}$$

The right-hand sides of (2.68), (2.69) *and* (2.70) *tend to* 0 *as $t \to \infty$.*

Proof. We set $\alpha = \sqrt{\lambda_1 - C_\Lambda}$ and, as before, represent u as $u = \overline{u} + v$. Let us first study the term $\overline{u}$ which satisfies the equation (2.18). We denote the right-hand side of (2.18) by g, i.e.

$$g(x_n) = \overline{h}(x_n) + K(\overline{u}, v)(x_n)$$

with K given by (2.17). Since

$$K = \frac{1}{|\Omega|}\int_\Omega (q(\overline{u})\overline{u} - q(u)u)\,dx',$$

it follows from (1.4) that $|K(x_n)| \le C\|v(\cdot, x_n)\|_{L^1(\Omega)}$. From (2.61) we get

$$|g(x_n)| \le C\left(|\overline{h}(x_n)| + \int_{-\infty}^\infty e^{-\alpha|x_n - s|}\|h_1(\cdot, s)\|_{L^2(\Omega)}\,ds\right). \tag{2.71}$$

Let us find a bound for $\int_t^\infty |sg(s)|\,ds$ for $t \geq 1$. Obviously,

$$\int_t^\infty |s\bar{h}(s)|\,ds \leq \frac{1}{|\Omega|}\int_t^\infty s\|h(\cdot,s)\|_{L^1(\Omega)}\,ds \leq C\int_t^\infty s\|h(\cdot,s)\|_{L^2(\Omega)}\,ds, \tag{2.72}$$

and

$$\int_t^\infty s\int_{-\infty}^\infty e^{-\alpha|s-\tau|}\|h_1(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau\,ds \leq C\int_{-\infty}^\infty I_1(t,\tau)\|h(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau, \tag{2.73}$$

where

$$I_1(t,\tau) = \int_t^\infty se^{-\alpha|s-\tau|}\,ds. \tag{2.74}$$

For $\tau \leq t$, a direct computation yields

$$I_1 = e^{\alpha\tau}\left(\frac{te^{-\alpha t}}{\alpha} + \frac{e^{-\alpha t}}{\alpha^2}\right) \leq Cte^{-\alpha(t-\tau)}. \tag{2.75}$$

If $\tau \geq t$, we make the substitution $s-\tau \mapsto s$ in (2.74) and obtain

$$I_1(t,\tau) \leq \int_{-\infty}^\infty |s|e^{-\alpha|s|}\,ds + \tau\int_{-\infty}^\infty e^{-\alpha|s|}\,ds \leq 2(\alpha^{-2}+\alpha^{-1})\tau. \tag{2.76}$$

The combination of (2.71), (2.72), (2.73) and the use of the estimates (2.75) and (2.76) for I_1 give

$$\int_t^\infty |sg(s)|\,ds \leq C\left(\int_t^\infty s\|h(\cdot,s)\|_{L^2(\Omega)}\,ds + t\int_{-\infty}^t e^{-\alpha(t-s)}\|h(\cdot,s)\|_{L^2(\Omega)}\,ds\right). \tag{2.77}$$

Next we show that it is possible to replace $\|h(\cdot,s)\|_{L^2(\Omega)}$ by $\|h\|_{L^2(\mathcal{C}_s)}$ in (2.77). Denote the right hand side of (2.77) by $G(t)$ and fix a $\tau \in [0,1]$. Making the variable substitution $s \mapsto s+\tau$ we get

$$G(t) \leq C\left(\int_{t-\tau}^\infty (s+\tau)\|h(\cdot,s+\tau)\|_{L^2(\Omega)}\,ds + t\int_{-\infty}^{t-\tau} e^{-\alpha(t-s)}\|h(\cdot,s+\tau)\|_{L^2(\Omega)}\,ds\right).$$

Since $\tau \in [0,1]$, this implies that

$$G(t) \leq C\left(t\int_{-\infty}^t e^{-\alpha(t-s)}\|h(\cdot,s+\tau)\|_{L^2(\Omega)}\,ds + \int_t^\infty s\|h(\cdot,s+\tau)\|_{L^2(\Omega)}\,ds\right).$$

Therefore

$$\int_t^\infty |sg(s)|\,ds \le C\int_{-\infty}^\infty k(t,s)\|h(\cdot,s+\tau)\|_{L^2(\Omega)}\,ds, \tag{2.78}$$

where

$$k(t,s) = \begin{cases} te^{-\alpha(t-s)} & \text{if } s < t \\ s & \text{if } s \ge t. \end{cases}$$

Integrating (2.78) with respect to τ over $[0,1]$ and using the inequality

$$\int_0^1 \|h(\cdot,s+\tau)\|_{L^2(\Omega)}\,d\tau \le C\|h\|_{L^p(\mathcal{C}_s)},$$

we obtain

$$\int_t^\infty |sg(s)|\,ds \le C\left(\int_t^\infty s\|h\|_{L^p(\mathcal{C}_s)}\,ds + t\int_{-\infty}^t e^{-\alpha(t-s)}\|h\|_{L^p(\mathcal{C}_s)}\,ds\right). \tag{2.79}$$

The right-hand side is bounded because of (2.12). Therefore the assumption (2.23) is verified and we can apply Theorem 2.4 on the function $\overline{u}$. Hence either

1. $\overline{u}(x_n) = u_h(x_n) + w_1(x_n)$, where u_h is a nonzero periodic solution of (2.67) and w_1 satisfies (2.27)

or

2. $\overline{u}(x_n) \to 0$ as $x_n \to \infty$ and (2.28) is valid.

Let us estimate the right-hand sides in (2.27) and (2.28). From (2.79) we get that

$$|w_1(t)| \le C\left(\int_t^\infty s\|h\|_{L^p(\mathcal{C}_s)}\,ds + t\int_{-\infty}^t e^{-\alpha(t-s)}\|h\|_{L^p(\mathcal{C}_s)}\,ds\right)$$

and to obtain the estimate (2.69), we can use that

$$\int_t^\infty |g(s)|\,ds \le C\left(\int_t^\infty \|h\|_{L^p(\mathcal{C}_s)}\,ds + \int_{-\infty}^t e^{-\alpha(t-s)}\|h\|_{L^p(\mathcal{C}_s)}\,ds\right).$$

This follows from (2.71) and calculations similar to those done to obtain the estimate (2.79).

Setting $w(x) = w_1(x_n) + v(x)$ in the first case and using Lemma 2.6 to estimate $\|v\|_{L^\infty(\mathcal{C}_t)}$, we arrive at (2.68). The estimate (2.70) follows directly from Lemma 2.6.

Let us show that the right-hand sides of (2.68), (2.69) and (2.70) tend to 0 as $t \to \infty$. In fact, it is enough to show this for (2.68). Due to (2.12), the first term in the right-hand side of (2.68) tends to 0 as $t \to \infty$. The second term is estimated by

$$
\begin{aligned}
t\int_{-\infty}^{t} e^{-\alpha(t-s)}\|h\|_{L^p(\mathcal{C}_s)}\,ds &\le C\Big(\int_{-\infty}^{M} e^{-\frac{\alpha}{2}(t-s)}(1+|s|)\|h\|_{L^p(\mathcal{C}_s)}\,ds \\
&\quad + \int_{M}^{t} e^{-\frac{\alpha}{2}(t-s)}(1+|s|)\|h\|_{L^p(\mathcal{C}_s)}\,ds\Big) \qquad (2.80)\\
&\le C\Big(\int_{-\infty}^{M} e^{-\frac{\alpha}{2}(t-s)}(1+|s|)\|h\|_{L^p(\mathcal{C}_s)}\,ds + \int_{M}^{\infty}(1+|s|)\|h\|_{L^p(\mathcal{C}_s)}\,ds\Big)
\end{aligned}
$$

for any $M \le t$. Given $\epsilon > 0$, we can choose M so large that the last integral is less than $\epsilon/2$. The integral

$$
\int_{-\infty}^{M} e^{-\frac{\alpha}{2}(t-s)}(1+|s|)\|h\|_{L^p(\mathcal{C}_s)}\,ds
$$

is then majorized by $C\exp(-\alpha t/2)$ (where C depends on M and h) and the expression in the right-hand side of (2.80) is less than ϵ if t is large enough. Hence the right-hand side of (2.68) tends to 0 as $t \to \infty$ and the proof is complete.

□

2.9 End of the proof of Theorem 2.1

We are now in position to complete the proof of Theorem 2.1.

Proof of Theorem 2.1: Let $\eta(t)$ be a smooth function with $0 \le \eta \le 1$, $\eta(t) = 0$ if $t \le 1$ and $\eta(t) = 1$ if $t \ge 2$. We set $u(x) = \eta(x_n)U(x)$. Then

$$
\begin{cases} \Delta u + q(u)u = h & \text{in } \mathcal{C} \\ \partial_\nu u = 0 & \text{on } \partial\mathcal{C}, \end{cases}
$$

where

$$
h = \eta H + q(u)u - \eta q(U)U + 2\eta' U_{x_n} + \eta'' U.
$$

Obviously, $\eta H \in L^p_{\mathrm{loc}}(\mathcal{C})$. The functions $\eta'' U$ and $\chi = q(u)u - \eta q(U)U$ are bounded and equal to 0 for $x_n \le 1$ and $x_n \ge 2$. Furthermore, we have that the function $\eta' U_{x_n}$ belongs to $L^p(\mathcal{C})$ and is also equal to 0 for $x_n \le 1$ and $x_n \ge 2$. Thus, the inequality (2.12) follows from (1.5). Now we can apply Lemma 2.7 on u. In the first case we have the representation $u = u_h + w_1$ where u_h and w_1 are subject to (2.67) and (2.68), respectively. We set

$$
w = \begin{cases} w_1 & \text{if } x_n \ge 2 \\ U - u_h & \text{if } 1 \le x_n < 2 \\ 0 & \text{otherwise.} \end{cases}
$$

Then $w \in L^\infty(\mathcal{C}_+)$ and for $t > 2$,

$$\begin{aligned}\|w\|_{L^\infty(\mathcal{C}_t)} &\le C\left(\int_t^\infty s\|H\|_{L^p(\mathcal{C}_s)}\,ds + t\int_2^t e^{-\alpha(t-s)}\|H\|_{L^p(\mathcal{C}_s)}\,ds\right.\\ &\quad \left.+t\int_0^2 e^{-\alpha(t-s)}\|h\|_{L^p(\mathcal{C}_s)}\,ds\right)\\ &\le C\left(\int_t^\infty s\|H\|_{L^p(\mathcal{C}_s)}\,ds + t\int_0^t e^{-\alpha(t-s)}\|H\|_{L^p(\mathcal{C}_s)}\,ds + te^{-\alpha t}\right).\end{aligned}$$

It is easy to see that this estimate is valid also for $1 \le t \le 2$ so the first case in Theorem 2.1 is proved.

In the second case of Lemma 2.7, we have $u = \overline{u} + v$, where $\overline{u}$ and v are subject to (2.69) and (2.70), respectively. We set $u_0 = \overline{u}$ and

$$w = \begin{cases} v & \text{if } x_n \ge 2 \\ U - \overline{u} & \text{if } 1 \le x_n < 2 \\ 0 & \text{otherwise.} \end{cases}$$

The inequalities (2.2) and (2.3) now follow from (2.69) and (2.70) and the proof is complete.

□

3 The Dirichlet problem

3.1 Problem formulation and assumptions

In Section 3 we study bounded solutions of the Dirichlet problem

$$\begin{cases} \Delta U + q(U)U = H & \text{in } \mathcal{C}_+ \\ U = 0 & \text{on } \partial\Omega \times (0,\infty). \end{cases} \tag{3.1}$$

We assume that U fulfills

$$\sup_{x\in\mathcal{C}_+} |U(x)| \le \Lambda \tag{3.2}$$

and that q is continuous. Let λ_D be the first eigenvalue of the Dirichlet problem for $-\Delta'$ in Ω. We suppose that there is a constant $C_\Lambda < \lambda_D$ such that

$$|v| \le \Lambda \Rightarrow |q(v)| \le C_\Lambda. \tag{3.3}$$

Finally, we assume that $H \in L^p_{\text{loc}}(\mathcal{C}_+)$, where p satisfies (1.6), and that $\|H\|_{L^p(\mathcal{C}_t)}$ is a bounded function of t, $t \ge 0$.

3.2 The main asymptotic result

The main result of Section 3 is the following theorem concerning the asymptotic behaviour of solutions U of (3.1).

Theorem 3.1 *Assume that* $U \in W^{2,p}_{\text{loc}}(\mathcal{C}_+)$*, where* p *satisfies* (1.6)*, is a solution of* (3.1) *subject to* (3.2)*. Assume further that* q *is continuous and that* (3.3) *is satisfied. Also, assume that* $H \in L^p_{\text{loc}}(\mathcal{C}_+)$ *and that* $\|H\|_{L^p(\mathcal{C}_t)}$ *is a bounded function of* t *for* $t \geq 0$*. Then*

$$\|U\|_{L^\infty(\mathcal{C}_t)} \leq C\left(\int_0^\infty e^{-\sqrt{\lambda_D - C_\Lambda}|t-s|}\|H\|_{L^p(\mathcal{C}_s)}\,ds + e^{-\sqrt{\lambda_D - C_\Lambda}t}\right), \quad (3.4)$$

where C *is independent of* t*. In particular, if* $H = 0$ *and* $q(0) = 0$*, then*

$$\|U\|_{L^\infty(\mathcal{C}_t)} = O\left(e^{-\sqrt{\lambda_D - \epsilon}\,t}\right), \quad (3.5)$$

where $\epsilon > 0$ *is arbitrary.*

Remark 3.2 If $\|H\|_{L^p(\mathcal{C}_t)} \to 0$ as $t \to \infty$, then, by the same reasoning as in the end of the proof of Lemma 2.7, we see that the right-hand side of (3.4) tends to 0 as $t \to \infty$.

The proof of Theorem 3.1 is similar to the proof of Theorem 2.1 but shorter. It is contained in Section 3.3 and 3.4.

Theorem 3.1 is a generalization of Theorem 2(i) in Kozlov [14]. As in the Neumann problem, the case $q(U) = |U|^{p-1}$ is studied in [14].

3.3 The corresponding problem in $\mathcal{C}$

As in the Neumann problem in Section 2, we first consider the problem in the whole cylinder. Suppose that $u \in W^{2,p}_{\text{loc}}(\mathcal{C})$ is a solution of the problem

$$\begin{cases} \Delta u + q(u)u = h & \text{in } \mathcal{C} \\ u = 0 & \text{on } \partial\mathcal{C} \end{cases} \quad (3.6)$$

satisfying

$$\sup_{x\in\mathcal{C}} |u(x)| \leq \Lambda \quad (3.7)$$

with the same Λ as in (3.2). We have the following:

Lemma 3.3 *Let* u *be a solution of* (3.6) *fulfilling* (3.7) *and suppose that* $h \in L^p_{\text{loc}}(\mathcal{C})$ *with* $\|h\|_{L^p(\mathcal{C}_t)}$ *bounded on* $\mathbf{R}$*. Then we have the estimate*

$$\|u\|_{L^\infty(\mathcal{C}_t)} \leq C\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_D - C_\Lambda}|t-s|}\|h\|_{L^p(\mathcal{C}_s)}\,ds. \quad (3.8)$$

Proof. We consider the boundary value problem

$$\begin{cases} -\Delta v = g & \text{in } \mathcal{C} \\ v = 0 & \text{on } \partial\mathcal{C}, \end{cases}$$

where

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \|g\|_{L^2(\mathcal{C}_s)}\, ds < \infty.$$

From Section A.2 it follows that it has a unique bounded solution $v \in W^{1,2}_{\text{loc}}(\mathcal{C})$ with

$$\|v(\cdot, x_n)\|_{L^2(\Omega)} \le \frac{1}{2\sqrt{\lambda_D}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_D}|x_n - s|} \|g(\cdot, s)\|_{L^2(\Omega)}\, ds.$$

Applying this estimate to the solution u of the problem (3.6), we obtain

$$\|u(\cdot, x_n)\|_{L^2(\Omega)} \le \frac{1}{2\sqrt{\lambda_D}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_D}|x_n - s|} \big(\|h(\cdot, s)\|_{L^2(\Omega)} + C_\Lambda \|u(\cdot, s)\|_{L^2(\Omega)}\big)\, ds.$$

Iterating this estimate in the same way as in the proof of Lemma 2.6 in Section 2.7, we get

$$\|u(\cdot, x_n)\|_{L^2(\Omega)} \le \frac{1}{2\sqrt{\lambda_D - C_\Lambda}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_D - C_\Lambda}|x_n - s|} \|h(\cdot, s)\|_{L^2(\Omega)}\, ds. \tag{3.9}$$

Now use the local estimate

$$\|u\|_{L^\infty(\mathcal{C}'_t)} \le C(\|u\|_{L^2(\mathcal{C}_t)} + \|h\|_{L^p(\mathcal{C}_t)})$$

(compare with (2.65)) and estimate the term $\|u\|_{L^2(\mathcal{C}_t)}$ by (3.9). Doing the same calculations as in Section 2.7 we arrive at (3.8). □

3.4 End of the proof of Theorem 3.1

We now complete the proof of Theorem 3.1.

Proof of Theorem 3.1: We use the same smooth function η as in the proof of Theorem 2.1 (page 31) and get, for $u = \eta U$, the problem

$$\begin{cases} \Delta u + q(u)u = h & \text{in } \mathcal{C} \\ u = 0 & \text{on } \partial\mathcal{C} \end{cases}$$

where

$$h = \eta H + q(u)u - \eta q(U)U + 2\eta' U_{x_n} + \eta'' U.$$

Clearly, $\|h\|_{L^p(\mathcal{C}_t)}$ is bounded and the estimate (3.4) follows from (3.8).

Suppose now that $H = 0$ and $q(0) = 0$. From (3.4) it follows that

$$\|U\|_{L^\infty(\mathcal{C}_t)} = O\left(e^{-\sqrt{\lambda_D - C_\Lambda} t}\right).$$

By choosing T large enough and considering (3.1) in $\Omega \times (T, \infty)$ instead of $\mathcal{C}_+$, we can make Λ and, since $q(u) \to 0$ as $u \to 0$, also C_Λ arbitrary small. From this (3.5) follows. □

3.5 The case of a star-shaped cross-section

In this section we show that under some special assumptions on Ω and q, every bounded solution of (3.1) with $H = 0$ will satisfy (3.5). This is a generalization of Theorem 2(iii) in [14] where the case $q(U) = |U|^{p-1}$ is studied.

Theorem 3.4 *Suppose that $n \geq 4$ and Ω is star-shaped with respect to the origin and has C^2-boundary. Also assume that q is continuous with $q(0) = 0$, and $q(u) > 0$ otherwise, and that*

$$\frac{n-3}{2} q(u) u^2 - (n-1) \int_0^u q(v) v \, dv \geq \epsilon \, q(u) u^2 \tag{3.10}$$

for some $\epsilon > 0$. Then every bounded solution of (3.1) *with $H = 0$ is subject to* (3.5).

Before proving this theorem we give some examples of functions q satisfying (3.10). Let us check that all functions of the form

$$q(u) = f(|u|)|u|^{a+\delta} \tag{3.11}$$

with $a = 4/(n-3)$, $\delta > 0$ and f being a nondecreasing function satisfy (3.10). Obviously, the function q in (3.11) is even. Therefore also both sides of the inequality (3.10) are even and we can assume that $u \geq 0$. We have

$$\int_0^u q(v) v \, dv \leq f(u) \int_0^u v^{a+1+\delta} \, dv = \frac{f(u) u^{a+2+\delta}}{a+2+\delta}$$

and by using this inequality we obtain

$$\begin{aligned} &\frac{n-3}{2} q(u) u^2 - (n-1) \int_0^u q(v) v \, dv \geq \\ &\frac{n-3}{2} f(u) u^{a+2+\delta} - \frac{(n-1) f(u) u^{a+2+\delta}}{a+2+\delta} = \\ &b f(u) u^{a+2+\delta}, \end{aligned}$$

where

$$b = \frac{\delta(n-3)}{2(a+2+\delta)}.$$

Obviously, $b > 0$ for every $\delta > 0$ since $n \geq 4$. Choosing $\epsilon = b$, we see that (3.10) is fulfilled.

Here are some examples of functions satisfying (3.11):

(i) $q(u) = |u|^p, \quad p > \frac{4}{n-3}$.

(ii) $q(u) = |u|^p e^{|u|}, \quad p > \frac{4}{n-3}$.

(iii) $q(u) = |u|^p (e^{|u|} - 1), \quad p > \frac{7-n}{n-3}$.

(iv) Linear combinations with positive coefficients of functions from (i) - (iii).

Proof of Theorem 3.4: The function u is a solution of the problem

$$\begin{cases} \Delta u + q(u)u = 0 & \text{in } \mathcal{C}_+ \\ u = 0 & \text{on } \partial\Omega \times (0, \infty). \end{cases} \tag{3.12}$$

As before, we set $G(u) = \int_0^u q(v)v\,dv$. Using Pohožaev's identity with respect to the x'-variables, compare with Section 3.2, Kozlov [14], we get

$$\begin{aligned} 0 &= (\Delta' u + f(u))(x' \cdot \nabla' u) + u_{x_n x_n}(x' \cdot \nabla' u) \\ &= \operatorname{div}' \left(\nabla' u (x' \cdot \nabla' u) - x' \frac{|\nabla' u|^2}{2} + x' G(u) \right) + \frac{n-3}{2} |\nabla' u|^2 \\ &- (n-1) G(u) + u_{x_n x_n}(x' \cdot \nabla' u). \end{aligned}$$

This implies that

$$\begin{aligned} 0 &= \operatorname{div} (\nabla u (x' \cdot \nabla' u)) + \operatorname{div}' \left(-\frac{x' |\nabla u|^2}{2} + x' G(u) \right) + u_{x_n}^2 \\ &+ \frac{n-3}{2} |\nabla u|^2 - (n-1) G(u). \end{aligned} \tag{3.13}$$

Set $\mathcal{C}_N = \Omega \times (1, N)$ and $\Gamma_N = \partial\Omega \times (1, N)$ and observe that

$$\nabla u = \frac{\partial u}{\partial \nu}\, \nu \quad \text{on } \Gamma_N.$$

This follows by representing the vector ∇u at a certain point on Γ_N as the sum of a normal and a tangent vector. The tangent vector is then zero because of the homogeneous Dirichlet boundary condition.

Using that $G(u) = 0$ on Γ_N and integrating (3.13) over $\mathcal{C}_N$, we obtain

$$\begin{aligned} 0 = &\int_{\mathcal{C}_N} \left(\frac{n-3}{2} |\nabla u|^2 + u_{x_n}^2 - (n-1) G(u) \right) dx \\ &+ \frac{1}{2} \int_{\Gamma_N} (x \cdot \nu) \left(\frac{\partial u}{\partial \nu} \right)^2 dS + F_1(N) - F_1(1), \end{aligned} \tag{3.14}$$

where

$$F_1(t) = \int_\Omega u_{x_n}(x' \cdot \nabla' u)\, dx' \Big|_{x_n=t}.$$

Multiplying (3.12) by u and using Green's formula, we have

$$0 = \int_{\mathcal{C}_N} \left(-|\nabla u|^2 + q(u)u^2\right) dx + F_2(N) - F_2(1), \tag{3.15}$$

where

$$F_2(t) = \int_\Omega u_{x_n} u\, dx' \Big|_{x_n=t}.$$

A linear combination of (3.14) and (3.15) gives

$$\int_{\mathcal{C}_N} \left(\left(\frac{\partial u}{\partial x_n}\right)^2 + \frac{n-3}{2} q(u)u^2 - (n-1)G(u)\right) dx + \frac{1}{2}\int_{\Gamma_N} \left(\frac{\partial u}{\partial \nu}\right)^2 (x\cdot\nu)\, dS = F(1) - F(N), \tag{3.16}$$

where

$$F(t) = F_1(t) + \frac{n-3}{2} F_2(t).$$

Let us show that F is bounded on $[1,\infty)$. For given $t > 0$, choose $r > n$. Using the Sobolev embedding theorem, we obtain

$$\|\nabla u\|_{L^\infty(\mathcal{C}_t')} \le C\|\nabla u\|_{W^{1,r}(\mathcal{C}_t')} \le C\|u\|_{W^{2,r}(\mathcal{C}_t')}.$$

Now, Lemma A.16 in Section A.3 gives

$$\|u\|_{W^{2,r}(\mathcal{C}_t')} \le C(\|q(u)u\|_{L^r(\mathcal{C}_t)} + \|u\|_{L^2(\mathcal{C}_t)}) \le C_1.$$

Therefore, $\|\nabla u\|_{L^\infty(\mathcal{C}_t')}$ is a bounded function. This implies that F_1 and F_2 are bounded functions on $[1,\infty)$.

Since Ω is star-shaped, it follows that $x'\cdot\nu' \ge 0$. This and (3.10) imply that every term in (3.16) is non-negative and, from the previous analysis, bounded in N. Thus

$$\int_{\mathcal{C}_+} q(u)u^2\, dx < \infty \tag{3.17}$$

and from (3.15) it therefore follows that

$$\int_{\mathcal{C}_t} |\nabla u|^2\, dx \to 0 \quad \text{as } t\to\infty. \tag{3.18}$$

Corollary A.18 in Section A.3 gives

$$\|u\|_{L^\infty(\mathcal{C}_t')} \le C\left(\|q(u)u\|_{L^p(\mathcal{C}_t)} + \|u\|_{L^2(\mathcal{C}_t)}\right). \tag{3.19}$$

From Poincaré's inequality it follows that

$$\|u\|_{L^2(\mathcal{C}_t)} \leq C\|\nabla u\|_{L^2(\mathcal{C}_t)}$$

and using Hölder's inequality, we get

$$\|q(u)u\|^p_{L^p(\mathcal{C}_t)} \leq \|q(u)^{1/2}u\|_{L^2(\mathcal{C}_t)}\|q(u)^{p-1/2}|u|^{p-1}\|_{L^2(\mathcal{C}_t)}.$$

Since u is bounded and q continuous, the right-hand side is also bounded. Applying the last two inequalities for estimating the right-hand side in (3.19), we obtain

$$\|u\|_{L^\infty(\mathcal{C}'_t)} \leq C\left(\|q(u)u^2\|^{1/(2p)}_{L^1(\mathcal{C}_t)} + \|\nabla u\|_{L^2(\mathcal{C}_t)}\right).$$

Using (3.17) and (3.18) we see that $\|u\|_{L^\infty(\mathcal{C}_t)} \to 0$ as $t \to \infty$.

Applying Theorem 3.1 on $v(x', x_n) = u(x', x_n + N)$ for N large enough completes the proof.

□

3.6 An estimate for solutions of a nonlinear ordinary differential equation

The estimate (3.5) contains an arbitrary small parameter ϵ. Is it possible to remove ϵ from this estimate? In order to see what kind of requirements we need on the function q, we consider here the model equation

$$u'' - \lambda u + q(u)u = 0. \tag{3.20}$$

Theorem 3.5 *Let u be a solution of* (3.20) *subject to the condition*

$$u(t) \to 0 \quad \text{as } t \to \infty. \tag{3.21}$$

Suppose that $\lambda > 0$, that the function $f(u) = q(u)u$ is Lipschitz continuous with $f(0) = 0$ and that $q(u) > 0$ if $u \neq 0$. Suppose also that

$$\int_{-1}^{1} \frac{q(u)}{|u|}\,du < \infty. \tag{3.22}$$

Then

$$|u(t)| + |u'(t)| = O\left(e^{-\sqrt{\lambda}t}\right).$$

Proof. We may assume that u is not identically zero. Using (3.21), we see from (3.20) that $u''(t) \to 0$ as $t \to \infty$. Since

$$u(t+1) = u(t) + u'(t) - \int_t^{t+1} (s - t - 1)u''(s)\,ds,$$

we have

$$|u'(t)| \le |u(t)| + |u(t+1)| + \frac{1}{2} \sup_{[t,t+1]} |u''(s)|.$$

Therefore also $u'(t) \to 0$ as $t \to \infty$.

As before, we use the notation

$$G(u) = \int_0^u q(v)v\,dv.$$

We multiply (3.20) by u' and obtain

$$\big((u')^2 - \lambda u^2 + 2G(u)\big)' = 0.$$

Therefore $(u')^2 - \lambda u^2 + 2G(u)$ is a constant. Since the functions u and u' vanish at ∞ we conclude that this constant is 0. Thus

$$u' = \pm\sqrt{\lambda u^2 - 2G(u)}. \tag{3.23}$$

Let us show that the function $\lambda u^2(t) - 2G(u(t))$ has no zeros for t large enough. First, suppose that $u(t_0) = 0$ for some $t_0 \in \mathbf{R}$. Then (3.23) implies that $u'(t_0) = 0$ and we get $u(t) \equiv 0$ by the uniqueness of solutions to the Cauchy problem of equation (3.20). Furthermore, define

$$Q(u) = \frac{1}{\lambda u^2} \int_0^u q(v)v\,dv.$$

Since

$$0 \le Q(u) \le \frac{1}{\lambda} \int_0^u \frac{q(v)}{v}\,dv,$$

we have, because of (3.21) and (3.22), that

$$Q(u(t)) \to 0 \text{ as } t \to \infty.$$

This implies that $\lambda u^2 - 2G(u)$ is positive for large t. From (3.23) it follows that $u'(t)$ has no zeros for large t.

Since $u(t) \to 0$ as $t \to \infty$, we meet here one of two possibilities: Either $u > 0$ and $u' < 0$ or else $u < 0$ and $u' > 0$. We consider the first possibility. The second one is considered analogously. We have

$$u' = -\sqrt{\lambda}\,u\sqrt{1 - 2Q(u)}. \tag{3.24}$$

Power series expansion yields that there exists an $\epsilon > 0$ such that

$$\left|\frac{1}{\sqrt{1-2x}} - 1\right| \le 2|x| \tag{3.25}$$

if $|x| \le \epsilon$. Furthermore, since $Q(v) \to 0$ as $v \to 0$, there exists a $\delta > 0$ such that $|Q(v)| \le \epsilon$ if $|v| \le \delta$. Choose t so large that $|u(t)| \le \delta$. Integrating (3.24), we obtain

$$\frac{1}{\sqrt{\lambda}} \int_u^\delta \frac{dv}{v\sqrt{1-2Q(v)}} = t - C_1. \tag{3.26}$$

The left-hand side can be written as

$$\frac{1}{\sqrt{\lambda}}\int_u^\delta \frac{dv}{v} + \frac{1}{\sqrt{\lambda}}\int_u^\delta \frac{1}{v}\left(\frac{1}{\sqrt{1-2Q(v)}} - 1\right) dv. \tag{3.27}$$

Furthermore, we get from (3.25) that

$$\left|\int_u^\delta \frac{1}{v}\left(\frac{1}{\sqrt{1-2Q(v)}} - 1\right) dv\right| \leq 2\int_{-1}^1 \frac{Q(v)}{|v|}\,dv < \infty, \tag{3.28}$$

where the last inequality follows from (3.22). From (3.26), (3.27) and (3.28) it follows that

$$\ln u = B(t) - \sqrt{\lambda}\, t,$$

where B is a bounded function. Therefore $u(t) = O\left(e^{-\sqrt{\lambda}t}\right)$. Using (3.24), we also obtain that $u'(t) = O\left(e^{-\sqrt{\lambda}t}\right)$. □

A Some results from functional analysis

A.1 Eigenvalues and eigenvectors of $-\Delta$

Here we show that if $\partial\Omega$ is smooth enough there exists an ON-basis of $L^2(\Omega)$ consisting of eigenvectors of the operator $-\Delta$. For the Dirichlet problem this result is often proved in textbooks in partial differential equations so we focus on the Neumann problem and prove it since in this case the proof is not easily found. On our way we need some lemmas which will be stated without proof.

Throughout this appendix, let H denote a Hilbert space and $\langle\cdot,\cdot\rangle_H$ the inner product in H. If A is a bounded linear operator on H, we let $\sigma(A)$ denote the spectrum of A and $\sigma_p(A)$ the set of eigenvalues of A, i.e. the point spectrum.

The following lemmas are well-known results from functional analysis.

Lemma A.1 *Let Ω be an open set in $\mathbf{R}$. Then $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$.*

Definition A.2 If X is a normed space we say that a subset $M \subset X$ is *total* in X if the span of M is dense in X.

Lemma A.3 *If M is total in X and for some $x \in X$ we have that $x \perp M$, then $x = 0$. Conversely, if X is complete and $x \perp M$ implies $x = 0$, then M is total in X.*

Lemma A.4 *Suppose K is a compact linear operator on H. Then $N(I-K)$ is finite dimensional.*

If $\dim H = \infty$ then also

(i) $0 \in \sigma(K)$.

(ii) $\sigma(K) - \{0\} = \sigma_p(K) - \{0\}$.

(iii) $\sigma(K) - \{0\}$ *is either finite or countable with the unique limit point* 0.

Lemma A.5 *Suppose A is a linear, bounded and symmetric operator on H and define*

$$\begin{aligned} m &= \inf\{\langle Au, u\rangle_H : u \in H, \|u\| = 1\} \\ M &= \sup\{\langle Au, u\rangle_H : u \in H, \|u\| = 1\}. \end{aligned}$$

Then m, $M \in \sigma(A) \subset [m, M]$.

Remark A.6 Since A is symmetric, $\langle Au, u\rangle_H$ is always real.

Lemma A.7 *Suppose H is separable and $A : H \to H$ is linear, bounded, symmetric and compact. Then there exists a countable orthonormal basis of H consisting of eigenvectors of A.*

Theorem A.8 *Let Ω be an open, bounded region in $\mathbf{R}^n$ with C^1-boundary. Then there exists an ON-basis $\{\phi_k\}_{k=0}^{\infty}$ of $L^2(\Omega)$ consisting of eigenfunctions of the operator $-\Delta$ for the Neumann problem, i.e.*

$$\int_\Omega \nabla\phi_k \cdot \overline{\nabla v}\, dx = \lambda_k \int_\Omega \phi_k \overline{v}\, dx, \qquad \forall v \in H^1(\Omega), \tag{A.1}$$

and the eigenvalues λ_k are subject to

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \ldots, \qquad \lambda_k \to \infty \text{ as } k \to \infty, \tag{A.2}$$

where each eigenvalue is repeated according to its multiplicity.

Moreover, $\{\phi_k\}_{k=0}^{\infty}$ is also an orthogonal basis of $H^1(\Omega)$.

Proof. Given $u \in H^1(\Omega)$, the functional $v \mapsto \int_\Omega v\overline{u}\, dx$ is linear and bounded on $H^1(\Omega)$. According to Riesz representation theorem, there exists a unique $Tu \in H^1(\Omega)$ such that

$$\int_\Omega v\overline{u}\, dx = \langle v, Tu\rangle_{H^1(\Omega)}, \qquad \forall v \in H^1(\Omega). \tag{A.3}$$

We therefore define the operator $T : H^1(\Omega) \to H^1(\Omega)$ by the relation (A.3). It is easy to see that T is linear and from the closed graph theorem it follows that T is bounded. Namely, let $\Gamma = \{(x, Tx) : x \in H^1(\Omega)\}$ be the graph of T and suppose that $(x_n, y_n) \in \Gamma$, $(x_n, y_n) \to (x, y)$ in $H^1(\Omega) \times H^1(\Omega)$. From (A.3) it follows that $\langle y_n - Tx, y_n - Tx\rangle_{H^1(\Omega)} = \int_\Omega (y_n - Tx)(\overline{x_n} - \overline{x})$ which, together with Hölder's inequality, shows that $y_n \to Tx$ and thus $y = Tx$. Hence the graph is closed.

T is also symmetric, because the equality $\langle Tu, v\rangle_{H^1(\Omega)} = \langle u, Tv\rangle_{H^1(\Omega)}$ follows by combining the fact that $\langle Tu, v\rangle_{H^1(\Omega)} = \overline{\langle v, Tu\rangle_{H^1(\Omega)}}$ with the relation (A.3). Finally, T is compact: let $i : H^1(\Omega) \to L^2(\Omega)$ be the inclusion of $H^1(\Omega)$ in $L^2(\Omega)$ and $T_1 : L^2(\Omega) \to H^1(\Omega)$ be the extension of T to $L^2(\Omega)$ as defined by the left-hand side of (A.3). Since $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$ and T_1, as T, is bounded, it follows that $T = T_1 \circ i$ is compact.

Lemma A.7 now gives that there exists a countable orthonormal basis $\{\psi_j\}_{j=0}^{\infty}$ of $H^1(\Omega)$ consisting of eigenfunctions of T and we will now prove that, after normalization, the same set is an ON-basis for $L^2(\Omega)$. Let μ_j be the eigenvalue corresponding to ψ_j. From (A.3) we get that $\int_\Omega \psi_j \overline{\psi_k}\, dx = \overline{\mu_k}\langle \psi_j, \psi_k\rangle_{H^1(\Omega)}$ so the eigenfunctions are orthogonal to each other also in $L^2(\Omega)$. By setting

$$\phi_j = \frac{\psi_j}{\|\psi_j\|_{L^2(\Omega)}},$$

we thus get an orthonormal sequence $\{\phi_j\}_{j=0}^{\infty}$ in $L^2(\Omega)$, each ϕ_j also an eigenfunction of T with the same eigenvalue as ψ_j. It remains to see that each function in $L^2(\Omega)$ can be written as a (finite or infinite) linear combination of them.

Set $M = \{\phi_j\}_{j=0}^{\infty}$. The fact that M is a basis of $H^1(\Omega)$ implies that M is total in $H^1(\Omega)$. Since $H^1(\Omega)$ is dense in $L^2(\Omega)$ according to Lemma A.1, M is also total in $L^2(\Omega)$. For $f \in L^2(\Omega)$ we construct

$$g = \sum_{k=0}^{\infty} \langle f, \phi_k\rangle_{L^2(\Omega)} \phi_k,$$

which is convergent in $L^2(\Omega)$. We get immediately $\langle f - g, \phi_j\rangle_{L^2(\Omega)} = 0$ for every j so Lemma A.3 gives that $f = g$. Thus $\{\phi_j\}_{j=0}^{\infty}$ is an ON-basis for $L^2(\Omega)$ and the coordinates of f are its usual Fourier coefficients.

The final step is to investigate the eigenvalues of T. From (A.3) it follows that

$$\langle Tu, u\rangle_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} \tag{A.4}$$

so Lemma A.5 gives that $\sigma(T) \subset [0, 1]$ with $0, 1 \in \sigma(T)$. (A.4) also shows that 0 is not an eigenvalue. Furthermore, if λ is an eigenvalue, Lemma A.4 shows that $N(\lambda I - T) = N(I - \lambda^{-1}T)$ is finite dimensional. But since $\dim H^1(\Omega) = \infty$ and the eigenfunctions form a basis, we conclude that T has infinitely many eigenvalues. Lemma A.4 also gives that 1 really is an eigenvalue (with constants as eigenfunctions) and that we can label the eigenvalues in decreasing order as

$$1 = \mu_0 \geq \mu_1 \geq \ldots > 0, \quad \mu_k \to 0 \text{ as } k \to \infty. \tag{A.5}$$

We see that

$$\langle u, v\rangle_{H^1(\Omega)} = \int_\Omega f\overline{v}\, dx, \quad \forall v \in H^1(\Omega)$$

if and only if

$$u = Tf.$$

This shows that if ϕ_k is an eigenfunction of T with eigenvalue μ_k, then it is also a solution of (A.1) with

$$\lambda_k = \frac{1}{\mu_k} - 1$$

and thus an eigenfunction of the operator $-\Delta$ for the Neumann problem. We also see from (A.1) that the only eigenfunctions to $\lambda_0 = 0$ are the constant functions. Hence λ_0 has single multiplicity and (A.2) follows from (A.5). □

Theorem A.9 *Let Ω be a bounded domain in $\mathbf{R}^n$. Then there exists an ON-basis $\{\phi_k\}_{k=1}^{\infty}$ of $L^2(\Omega)$ where $\phi_k \in H_0^1(\Omega)$ are eigenfunctions of the operator $-\Delta$ for the Dirichlet problem, i.e.*

$$\int_\Omega \nabla\phi_k \cdot \overline{\nabla v}\, dx = \lambda_k \int_\Omega \phi_k \overline{v}\, dx, \qquad \forall v \in H_0^1(\Omega),$$

and the eigenvalues λ_k are subject to

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots, \quad \lambda_k \to \infty \text{ as } k \to \infty,$$

where each eigenvalue is repeated according to its multiplicity.

Also, $\{\phi_k\}_{k=0}^{\infty}$ is an orthogonal basis of $H_0^1(\Omega)$.

Proof. We prove only the last part of the theorem because the remaining parts of the complete proof can be found in Evans [8]. For $u \in H_0^1(\Omega)$ we have the equality

$$\langle u, \phi_j \rangle_{H_0^1(\Omega)} = (1 + \lambda_j)\langle u, \phi_j \rangle_{L^2(\Omega)}$$

so if $u \in H_0^1(\Omega)$ is orthogonal to $\{\phi_j\}_1^{\infty}$ in $H_0^1(\Omega)$, it is also orthogonal to $\{\phi_j\}_1^{\infty}$ in $L^2(\Omega)$. From this and Lemma A.3 it now follows that $\{\phi_j\}_1^{\infty}$ is total in $H_0^1(\Omega)$ and that it in fact is an orthogonal set. □

A.2 Existence and uniqueness of bounded solutions of Poisson's equation in $\mathcal{C}$

In this section we will see that under some conditions the problem

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{C} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{C} \end{cases}$$

has a unique bounded solution with mean value 0 over Ω and also derive a bound for its $L^2(\Omega)$-norm. We then turn to the problem

$$\begin{cases} -\Delta u = f & \text{in } \mathcal{C} \\ u = 0 & \text{on } \partial\mathcal{C} \end{cases} \tag{A.6}$$

and state that practically the same results hold in this case. We use the theory of eigenvalues of the operator $-\Delta'$ in Ω presented in Section A.1 and construct the solution in each case as an infinite linear combination of the eigenfunctions. We denote the first positive eigenvalue by λ_1 in the Neumann case and by λ_D in the Dirichlet case.

Lemma A.10 *Suppose $f \in L^2_{\text{loc}}(\mathcal{C})$ fulfills*

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \|f\|_{L^2(\mathcal{C}_s)}\, ds < \infty.$$

Then also

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \|f(\cdot, s)\|_{L^2(\Omega)}\, ds < \infty.$$

Proof. First note that if $a \geq 0$ and $|x - y| \leq c$ for some c, then

$$e^{-a|y|} \leq e^{ac} e^{-a|x|}. \tag{A.7}$$

This follows from the fact that $|x| - |y| \leq |x - y|$.

We write $g(t) = \|f(\cdot, t)\|_{L^2(\Omega)}$ and observe that for $a \geq 0$, Hölder's inequality gives

$$\int_0^a \|f(\cdot, t + s)\|_{L^2(\Omega)}\, ds = \|g\|_{L^1(t,t+a)} \leq \sqrt{a} \|f\|_{L^2(\Omega \times (t,t+a))}. \tag{A.8}$$

It is easy to see that

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \|f(\cdot, s)\|_{L^2(\Omega)} ds = \int_0^1 \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s+t|} \|f(\cdot, s + t)\|_{L^2(\Omega)}\, ds\, dt$$

by noting that the value of the inner integral on the right-hand side is independent of t. By applying (A.7) and (A.8) on the right-hand side we arrive at

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \|f(\cdot, s)\|_{L^2(\Omega)} ds \leq e^{\sqrt{\lambda_1}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \|f\|_{L^2(\mathcal{C}_s)}\, ds$$

from which the lemma now directly follows. □

Remark A.11 It follows from (A.7) with $a = \sqrt{\lambda_1}$, $c = |x_n|$, $x = s$ and $y = s - x_n$ that

$$e^{-\sqrt{\lambda_1}|x_n - s|} \leq e^{\sqrt{\lambda_1}|x_n|} e^{-\sqrt{\lambda_1}|s|}.$$

Under the same conditions as in Lemma A.10 we therefore also have that

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|x_n - s|} \|f(\cdot, s)\|_{L^2(\Omega)}\, ds < \infty$$

for every $x_n \in \mathbf{R}$.

For functions $v \in H^1(\Omega)$ subject to

$$\int_\Omega v\,dx = 0 \tag{A.9}$$

the following inequality holds:

$$\|\nabla' v\|^2_{L^2(\Omega)} \geq \lambda_1 \|v\|^2_{L^2(\Omega)}. \tag{A.10}$$

In order to see this, let us consider the ON-basis $\{\phi_j\}_{j=0}^\infty$ of $L^2(\Omega)$ consisting of eigenfunctions of the Neumann problem for $-\Delta'$ in Ω corresponding to the eigenvalues $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$ (see Theorem A.8 in Section A.1). We have that

$$v = \sum_{j=0}^\infty v_j,$$

where $v_j = \langle v, \phi_j\rangle_{L^2(\Omega)}\phi_j$. Since $\phi_0 = \text{const}$, it follows from (A.9) that $v_0 = 0$. The set $\{\phi_j\}$ also forms an orthogonal basis of $H^1(\Omega)$ so

$$\nabla' v = \sum_{j=1}^\infty \nabla' v_j$$

and (A.10) now follows from Parseval's identity together with the identity

$$\langle \nabla' v_i, \nabla' v_j\rangle_{L^2(\Omega)} = \lambda_i \langle v_i, v_j\rangle_{L^2(\Omega)}$$

obtained from Greens formula.

Lemma A.12 *Suppose $f \in L^2_{\text{loc}}(\mathcal{C})$ satisfies*

$$\int_\Omega f(x', x_n)\,dx' = 0 \quad \textit{for a.e.} \quad x_n \in \mathbf{R} \tag{A.11}$$

and

$$\int_{-\infty}^\infty e^{-\sqrt{\lambda_1}|s|}\|f\|_{L^2(\mathcal{C}_s)}\,ds < \infty.$$

Then the problem

$$\begin{cases} -\Delta u = & f \quad \textit{in } \mathcal{C} \\ \dfrac{\partial u}{\partial \nu} = & 0 \quad \textit{on } \partial\mathcal{C} \end{cases} \tag{A.12}$$

has a bounded solution $u \in W^{1,2}_{\text{loc}}(\mathcal{C})$ with

$$\int_\Omega u(x', x_n)\,dx' = 0 \quad \textit{for a.e. } x_n \in \mathbf{R} \tag{A.13}$$

and

$$\|u(\cdot, x_n)\|_{L^2(\Omega)} \leq \frac{1}{2\sqrt{\lambda_1}} \int_{-\infty}^\infty e^{-\sqrt{\lambda_1}|x_n - s|}\|f(\cdot, s)\|_{L^2(\Omega)}\,ds, \tag{A.14}$$

the last expression being finite. This solution is unique up to a constant.

Remark A.13 Since $\partial\Omega \in C^2$, it follows from Section A.3 that if $f \in L^r_{\mathrm{loc}}(\mathcal{C})$ for some $r \in [2,\infty)$, then the fact that $u \in W^{1,2}_{\mathrm{loc}}(\mathcal{C})$ implies that $u \in W^{2,r}_{\mathrm{loc}}(\mathcal{C})$, i.e. there is no difference between a weak and a strong solution of (A.12).

Proof. Lemma A.10 gives that the right hand side of (A.14) is finite (see the remark). We begin by finding a solution of (A.12). According to Theorem A.8 in Section A.1 there exists an orthonormal basis $\{\phi_j\}_{j=0}^{\infty}$ of $L^2(\Omega)$ consisting of eigenfunctions of the Neumann problem in Ω, i.e.

$$\begin{cases} -\Delta'\phi_j = \lambda_j\phi_j & \text{in } \Omega \\ \dfrac{\partial\phi_j}{\partial\nu'} = 0 & \text{on } \partial\Omega \end{cases} \tag{A.15}$$

with $\lambda_0 = 0$ and all other $\lambda_j > 0$. Defining

$$f_j = \int_\Omega f\phi_j\,dx', \quad j = 1, 2, \ldots,$$

we get, since $f_0 = 0$ by (A.11),

$$f(x', x_n) = \sum_{j=1}^{\infty} f_j(x_n)\phi_j(x').$$

We also define

$$u_j(t) = \frac{1}{2\sqrt{\lambda_j}}\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_j}|t-s|} f_j(s)\,ds, \quad j = 1, 2, \ldots$$

and set for $N \geq 1$

$$u^{(N)}(x', x_n) = \sum_{j=1}^{N} u_j(x_n)\phi_j(x'),$$

$$f^{(N)} = \sum_{j=1}^{N} f_j(x_n)\phi_j(x').$$

It then follows that

$$-\Delta u^{(N)} = f^{(N)}.$$

Then, using Parseval's identity, Minkowski's inequality (see for example Section 6.3, Folland [9] or Section 2.4, Lieb and Loss [16]) and Parseval again yields

$$
\begin{aligned}
\|u^{(N)}(\cdot,x_n)\|_{L^2(\Omega)} &= \left(\sum_{j=1}^{N} u_j(x_n)^2\right)^{1/2} \\
&\le \left(\sum_{j=1}^{N}\left(\frac{1}{2\sqrt{\lambda_j}}\right)^2 \left(\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_j}|x_n-s|}|f_j(s)|\,ds\right)^2\right)^{1/2} \\
&\le \frac{1}{2\sqrt{\lambda_1}}\int_{-\infty}^{\infty}\left(e^{-2\sqrt{\lambda_1}|x_n-s|}\sum_{j=1}^{N} f_j(s)^2\right)^{1/2} ds \\
&\le \frac{1}{2\sqrt{\lambda_1}}\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|x_n-s|}\|f(\cdot,s)\|_{L^2(\Omega)}\,ds \qquad \text{(A.16)}
\end{aligned}
$$

which shows that $\{u^{(N)}\}_N$ is a Cauchy sequence in $L^2(\Omega)$. When working with u_j' instead of u_j and noting that

$$
u_j'(t) \le \frac{1}{2}\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|t-s|}|f_j(s)|\,ds,
$$

it can be proved in the same way that also $\{\partial u^{(N)}/\partial x_n\}_N$ is Cauchy in $L^2(\Omega)$.

Write

$$
\begin{aligned}
\|\nabla' u^{(N)}(\cdot,x_n)\|_{L^2(\Omega)} &= \langle -\Delta' u^{(N)}(\cdot,x_n), u^{(N)}(\cdot,x_n)\rangle_{L^2(\Omega)}^{1/2} \\
&= \left(\sum_{j=1}^{N} \lambda_j u_j(x_n)^2\right)^{1/2} \\
&\le \left(\sum_{j=1}^{N}\frac{1}{4}\left(\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|x_n|}|f_j(s)|\,ds\right)^2\right)^{1/2}
\end{aligned}
$$

and proceed as before to obtain

$$
\|\nabla u^{(N)}(\cdot,x_n)\|_{L^2(\Omega)} \le \frac{1}{2}\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|x_n-s|}\|f(\cdot,s)\|_{L^2(\Omega)}\,ds. \qquad \text{(A.17)}
$$

This proves that also $\{\nabla' u^{(N)}\}_N$ is Cauchy in $L^2(\Omega)$. The conclusion is that $u^{(N)}$ converges in $W^{1,2}(\Omega)$. To show the convergence in $W^{1,2}_{\mathrm{loc}}(\mathcal{C})$, we estimate $\|u^{(N)}\|_{W^{1,2}(\mathcal{C}_t)}$ for a fixed t in the same way as above. Using

(A.16), Minkowski's inequality and (A.7), we get

$$\begin{aligned}
\|u^{(N)}\|_{L^2(\mathcal{C}_t)} &= \left(\int_t^{t+1} \|u^{(N)}(\cdot,x_n)\|^2_{L^2(\Omega)}\,dx_n\right)^{1/2} \\
&\leq \frac{1}{2\sqrt{\lambda_1}} \int_{-\infty}^{\infty} \left(\int_t^{t+1} e^{-2\sqrt{\lambda_1}|x_n-s|}\|f(\cdot,s)\|^2_{L^2(\Omega)}\,dx_n\right)^{1/2} ds \\
&\leq C\int_{-\infty}^{\infty} \|f(\cdot,s)\|_{L^2(\Omega)} \left(\int_t^{t+1} e^{-2\sqrt{\lambda_1}|t-s|}\,dx_n\right)^{1/2} ds \\
&\leq Ce^{\sqrt{\lambda_1}|t|}\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|}\|f(\cdot,s)\|_{L^2(\Omega)}\,ds.
\end{aligned}$$

The right-hand side is finite from Lemma A.10 so $\{u^{(N)}\}$ is Cauchy also in $L^2_{\mathrm{loc}}(\mathcal{C})$.

In the same way it follows from (A.17) that $\{\nabla u^{(N)}\}$ is Cauchy in $L^2_{\mathrm{loc}}(\mathcal{C})$ and that $u^{(N)}$ converges to some function u in $W^{1,2}_{\mathrm{loc}}(\mathcal{C})$. Regularity theory gives that u solves (A.12).

Furthermore, ϕ_0 is constant so due to the orthogonality of $\{\phi_j\}$ we have $\int_\Omega u^{(N)}\,dx' = 0$. Thus, the property (A.13) follows from the convergence in $L^2(\Omega)$.

To prove the assertion about uniqueness, we now show that the only bounded solutions of

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{C} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{C} \end{cases}$$

are the constant functions. Using the decomposition

$$u(x) = \sum_{j=0}^{\infty} u_j(x_n)\phi_j(x')$$

and (A.15), we get

$$\begin{aligned}
0 &= -\int_\Omega \phi_j \Delta u\,dx' = -\int_\Omega u\Delta'\phi_j\,dx' - \int_\Omega \phi_j u_{x_n x_n}\,dx' \\
&= \lambda_j u_j(x_n) - u_j''(x_n), \quad j = 0,1,2,\ldots. \qquad \text{(A.18)}
\end{aligned}$$

If $j \neq 0$, then $\lambda_j > 0$ and in these cases the only bounded solution of (A.18) is $u_j = 0$. Furthermore, $\lambda_0 = 0$, so in the case $j = 0$ we get the only bounded solutions as $u_0(x_n) = A$, where A is constant. Since also ϕ_0 is constant, we get that u is a constant function.

We now prove (A.14). We assume that u is non-vanishing (so that $\|u(\cdot,x_n)\|_{L^2(\Omega)}$ is twice differentiable). In other case, we can replace $\|u(\cdot,x_n)\|_{L^2(\Omega)}$ by $(\|u(\cdot,x_n)\|^2_{L^2(\Omega)} + \epsilon)^{1/2}$ in the calculations below and then let $\epsilon \to 0$ to obtain the same result.

For clarity, we skip the index $L^2(\Omega)$ in the norm and inner product notations. Since $u \in W^{1,2}_{\text{loc}}(\mathcal{C})$ we get that $u(\cdot, x_n) \in W^{1,2}(\Omega)$ for almost every x_n. When multiplying (A.12) by u, integrating over Ω and using Green's theorem, we arrive at

$$\int_\Omega |\nabla' u|^2 \, dx' - \int_\Omega u \frac{\partial^2 u}{\partial x_n^2} \, dx = \int_\Omega f u \, dx.$$

It follows, by (A.10) together with Cauchy-Schwartz inequality, that

$$\left\langle \frac{\partial^2 u}{\partial x_n^2}, u \right\rangle \geq \lambda_1 \|u\|^2 - \|f\| \|u\|. \tag{A.19}$$

Differentiating $\|u(\cdot, x_n)\|$ twice gives

$$\|u\| \frac{d^2 \|u\|}{dx_n^2} = \left\langle \frac{\partial^2 u}{\partial x_n^2}, u \right\rangle - \|u\|^{-2} \left(\left\langle \frac{\partial u}{\partial x_n}, u \right\rangle^2 - \left\| \frac{\partial u}{\partial x_n} \right\|^2 \|u\|^2 \right).$$

The Cauchy-Schwartz inequality gives that $\langle \partial_{x_n} u, u \rangle^2 - \|\partial_{x_n} u\|^2 \|u\|^2 \leq 0$ so from (A.19)

$$\frac{d^2 \|u\|}{dx_n^2} \geq \lambda_1 \|u\| - \|f\|. \tag{A.20}$$

Consider the equation

$$-\frac{d^2}{dx_n^2} \|u(\cdot, x_n)\| + \lambda_1 \|u(\cdot, x_n)\| = g(x_n). \tag{A.21}$$

By using a Green function and the fact that u is bounded, we get

$$\|u(\cdot, x_n)\| = \frac{1}{2\sqrt{\lambda_1}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|x_n - s|} g(s) \, ds.$$

But from (A.20) and (A.21) we have $g(x_n) \leq \|f(\cdot, x_n)\|$ so (A.14) now directly follows. $\square$

We now turn to the Dirichlet problem (A.6) and have the following analogue of Lemma A.12:

Lemma A.14 *Suppose $f \in L^2_{\text{loc}}(\mathcal{C})$ is subject to*

$$\int_{-\infty}^{\infty} e^{-\sqrt{\lambda_1}|s|} \|f\|_{L^2(\mathcal{C}_s)} \, ds < \infty.$$

Then the problem (A.6) *has a unique, bounded solution $u \in W^{1,2}_{\text{loc}}(\mathcal{C})$ and*

$$\|u(\cdot, x_n)\|_{L^2(\Omega)} \leq \frac{1}{2\sqrt{\lambda_D}} \int_{-\infty}^{\infty} e^{-\sqrt{\lambda_D}|x_n - s|} \|f(\cdot, s)\|_{L^2(\Omega)} \, ds,$$

where the last expression is finite.

Remark A.15 Since all eigenvalues of the Dirichlet problem are positive, we do not have the conditions that f and u are orthogonal to constants in $L^2(\Omega)$ as in the Neumann problem.

Proof. The proof is similar to the proof of Lemma A.12. A difference is however that the estimate (A.10), with λ_1 replaced by λ_D, is now valid for all $v \in W_0^{1,2}(\mathcal{C})$. □

A.3 A local estimate for solutions of Poisson's equation

We say that D is a *cylindrical type domain (CTD)* if $D = \Omega \times (a, b)$, where $-\infty < a < b < \infty$. Furthermore, if $D = \Omega \times (a, d)$ and $D' = \Omega \times (b, c)$ are two CTD:s, we say that *D' is compactly contained in D in the x_n-direction* if $a < b < c < d$.

Given a solution $u \in W^{1,2}_{\mathrm{loc}}(\mathcal{C})$ to some of the problems

$$\begin{cases} \Delta u = f & \text{in } \mathcal{C} \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{C} \end{cases} \tag{A.22}$$

or

$$\begin{cases} \Delta u = f & \text{in } \mathcal{C} \\ \quad u = 0 & \text{on } \partial\mathcal{C}, \end{cases} \tag{A.23}$$

where $f \in L^p_{\mathrm{loc}}(\mathcal{C})$, we will see that in fact $u \in W^{2,p}_{\mathrm{loc}}(\mathcal{C})$ and find a bound for $\|u\|_{W^{2,p}(\mathcal{C}'_t)}$ expressed in terms of $\|u\|_{L^2(\mathcal{C}_t)}$ and $\|f\|_{L^p(\mathcal{C}_t)}$.

Lemma A.16 *Suppose that p fulfills* (1.6) *and that $u \in W^{1,2}_{\mathrm{loc}}(\mathcal{C})$ is a solution of* (A.22) *or* (A.23)*. Suppose further that E_1 and E_2 are CTD:s and that E_1 is compactly contained in E_2 in the x_n-direction. Suppose also that $E_2 \subset \mathcal{C}_t$ for some $t \in \mathbf{R}$. Then $u \in W^{2,p}_{\mathrm{loc}}(\mathcal{C})$ and*

$$\|u\|_{W^{2,p}(E_1)} \le C_0(\|u\|_{L^2(E_2)} + \|f\|_{L^p(E_2)}), \tag{A.24}$$

for some constant C_0 depending on p, n, E_1 and E_2 but not on t.

Remark A.17 The assumption that $E_2 \subset \mathcal{C}_t$ for some $t \in \mathbf{R}$ is not essential for the result but indicates that the constant C_0 is independent of t.

Proof. Given $t \in \mathbf{R}$, make first the transformations $v(x', x_n) = u(x', x_n + t)$, $g(x', x_n) = f(x', x_n + t)$ and note that the equation $\Delta u = f$ on $\mathcal{C}_t$ is equivalent to the equation $\Delta v = g$ on $\mathcal{C}_0$. From Theorem 15.1" in Agmon, Douglis and Nirenberg, [1], we get the interior estimate

$$\|v\|_{W^{2,r}(D')} \le C(\|v\|_{L^r(D)} + \|g\|_{L^r(D)}) \tag{A.25}$$

if $r \in (1, \infty)$ and $D' \subset\subset D \subset \mathcal{C}_0$. The constant C depends only on D, D' and r. Furthermore, suppose x is a point on $\partial\Omega \times (0, 1)$. After straightening the boundary in a neighborhood of x, Theorem 15.3 [1] and the remark immediately after it gives the estimate (A.25) for D' and D equal to hemispheres located at x. Using a partition of unity we finally get the estimate

$$\|v\|_{W^{2,r}(D_1)} \leq C(\|v\|_{L^r(D_2)} + \|g\|_{L^r(D_2)}) \tag{A.26}$$

for any D_1, D_2 such that D_1 is compactly contained in $D_2 \subset \mathcal{C}_0$ in the x_n-direction. The constant C depends on r, D_1 and D_2 but not on v and g. We can therefore revert to the functions u and f and get the inequality

$$\|u\|_{W^{2,r}(D_{1,t})} \leq C(\|u\|_{L^r(D_{2,t})} + \|f\|_{L^r(D_{2,t})}), \tag{A.27}$$

where $D_{1,t}$ and $D_{2,t}$ are equal to D_1 and D_2 translated t steps in the x_n-direction and C is the same constant as in (A.26), i.e. independent of t.

If $n = 2$ or 3, the lemma follows from (A.27) with $r = 2$, since $p = 2$ in these cases. Now suppose ω is any bounded open subset of $\mathbf{R}^n$ with Lipschitz boundary and introduce $\phi(x) = nx/(n-2x)$ with inverse $\psi(x) = nx/(n+2x)$. A well-known Sobolev inequality states that if $r < n/2$ and $s = \phi(r)$, then

$$\|u\|_{L^s(\omega)} \leq C\|u\|_{W^{2,r}(\omega)} \tag{A.28}$$

and if $r > n/2$

$$\|u\|_{L^\infty(\overline{\omega})} \leq C\|u\|_{W^{2,r}(\omega)}. \tag{A.29}$$

Set $E_1^0 = E_1$ and choose a sequence $\{E_1^k\}_{k=1}^\infty$ such that E_1^k is compactly contained in E_1^{k+1} as well as in E_2 in the x_n-direction for $k \geq 0$. We set $p_1 = p$, $p_2 = \psi(p)$ and notice that $p_2 < n/2$. Using (A.27), (A.28) and then (A.27) again we obtain

$$\begin{aligned}\|u\|_{W^{2,p}(E_1)} &\leq C_1(\|u\|_{L^p(E_1^1)} + \|f\|_{L^p(E_2)}) \\ &\leq C_2(\|u\|_{W^{2,p_2}(E_1^1)} + \|f\|_{L^p(E_2)}) \\ &\leq C_3(\|u\|_{L^{p_2}(E_1^2)} + \|f\|_{L^p(E_2)}).\end{aligned} \tag{A.30}$$

We now continue the iteration indicated in (A.30) in the following way: for $k \geq 2$, set $p_{k+1} = \psi(p_k)$ and combine again (A.27) and (A.28) to obtain

$$\|u\|_{L^{p_k}(E_1^k)} \leq C(\|u\|_{L^{p_{k+1}}(E_1^{k+1})} + \|f\|_{L^p(E_2)}).$$

Obviously, $p_1 > p_2 > \ldots$ and eventually $p_K \leq 2$ for some K. We can assume $p_K > 1$, since if $p_K = \psi(p_{K-1}) \leq 1$, then p_{K-1} can be increased so that the condition $p_K > 1$ is met. We get

$$\|u\|_{W^{2,p}(E_1)} \leq C(\|u\|_{L^{p_K}(E_1^K)} + \|f\|_{L^p(E_2)}),$$

from which (A.24) follows.

□

Corollary A.18 *Under the same conditions as in Lemma* A.16

$$\|u\|_{L^\infty(E_1)} \leq C(\|u\|_{L^2(E_2)} + \|f\|_{L^p(E_2)}).$$

Proof. Since $p > n/2$, the corollary follows by combining Lemma A.16 with the Sobolev inequality (A.29). □

Remark A.19 The lemma as well as the corollary also hold with $\mathcal{C}$ and $\partial\mathcal{C}$ replaced by $\mathcal{C}_+$ and $\partial\Omega \times (0, \infty)$, respectively, but with the additional condition $t \geq 0$.

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Paper 2

Asymptotic analysis of solutions to parabolic systems

Vladimir Kozlov *Mikael Langer* *Peter Rand*

Abstract

We study asymptotics as $t \to \infty$ of solutions to a linear, parabolic system of equations with time-dependent coefficients in $\Omega \times (0, \infty)$, where Ω is a bounded domain. On $\partial\Omega \times (0, \infty)$ we prescribe the homogeneous Dirichlet boundary condition. For large values of t, the coefficients in the elliptic part are close to time-independent coefficients in an integral sense which is described by a certain function $\kappa(t)$. This includes in particular situations when the coefficients may take different values on different parts of Ω and the boundaries between them can move with t but stabilize as $t \to \infty$. The main result is an asymptotic representation of solutions for large t. As a corollary, it is proved that if $\kappa \in L^1(0, \infty)$, then the solution behaves asymptotically as the solution to a parabolic system with time-independent coefficients.

1 Introduction

Let Ω denote an open, bounded region in $\mathbf{R}^n$ with Lipschitz boundary and introduce $Q = \Omega \times (0, \infty)$. By $x = (x_1, \dots, x_n)$ we denote the variables in Ω and by t the unbounded variable. We consider the parabolic system

$$u_t - \sum_{i,j=1}^{n} (A_{ij} u_{x_j})_{x_i} + Au = 0 \quad \text{in } Q, \tag{1.1}$$

where $u = (u_1, \dots, u_N)$ is a function from Q to $\mathbf{C}^N$ and A_{ij}, $i, j = 1, \dots, n$, and A are quadratic matrices of size $N \times N$ whose elements are functions from Q to $\mathbf{C}$. We will assume that u satisfies the Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{if } x \in \partial\Omega,\ t > 0 \tag{1.2}$$

and that

$$u(x,0) = \psi(x), \tag{1.3}$$

where ψ is a function from $\left(L^2(\Omega)\right)^N$.

For general theory of parabolic equations and systems, which include in particular solvability and uniqueness results, we refer to Ladyženskaja et al [12], Dautray, Lions [1], Lions, Magenes [13] and Eidel'man [2]. Evolution problems of the above type appear for example in biology and chemistry when studying reaction diffusion problems, see for example Murray [14] or Fife [3]. Another application can be found in multigroup diffusion in neutron physics, see Example 3, Chapter XVIIB, §3, Dautray, Lions [1]. We are concerned only with the asymptotic behaviour of solutions as $t \to \infty$. Therefore, we suppose that the matrices A_{ij} and A can be written as

$$A_{ij}(x,t) = A_{ij}^{(0)}(x) + A_{ij}^{(1)}(x,t) \tag{1.4}$$

and

$$A(x,t) = A^{(0)}(x) + A^{(1)}(x,t), \tag{1.5}$$

where $A_{ij}^{(1)}$ and $A^{(1)}$ are considered as perturbations. We assume that the relation

$$\big(A_{ij}^{(0)}\big)^* = A_{ji}^{(0)}, \tag{1.6}$$

where A^* denotes the adjoint matrix of A, holds for every pair (i,j) and that $A^{(0)}$ is hermitian, i.e.

$$\big(A^{(0)}\big)^* = A^{(0)}.$$

The matrices $A_{ij}^{(0)}$ fulfill the two-sided inequality

$$\nu \sum_{i=1}^{n} |\xi_i|^2 \le \sum_{i,j=1}^{n} (A_{ij}^{(0)}\xi_j, \xi_i) \le \nu^{-1} \sum_{i=1}^{n} |\xi_i|^2 \tag{1.7}$$

for all ξ_i, $\xi_j \in \mathbf{C}^N$ and some positive constant ν. Here we use the notations

$$(u,v) = \sum_{k=1}^{N} u_k \overline{v_k}$$

and $|u| = (u,u)^{1/2}$ for $u, v \in \mathbf{C}^N$. The matrix $A^{(0)}$ is supposed to belong to $\big(L^q(\Omega)\big)^{N^2}$, where

$$\begin{cases} q \in [n,\infty] & \text{if } n \ge 3 \\ q \in (2,\infty] & \text{if } n = 2 \\ q \in [2,\infty] & \text{if } n = 1. \end{cases} \tag{1.8}$$

Writing

$$A^{(0)} = A_+^{(0)} - A_-^{(0)}, \tag{1.9}$$

where both matrices $A_+^{(0)}$ and $A_-^{(0)}$ are positive, we require that $A_-^{(0)}$ is bounded. This means that there exists a constant ν_1 such that

$$\|A_-^{(0)}\|_{L^\infty(Q)} \le \nu_1. \tag{1.10}$$

Furthermore, we also assume that there exists a constant ν_2 such that

$$\int_\Omega \big(A^{(0)}u, u\big)\, dx \le \nu_2 \|\nabla u\|^2_{L^2(\Omega)} \tag{1.11}$$

for every $u \in \big(L^2(\Omega)\big)^N$.

Under the above conditions on the matrices $A^{(0)}_{ij}$ and $A^{(0)}$, there exists an ON-basis of $\big(L^2(\Omega)\big)^N$ consisting of eigenfunctions of the time-independent operator

$$-\sum_{i,j=1}^{n} (A^{(0)}_{ij} u_{x_j})_{x_i} + A^{(0)} u$$

with the Dirichlet boundary condition. Let λ_k, $k = 1, 2, \dots$, denote the eigenvalues in increasing order and J be the multiplicity of λ_1. This means that

$$\lambda_1 = \ldots = \lambda_J < \lambda_{J+1} \le \lambda_{J+2} \le \ldots, \quad \lambda_k \to \infty \text{ as } k \to \infty.$$

Let furthermore $\phi_1, \phi_2, \dots, \phi_J$ be a basis in the eigenspace of λ_1 which is orthogonal in L^2-sense. The conditions on $A^{(0)}_{ij}$, $A^{(0)}$ and Ω imply that $\phi_k \in \big(W^{1,p}(\Omega)\big)^N$ for some $p > 2$ and $k = 1, 2, \dots, J$, see Theorem A.2 in Appendix A.

We also assume some similar conditions on A_{ij} and A, see Section 2.2. The main characteristic of our perturbation is the function

$$\kappa(t) = \sum_{i,j=1}^{n} \|A^{(1)}_{ij}\|_{L^{s_1,2}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^{s_2,1}(\mathcal{C}_t)}, \tag{1.12}$$

where

$$\mathcal{C}_t = \Omega \times (t, t+1),$$
$$s_1 = \frac{2p}{p-2}, \tag{1.13}$$

and

$$s_2 = \begin{cases} 2 & \text{if } n < p \\ s_2' & \text{if } n = p \\ \frac{2np}{np-2(n-p)} & \text{if } n > p, \end{cases} \tag{1.14}$$

where s_2' denotes an arbitrary number in $(2, n]$. In (1.12) we have extended the matrices $A^{(1)}_{ij}$, $i, j = 1, \dots, n$, and $A^{(1)}$ to $\Omega \times \mathbf{R}$ by setting $A^{(1)}_{ij}(x,t) = A^{(1)}(x,t) = 0$ for $t < 0$, so $\kappa(t)$ is defined for every $t \in \mathbf{R}$. We set

$$\kappa_0 = \sup_{t \ge 0} \kappa(t), \tag{1.15}$$

and consider perturbations subject to

$$\kappa_0 \le \varkappa, \tag{1.16}$$

where $\varkappa$ is a sufficiently small constant depending on n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2. An exact value of $\varkappa$ is difficult to give but the requirement is that $\varkappa$ is so small that some inequality type conditions appearing in the proof of Theorem 1.1 are satisfied. Note that $\varkappa$ does not depend on $A_{ij}^{(1)}$ or $A^{(1)}$.

We define $\big(W_{0,\mathrm{loc}}^{1,0;2}(Q)\big)^N$ as the space consisting of functions u vanishing on $\partial\Omega \times (0,\infty)$ such that u has a weak derivate with respect to every x_k and every such derivate, together with u itself, belong to $\big(L^2(\mathcal{C}_t)\big)^N$ for every $t \ge 0$. See further in Section 2.1.2. We let ∇ denote the gradient with respect to the x-variables and define $\big(V_{0,\mathrm{loc}}^2(Q)\big)^N$ as the subspace of $\big(W_{0,\mathrm{loc}}^{1,0;2}(Q)\big)^N$ consisting of functions u such that

$$|u|_{\mathcal{C}_t} = \operatorname*{ess\,sup}_{t<s<t+1} \|u(\cdot,s)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\mathcal{C}_t)}$$

is finite for every $t \ge 0$. It can be proved that problem (1.1)–(1.3) has a unique solution in $\big(V_{0,\mathrm{loc}}^2(Q)\big)^N$. The main result of the paper is the following theorem.

Theorem 1.1 *If the constant $\varkappa$ introduced in* (1.16) *is small enough, then the unique solution u in $\big(V_{0,\mathrm{loc}}^2(Q)\big)^N$ of* (1.1)–(1.3) *can be represented as*

$$u(x,t) = e^{-\lambda_1 t + \int_0^t (-f(s)+\Lambda(s))\,ds} \left(w_0 \sum_{k=1}^{J} \theta_k(t)\phi_k(x) + V(x,t) \right), \tag{1.17}$$

where w_0 is a constant, $\Theta = (\theta_1, \ldots, \theta_J)$ is an absolutely continuous unit vector and

$$f = (\mathcal{R}\Theta, \Theta). \tag{1.18}$$

Here, $\mathcal{R}$ denotes the $J \times J$ matrix with entry (k,l) equal to

$$\mathcal{R}_{kl} = \int_\Omega \left[\sum_{i,j=1}^{n} (A_{ij}^{(1)} \phi_{l x_j}, \phi_{k x_i}) + (A^{(1)}\phi_l, \phi_k) \right] dx. \tag{1.19}$$

Furthermore, the following estimates are valid:

$$|w_0| \le C\|\psi\|_{L^2(\Omega)} \tag{1.20}$$

and

$$\|\Lambda\|_{L^1(t,t+1)} \le C\kappa(t) \left(\int_{-1}^{t} e^{-b_0(t-s)}\kappa(s)\,ds + \kappa(t) \right), \tag{1.21}$$

$$\|\Theta'\|_{L^1(t,t+1)} \le C\kappa(t), \tag{1.22}$$

$$|V|_{\mathcal{C}_t} \le C\|\psi\|_{L^2(\Omega)} \left(e^{-b_0 t} + \int_{-1}^{t} e^{-b_0(t-s)}\kappa(s)\,ds + \kappa(t) \right) \tag{1.23}$$

for $t \geq 0$. *Here,* $b_0 = \lambda_{J+1} - \lambda_1 - C_1\kappa_0$ *and* C *and* C_1 *denote constants depending on* n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 *and* ν_2.

Remark 1.2 Since we will study the asymptotic behaviour of solutions for large t, it suffices to require that

$$\sup_{t \geq T} \kappa(t) \leq \varkappa$$

for some $T \geq 0$, because the condition (1.16) will then be fulfilled after a translation of the time variable.

In Corollary 9.2, we prove that the asymptotic formula (1.17) implies the estimate

$$|u|_{\mathcal{C}_t} \leq C_1 \|\psi\|_{L^2(\Omega)} e^{-\lambda_1 t + \int_0^t (-f(s) + C_2 \kappa(s)^2)\, ds}.$$

If, in addition, $\kappa \in L^1(0, \infty)$, Corollary 9.1 states that

$$u(x,t) = e^{-\lambda_1 t}\left(\sum_{k=1}^{J} b_k \phi_k(x) + \omega(x,t)\right),$$

where b_k, $k = 1, \ldots, J$, are constants and $|\omega|_{\mathcal{C}_t} \to 0$ as $t \to \infty$. We have here the same leading term as in the case when $A_{ij}^{(1)} = 0$, $i, j = 1, \ldots, n$, $A^{(1)} = 0$. If, instead, $A^{(1)} = 0$,

$$\sum_{i,j=1}^{n} \int_0^\infty \int_\Omega |A_{ij}^{(1)}(x,t)|\, dx\, dt < \infty$$

and $p = \infty$, i.e. the gradients of the eigenfunctions belong to $L^\infty(\Omega)$, we get from Corollary 9.3 that

$$u(x,t) = e^{-\lambda_1 t}\left(b \sum_{k=1}^{J} \theta_k(t) \phi_k(x) + \omega(x,t)\right),$$

where $|\omega|_{\mathcal{C}_t} \to 0$ as $t \to \infty$ and b is a constant which may depend on $A_{ij}^{(1)}$, i, $j = 1, \ldots, n$, and ψ.

As can be seen from (1.18), the function f in (1.17) is not given exactly since its definition contains the unknown vector-valued function Θ. If the eigenvalue λ_1 is simple, i.e. $J = 1$, and A and A_{ij} are real-valued, then $\Theta = 1$ and we arrive at the following asymptotic expansion for u:

$$u(x,t) = e^{-\lambda_1 t + \int_0^t (-\mathcal{R}(s) + \Lambda(s))\, ds}\big(w_0 \phi_1(x) + V(x,t)\big).$$

See Theorem 9.4.

Asymptotics for solutions of (1.1) with a non-zero right-hand side stabilizing at infinity has been studied by Friedman, [4], [5], [6], and

Pazy, [15], [16]. Ordinary differential equations with unbounded operator coefficients which include parabolic ones are studied in Kozlov, Maz'ya [9]. In particular, asymptotic results from Part III can give the asymptotic formula (1.17) under the restriction that λ_1 is simple and the quantity

$$\sum_{i,j=1}^{n} \|A_{ij}^{(1)}\|_{L^\infty(\mathcal{C}_t)} + \|\rho^{-2}A^{(1)}\|_{L^\infty(\mathcal{C}_t)},$$

where $\rho(x)$ denotes the distance to $\partial\Omega$, is small.

The proof of Theorem 1.1 can very briefly be outlined in the following way. Using spectral splitting, we write

$$u(x,t) = \sum_{k=1}^{J} h_k(t)\phi_k(x) + w(x,t), \tag{1.24}$$

where $h_k = \int_\Omega (u, \phi_k)\, dx$ and $w(x,t)$ is the remainder term. The most part of the proof is devoted to derivation of a system of first order ordinary differential equations for $h_1, \dots, h_J$ perturbed by a small integro-differential term and to estimation of w. An important role here plays a preliminary spectral splitting with J in (1.24) replaced by M, where M is sufficiently large, see Section 3. After this, the proof is completed by study of asymptotic behaviour of solutions to the perturbed system of ordinary differential equations.

2 Problem formulation and elementary properties

2.1 Notation

Throughout this paper Ω denotes an open, bounded region in $\mathbf{R}^n$ with Lipschitz boundary. We let $x = (x_1, \dots, x_n)$ denote a point in Ω and t be a real variable (time).

For two vectors $u = (u_1, u_2, \dots, u_N)$, $v = (v_1, v_2, \dots, v_N) \in \mathbf{C}^N$ (depending on the context, vectors will also be regarded as column vectors), we define the scalar product

$$(u, v) = \sum_{i=1}^{N} u_i \overline{v_i}.$$

If ξ is a vector in $\mathbf{C}^N$ where the k:th component is denoted by ξ_k, we write

$$|\xi| = \left(\sum_{k=1}^{N} |\xi_k|^2 \right)^{1/2}.$$

For a matrix M with k columns, let m_{ij} denote the element of M on position (i,j) and define $|M|$ as the matrix norm corresponding to the vector norm $|\cdot|$, i.e.

$$|M| = \sup_{\substack{u\in\mathbf{C}^k \\ |u|=1}} |Mu|.$$

The adjoint matrix of M is denoted by M^*. We say that M is *positive* if $(Mu,u) \geq 0$ for every complex vector u of the appropriate dimension.

2.1.1 Spaces not involving time

Given a function $u = (u_1,\dots,u_N) : \Omega \to \mathbf{C}^N$ and a multiindex $\alpha = (\alpha_1,\dots,\alpha_n)$ with $|\alpha| = \alpha_1 + \dots + \alpha_n$, we set

$$\partial^\alpha u(x) = \frac{\partial^{|\alpha|}u(x)}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}},$$

where the differentiation acts on u componentwise. If α is the k:th unit vector, we often write u_{x_k} or $\partial_{x_k}u$. By ∇u we mean the collection of vectors $(\nabla u_1,\dots,\nabla u_N)$ and we set

$$\|\nabla u\|_{L^2(\Omega)} = \left(\sum_{k=1}^{N}\sum_{j=1}^{n}\int_\Omega \left|(u_k)_{x_j}\right|^2\,dx\right)^{1/2}.$$

We say that u belongs to $\left(L^p(\Omega)\right)^N$ or $\left(W^{k,p}(\Omega)\right)^N$, $k = 0,1,2,\dots$ and $1\leq p\leq\infty$, if every component of u belongs to $L^p(\Omega)$ or $W^{k,p}(\Omega)$, respectively. The norms in these spaces are defined as

$$\|u\|_{L^p(\Omega)} = \begin{cases} \left(\int_\Omega |u|^p\,dx\right)^{1/p} & \text{if } 1\leq p<\infty \\ \text{ess sup}_\Omega|u| & \text{if } p=\infty \end{cases}$$

and

$$\|u\|_{W^{k,p}(\Omega)} = \begin{cases} \left(\sum_{|\alpha|\leq k}\|\partial^\alpha u\|^p_{L^p(\Omega)}\right)^{1/p} & \text{if } 1\leq p<\infty \\ \sum_{|\alpha|\leq k}\text{ess sup}_\Omega|\partial^\alpha u| & \text{if } p=\infty. \end{cases}$$

The space $\left(W^{k,2}(\Omega)\right)^N$ becomes a Hilbert space, denoted by $\left(H^k(\Omega)\right)^N$, by introducing the scalar product

$$\langle u,v\rangle_{H^k(\Omega)} = \sum_{j=1}^{N}\sum_{|\alpha|\leq k}\int_\Omega \left(\partial^\alpha u_j\right)\left(\overline{\partial^\alpha v_j}\right)dx.$$

By $\left(W_0^{k,p}(\Omega)\right)^N$ we denote the closure of $\left(C_c^\infty(\Omega)\right)^N$ in $\left(W^{k,p}(\Omega)\right)^N$.

For a matrix M with elements belonging to $L^p(\Omega)$, we define $\|M\|_{L^p(\Omega)}$ as $\||M|\|_{L^p(\Omega)}$.

2.1.2 Spaces involving time

We introduce

$$Q = \Omega \times (0, \infty)$$

and, for a given $T > 0$,

$$Q_T = \Omega \times (0, T).$$

For $t \in \mathbf{R}$, we set

$$\mathcal{C}_t = \Omega \times (t, t+1).$$

Let u denote a measurable function from $\mathcal{C}_t$ to $\mathbf{C}^N$. By $\big(L^{q,r}(\mathcal{C}_t)\big)^N$ we mean the space of all such functions with the norm

$$\|u\|_{L^{q,r}(\mathcal{C}_t)} = \begin{cases} \left(\int_t^{t+1} \|u(\cdot,s)\|^r_{L^q(\Omega)}\, ds\right)^{1/r} & \text{if } 1 \le r < \infty \\ \operatorname{ess\,sup}_{t<s<t+1} \|u(\cdot,s)\|_{L^q(\Omega)} & \text{if } r = \infty \end{cases} \tag{2.1}$$

being finite. Instead of $\big(L^{q,q}(\mathcal{C}_t)\big)^N$ we write $\big(L^{q}(\mathcal{C}_t)\big)^N$.

Furthermore, we say that u belongs to the Hilbert space $\big(W^{1,0;2}(\mathcal{C}_t)\big)^N$ if u has a weak derivative with respect to every x_k and every such derivative, together with u itself, belongs to $\big(L^{2}(\mathcal{C}_t)\big)^N$. The scalar product in $\big(W^{1,0;2}(\mathcal{C}_t)\big)^N$ is defined as

$$\langle u, v\rangle_{W^{1,0;2}(\mathcal{C}_t)} = \int_{\mathcal{C}_t} \left[(u,v) + \sum_{k=1}^{n} (u_{x_k}, v_{x_k})\right] dx\, ds.$$

Analogously, u belongs to $\big(W^{1,1;2}(\mathcal{C}_t)\big)^N$ if u belongs to $\big(W^{1,0;2}(\mathcal{C}_t)\big)^N$ and in addition has a weak derivative with respect to t, belonging to $\big(L^{2}(\mathcal{C}_t)\big)^N$. The space $\big(W^{1,1;2}(\mathcal{C}_t)\big)^N$ is also a Hilbert space and the scalar product is given by

$$\langle u, v\rangle_{W^{1,1;2}(\mathcal{C}_t)} = \int_{\mathcal{C}_t} \left[(u,v) + \sum_{k=1}^{n} (u_{x_k}, v_{x_k}) + (u_t, v_t)\right] dx\, ds.$$

We define $\big(V^{2}(\mathcal{C}_t)\big)^N$ as the subspace of $\big(W^{1,0;2}(\mathcal{C}_t)\big)^N$ consisting of functions u with the norm

$$|u|_{\mathcal{C}_t} = \operatorname*{ess\,sup}_{t<s<t+1} \|u(\cdot,s)\|_{L^2(\Omega)} + \|\nabla u\|_{L^2(\mathcal{C}_t)} \tag{2.2}$$

being finite. Here, we define $\|\nabla u\|_{L^2(\mathcal{C}_t)}$ as

$$\|\nabla u\|_{L^2(\mathcal{C}_t)} = \left(\sum_{k=1}^{N}\sum_{j=1}^{n} \int_t^{t+1}\int_\Omega \left|(u_k)_{x_j}\right|^2 dx\, ds\right)^{1/2}.$$

The spaces $\left(W_0^{1,0;2}(\mathcal{C}_t)\right)^N$, $\left(W_0^{1,1;2}(\mathcal{C}_t)\right)^N$ and $\left(V_0^2(\mathcal{C}_t)\right)^N$ are defined as the sets of functions in respective space without a zero subindex which vanish on $\partial\Omega \times (t, t+1)$. If $u \in \left(L^{q,r}(\mathcal{C}_t)\right)^N$ for every $t \geq 0$, we say that $u \in \left(L^{q,r}_{\text{loc}}(Q)\right)^N$. Other function spaces with the subscript "loc" are defined similarly.

All function spaces defined over $\mathcal{C}_t$ can equally well be defined over Q_T for some $T > 0$ by changing the domain of integration in the norms and scalar products accordingly.

As in Section 2.1.1, the definition (2.1) can easily be generalized to be valid also for matrices.

If f is a function of one variable and $-\infty \leq a < b \leq \infty$, we let $\|f\|_{L^p(a,b)}$ denote the L^p-norm of f on the interval (a, b).

2.2 Problem formulation and assumptions

We will study asymptotics of weak solutions of the equation

$$u_t - \sum_{i,j=1}^{n} (A_{ij} u_{x_j})_{x_i} + Au = 0 \quad \text{in } Q. \tag{2.3}$$

Here, u is a function from Q to $\mathbf{C}^N$ and A_{ij}, $i, j = 1, \ldots, n$, and A are quadratic matrices of size $N \times N$ whose elements are functions from Q to $\mathbf{C}$. We assume that the boundary condition

$$u(x,t) = 0 \quad \text{if } x \in \partial\Omega,\ t > 0 \tag{2.4}$$

and the initial condition

$$u(x, 0) = \psi(x), \tag{2.5}$$

where $\psi \in \left(L^2(\Omega)\right)^N$, are valid. We suppose that the matrices A_{ij} and A can be written as (1.4) and (1.5), where $A_{ij}^{(1)}$ and $A^{(1)}$ are small perturbations in a sense described later.

Let us now give the assumptions on $A^{(0)}$ and $A_{ij}^{(0)}$. We require that the symmetry condition (1.6) for matrices $A_{ij}^{(0)}$ holds and the two-sided inequality (1.7) is fulfilled. The upper inequality in (1.7) implies in particular that $A_{ij}^{(0)}$ all are bounded. The matrix $A^{(0)}$ is supposed to be hermitian and belong to $\left(L^q(\Omega)\right)^{N^2}$, where q is given by (1.8). We suppose also that the matrix $A^{(0)}$ admits the representation (1.9), where both matrices $A_+^{(0)}$ and $A_-^{(0)}$ are positive, and that the conditions (1.10) and (1.11) are satisfied.

Let us introduce the operator

$$L^{(0)}u = -\sum_{i,j=1}^{n} \partial_{x_i}\left(A_{ij}^{(0)} u_{x_j}\right) + A^{(0)}u$$

defined on $\left(W_0^{1,2}(\Omega)\right)^N$. This operator has an infinite number of eigenvalues

$$\lambda_1 \le \lambda_2 \le \dots, \quad \lambda_k \to \infty \text{ as } k \to \infty,$$

and the corresponding eigenfunctions $\{\phi_k\}_{k=1}^\infty$ form an ON-basis in $\left(L^2(\Omega)\right)^N$, see Theorem A.1 in Appendix A. We denote by $p \in (2,\infty]$ some number for which

$$\phi_k \in \left(W_0^{1,p}(\Omega)\right)^N, \quad k = 1, 2, \dots. \tag{2.6}$$

Relation (2.6) is always true for p sufficiently close to 2, see Theorem A.2 in Appendix A.

We suppose that the matrices A_{ij} and A satisfy the following conditions. The relations

$$A_{ij}^* = A_{ji}, \quad i, j = 1, \dots, n,$$

hold and the matrix A is hermitian. With ν the same as in (1.7), we assume further that

$$\nu \sum_{i=1}^n |\xi_i|^2 \le \sum_{i,j=1}^n (A_{ij}\xi_j, \xi_i) \le \nu^{-1} \sum_{i=1}^n |\xi_i|^2 \tag{2.7}$$

for every set of N-dimensional vectors $\xi_1, \dots, \xi_n$. We also assume that

$$A \in \left(L_{\text{loc}}^{q,r}(Q)\right)^{N^2},$$

where q is the same as in (1.8) and

$$r = \frac{2q}{2q - n}.$$

Furthermore, writing

$$A = A_+ - A_-, \tag{2.8}$$

where A_+ and A_- are positive, we assume that A_- is bounded by ν_1 from (1.10), i.e.

$$\|A_-\|_{L^\infty(Q)} \le \nu_1. \tag{2.9}$$

Let us show that the function $\kappa(t)$ introduced in (1.12) is finite. It can be checked that

$$\frac{2np}{np - 2(n - p)} \le n$$

for $n > p \ge 2$ and since $n \le q$, it follows from (1.14) that $2 \le s_2 \le q$. Since $A^{(1)} \in \left(L_{\text{loc}}^{q,r}(Q)\right)^{N^2}$ and $A_{ij}^{(1)}$ is bounded, the function $\kappa(t)$ is finite for every t.

We define a weak solution of (2.3) under conditions (2.4) and (2.5). We begin with studying the problem

$$u_t - \sum_{i,j=1}^n (A_{ij} u_{x_j})_{x_i} + Au = 0 \quad \text{in } Q_T, \tag{2.10}$$

$$u(x,t)=0 \quad \text{if } x\in\partial\Omega,\ t\in(0,T), \tag{2.11}$$

$$u(x,0)=\psi(x) \tag{2.12}$$

for some fixed, positive T. We introduce

$$\mathcal{L}_1(u,\eta)=\int_\Omega\Big[\sum_{i,j=1}^n (A_{ij}u_{x_j},\eta_{x_i})+(Au,\eta)\Big]\,dx \tag{2.13}$$

and say that $u\in\big(V_0^2(Q_T)\big)^N$ is a *weak solution* of problem (2.10)–(2.12) if

$$-\int_{Q_T}(u,\eta_t)\,dx\,dt+\int_0^T\mathcal{L}_1(u,\eta)\,dt=\int_\Omega\big(\psi(x),\eta(x,0)\big)\,dx$$

for all $\eta\in\big(W_0^{1,1;2}(Q_T)\big)^N$ such that $\eta(x,T)=0$. It is a well-known result that problem (2.10)–(2.12) has a unique weak solution $u=u_T$ from $\big(V_0^2(Q_T)\big)^N$. Indeed, this result for a single equation and a system where A_{ij} are scalars can be found in Ladyženskaja et al [12], sections III.1-III.4 and VII.1. The generalization to systems of equations where A_{ij} are not necessarily scalars is straightforward. Since this can be done for any $T>0$, and the solution is unique, we obtain a unique function u defined on Q. We define this function as the weak solution from $\big(V_{0,\mathrm{loc}}^2(Q)\big)^N$ of equation (2.3) under conditions (2.4) and (2.5).

Equivalently, one can say that u is a weak solution of the problem (2.3)–(2.5) if $u\in\big(V_{0,\mathrm{loc}}^2(Q)\big)^N$ is the unique function satisfying the relation

$$-\int_Q(u,\eta_t)\,dx\,dt+\int_0^\infty\mathcal{L}_1(u,\eta)\,dt=\int_\Omega\big(\psi(x),\eta(x,0)\big)\,dx \tag{2.14}$$

for all $\eta\in\big(W_0^{1,1;2}(Q)\big)^N$ with bounded support.

The symbol C, possibly with a subscript, is frequently used to denote a constant depending only on n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2. The lower-case letter c is used for constants depending on the same quantities but in situations where the symbol denotes a specific constant which may be referred to in some other part of the paper.

The remaining part of the paper except Section 9 is devoted to the proof of Theorem 1.1. Without loss of generality, we can assume that $\lambda_1=0$. Namely, if $\lambda_1\neq 0$, we set

$$U(x,t)=e^{\lambda_1 t}u(x,t).$$

Then U satisfies the equation

$$U_t-\sum_{i,j=1}^n (A_{ij}U_{x_j})_{x_i}+A'U=0 \quad \text{in } Q \tag{2.15}$$

and the initial and boundary conditions (2.4), (2.5). In (2.15) we have $A'=A^{(0)'}+A^{(1)}$, where $A^{(0)'}=A^{(0)}-\lambda_1 I$. Clearly, the matrix $A^{(0)'}$ satisfies (1.10) and (1.11), possibly with other constants ν_1 and ν_2.

2.3 An estimate for u

In this section, an estimate for u, to be used later in the paper, is derived.

Lemma 2.1 *There exists a constant a_0 depending on ν, ν_1 and diam Ω such that the inequality*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \leq \|\psi\|_{L^2(\Omega)} e^{a_0 t} \tag{2.16}$$

is valid a.e.

Proof. From (2.7) and Poincaré's inequality it follows for an arbitrary function $w \in \left(W_0^{1,2}(\Omega)\right)^N$ that

$$\int_\Omega \sum_{i,j=1}^n (A_{ij} w_{x_j}, w_{x_i})\, dx \geq \nu \|\nabla w\|_{L^2(\Omega)}^2 \geq C_1 \|w\|_{L^2(\Omega)}^2. \tag{2.17}$$

Furthermore, using the decomposition (2.8) and inequality (2.9) together with the fact that A_+ is positive, we get

$$(Aw, w) = (A_+ w, w) - (A_- w, w) \geq -\nu_1 |w|^2. \tag{2.18}$$

From (2.13), (2.17) and (2.18) the inequality

$$\mathcal{L}_1(w,w) \geq -a_0 \|w\|_{L^2(\Omega)}^2 \tag{2.19}$$

follows, where $a_0 = \nu_1 - C_1$.

In the same way as in Ladyženskaja et al [12] Section III.2, we can derive the equality

$$\frac{1}{2} \int_\Omega |u(x,s)|^2\, dx \Big|_{s=0}^{t} + \int_0^t \mathcal{L}_1(u,u)\, ds = 0 \tag{2.20}$$

for a.e. $t > 0$. This is equivalent to

$$\frac{1}{2} \|u(\cdot,t)\|_{L^2(\Omega)}^2 = \frac{1}{2} \|\psi\|_{L^2(\Omega)}^2 - \int_0^t \mathcal{L}_1(u,u)\, ds. \tag{2.21}$$

Since A_{ij} is bounded and $u \in \left(V_{0,\mathrm{loc}}^2(Q)\right)^N$, we see by Hölder's inequality that the term $(A_{ij} u_{x_j}, u_{x_i})$ occurring in the expression of $\mathcal{L}_1(u,u)$ is an element of $L^1(\Omega)$ for almost every t. The same is valid for (Au, u); this can be seen from the derivation of (A.3) by replacing $A^{(0)}$ by A. Hence we can differentiate (2.21) so that

$$\frac{1}{2} \frac{d}{dt} \left(\|u(\cdot,t)\|_{L^2(\Omega)}^2 \right) = -\mathcal{L}_1(u,u)$$

and it follows from (2.19) that

$$\frac{1}{2} \frac{d}{dt} \left(\|u(\cdot,t)\|_{L^2(\Omega)}^2 \right) \leq a_0 \|u(\cdot,t)\|_{L^2(\Omega)}^2.$$

Hence

$$\frac{d}{dt}\left(\|u(\cdot,t)\|^2_{L^2(\Omega)}e^{-2a_0t}\right) \le 0$$

so

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} \le \|\psi\|^2_{L^2(\Omega)}e^{2a_0t}.$$

After taking the square root, inequality (2.16) follows. □

3 Spectral splitting of the solution u

Let u denote the unique weak solution of (2.3) under the conditions (2.4) and (2.5) in Q as defined in Section 2.2. Setting

$$h_k(t) = \int_\Omega \big(u(\cdot,t),\phi_k\big)\,dx, \quad k = 1,\ldots,M,$$

we obtain the representation

$$u(x,t) = \sum_{k=1}^{M} h_k(t)\phi_k(x) + v(x,t), \tag{3.1}$$

where $v(\cdot,t)$ is orthogonal to ϕ_k in $\big(L^2(\Omega)\big)^N$ for $k = 1,2,\ldots,M$. The integer M will be chosen later.

Recall that it is assumed that $\lambda_1 = 0$ and let J denote the multiplicity of the eigenvalue 0. This means that

$$0 = \lambda_1 = \ldots = \lambda_J < \lambda_{J+1} \le \ldots. \tag{3.2}$$

To get an equation for h_k, we choose the functions η in (2.14) as

$$\eta(x,t) = \xi(t)\phi_k(x),$$

where $1 \le k \le M$ and the scalar function ξ belongs $C_c^1(0,\infty)$. This gives, after using the orthogonality between the elements $\phi_1,\ldots,\phi_M$ and v, that

$$-\int_0^\infty \xi' h_k\,dt + \int_0^\infty \xi\Big(\sum_{l=1}^{M}\widetilde{\mathcal{R}_{kl}}h_l + \widetilde{g_k}(v)\Big)dx = 0,$$

where

$$\widetilde{\mathcal{R}_{kl}} = \int_\Omega\Big[\sum_{i,j=1}^{n}(A_{ij}\phi_{l_{x_j}},\phi_{k_{x_i}}) + (A\phi_l,\phi_k)\Big]\,dx$$

are known functions of t and

$$\widetilde{g_k}(v) = \int_\Omega\Big[\sum_{i,j=1}^{n}(A_{ij}v_{x_j},\phi_{k_{x_i}}) + (Av,\phi_k)\Big]\,dx.$$

This implies that h_k has a (distributional) derivate h'_k and we get the system of equations

$$h'_k + \sum_{l=1}^{M} \widetilde{\mathcal{R}_{kl}} h_l + \widetilde{g_k}(v) = 0, \quad k = 1, 2, \ldots, M. \tag{3.3}$$

Introducing

$$\mathcal{R}_{kl} = \int_\Omega \left[\sum_{i,j=1}^{n} (A_{ij}^{(1)} \phi_{l x_j}, \phi_{k x_i}) + (A^{(1)} \phi_l, \phi_k) \right] dx \tag{3.4}$$

and

$$g_k(v) = \int_\Omega \left[\sum_{i,j=1}^{n} (A_{ij}^{(1)} v_{x_j}, \phi_{k x_i}) + (A^{(1)} v, \phi_k) \right] dx, \tag{3.5}$$

we see, by using (1.4) and (1.5), that

$$\widetilde{\mathcal{R}_{kl}} = B[\phi_l, \phi_k] + \mathcal{R}_{kl}$$

and

$$\widetilde{g_k}(v) = B[v, \phi_k] + g_k(v),$$

with B as given in (A.2). Since

$$B[\phi_l, \phi_k] = \begin{cases} 0 & \text{if } k \neq l \\ \lambda_k & \text{if } k = l \end{cases}$$

and

$$B[v, \phi_k] = B[\phi_k, v] = \lambda_k \int_\Omega (\phi_k, v)\, dx = 0$$

for $k = 1, \ldots, M$, it follows that (3.3) can be rewritten as

$$h'_k + \lambda_k h_k + \sum_{l=1}^{M} \mathcal{R}_{kl} h_l + g_k(v) = 0, \quad k = 1, 2, \ldots, M. \tag{3.6}$$

We also have the initial values

$$h_k(0) = \int_\Omega (\psi, \phi_k)\, dx, \quad k = 1, 2, \ldots, M. \tag{3.7}$$

The system of equations (3.6), (3.7) will be analyzed further in Sections 6 and 7.

In order to find an equation for v we use representation (3.1) in (2.14). This time we suppose that $\eta(\cdot, t)$ is orthogonal to ϕ_k, $k = 1, \ldots, M$, in

$\big(L^2(\Omega)\big)^N$ for almost every $t \in (0,\infty)$. This implies that the same orthogonality relation holds between $\eta_t(\cdot,t)$ and ϕ_k and it follows that

$$-\int_Q (v,\eta_t)\,dx\,dt + \int_0^\infty \left[\mathcal{L}_1(v,\eta) + \sum_{k=1}^M h_k \mathcal{L}_1(\phi_k,\eta)\right] dt = \int_\Omega \big(\psi(x),\eta(x,0)\big)\,dx. \quad (3.8)$$

Owing to decompositions (1.4) and (1.5), we can write

$$\mathcal{L}_1(\phi_k,\eta) = B[\phi_k,\eta] + \mathcal{L}_1^{(1)}(\phi_k,\eta),$$

where

$$\mathcal{L}_1^{(1)}(\phi_k,\eta) = \int_\Omega \left[\sum_{i,j=1}^n (A_{ij}^{(1)}\phi_{k_{x_j}},\eta_{x_i}) + (A^{(1)}\phi_k,\eta)\right] dx.$$

Because of the orthogonality between ϕ_k and η, it follows from (A.4) that $B[\phi_k,\eta]=0$ for a.e. t and hence we can rewrite (3.8) as

$$-\int_Q (v,\eta_t)\,dx\,dt + \int_0^\infty \left[\mathcal{L}_1(v,\eta) + \sum_{k=1}^M h_k \mathcal{L}_1^{(1)}(\phi_k,\eta)\right] dt = \int_\Omega \big(\psi(x),\eta(x,0)\big)\,dx. \quad (3.9)$$

4 Estimating the function v

4.1 A general estimate

We will now study solutions $w \in \big(V_{0,\mathrm{loc}}^2(Q)\big)^N$ of the equation

$$-\int_Q (w,\eta_t)\,dx\,dt + \int_0^\infty [\mathcal{L}_1(w,\eta) + \mathcal{L}_2(\mathbf{f},\eta)]\,dt = \int_\Omega \big(\psi(x),\eta(x,0)\big)\,dx \quad (4.1)$$

subject to the orthogonality condition

$$\int_\Omega \big(w(x,t),\phi_k(x)\big)\,dx = 0 \quad (4.2)$$

for $k=1,\dots,M$ and a.e. t. The operator $\mathcal{L}_1$ was defined in (2.13) and

$$\mathcal{L}_2(\mathbf{f},\eta) = \int_\Omega \left[\sum_{i=1}^n (f_i,\eta_{x_i}) + (f,\eta)\right] dx, \quad (4.3)$$

where $f \in \left(L^{2,1}_{\text{loc}}(Q)\right)^N$ and $f_i \in \left(L^2_{\text{loc}}(Q)\right)^N$, $i = 1, 2, \dots, n$. Equality (4.1) should be satisfied for every η from the same class of functions $\left(W^{1,1;2}_0(Q)\right)^N$ with bounded support as in (2.14) such that

$$\int_\Omega \big(\eta(x,t), \phi_k(x)\big)\, dx = 0$$

for $k = 1, \dots, M$ and a.e. t.

In the proof of the next theorem we use the estimate

$$\int_0^t |f(s)|\, ds \le \int_{-1}^t \|f\|_{L^1(s,s+1)}\, ds \tag{4.4}$$

for a function $f \in L^1_{\text{loc}}(\mathbf{R})$. This is obtained by noting that, for $\tau \in [0,1]$, we have

$$\begin{aligned} \int_0^t |f(s)|\, ds &= \int_{-\tau}^{t-\tau} |f(s+\tau)|\, ds = \int_0^1 \int_{-\tau}^{t-\tau} |f(s+\tau)|\, ds\, d\tau \\ &\le \int_{-1}^t \int_0^1 |f(s+\tau)|\, d\tau\, ds = \int_{-1}^t \|f\|_{L^1(s,s+1)}\, ds. \end{aligned}$$

Proposition 4.1 *Suppose that* $f \in \left(L^{2,1}_{\text{loc}}(Q)\right)^N$ *and* $f_i \in \left(L^2_{\text{loc}}(Q)\right)^N$ *for* $i = 1, 2, \dots, n$. *Then, for any* $M \ge 0$, *there exists a unique function* $w \in \left(V^2_{0,\text{loc}}(Q)\right)^N$ *satisfying* (4.1) *and fulfilling* (4.2) *for* $k = 1, \dots, M$ *and a.e. positive* t. *For every* $b > 0$, *there exists an integer* M *and a constant* C, *both depending on* b, n, N, Ω, $A^{(0)}_{ij}$, $A^{(0)}$, p, s_2', ν, ν_1 *and* ν_2, *such that the solution* w *is subject to the inequality*

$$|w|_{\mathcal{C}_t} \le \mathsf{C}\left(\|\psi\|_{L^2(\Omega)} e^{-bt} + \int_{-1}^t e^{-b(t-s)} \chi(s)\, ds + \chi(t) \right), \tag{4.5}$$

where the norm in the left-hand side is defined by (2.2) *and*

$$\chi(t) = \sum_{i=1}^n \|f_i\|_{L^2(\mathcal{C}_t)} + \|f\|_{L^{2,1}(\mathcal{C}_t)}. \tag{4.6}$$

Here we have extended f *and* f_i *by* 0 *for negative values of* t. *By extending also* w *by* 0 *for* $t < 0$, *estimate* (4.5) *becomes valid for all* $t \ge -1$.

The existence and uniqueness of the solution can be proved in the same way as in the case of a single equation, treated in sections III.3 and III.4 in Ladyženskaja et al [12]. We confine ourself to prove inequality (4.5) under an appropriate choice of M. The proof will be divided into several steps.

Proof of estimate (4.5): *Step 1. Deriving a differential inequality for* $\|w(\cdot,t)\|^2_{L^2(\Omega)}$. Analogously to (2.20), we can derive the equality

$$\frac{1}{2} \int_\Omega |w(x,s)|^2\, dx \Big|_{s=t_0}^t + \int_{t_0}^t [\mathcal{L}_1(w,w) + \mathcal{L}_2(\mathbf{f}, w)]\, ds = 0 \tag{4.7}$$

for any t_0 and t such that $0 \leq t_0 < t < \infty$. This corresponds to (2.13), Chapter III in Ladyženskaja et al. Since $\mathcal{L}_1(w,w)$ and $\mathcal{L}_2(\mathbf{f},w)$ belong to $L^1_{\text{loc}}(0,\infty)$, it follows that $\|w(\cdot,t)\|^2_{L^2(\Omega)}$ is differentiable for $t > t_0$ and

$$\frac{1}{2}\frac{d}{dt}\left(\|w(\cdot,t)\|^2_{L^2(\Omega)}\right) = \\ -\int_\Omega \left[\sum_{i,j=1}^n \left(A_{ij}w_{x_j}, w_{x_i}\right) + (Aw,w) + \sum_{i=1}^n (f_i, w_{x_i}) + (f,w)\right] dx. \quad (4.8)$$

We now estimate the terms in the right-hand side of (4.8). The decomposition $A = A_+ - A_-$, where A_+ is positive and A_- is bounded, and inequality (2.9) imply that

$$\begin{aligned} -\int_\Omega (Aw,w)\; dx &= -\int_\Omega (A_+w,w)\; dx + \int_\Omega (A_-w,w)\; dx \\ &\leq \nu_1\|w(\cdot,t)\|^2_{L^2(\Omega)}. \end{aligned} \quad (4.9)$$

Together with (2.7) this yields

$$-\int_\Omega \left[\sum_{i,j=1}^n \left(A_{ij}w_{x_j}, w_{x_i}\right) + (Aw,w)\right] dx \\ \leq -\nu\|\nabla w(\cdot,t)\|^2_{L^2(\Omega)} + \nu_1\|w(\cdot,t)\|^2_{L^2(\Omega)}. \quad (4.10)$$

Since w is orthogonal to $\phi_1,\ldots,\phi_M$, we have

$$\int_\Omega \left[\sum_{i,j=1}^n \left(A^{(0)}_{ij}w_{x_j}, w_{x_i}\right) + \left(A^{(0)}w,w\right)\right] dx \geq \lambda_{M+1}\|w(\cdot,t)\|^2_{L^2(\Omega)},$$

cf. (A.5) in Appendix A. From this fact, together with the upper inequality in (1.7) and property (1.11), we obtain

$$\begin{aligned} \|\nabla w(\cdot,t)\|^2_{L^2(\Omega)} &\geq \nu\int_\Omega \left[\sum_{i,j=1}^n \left(A^{(0)}_{ij}w_{x_j}, w_{x_i}\right) + \left(A^{(0)}w,w\right)\right] dx \\ &\quad - \nu\int_\Omega \left(A^{(0)}w,w\right) dx \\ &\geq \nu\lambda_{M+1}\|w(\cdot,t)\|^2_{L^2(\Omega)} - \nu\nu_2\|\nabla w(\cdot,t)\|^2_{L^2(\Omega)} \end{aligned}$$

and hence

$$\|\nabla w(\cdot,t)\|^2_{L^2(\Omega)} \geq \frac{\nu\lambda_{M+1}}{1+\nu\nu_2}\|w(\cdot,t)\|^2_{L^2(\Omega)}.$$

This gives together with (4.10) the estimate

$$-\int_\Omega \left[\sum_{i,j=1}^n \left(A_{ij}w_{x_j}, w_{x_i}\right) + (Aw,w)\right] dx \\ \leq -\frac{\nu}{2}\|\nabla w(\cdot,t)\|^2_{L^2(\Omega)} - b\|w(\cdot,t)\|^2_{L^2(\Omega)},$$

where

$$b = \frac{\nu^2 \lambda_{M+1}}{2(1+\nu\nu_2)} - \nu_1.$$

Since $\lambda_{M+1} \to \infty$ as $M \to \infty$, the constant b can be made arbitrarily large.

We continue to estimate the terms in the right-hand side of (4.8) and get

$$\begin{aligned} -\int_\Omega \sum_{i=1}^n (f_i, w_{x_i})\,dx &\le \left(\sum_{i=1}^n \|f_i(\cdot,t)\|_{L^2(\Omega)}\right)\|\nabla w(\cdot,t)\|_{L^2(\Omega)} \\ &\le \frac{\nu}{2}\|\nabla w(\cdot,t)\|^2_{L^2(\Omega)} + \frac{1}{2\nu}\left(\sum_{i=1}^n \|f_i(\cdot,t)\|_{L^2(\Omega)}\right)^2, \end{aligned} \tag{4.11}$$

where we have used the elementary inequality

$$ab \le \frac{\alpha}{2}a^2 + \frac{1}{2\alpha}b^2, \quad \alpha > 0,$$

in the last line. Also,

$$-\int_\Omega (f, w)\,dx \le \|f(\cdot,t)\|_{L^2(\Omega)}\|w(\cdot,t)\|_{L^2(\Omega)}$$

and when combining all estimates with (4.8), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|w(\cdot,t)\|^2_{L^2(\Omega)}) &\le -b\|w(\cdot,t)\|^2_{L^2(\Omega)} \\ &+ \|f(\cdot,t)\|_{L^2(\Omega)}\|w(\cdot,t)\|_{L^2(\Omega)} + \frac{1}{2\nu}\left(\sum_{i=1}^n \|f_i(\cdot,t)\|_{L^2(\Omega)}\right)^2. \end{aligned} \tag{4.12}$$

Choose M so large that b becomes positive and set

$$h(t) = \|w(\cdot,t)\|^2_{L^2(\Omega)}$$

and

$$G(t) = \frac{1}{\nu}\left(\sum_{i=1}^n \|f_i(\cdot,t)\|_{L^2(\Omega)}\right)^2. \tag{4.13}$$

Then, from (4.12), we obtain the differential inequality

$$h'(t) + 2bh(t) \le 2\|f(\cdot,t)\|_{L^2(\Omega)}h(t)^{1/2} + G(t), \quad t > 0. \tag{4.14}$$

Step 2. Finding an estimate for $\|w(\cdot,t)\|^2_{L^2(\Omega)}$. We set

$$\begin{aligned} h_+(t) =& 2h(0)e^{-2bt} + \frac{3}{2}\int_0^t e^{-2b(t-\tau)}G(\tau)\,d\tau \\ &+ 12\left(\int_0^t e^{-b(t-\tau)}\|f(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau\right)^2. \end{aligned} \tag{4.15}$$

Let us show that $h(t) \le h_+(t)$. We begin by proving that

$$h_+(t) \ge h(0)e^{-2bt} + \int_0^t e^{-2b(t-\tau)} \Big(2\|f(\cdot,\tau)\|_{L^2(\Omega)} h_+(\tau)^{1/2} + G(\tau)\Big)\, d\tau. \tag{4.16}$$

Let us denote the right-hand side by $r(t)$. When inserting the expression for h_+ in $r(t)$ and using the inequality $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$ for $a, b \ge 0$, we obtain

$$\begin{aligned} r(t) \le & h(0)e^{-2bt} + \int_0^t e^{-2b(t-\tau)} \Big[2\sqrt{2h(0)}\|f(\cdot,\tau)\|_{L^2(\Omega)} e^{-b\tau} \\ & + \sqrt{6}\|f(\cdot,\tau)\|_{L^2(\Omega)} \left(\int_0^\tau e^{-2b(\tau-s)} G(s)\, ds \right)^{1/2} \\ & + 4\sqrt{3}\|f(\cdot,\tau)\|_{L^2(\Omega)} \int_0^\tau e^{-b(\tau-s)} \|f(\cdot,s)\|_{L^2(\Omega)}\, ds + G(\tau) \Big]\, d\tau. \end{aligned} \tag{4.17}$$

We estimate the terms in (4.17) one by one, beginning with

$$\begin{aligned} & 2\sqrt{2h(0)} \int_0^t e^{-2b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)} e^{-b\tau}\, d\tau \\ & \qquad = 2\sqrt{2h(0)} e^{-bt} \int_0^t e^{-b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)}\, d\tau \\ & \qquad \le h(0)e^{-2bt} + 2\left(\int_0^t e^{-b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)}\, d\tau \right)^2. \end{aligned} \tag{4.18}$$

Since $\tau \le t$, we have

$$\begin{aligned} & \sqrt{6} \int_0^t e^{-2b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)} \left(\int_0^\tau e^{-2b(\tau-s)} G(s)\, ds \right)^{1/2} d\tau \\ & \quad \le \sqrt{6} \left(\int_0^t e^{-b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)}\, d\tau \right) \left(\int_0^t e^{-2b(t-s)} G(s)\, ds \right)^{1/2} \\ & \quad \le 3 \left(\int_0^t e^{-b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)}\, d\tau \right)^2 + \frac{1}{2} \int_0^t e^{-2b(t-s)} G(s)\, ds \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} & 4\sqrt{3} \int_0^t e^{-2b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)} \left(\int_0^\tau e^{-b(\tau-s)} \|f(\cdot,s)\|_{L^2(\Omega)}\, ds \right) d\tau \\ & \quad \le 4\sqrt{3} \int_0^t e^{-b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)} \left(\int_0^t e^{-b(t-s)} \|f(\cdot,s)\|_{L^2(\Omega)}\, ds \right) d\tau \\ & \quad = 4\sqrt{3} \left(\int_0^t e^{-b(t-\tau)} \|f(\cdot,\tau)\|_{L^2(\Omega)}\, d\tau \right)^2. \end{aligned} \tag{4.20}$$

Using (4.18), (4.19) and (4.20) in (4.17), we arrive at

$$\begin{aligned} r(t) &\le 2h(0)e^{-2bt} + \frac{3}{2}\int_0^t e^{-2b(t-\tau)}G(\tau)\,d\tau \\ &\quad + \left(5+4\sqrt{3}\right)\left(\int_0^t e^{-b(t-\tau)}\|f(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau\right)^2 \\ &\le h_+(t). \end{aligned}$$

This implies (4.16).

We now prove that $h(t) \le h_+(t)$ and conclude first that both of the functions h and h_+ are continuous. For h_+, this fact follows from (4.15) since G as well as $\|f\|_{L^2(\Omega)}$ belong to $L^1_{\mathrm{loc}}(0,\infty)$. Furthermore, the continuity of h follows from (4.7).

Suppose first that $h(0) > 0$. Then $h_+(0) = 2h(0) > h(0)$ and let us prove that $h_+(t) > h(t)$ for all $t \in [0,\infty)$. Namely, suppose that $h(\tau) < h_+(\tau)$ if $\tau \in [0,t)$ for some $t > 0$ (obviously, such a t exists because of the continuity of h and h_+). From (4.14) it follows that

$$\left(he^{2bt}\right)' \le 2e^{2bt}\|f(\cdot,t)\|_{L^2(\Omega)}h(t)^{1/2} + e^{2bt}G(t)$$

and hence

$$h(t) \le h(0)e^{-2bt} + \int_0^t e^{-2b(t-\tau)}\left(2\|f(\cdot,\tau)\|_{L^2(\Omega)}h(\tau)^{1/2} + G(\tau)\right)d\tau. \tag{4.21}$$

There are two possibilities. Either $\|f(\cdot,\tau)\|_{L^2(\Omega)} = 0$ for almost every $\tau \in [0,t]$ or else $\|f(\cdot,\tau)\|_{L^2(\Omega)} > 0$ on a subset of $[0,t]$ with positive measure. In the first case it follows from (4.21) that

$$h(t) \le h(0)e^{-2bt} + \int_0^t e^{-2b(t-\tau)}G(\tau)\,d\tau$$

and when comparing this with (4.15), we see that $h(t) < h_+(t)$.

Let us now consider the case when $\|f(\cdot,\tau)\|_{L^2(\Omega)}$ is not identically 0 on $[0,t]$, still assuming that $h(0) > 0$ and $h < h_+$ on $[0,t)$. From (4.21) and (4.16) we get that

$$h_+(t) - h(t) \ge 2\int_0^t e^{-2b(t-\tau)}\|f(\cdot,\tau)\|_{L^2(\Omega)}\left(h_+(\tau)^{1/2} - h(\tau)^{1/2}\right)d\tau > 0,$$

so also in this case $h(t) < h_+(t)$.

The conclusion is that the set

$$\{t \ge 0 : h(\tau) < h_+(\tau) \text{ for } \tau \in [0,t]\}$$

is non-empty, open (because of the continuity of h and h_+) and closed in $[0,\infty)$. The last statement follows from the preceding analysis. Hence, it must be equal to $[0,\infty)$ and we have proved that if $h(0) > 0$, then $h(t) < h_+(t)$ for every $t \geq 0$.

We now consider the case $h(0) = 0$. For a given $\epsilon > 0$, introduce

$$h^\epsilon(t) = h(t) + \epsilon e^{-2bt}$$

and

$$\begin{aligned} h_+^\epsilon(t) =& 2h^\epsilon(0)e^{-2bt} + \frac{3}{2}\int_0^t e^{-2b(t-\tau)}G(\tau)\,d\tau \\ &+ 12\left(\int_0^t e^{-b(t-\tau)}\|f(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau\right)^2. \end{aligned} \tag{4.22}$$

Since $h^{1/2} \leq (h^\epsilon)^{1/2}$, it follows from (4.21) that

$$h^\epsilon(t) \leq h^\epsilon(0)e^{-2bt} + \int_0^t e^{-2b(t-\tau)}\left(2\|f(\cdot,\tau)\|_{L^2(\Omega)}h^\epsilon(\tau)^{1/2} + G(\tau)\right)\,d\tau. \tag{4.23}$$

The relations (4.22) and (4.23) are the same as (4.15) and (4.21) with the only difference that h and h_+ are replaced by h^ϵ and h_+^ϵ, respectively and also (4.16) holds with the same substitutions. Obviously, $h^\epsilon(0) = \epsilon > 0$, so the previous case $h(0) > 0$ shows that $h^\epsilon(t) < h_+^\epsilon(t)$ for all $t \geq 0$. The last inequality says that

$$h(t) + \epsilon e^{-2bt} < h_+(t) + 2\epsilon e^{-2bt}$$

and by letting ϵ tend to 0 we see that $h(t) \leq h_+(t)$ for every $t \geq 0$.

We have thus obtained the inequality

$$\begin{aligned} \|w(\cdot,t)\|_{L^2(\Omega)}^2 \leq& 2\|\psi\|_{L^2(\Omega)}^2 e^{-2bt} + \frac{3}{2}\int_0^t e^{-2b(t-s)}G(s)\,ds \\ &+ 12\left(\int_0^t e^{-b(t-s)}\|f(\cdot,s)\|_{L^2(\Omega)}\,ds\right)^2. \end{aligned} \tag{4.24}$$

Step 3. Finding an estimate for $\|w(\cdot,t)\|_{L^2(\Omega)}$. Our next aim is to find an estimate for $\|w(\cdot,t)\|_{L^2(\Omega)}$. This will be done departing from (4.24). Before doing this, we set $\gamma(t) = \sqrt{G(t)}$ and prove that the inequality

$$\left(\int_0^t e^{-2b(t-s)}\gamma(s)^2\,ds\right)^{1/2} \leq e^b\int_{-1}^t e^{-b(t-s)}\|\gamma\|_{L^2(s,s+1)}\,ds, \tag{4.25}$$

where γ has been extended by 0 for $t < 0$, holds. It is readily verified that if $\tau \in [0,1]$, then

$$\left(\int_0^t e^{-2b(t-s)}\gamma(s)^2\,ds\right)^{1/2} \leq \left(\sum_{k=0}^{\lceil t\rceil}\int_{k-\tau}^{k+1-\tau} e^{-2b(t-s)}\gamma(s)^2\,ds\right)^{1/2},$$

where $\lceil t \rceil$ denotes the least integer larger than or equal to t. Since the left-hand side is independent of τ, we integrate the inequality from 0 to 1 with respect to τ and obtain

$$\left(\int_0^t e^{-2b(t-s)}\gamma(s)^2\,ds\right)^{1/2} \le \int_0^1 \left(\sum_{k=0}^{\lceil t \rceil}\int_{k-\tau}^{k+1-\tau} e^{-2b(t-s)}\gamma(s)^2\,ds\right)^{1/2} d\tau.$$

By using the inequality $\sqrt{a+b} \le \sqrt{a}+\sqrt{b}$ for $a, b \ge 0$, we can move the sum outside the parentheses. This gives the inequality

$$\left(\int_0^t e^{-2b(t-s)}\gamma(s)^2\,ds\right)^{1/2} \le e^b \sum_{k=0}^{\lceil t \rceil}\int_0^1 e^{-b(t+\tau-k)}\|\gamma\|_{L^2(k-\tau,k-\tau+1)}\,d\tau. \tag{4.26}$$

In the last integral we make the substitution $s = k - \tau$. This yields

$$\begin{aligned}&\sum_{k=0}^{\lceil t \rceil}\int_0^1 e^{-b(t+\tau-k)}\|\gamma\|_{L^2(k-\tau,k-\tau+1)}\,d\tau \\ &= \sum_{k=0}^{\lceil t \rceil}\int_{k-1}^{k} e^{-b(t-s)}\|\gamma\|_{L^2(s,s+1)}\,ds = \int_{-1}^{\lceil t \rceil} e^{-b(t-s)}\|\gamma\|_{L^2(s,s+1)}\,ds. \end{aligned}\tag{4.27}$$

The left-hand side of (4.25) shows that we can set $\gamma(s) = 0$ for $s > t$. Thus, (4.25) follows from (4.26) and (4.27).

In the remaining part of the proof, we let C denote a constant which may depend on n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2 and, additionally, b. We use (4.24) and get the estimate

$$\begin{aligned}\|w(\cdot,t)\|_{L^2(\Omega)} \le& \sqrt{2}\|\psi\|_{L^2(\Omega)}e^{-bt} + \sqrt{\frac{3}{2}}\left(\int_0^t e^{-2b(t-s)}G(s)\,ds\right)^{1/2} \\ &+ 2\sqrt{3}\int_0^t e^{-b(t-s)}\|f(\cdot,s)\|_{L^2(\Omega)}\,ds.\end{aligned}$$

An application of (4.25) yields, after using (4.13), that

$$\begin{aligned}\|w(\cdot,t)\|_{L^2(\Omega)} \le \mathsf{C}\Bigg(&\|\psi\|_{L^2(\Omega)}e^{-bt} + \sum_{i=1}^{n}\int_{-1}^{t} e^{-b(t-s)}\|f_i\|_{L^2(\mathcal{C}_s)}\,ds \\ &+ \int_0^t e^{-b(t-s)}\|f(\cdot,s)\|_{L^2(\Omega)}\,ds\Bigg).\end{aligned}\tag{4.28}$$

By using inequality (4.4), we obtain

$$\int_0^t e^{-b(t-s)}\|f(\cdot,s)\|_{L^2(\Omega)}\,ds \le e^b\int_{-1}^{t} e^{-b(t-s)}\|f\|_{L^{2,1}(\mathcal{C}_s)}\,ds \tag{4.29}$$

and (4.28) becomes

$$\|w(\cdot,t)\|_{L^2(\Omega)} \leq \mathsf{C}\left(\|\psi\|_{L^2(\Omega)}e^{-bt} + \int_{-1}^{t} e^{-b(t-s)}\chi(s)\,ds\right), \tag{4.30}$$

with χ as defined in (4.6). The convention that $w(x,t)=0$ for $t<0$ makes (4.30) valid for all $t \geq -1$.

Step 4. Completing the proof of (4.5). We now go on with deriving the estimate (4.5). Fix a value of $t > -1$ and introduce the matrices

$$\tilde{A}_{ij}(x,s) = \begin{cases} A_{ij}(x,s) & \text{if } s \leq t+1 \\ 0 & \text{if } s > t+1 \end{cases}$$

and

$$\tilde{A}(x,s) = \begin{cases} A(x,s) & \text{if } s \leq t+1 \\ 0 & \text{if } s > t+1. \end{cases}$$

The matrices $\tilde{A}_{ij}^{(0)}$, $\tilde{A}_{ij}^{(1)}$, $\tilde{A}^{(0)}$, $\tilde{A}^{(1)}$ and the functions $\tilde{f}_i$, $\tilde{f}$ are defined analogously. By replacing A_{ij}, A, f_i, f with $\tilde{A}_{ij}$, $\tilde{A}$, $\tilde{f}_i$, $\tilde{f}$ in (4.1) and (4.3), we see that the solution w is unchanged on the interval $[0,t+1]$.

Introduce

$$\tilde{\chi}(s) = \sum_{i=1}^{n} \|\tilde{f}_i\|_{L^2(\mathcal{C}_s)} + \|\tilde{f}\|_{L^{2,1}(\mathcal{C}_s)}.$$

For $\epsilon \in [0,1]$, inequality (4.30) yields that

$$\|w(\cdot,t+\epsilon)\|_{L^2(\Omega)} \leq \mathsf{C}\Bigg(\|\psi\|_{L^2(\Omega)}e^{-bt} + \int_{-1}^{t} e^{-b(t-s)}\tilde{\chi}(s)\,ds + \int_{t}^{t+\epsilon} e^{-b(t-s)}\tilde{\chi}(s)\,ds\Bigg).$$

Since obviously $\tilde{\chi}(s) \leq \chi(t)$ for $t \leq s \leq t+1$, it follows that

$$\operatorname*{ess\,sup}_{t<s<t+1} \|w(\cdot,s)\|_{L^2(\Omega)} \leq \mathsf{C}\left(\|\psi\|_{L^2(\Omega)}e^{-bt} + \int_{-1}^{t} e^{-b(t-s)}\chi(s)\,ds + \chi(t)\right). \tag{4.31}$$

In estimating $\|\nabla w\|_{L^2(\mathcal{C}_t)}$, we use that it follows from (4.7) that

$$\|w(\cdot,t+1)\|^2_{L^2(\Omega)} - \|w(\cdot,t)\|^2_{L^2(\Omega)} + 2\int_t^{t+1} \left[\mathcal{L}_1(w,w) + \mathcal{L}_2(\mathbf{f},w)\right]\,ds = 0.$$

This, together with (2.7), gives that

$$\begin{aligned} 2\nu\int_t^{t+1} \|\nabla w(\cdot,s)\|^2_{L^2(\Omega)}\,ds &\leq 2\int_t^{t+1}\int_\Omega \sum_{i,j=1}^{n} (A_{ij}w_{x_j}, w_{x_i})\,dx\,ds \\ &\leq \|w(\cdot,t)\|^2_{L^2(\Omega)} - 2\int_t^{t+1}\int_\Omega (Aw,w)\,dx\,ds \\ &\quad - 2\int_t^{t+1}\int_\Omega \sum_{i=1}^{n} (f_i, w_{x_i})\,dx\,ds - 2\int_t^{t+1}\int_\Omega (f,w)\,dx\,ds. \end{aligned} \tag{4.32}$$

As in (4.9) and (4.11), we obtain the inequalities

$$-2\int_t^{t+1}\int_\Omega (Aw,w)\,dx\,ds \le 2\nu_1\|w\|^2_{L^2(\mathcal{C}_t)} \tag{4.33}$$

and

$$-2\int_t^{t+1}\int_\Omega \sum_{i=1}^n (f_i, w_{x_i})\,dx\,ds \le \nu\|\nabla w\|^2_{L^2(\mathcal{C}_t)} + \frac{1}{\nu}\Big(\sum_{i=1}^n \|f_i\|_{L^2(\mathcal{C}_t)}\Big)^2. \tag{4.34}$$

Furthermore, from Hölder's inequality it follows that

$$\begin{aligned} -2\int_t^{t+1}\int_\Omega (f,w)\,dx\,ds &\le 2\|f\|_{L^{2,1}(\mathcal{C}_t)} \operatorname*{ess\,sup}_{t<s<t+1} \|w(\cdot,s)\|_{L^2(\Omega)} \\ &\le \|f\|^2_{L^{2,1}(\mathcal{C}_t)} + \Big(\operatorname*{ess\,sup}_{t<s<t+1} \|w(\cdot,s)\|_{L^2(\Omega)}\Big)^2. \end{aligned} \tag{4.35}$$

By inserting (4.33), (4.34) and (4.35) in (4.32) and using that

$$\|w(\cdot,t)\|^2_{L^2(\Omega)} \le \operatorname*{ess\,sup}_{t<s<t+1} \|w(\cdot,s)\|^2_{L^2(\Omega)},$$

it follows that

$$\begin{aligned} 2\nu\|\nabla w\|^2_{L^2(\mathcal{C}_t)} \le & 2\nu_1\|w\|^2_{L^2(\mathcal{C}_t)} + 2\Big(\operatorname*{ess\,sup}_{t<s<t+1} \|w(\cdot,s)\|_{L^2(\Omega)}\Big)^2 \\ & + \nu\|\nabla w\|^2_{L^2(\mathcal{C}_t)} + \frac{1}{\nu}\Big(\sum_{i=1}^n \|f_i\|_{L^2(\mathcal{C}_t)}\Big)^2 + \|f\|^2_{L^{2,1}(\mathcal{C}_t)}, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \nu\|\nabla w\|^2_{L^2(\mathcal{C}_t)} \le & 2\nu_1\|w\|^2_{L^2(\mathcal{C}_t)} + 2\Big(\operatorname*{ess\,sup}_{t<s<t+1} \|w(\cdot,s)\|_{L^2(\Omega)}\Big)^2 \\ & + \frac{1}{\nu}\Big(\sum_{i=1}^n \|f_i\|_{L^2(\mathcal{C}_t)}\Big)^2 + \|f\|^2_{L^{2,1}(\mathcal{C}_t)}. \end{aligned}$$

After taking the square root, this yields

$$\|\nabla w\|_{L^2(\mathcal{C}_t)} \le C\Big(\|w\|_{L^2(\mathcal{C}_t)} + \operatorname*{ess\,sup}_{t<s<t+1} \|w(\cdot,s)\|_{L^2(\Omega)} + \chi(t)\Big). \tag{4.36}$$

We now estimate the terms in the right-hand side and begin by studying $\|w\|_{L^2(\mathcal{C}_t)}$. Since

$$\|w\|^2_{L^2(\mathcal{C}_t)} = \int_t^{t+1} \|w(\cdot,s)\|^2_{L^2(\Omega)}\,ds,$$

we get from (4.24) and (4.13) the estimate

$$\begin{aligned}\|w\|^2_{L^2(\mathcal{C}_t)} \le C\bigg(&\|\psi\|^2_{L^2(\Omega)}\frac{e^{-2bt}}{2b}\\ &+\sum_{i=1}^n\int_t^{t+1}\int_0^s e^{-2b(s-\tau)}\|f_i(\cdot,\tau)\|^2_{L^2(\Omega)}\,d\tau\,ds\\ &+\int_t^{t+1}\Big(\int_0^s e^{-b(s-\tau)}\|f(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau\Big)^2 ds\bigg).\end{aligned}\tag{4.37}$$

We get an estimate for $\|w\|_{L^2(\mathcal{C}_t)}$ by taking the square root of each term. Indeed,

$$\begin{aligned}\Big(\int_t^{t+1}&\int_0^s e^{-2b(s-\tau)}\|f_i(\cdot,\tau)\|^2_{L^2(\Omega)}\,d\tau\,ds\Big)^{1/2}\\ &\le\Big(\int_t^{t+1}\int_0^{t+1} e^{-2b(t-\tau)}\|f_i(\cdot,\tau)\|^2_{L^2(\Omega)}\,d\tau\,ds\Big)^{1/2}\\ &=\Big(\int_0^{t+1} e^{-2b(t-\tau)}\|f_i(\cdot,\tau)\|^2_{L^2(\Omega)}\,d\tau\Big)^{1/2}\\ &\le e^b\int_{-1}^{t+1} e^{-b(t-\tau)}\|\tilde{f}_i\|_{L^2(\mathcal{C}_\tau)}\,d\tau\\ &\le e^b\Big(\int_{-1}^{t} e^{-b(t-\tau)}\|f_i\|_{L^2(\mathcal{C}_\tau)}\,d\tau+e^b\|f_i\|_{L^2(\mathcal{C}_t)}\Big),\end{aligned}\tag{4.38}$$

where we have used (4.25) with $\gamma=\|\tilde{f}_i\|_{L^2(\Omega)}$ in the second inequality. By extending the interval of integration as above, we have that

$$\begin{aligned}\Big(\int_t^{t+1}&\Big(\int_0^s e^{-b(s-\tau)}\|f(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau\Big)^2 ds\Big)^{1/2}\\ &=\int_0^{t+1} e^{-b(t-\tau)}\|f(\cdot,\tau)\|_{L^2(\Omega)}\,d\tau\\ &\le e^b\Big(\int_{-1}^{t} e^{-b(t-\tau)}\|f\|_{L^{2,1}(\mathcal{C}_\tau)}\,d\tau+\|f\|_{L^{2,1}(\mathcal{C}_t)}\Big),\end{aligned}\tag{4.39}$$

where (4.29) was used in the last inequality. We use the inequalities (4.38) and (4.39) in (4.37) and get

$$\|w\|_{L^2(\mathcal{C}_t)}\le \mathsf{C}\Big(\|\psi\|_{L^2(\Omega)}e^{-bt}+\int_{-1}^t e^{-b(t-s)}\chi(s)\,ds+\chi(t)\Big).$$

This, together with (4.31), used in (4.36) gives that

$$\|\nabla w\|_{L^2(\mathcal{C}_t)}\le \mathsf{C}\Big(\|\psi\|_{L^2(\Omega)}e^{-bt}+\int_{-1}^t e^{-b(t-s)}\chi(s)\,ds+\chi(t)\Big).\tag{4.40}$$

By combining (4.31) and (4.40), we finally obtain (4.5). □

4.2 Estimate for v

A consequence of Proposition 4.1 is that equation (3.9) has a unique solution in $\left(V^2_{0,\mathrm{loc}}(Q)\right)^N$. Let us fix $b = 2\lambda_{J+1}$, with J defined in (3.2). According to Proposition 4.1, it is possible to find an integer M such that the solution satisfies the estimate (4.5). Observe that after this choice of M, the constant C depends only on n, N, Ω, $A^{(0)}_{ij}$, $A^{(0)}$, p, s'_2, ν, ν_1 and ν_2. We introduce

$$H(t) = \begin{cases} \operatorname{ess\,sup}_{t<s<t+1} \sum_{k=1}^M |h_k(s)| & \text{if } t \geq 0 \\ \operatorname{ess\,sup}_{0<s<t+1} \sum_{k=1}^M |h_k(s)| & \text{if } -1 < t < 0 \\ 0 & \text{if } t \leq -1. \end{cases} \tag{4.41}$$

Then Proposition 4.1 yields the following corollary.

Corollary 4.2 *For $k = 1, \ldots, M$, let h_k be arbitrary functions from $L^\infty_{\mathrm{loc}}(0,\infty)$. Then equation (3.9) has a unique solution $v \in \left(V^2_{0,\mathrm{loc}}(Q)\right)^N$, orthogonal to $\phi_1, \ldots, \phi_m$. After extending v by 0 for $t < 0$, the estimate*

$$|v|_{\mathcal{C}_t} \leq C\left(\|\psi\|_{L^2(\Omega)} e^{-2\lambda_{J+1}t} + \int_{-1}^t e^{-2\lambda_{J+1}(t-s)}\kappa(s)H(s)\,ds + \kappa(t)H(t)\right) \tag{4.42}$$

is valid for all $t \geq -1$.

Proof. We apply Proposition 4.1 on (3.9). With $b = 2\lambda_{J+1}$, we get, from (4.5), the estimate

$$|v|_{\mathcal{C}_t} \leq C\left(\|\psi\|_{L^2(\Omega)} e^{-2\lambda_{J+1}t} + \int_{-1}^t e^{-2\lambda_{J+1}(t-s)}\chi(s)\,ds + \chi(t)\right), \tag{4.43}$$

where

$$\chi(t) = \sum_{i=1}^n \|f_i\|_{L^2(\mathcal{C}_t)} + \|f\|_{L^{2,1}(\mathcal{C}_t)}, \tag{4.44}$$

for

$$f_i = \sum_{j=1}^n \sum_{k=1}^M A^{(1)}_{ij} h_k \phi_{k\,x_j}$$

and

$$f = \sum_{k=1}^M A^{(1)} h_k \phi_k.$$

Using that $\phi_k \in \left(W^{1,p}_0(\Omega)\right)^N$ for some $p > 2$, see (2.6), we derive that

$$\|f_i\|_{L^2(\mathcal{C}_t)} \leq C\kappa(t)H(t), \tag{4.45}$$

where $\kappa(t)$ was introduced in (1.12). Namely, let a denote an element of $A_{ij}^{(1)}$ and φ an element of $\{\phi_{kx_j}\}_{k=1}^{M}$. With s_1 as defined in (1.13), Hölder's inequality implies that

$$\begin{aligned}\|a\varphi\|_{L^2(\Omega)} &= \|a^2\varphi^2\|_{L^1(\Omega)}^{1/2} \le \|a^2\|_{L^{s_1/2}(\Omega)}^{1/2}\|\varphi^2\|_{L^{p/2}(\Omega)}^{1/2} \\ &= \|a\|_{L^{s_1}(\Omega)}\|\varphi\|_{L^p(\Omega)}\end{aligned}$$

and it is now easy to show (4.45).

Moreover, Sobolev's embedding theorem implies that $\phi_k \in \big(L^{p_1}(\Omega)\big)^N$, where

$$p_1 = \begin{cases} \infty & \text{if } n < p \\ p_1' & \text{if } n = p \\ \frac{np}{n-p} & \text{if } n > p, \end{cases}$$

where p_1' is an arbitrary number in $[1, \infty)$, and the estimate

$$\|\phi_k\|_{L^{p_1}(\Omega)} \le \mathsf{C}\|\phi_k\|_{W^{1,p}(\Omega)}$$

is valid. The constant C depends on n, N, Ω, p and p_1'. It follows that

$$\|f(\cdot,t)\|_{L^2(\Omega)} \le \mathsf{C}\|A^{(1)}(\cdot,t)\|_{L^{p_2}(\Omega)} \sum_{k=1}^{M} |h_k(t)|\|\phi_k\|_{W^{1,p}(\Omega)},$$

where

$$p_2 = \begin{cases} 1 & \text{if } n < p \\ \frac{p_1'}{p_1'-1} & \text{if } n = p \\ \frac{np}{np-n+p} & \text{if } n > p \end{cases}$$

is the conjugate exponent to p_1. In the case $n = p$, we set

$$p_1' = \frac{s_2'}{s_2' - 1},$$

where s_2' was introduced in (1.14). This means in particular that $p_2 = s_2'$. From (1.14) it then follows that $p_2 \le s_2$ for every value of n, so

$$\|f\|_{L^{2,1}(\mathcal{C}_t)} \le C\kappa(t)H(t). \tag{4.46}$$

Using the estimates (4.45) and (4.46) in (4.44) we get

$$\chi(t) \le C\kappa(t)H(t)$$

and (4.42) follows from (4.43). □

5 Norm estimates for $\mathcal{R}_{kl}$ and $g_k(w)$

In this section we find norm estimates for $\mathcal{R}_{kl}$ and $g_k(w)$, introduced in (3.4) and (3.5).

Lemma 5.1 *Suppose that w is a function in $V^2_{\rm loc}(Q)$. There exist constants c_1 and c_2 such that*

$$\|\mathcal{R}_{kl}\|_{L^1(t,t+1)} \le c_1\kappa(t) \tag{5.1}$$

and

$$\|g_k(w)\|_{L^1(t,t+1)} \le c_2\kappa(t)|w|_{\mathcal{C}_t}. \tag{5.2}$$

Here, c_1 and c_2 depend only on n, p, s_2' and Ω.

Proof. We begin with the case $n > p$. As in the proof of Corollary 4.2, it follows that $\phi_k \in \big(L^{p_1}(\Omega)\big)^N$ with $p_1 = np/(n-p)$ and the estimate

$$\|\phi_k\|_{L^{p_1}(\Omega)} \le C\|\phi_k\|_{W^{1,p}(\Omega)} \tag{5.3}$$

is valid. By applying Hölder's inequality on (3.4), the estimate

$$|\mathcal{R}_{kl}(\tau)| \le \sum_{i,j=1}^n \|A^{(1)}_{ij}(\cdot,\tau)\|_{L^{s_0}(\Omega)}\|\phi_{l x_j}\|_{L^p(\Omega)}\|\phi_{k x_i}\|_{L^p(\Omega)} + \|A^{(1)}(\cdot,\tau)\|_{L^{r_0}(\Omega)}\|\phi_l\|_{L^{p_1}(\Omega)}\|\phi_k\|_{L^{p_1}(\Omega)} \tag{5.4}$$

is obtained, where $s_0^{-1} + 2p^{-1} = 1$ and $r_0^{-1} + 2p_1^{-1} = 1$, i.e. $s_0 = p/(p-2)$ and

$$r_0 = \frac{np}{np - 2(n-p)}.$$

Integrating (5.4) from t to $t+1$ and using (5.3), we obtain

$$\|\mathcal{R}_{kl}\|_{L^1(t,t+1)} \le C\|\phi_k\|_{W^{1,p}(\Omega)}\|\phi_l\|_{W^{1,p}(\Omega)} \times \Big(\sum_{i,j=1}^n \|A^{(1)}_{ij}\|_{L^{s_0,1}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^{r_0,1}(\mathcal{C}_t)}\Big). \tag{5.5}$$

Analogously, we get from (3.5) that

$$\big|\big(g_k(w)\big)(\tau)\big| \le \sum_{i,j=1}^n \|A^{(1)}_{ij}(\cdot,\tau)\|_{L^{s_1}(\Omega)}\|w_{x_j}(\cdot,\tau)\|_{L^2(\Omega)}\|\phi_{k x_i}\|_{L^p(\Omega)} + \|A^{(1)}(\cdot,\tau)\|_{L^{s_2}(\Omega)}\|w(\cdot,\tau)\|_{L^2(\Omega)}\|\phi_k\|_{L^{p_1}(\Omega)},$$

with the same s_1 and s_2 as in (1.13), (1.14). After integrating from t to $t+1$, we obtain

$$\|g_k(w)\|_{L^1(t,t+1)} \le C\|\phi_k\|_{W^{1,p}(\Omega)}|w|_{\mathcal{C}_t} \times \Big(\sum_{i,j=1}^n \|A^{(1)}_{ij}\|_{L^{s_1,2}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^{s_2,1}(\mathcal{C}_t)}\Big). \tag{5.6}$$

Since $r_0 = s_2/2$ and $s_0 = s_1/2$, Lemma 5.1 follows for $n > p$ from (5.5) and (5.6).

In the case where $n < p$, it follows immediately from Sobolev's embedding theorem that $\phi_k \in \big(L^\infty(\Omega)\big)^N$ with the estimate

$$\|\phi_k\|_{L^\infty(\Omega)} \le C\|\phi_k\|_{W^{1,p}(\Omega)}.$$

Using this, we get analogously to the case where $n > p$ the estimates

$$\|\mathcal{R}_{kl}\|_{L^1(t,t+1)} \le C\|\phi_k\|_{W^{1,p}(\Omega)}\|\phi_l\|_{W^{1,p}(\Omega)} \times \Bigg(\sum_{i,j=1}^{n} \|A_{ij}^{(1)}\|_{L^{s_0,1}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^1(\mathcal{C}_t)}\Bigg)$$

and

$$\|g_k(w)\|_{L^1(t,t+1)} \le C\|\phi_k\|_{W^{1,p}(\Omega)}|w|_{\mathcal{C}_t} \times \Bigg(\sum_{i,j=1}^{n} \|A_{ij}^{(1)}\|_{L^{s_1,2}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^{2,1}(\mathcal{C}_t)}\Bigg)$$

with the same s_0 as before and the lemma follows in the case when $n < p$.

For $n = p$, the fact that $\phi_k \in \big(W^{1,p}(\Omega)\big)^N$ implies that

$$\|\phi_k\|_{L^{p_2}(\Omega)} \le \mathsf{C}\|\phi_k\|_{W^{1,p}(\Omega)} \tag{5.7}$$

for any $p_2 \in [1, \infty)$, where C depends on n, p, p_2 and Ω. We obtain in the same way as before

$$\|\mathcal{R}_{kl}\|_{L^1(t,t+1)} \le C\|\phi_k\|_{W^{1,p}(\Omega)}\|\phi_l\|_{W^{1,p}(\Omega)} \times \Bigg(\sum_{i,j=1}^{n} \|A_{ij}^{(1)}\|_{L^{s_0,1}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^{2,1}(\mathcal{C}_t)}\Bigg),$$

which implies (5.1). Finally,

$$\big|\big(g_k(w)\big)(\tau)\big| \le \sum_{i,j=1}^{n} \|A_{ij}^{(1)}(\cdot,\tau)\|_{L^{s_1}(\Omega)}\|w_{x_j}(\cdot,\tau)\|_{L^2(\Omega)}\|\phi_{k x_i}\|_{L^p(\Omega)} + \|A^{(1)}(\cdot,\tau)\|_{L^{s_2'}(\Omega)}\|w(\cdot,\tau)\|_{L^2(\Omega)}\|\phi_k\|_{L^{s_3}(\Omega)}, \tag{5.8}$$

where s_2' was introduced in (1.14) and

$$s_3 = \frac{2s_2'}{s_2' - 2}.$$

By using (5.7) with $p_2 = s_3$ and integrating (5.8) from t to $t+1$, we obtain

$$\|g_k(w)\|_{L^1(t,t+1)} \le C\|\phi_k\|_{W^{1,p}(\Omega)}|w|_{\mathcal{C}_t} \times \left(\sum_{i,j=1}^{n} \|A_{ij}^{(1)}\|_{L^{s_1,2}(\mathcal{C}_t)} + \|A^{(1)}\|_{L^{s_2',1}(\mathcal{C}_t)} \right)$$

and (5.2) follows. The proof is complete. □

6 Functions $h_{J+1}, \dots, h_M$

6.1 Definition of functions v_0, v_1 and v_2

For $k = 1, \dots, M$, let z_k denote an element of $L^\infty_{\text{loc}}(0,\infty)$ and introduce the vectors

$$z = (z_1, \dots, z_M),$$

$$\hat{z} = (z_1, \dots, z_J) \tag{6.1}$$

and

$$\check{z} = (z_{J+1}, \dots, z_M). \tag{6.2}$$

We define $v_0(\check{z})$, $v_1(\hat{z})$ and v_2 as solutions of the equations

$$-\int_Q (v_0, \eta_t)\, dx\, dt + \int_0^\infty \left[\mathcal{L}_1(v_0, \eta) + \sum_{k=J+1}^{M} z_k \mathcal{L}_1^{(1)}(\phi_k, \eta) \right] dt = 0,$$

$$-\int_Q (v_1, \eta_t)\, dx\, dt + \int_0^\infty \left[\mathcal{L}_1(v_1, \eta) + \sum_{k=1}^{J} z_k \mathcal{L}_1^{(1)}(\phi_k, \eta) \right] dt = 0$$

and

$$-\int_Q (v_2, \eta_t)\, dx\, dt + \int_0^\infty \mathcal{L}_1(v_2, \eta)\, dt = \int_\Omega \big(\psi(x), \eta(x,0)\big)\, dx,$$

respectively. Here we mean solutions in the sense of Section 4.1.

For a given z, the existence and uniqueness of $v_0(\check{z})$, $v_1(\hat{z})$ and v_2, considered as elements of $\big(V^2_{0,\text{loc}}(Q)\big)^N$, are guaranteed by Proposition 4.1. Furthermore, the elements v_0 and v_1 are linear operators on $\big(L^\infty_{\text{loc}}(0,\infty)\big)^{M-J}$ and $\big(L^\infty_{\text{loc}}(0,\infty)\big)^J$, respectively. Also g_k, as defined in (3.5), is a linear operator but on $\big(V^2_{\text{loc}}(Q)\big)^N$. It follows that

$$\Gamma_k(\check{z}) = g_k\big(v_0(\check{z})\big) \tag{6.3}$$

is a linear operator on $\big(L^\infty_{\text{loc}}(0,\infty)\big)^{M-J}$.

6.2 Integro-differential system for $h_{J+1}, \dots, h_M$

We introduce

$$\hat{h} = (h_1, \dots, h_J) \quad \text{and} \quad \check{h} = (h_{J+1}, \dots, h_M).$$

Because of linearity, we can split the function v in (3.1) as

$$v = v_0(\check{h}) + v_1(\hat{h}) + v_2. \tag{6.4}$$

Since also g_k in (3.5) is linear, the last $M - J$ equations in (3.6) can be rewritten as

$$h_k' + \lambda_k h_k + \sum_{l=J+1}^{M} \mathcal{R}_{kl} h_l + \Gamma_k(\check{h}) = F_k, \quad k = J+1, \dots, M, \tag{6.5}$$

where

$$F_k = -\sum_{l=1}^{J} \mathcal{R}_{kl} h_l - \gamma_k(\hat{h}) - g_k(v_2) \tag{6.6}$$

and

$$\gamma_k(\hat{h}) = g_k\big(v_1(\hat{h})\big),$$

which obviously is linear with respect to $\hat{h}$. Together with the boundary conditions

$$h_k(0) = \int_\Omega (\psi, \phi_k)\, dx, \quad k = J+1, \dots, M \tag{6.7}$$

this yields a new linear system of equations for $\check{h}$ with $\hat{h}$ considered as a given vector.

6.3 A general estimate

For some given functions $F_k \in L^1_{\text{loc}}(0, \infty)$, not necessarily coinciding with (6.6), consider the linear system of equations

$$z_k' + \lambda_k z_k + \sum_{l=J+1}^{M} \mathcal{R}_{kl} z_l + \Gamma_k(\check{z}) = F_k, \quad k = J+1, \dots, M \tag{6.8}$$

on the positive real half axis together with the initial conditions

$$z_k(0) = a_k, \quad k = J+1, \dots, M. \tag{6.9}$$

We introduce

$$a = \sum_{k=J+1}^{M} |a_k| \tag{6.10}$$

and

$$F(t) = \sum_{k=J+1}^{M} \|F_k\|_{L^1(t,t+1)}. \tag{6.11}$$

Under a few assumptions on F_k, we will prove existence and uniqueness of solutions of the problem (6.8), (6.9) in an appropriate Banach space. We will also find an estimate of the solution. But let us first introduce the space in which we are going to work.

Suppose that ξ_k, $k = J+1, \dots, M$, are measurable functions on $(0,\infty)$ and introduce

$$(\Xi_2\xi)(t) = \begin{cases} \operatorname{ess\,sup}_{t<s<t+1} \sum_{k=J+1}^{M} |\xi_k(s)| & \text{if } t \geq 0 \\ \operatorname{ess\,sup}_{0<s<t+1} \sum_{k=J+1}^{M} |\xi_k(s)| & \text{if } -1 < t < 0 \\ 0 & \text{if } t \leq -1. \end{cases}$$

For a given $\Upsilon \geq 0$, set

$$\|\xi\|_{\mathcal{A}_\Upsilon} = \sup_{t \geq -1} e^{-\Upsilon t} (\Xi_2 \xi)(t).$$

The reason for the subindex "2" will appear later. We define the Banach space $\mathcal{A}_\Upsilon$ as the set of measurable functions $\xi = (\xi_{J+1}, \dots, \xi_M)$ on $(0,\infty)$ such that $\|\xi\|_{\mathcal{A}_\Upsilon} < \infty$.

Before proving next result, we observe that if $f \in L^1_{\text{loc}}(\mathbf{R})$, it follows from (4.4) that

$$\begin{aligned} \int_0^{t+1} |f(s)|\, ds &= \int_0^t |f(s)|\, ds + \|f\|_{L^1(t,t+1)} \\ &\leq \int_{-1}^t \|f\|_{L^1(s,s+1)}\, ds + \|f\|_{L^1(t,t+1)} \end{aligned} \tag{6.12}$$

for $t \geq -1$.

Proposition 6.1 *For $k = J+1, \dots, M$, let a_k be arbitrary real numbers and assume that $F_k \in L^1_{\text{loc}}(\mathbf{R})$ where $F_k(t) = 0$ for $t < 0$. Suppose further that, for some positive Υ, the relations*

$$\sup_{t \geq -1} e^{-\Upsilon t} \|F_k\|_{L^1(t,t+1)} < \infty, \quad k = J+1, \dots, M, \tag{6.13}$$

hold. If the constant $\varkappa$ in (1.16) *is small enough, then there exists a unique solution $\check{z} \in \mathcal{A}_\Upsilon$ of* (6.8), (6.9) *and there exist constants C and c_0 such that*

$$(\Xi_2 \check{z})(t) \leq C\left(a e^{-\mu t} + \int_{-1}^{t} e^{-\mu(t-s)} F(s)\, ds + F(t) \right), \quad t \geq -1, \tag{6.14}$$

where

$$\mu = \lambda_{J+1} - c_0 \kappa_0 \tag{6.15}$$

and a is given by (6.10). *The constants C and c_0 do only depend on n, N, Ω, $A^{(0)}_{ij}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2 and not on Υ.*

Proof. We first prove the existence and uniqueness of solutions. Let us fix $k \in \{J+1, \dots, M\}$ and consider the equation corresponding to k in (6.8). It follows immediately that

$$\left(z_k e^{\lambda_k t}\right)' = e^{\lambda_k t}\left(F_k - \sum_{l=J+1}^{M} \mathcal{R}_{kl} z_l - \Gamma_k(\check{z})\right)$$

and after integrating from 0 to t we obtain

$$z_k(t) = a_k e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-s)}\left(F_k - \sum_{l=J+1}^{M} \mathcal{R}_{kl} z_l - \Gamma_k(\check{z})\right) ds. \quad (6.16)$$

We construct a linear operator S acting on $\mathcal{A}_\Upsilon$ so that the k:th component of S becomes

$$(S\check{z})_k(t) = \int_0^t e^{-\lambda_k(t-s)}\left(\sum_{l=J+1}^{M} \mathcal{R}_{kl} z_l + \Gamma_k(\check{z})\right) ds \quad (6.17)$$

for $t \geq 0$. If $t < 0$ we set $(S\check{z})(t) = 0$. The fact that $S\check{z} \in \mathcal{A}_\Upsilon$ if $\check{z} \in \mathcal{A}_\Upsilon$ will appear from the following analysis.

Equation (6.16) can be written as

$$z_k(t) = -(S\check{z})_k(t) + \alpha_k(t), \quad (6.18)$$

where

$$\alpha_k(t) = a_k e^{-\lambda_k t} + \int_0^t e^{-\lambda_k(t-s)} F_k(s)\, ds. \quad (6.19)$$

From this, the existence and uniqueness of z_k follows if we can prove that $\|S\| < 1$ and that

$$\alpha = (\alpha_{J+1}, \dots, \alpha_M) \quad (6.20)$$

belongs to $\mathcal{A}_\Upsilon$.

Let us prove that $\|S\| < 1$ if $\varkappa$ is small enough. We analyze (6.17) term by term and introduce S_{1l} through the relation

$$(S_{1l}\check{z})_k(t) = \int_0^t e^{-\lambda_k(t-s)} \mathcal{R}_{kl} z_l\, ds, \quad t \geq 0. \quad (6.21)$$

Let us prove that there exists a constant C such that

$$\|S_{1l}\check{z}\|_{\mathcal{A}_\Upsilon} \leq C\varkappa \|\check{z}\|_{\mathcal{A}_\Upsilon}. \quad (6.22)$$

To make clear that C depends only on n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2 and not on, for example, Υ, we will write out the exact values of all constants occurring in the calculations below. Using (6.12), it follows from (6.21) that, for $t \geq -1$,

$$
\begin{aligned}
\operatorname*{ess\,sup}_{t<s<t+1} \sum_{k=J+1}^{M} |(S_{1l}\check{z})_k(s)| &\leq \int_0^{t+1} e^{-\lambda_{J+1}(t-s)} \Bigg(\sum_{k=J+1}^{M} |\mathcal{R}_{kl}| \Bigg) |z_l|\, ds \\
&\leq \int_{-1}^{t} \Bigg\| e^{-\lambda_{J+1}(t-\cdot)} \Bigg(\sum_{k=J+1}^{M} |\mathcal{R}_{kl}| \Bigg) z_l \Bigg\|_{L^1(s,s+1)} ds \qquad (6.23) \\
&\quad + \Bigg\| e^{-\lambda_{J+1}(t-\cdot)} \Bigg(\sum_{k=J+1}^{M} |\mathcal{R}_{kl}| \Bigg) z_l \Bigg\|_{L^1(t,t+1)},
\end{aligned}
$$

where we have extended z_l by 0 for $t < 0$. After a use of Hölder's inequality and (5.1), this expression can be majorized by

$$
c_1 \kappa_0 (M-J) e^{\lambda_{J+1}} \left(\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} \|z_l\|_{L^\infty(s,s+1)}\, ds + \|z_l\|_{L^\infty(t,t+1)} \right).
$$

Using the fact that

$$
\|z_l\|_{L^\infty(s,s+1)} \leq e^{\Upsilon t} \|\check{z}\|_{\mathcal{A}_\Upsilon}
$$

for $-1 \leq s \leq t$ and the inequality

$$
\int_{-1}^{t} e^{-a(t-s)}\, d\tau \leq \frac{1}{a}, \quad a > 0,
$$

with $a = \lambda_{J+1}$, we finally get that

$$
\operatorname*{ess\,sup}_{t<s<t+1} \sum_{k=J+1}^{M} |(S_{1l}\check{z})_k(s)| \leq c_1 (M-J) e^{\lambda_{J+1}} \left(1 + \frac{1}{\lambda_{J+1}} \right) \kappa_0 e^{\Upsilon t} \|\check{z}\|_{\mathcal{A}_\Upsilon}.
$$

We remind that the constant M depends only on n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2, see Proposition 4.1 and the beginning of Section 4.2. Therefore, denoting by C the constant before κ_0 in the right-hand side, we arrive at (6.22).

We next introduce S_2 so that

$$
(S_2\check{z})_k(t) = \int_0^t e^{-\lambda_k(t-s)} \Gamma_k(\check{z})\, ds
$$

for $t \geq 0$ and prove the existence of a constant C satisfying the relation

$$
\|S_2\check{z}\|_{\mathcal{A}_\Upsilon} \leq C \varkappa^2 \|\check{z}\|_{\mathcal{A}_\Upsilon}. \qquad (6.24)
$$

From Corollary 4.2 it follows that

$$
|v_0(\check{z})|_{\mathcal{C}_t} \leq C \left(\int_{-1}^{t} e^{-2\lambda_{J+1}(t-s)} \kappa(s) Z(s)\, ds + \kappa(t) Z(t) \right), \qquad (6.25)
$$

where $Z = \Xi_2 \check{z}$. Using (5.2) together with this fact on $\Gamma_k(\check{z})$, as defined in (6.3), we obtain

$$\begin{aligned}\|\Gamma_k(\check{z})\|_{L^1(t,t+1)} &\le c_2\kappa(t)|v_0(\check{z})|_{\mathcal{C}_t} \\ &\le c_3\kappa(t)\left(\int_{-1}^{t} e^{-2\lambda_{J+1}(t-s)}\kappa(s)Z(s)\,ds + \kappa(t)Z(t)\right)\end{aligned} \tag{6.26}$$

for some constant c_3.

As in (6.23), we get the estimate

$$\begin{aligned}&\operatorname*{ess\,sup}_{t<s<t+1} \sum_{k=J+1}^{M} |(S_2\check{z})_k(s)| \\ &\qquad \le e^{\lambda_{J+1}} \sum_{k=J+1}^{M} \left(\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)}\|\Gamma_k(\check{z})\|_{L^1(s,s+1)}\,ds \right. \\ &\qquad\quad \left. + \|\Gamma_k(\check{z})\|_{L^1(t,t+1)}\right).\end{aligned} \tag{6.27}$$

We use (6.26) to estimate the terms in (6.27). For the integral, it follows that

$$\begin{aligned}&\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)}\|\Gamma_k(\check{z})\|_{L^1(s,s+1)}\,ds \\ &\quad \le c_3\kappa_0^2 \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)}\left(\int_{-1}^{s} e^{-2\lambda_{J+1}(s-\tau)}Z(\tau)\,d\tau + Z(s)\right)ds \\ &\quad \le \frac{c_3\kappa_0^2\|\check{z}\|_{\mathcal{A}_\Upsilon}}{\lambda_{J+1}+\Upsilon}\left(\frac{1}{2\lambda_{J+1}+\Upsilon}+1\right)e^{\Upsilon t},\end{aligned}$$

where we have used that $Z(t) \le e^{\Upsilon t}\|\check{z}\|_{\mathcal{A}_\Upsilon}$. In the same way we get that

$$\|\Gamma_k(\check{z})\|_{L^1(t,t+1)} \le c_3\kappa_0^2\|\check{z}\|_{\mathcal{A}_\Upsilon}\left(\frac{1}{2\lambda_{J+1}+\Upsilon}+1\right)e^{\Upsilon t}.$$

With these estimates, inequality (6.27) implies (6.24) with

$$C = c_3 e^{\lambda_{J+1}}(M-J)\left(1+\frac{1}{2\lambda_{J+1}}\right)\left(1+\frac{1}{\lambda_{J+1}}\right).$$

Since

$$S = \sum_{l=J+1}^{M} S_{1l} + S_2,$$

it follows from (6.22) and (6.24) that $S : \mathcal{A}_\Upsilon \to \mathcal{A}_\Upsilon$ and that $\|S\| < 1$, i.e. S is a contraction on $\mathcal{A}_\Upsilon$ if $\varkappa$ is small enough.

We now prove that α, as defined in (6.19) and (6.20), is an element of $\mathcal{A}_\Upsilon$. Obviously, the term $a_k e^{-\lambda_k t}$ belongs to $\mathcal{A}_\Upsilon$ since it is bounded for $t \geq -1$. What remains is to check that $\beta = (\beta_{J+1}, \ldots, \beta_M) \in \mathcal{A}_\Upsilon$, where

$$\beta_k(t) = \int_0^t e^{-\lambda_k(t-s)} F_k(s)\, ds.$$

We have

$$\begin{aligned}
&\operatorname*{ess\,sup}_{t<s<t+1} \sum_{k=J+1}^{M} |\beta_k(s)| \\
&\qquad \leq e^{\lambda_{J+1}} \sum_{k=J+1}^{M} \left(\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} \|F_k\|_{L^1(s,s+1)}\, ds + \|F_k\|_{L^1(t,t+1)} \right) \\
&\qquad \leq b e^{\lambda_{J+1}} (M-J) \left(\frac{1}{\lambda_{J+1}} + 1 \right) e^{\Upsilon t},
\end{aligned}$$

where

$$b = \max \left\{ \sup_{t \geq -1} e^{-\Upsilon t} \|F_k\|_{L^1(t,t+1)} : k \in \{J+1, \ldots, M\} \right\}$$

is finite because of (6.13). Hence $\beta \in \mathcal{A}_\Upsilon$ so the same is true for α.

We sum up what has been done: Since the relation (6.18) is equivalent to

$$(I+S)\check{z} = \alpha, \tag{6.28}$$

where $\alpha \in \mathcal{A}_\Upsilon$ and $\|S\| < 1$, it follows that (6.28) has exactly one solution $\check{z} \in \mathcal{A}_\Upsilon$. Hence, the existence and uniqueness of solutions in $\mathcal{A}_\Upsilon$ is proved.

We continue by proving the estimate (6.14). From (6.16) it follows that

$$|z_k(t+\epsilon)| \leq a_k e^{-\lambda_k t} + \int_0^{t+1} e^{-\lambda_k(t-s)} \left(|F_k| + \sum_{l=J+1}^{M} |\mathcal{R}_{kl} z_l| + |\Gamma_k(\check{z})| \right) ds$$

for $\epsilon \in [0,1]$ (if $-1 < t < 0$, we require that $\epsilon \in \big[|t|, 1\big]$). Making the same estimates as in the computations of $\|S_{1l}\check{z}\|_{\mathcal{A}_\Upsilon}$ and $\|S_2\check{z}\|_{\mathcal{A}_\Upsilon}$ and using Lemma 5.1, we obtain

$$Z(t) \leq a e^{-\lambda_{J+1} t} + C \left(\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} B_1(s)\, ds + B_1(t) \right), \tag{6.29}$$

where

$$B_1(t) = F(t) + \kappa(t)\big(Z(t) + |v_0(\check{z})|_{\mathcal{C}_t}\big).$$

The quantities a and $F(t)$ were defined in (6.10) and (6.11). Using (6.25), we get

$$Z(t) \leq a e^{-\lambda_{J+1} t} + C' \left(\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} B_2(s)\, ds + B_2(t) \right), \tag{6.30}$$

with

$$B_2(t) = F(t) + \kappa(t)\left(Z(t) + \int_{-1}^{t} e^{-2\lambda_{J+1}(t-s)}\kappa(s)Z(s)\,ds\right). \tag{6.31}$$

In the last parenthesis we have omitted a term $\kappa(t)Z(t)$, since, for small values of $\varkappa$, its contribution is covered by making C' larger than C in (6.29).

We rewrite the estimate contained in (6.30), (6.31). By changing the order of integration and using that $\kappa(s) \le \kappa_0$, it follows that

$$\begin{aligned}\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)}\kappa(s)\int_{-1}^{s} e^{-2\lambda_{J+1}(s-\tau)}\kappa(\tau)Z(\tau)\,d\tau\,ds \\ \le \kappa_0 e^{-\lambda_{J+1}t}\int_{-1}^{t} e^{2\lambda_{J+1}\tau}\kappa(\tau)Z(\tau)\int_{\tau}^{t} e^{-\lambda_{J+1}s}\,ds\,d\tau \\ \le \frac{\kappa_0}{\lambda_{J+1}}\int_{-1}^{t} e^{-\lambda_{J+1}(t-\tau)}\kappa(\tau)Z(\tau)\,d\tau.\end{aligned} \tag{6.32}$$

By inserting (6.31) into (6.30) and using (6.32), we get

$$Z(t) \le ae^{-\lambda_{J+1}t} + \frac{C_2}{2}\left(\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)}B_3(s)\,ds + B_3(t)\right) \tag{6.33}$$

for some constant C_2 and

$$B_3(t) = F(t) + \kappa(t)Z(t).$$

Assume that $\varkappa$ fulfills the inequality

$$\varkappa \le \frac{1}{C_2}. \tag{6.34}$$

It then follows from (6.33) that

$$\begin{aligned}Z(t) \le 2ae^{-\lambda_{J+1}t} + C_2\Bigg(\kappa_0\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)}Z(s)\,ds \\ + \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)}F(s)\,ds + F(t)\Bigg).\end{aligned} \tag{6.35}$$

We also assume that

$$\varkappa < \frac{\lambda_{J+1}}{2C_2} \tag{6.36}$$

and set

$$\epsilon = 2C_2\kappa_0 \quad \text{and} \quad \mu = \lambda_{J+1} - \epsilon. \tag{6.37}$$

Observe that, because of (6.36), we have $\mu > 0$. For $t \ge -1$, we define

$$Z_+(t) = b_1 ae^{-\mu t} + b_2\int_{-1}^{t} e^{-\mu(t-s)}F(s)\,ds + C_2F(t), \tag{6.38}$$

where $b_1 = 4e^{\epsilon}$ and

$$b_2 = 2C_2(C_2\kappa_0 + 1). \tag{6.39}$$

We are going to prove that $Z(t) \leq Z_+(t)$ and first step is to show that

$$2ae^{-\lambda_{J+1}t} + C_2\biggl(\kappa_0 \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} Z_+(s)\,ds + \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} F(s)\,ds + F(t)\biggr) \leq Z_+(t) \tag{6.40}$$

if $t \geq -1$. We expand the first integral in (6.40) and obtain

$$\begin{aligned}\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} Z_+(s)\,ds &= b_1 a e^{-\lambda_{J+1}t} \int_{-1}^{t} e^{\epsilon s}\,ds \\ &+ b_2 \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} \int_{-1}^{s} e^{-\mu(s-\tau)} F(\tau)\,d\tau\,ds \\ &+ C_2 \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} F(s)\,ds.\end{aligned} \tag{6.41}$$

Since

$$\int_{-1}^{t} e^{\epsilon s}\,ds \leq \frac{e^{\epsilon t}}{\epsilon}$$

and

$$\begin{aligned}&\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} \int_{-1}^{s} e^{-\mu(s-\tau)} F(\tau)\,d\tau\,ds \\ &\qquad = e^{-\lambda_{J+1}t} \int_{-1}^{t} e^{\mu\tau} F(\tau) \int_{\tau}^{t} e^{\epsilon s}\,ds\,d\tau \leq \frac{1}{\epsilon} \int_{-1}^{t} e^{-\mu(t-\tau)} F(\tau)\,d\tau,\end{aligned}$$

we get from (6.41) the inequality

$$\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} Z_+(s)\,ds \leq \frac{b_1 a}{\epsilon} e^{-\mu t} + \left(C_2 + \frac{b_2}{\epsilon}\right) \int_{-1}^{t} e^{-\mu(t-s)} F(s)\,ds.$$

Using this estimate, we obtain

$$\begin{aligned}&2ae^{-\lambda_{J+1}t} + C_2\kappa_0 \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} Z_+(s)\,ds \\ &\quad + C_2\biggl(\int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} F(s)\,ds + F(t)\biggr) \\ &\leq \left(2e^{\epsilon} + \frac{b_1}{2}\right) a e^{-\mu t} + \left(C_2^2\kappa_0 + \frac{b_2}{2} + C_2\right) \int_{-1}^{t} e^{-\mu(t-s)} F(s)\,ds \\ &\quad + C_2 F(t) \\ &= Z_+(t)\end{aligned}$$

and (6.40) is proved.

We now prove that $Z(t) \leq Z_+(t)$ for $t \geq -1$ and use an approach similar to the one used in Step 2 of the proof of Proposition 4.1 when proving that $h(t) \leq h_+(t)$. It follows immediately from (6.38) that $Z_+(-1) \geq 4a$, so $a = Z(-1) \leq Z_+(-1)$. We divide the problem into the cases when $Z_+(-1) > Z(-1)$ and $Z_+(-1) = Z(-1)$. Suppose first that $Z_+(-1) > Z(-1)$. Since Z and Z_+ are continuous, it follows that there exists a $t > -1$ such that $Z(\tau) < Z_+(\tau)$ if $\tau \in [-1, t)$. Then (6.35) and (6.40) imply that

$$Z_+(t) - Z(t) \geq C_2 \kappa_0 \int_{-1}^{t} e^{-\lambda_{J+1}(t-s)} \big(Z_+(s) - Z(s)\big)\, ds > 0$$

and the consequence is that the set

$$\{t \geq -1 : Z(\tau) < Z_+(\tau) \text{ for } \tau \in [-1, t]\}$$

is non-empty, open and closed in $[-1, \infty)$, i.e. equal to $[-1, \infty)$. Hence $Z(t) < Z_+(t)$ for every $t \geq -1$.

If $Z_+(-1) = Z(-1)$, we introduce, for a given $\delta > 0$, the functions

$$Z^\delta(t) = Z(t) + \delta e^{-\lambda_{J+1} t}$$

and

$$Z_+^\delta(t) = Z_+(t) + b_1 \delta e^{-\mu t}$$

for $t \geq -1$. From this, it is easy to see that Z^δ satisfies the inequality (6.35) with a replaced by $a + \delta$ and every occurrence of Z replaced by Z^δ. It is also seen that, for $t \geq -1$, the function Z_+^δ satisfies (6.38) with a replaced by $a + \delta$. As was the case for Z_+, we can hence prove that (6.40) is fulfilled with Z_+ replaced by Z_+^δ. From the facts that $Z(-1) = Z_+(-1)$ and $b_1 = 4e^\epsilon$ it follows that

$$\begin{aligned} Z^\delta(-1) = Z(-1) + \delta e^{\lambda_{J+1}} &= Z_+(-1) + \delta e^{\mu+\epsilon} \\ &< Z_+(-1) + b_1 \delta e^{\mu} = Z_+^\delta(-1). \end{aligned}$$

The previously treated case gives that $Z^\delta(t) < Z_+^\delta(t)$ for $t \geq -1$, i.e.

$$Z(t) + \delta e^{-\lambda_{J+1} t} < Z_+(t) + b_1 \delta e^{-\mu t}.$$

By letting $\delta \to 0$, we see that $Z(t) \leq Z_+(t)$ for $t \geq -1$ also in the case where $Z(-1) = Z_+(-1)$. Since $b_1 < 4e^{\lambda_{J+1}}$ and $b_2 \leq 4C_2$, where we have used that $\kappa_0 \leq \varkappa$ and (6.34) in (6.39), we have thus proved the estimate

$$Z(t) \leq C \left(a e^{-\mu t} + \int_{-1}^{t} e^{-\mu(t-s)} F(s)\, ds + F(t) \right),$$

i.e. (6.14). From (6.37), we see that c_0 in (6.15) can be chosen as $2C_2$. $\square$

6.4 A particular case of equation (6.8)

For $k = 1, \dots, J$, let ξ_k be a measurable function on $(0, \infty)$. We introduce

$$(\Xi_1 \xi)(t) = \begin{cases} \operatorname{ess\,sup}_{t<s<t+1} \sum_{k=1}^{J} |\xi_k(s)| & \text{if } t \geq 0 \\ \operatorname{ess\,sup}_{0<s<t+1} \sum_{k=1}^{J} |\xi_k(s)| & \text{if } -1 < t < 0 \\ 0 & \text{if } t \leq -1 \end{cases} \tag{6.42}$$

and, for a given $\Upsilon \geq 0$, the Banach space $\mathcal{B}_\Upsilon$ consisting of functions $\xi = (\xi_1, \dots, \xi_J)$ measurable on $(0, \infty)$ such that the norm

$$\|\xi\|_{\mathcal{B}_\Upsilon} = \sup_{t \geq -1} e^{-\Upsilon t} (\Xi_1 \xi)(t)$$

is finite. Let $\hat{z}$ and $\check{z}$ be as defined in (6.1), (6.2) and set

$$\mathfrak{F}_k(\hat{z}) = -\sum_{l=1}^{J} \mathcal{R}_{kl} z_l - \gamma_k(\hat{z}) - g_k(v_2), \quad k = J+1, \dots, M. \tag{6.43}$$

We consider the system of equations

$$z_k' + \lambda_k z_k + \sum_{l=J+1}^{M} \mathcal{R}_{kl} z_l + \Gamma_k(\check{z}) = \mathfrak{F}_k(\hat{z}), \quad k = J+1, \dots, M, \tag{6.44}$$

on $(0, \infty)$ together with the boundary conditions

$$z_k(0) = \int_\Omega (\psi, \phi_k), \quad k = J+1, \dots, M, \tag{6.45}$$

and have the following result.

Corollary 6.2 *Assume that $\hat{z} \in \mathcal{B}_\Upsilon$ for some $\Upsilon > 0$. Then, the system* (6.44), (6.45) *has a unique solution $\check{z}$ from $\mathcal{A}_\Upsilon$. The solution satisfies the estimate*

$$(\Xi_2 \check{z})(t) \leq C\bigg(\|\psi\|_{L^2(\Omega)} e^{-\mu t} + \int_{-1}^{t} e^{-\mu(t-s)} \kappa(s) (\Xi_1 \hat{z})(s)\, ds \\ + \kappa(t)(\Xi_1 \hat{z})(t) \bigg), \quad t \geq -1, \tag{6.46}$$

with the same μ as in Proposition 6.1.

Proof. We prove that the assertion (6.13) is true for given Υ. Inequality (5.1) implies that

$$\sum_{l=1}^{J} \|\mathcal{R}_{kl} z_l\|_{L^1(t,t+1)} \leq C \kappa(t) (\Xi_1 \hat{z})(t), \tag{6.47}$$

where z_l in the left-hand side has been extended by 0 for $t < 0$. From the assumption that $\hat{z} \in \mathcal{B}_\Upsilon$, it follows that

$$(\Xi_1 \hat{z})(t) \le \|\hat{z}\|_{\mathcal{B}_\Upsilon} e^{\Upsilon t} \tag{6.48}$$

and when using this in (6.47), we obtain

$$\sum_{l=1}^{J} \|\mathcal{R}_{kl} z_l\|_{L^1(t,t+1)} \le C \|\hat{z}\|_{\mathcal{B}_\Upsilon} \kappa(t) e^{\Upsilon t}. \tag{6.49}$$

From (5.2), we get

$$\|\gamma_k(\hat{z})\|_{L^1(t,t+1)} \le c_2 \kappa(t) |v_1(\hat{z})|_{\mathcal{C}_t},$$

which together with Corollary 4.2 shows that

$$\begin{aligned} &\|\gamma_k(\hat{z})\|_{L^1(t,t+1)} \\ &\quad \le C\kappa(t) \left(\int_{-1}^{t} e^{-2\lambda_{J+1}(t-s)} \kappa(s) (\Xi_1 \hat{z})(s)\, ds + \kappa(t)(\Xi_1 \hat{z})(t) \right). \end{aligned} \tag{6.50}$$

We use (6.48) in (6.50) and obtain the inequality

$$\|\gamma_k(\hat{z})\|_{L^1(t,t+1)} \le C \varkappa \|\hat{z}\|_{\mathcal{B}_\Upsilon} \kappa(t) e^{\Upsilon t}. \tag{6.51}$$

When considering the term $g_k(v_2)$, we see from Corollary 4.2 that

$$|v_2|_{\mathcal{C}_t} \le C \|\psi\|_{L^2(\Omega)} e^{-2\lambda_{J+1} t},$$

so (5.2) implies that

$$\|g_k(v_2)\|_{L^1(t,t+1)} \le C\kappa(t) \|\psi\|_{L^2(\Omega)} e^{-2\lambda_{J+1} t}. \tag{6.52}$$

When using (6.34) in (6.51) and combining (6.49), (6.51) and (6.52), it follows that $\mathfrak{F}_k(\hat{z})$, as defined in (6.43), is subject to

$$\|\mathfrak{F}_k(\hat{z})\|_{L^1(t,t+1)} \le C \left(\|\hat{z}\|_{\mathcal{B}_\Upsilon} + \|\psi\|_{L^2(\Omega)} \right) \kappa(t) e^{\Upsilon t}.$$

Since $\kappa(t) \le \varkappa$, we can again use (6.34) and conclude that $F_k = \mathfrak{F}_k(\hat{z})$ fulfills (6.13).

From (6.47), (6.50) and (6.52) we find the estimate

$$\begin{aligned} F(t) \le C\kappa(t) \Bigg(&\|\psi\|_{L^2(\Omega)} e^{-2\lambda_{J+1} t} \\ &+ \int_{-1}^{t} e^{-2\lambda_{J+1}(t-s)} \kappa(s) (\Xi_1 \hat{z})(s)\, ds + (\Xi_1 \hat{z})(t) \Bigg), \end{aligned} \tag{6.53}$$

where

$$F(t) = \sum_{k=J+1}^{M} \|\mathfrak{F}_k(\hat{z})\|_{L^1(t,t+1)}.$$

Furthermore, from (6.45) it follows that the quantity

$$a = \sum_{k=J+1}^{M} |z_k(0)|$$

occurring in (6.14) can be majorized by a constant times $\|\psi\|_{L^2(\Omega)}$. Hence we obtain from (6.14) the estimate

$$(\Xi_2 \check{z})(t) \le C\Big(\|\psi\|_{L^2(\Omega)} e^{-\mu t} + \int_{-1}^{t} e^{-\mu(t-s)} F(s)\, ds + F(t) \Big) \tag{6.54}$$

and we will now find an estimate for the term

$$\int_{-1}^{t} e^{-\mu(t-s)} F(s)\, ds$$

by using (6.53). Since

$$\int_{-1}^{t} e^{-(2\lambda_{J+1}-\mu)s}\, ds \le \frac{e^{2\lambda_{J+1}-\mu}}{2\lambda_{J+1}-\mu} < \frac{e^{2\lambda_{J+1}}}{\lambda_{J+1}},$$

we see that

$$\int_{-1}^{t} e^{-\mu(t-s)} e^{-2\lambda_{J+1}s}\, ds \le C e^{-\mu t}.$$

As in (6.32) we also get the inequalities

$$\begin{aligned} \int_{-1}^{t} e^{-\mu(t-s)} \int_{-1}^{s} & e^{-2\lambda_{J+1}(s-\tau)} \kappa(\tau)(\Xi_1 \hat{z})(\tau)\, d\tau\, ds \\ & \le \frac{1}{2\lambda_{J+1}-\mu} \int_{-1}^{t} e^{-\mu(t-\tau)} \kappa(\tau)(\Xi_1 \hat{z})(\tau)\, d\tau \\ & < \frac{1}{\lambda_{J+1}} \int_{-1}^{t} e^{-\mu(t-\tau)} \kappa(\tau)(\Xi_1 \hat{z})(\tau)\, d\tau. \end{aligned}$$

Combining these results with (6.53), it follows that

$$\begin{aligned} \int_{-1}^{t} & e^{-\mu(t-s)} F(s)\, ds \\ & \le C\left(\kappa_0 \|\psi\|_{L^2(\Omega)} e^{-\mu t} + \int_{-1}^{t} e^{-\mu(t-s)} \kappa(s)(\Xi_1 \hat{z})(s)\, ds \right). \end{aligned}$$

From this inequality, it is now easy to see that (6.46) follows from (6.54). □

6.5 Estimate for $\check{h}$

We return to the study of $\check{h}$ and introduce

$$H_1 = \Xi_1 \hat{h} \quad \text{and} \quad H_2 = \Xi_2 \check{h}. \tag{6.55}$$

Corollary 6.3 *The vectors $\check{h}$ and $\hat{h}$ are elements of $\mathcal{A}_{a_0}$ and $\mathcal{B}_{a_0}$, respectively, where a_0 was introduced in Lemma* 2.1*, and the estimate*

$$H_2(t) \le C\Big(\|\psi\|_{L^2(\Omega)} e^{-\mu t} + \int_{-1}^{t} e^{-\mu(t-s)} \kappa(s) H_1(s)\, ds \\ + \kappa(t) H_1(t) \Big), \quad t \ge -1,$$

with the same μ as in Proposition 6.1*, is valid.*

Proof. We are going to use Corollary 6.2 applied on the system (6.5), (6.7) with $\mathfrak{F}_k(\hat{h})$ being the functions F_k, $k = J+1, \dots, M$, as defined in (6.6). Since

$$h_k = \int_\Omega (u, \phi_k)\, dx$$

and $\|\phi_k\|_{L^2(\Omega)} = 1$, it follows from Hölder's inequality and (2.16) that

$$|h_k(t)| \le \|\psi\|_{L^2(\Omega)} e^{a_0 t}. \tag{6.56}$$

This implies that $\check{h} \in \mathcal{A}_{a_0}$ and $\hat{h} \in \mathcal{B}_{a_0}$. Since $\check{h}$ obviously is a solution of (6.5), (6.7), Corollary 6.3 is a consequence of Corollary 6.2. □

6.6 A representation for $\check{h}$

The terms $\mathcal{R}_{kl} h_l$ and $\gamma_k(\hat{h})$ occurring in F_k are linear functions of $\hat{h}$ while $g_k(v_2)$ is independent of $\hat{h}$. This makes us introduce the vectors

$$\check{h}_0 = (h_{0,J+1}, \dots, h_{0,M}) \quad \text{and} \quad \check{h}_1 = (h_{1,J+1}, \dots, h_{0,M})$$

as the unique solutions in $\mathcal{A}_{a_0}$ of the systems of equations

$$h'_{0,k} + \lambda_k h_{0,k} + \sum_{l=J+1}^{M} \mathcal{R}_{kl} h_{0,l} + \Gamma_k(\check{h}_0) = -\sum_{l=1}^{J} \mathcal{R}_{kl} h_l - \gamma_k(\hat{h}),$$

$$h_{0,k}(0) = 0 \tag{6.57}$$

and

$$h'_{1,k} + \lambda_k h_{1,k} + \sum_{l=J+1}^{M} \mathcal{R}_{kl} h_{1,l} + \Gamma_k(\check{h}_1) = -g_k(v_2), \tag{6.58}$$

$$h_{1,k}(0) = \int_\Omega (\psi, \phi_k)\, dx \tag{6.59}$$

for $k = J+1,\dots,M$. We observe that the the solution $\check{h}_0$ depends linearly on $\hat{h}$ while $\check{h}_1$ is independent of $\hat{h}$. This motivates us to write $\check{h}_0(\hat{h})$ instead of $\check{h}_0$. We also realize that, in account of linearity and uniqueness of solutions in $\mathcal{A}_{a_0}$, the following representation for $\check{h}$ holds:

$$\check{h} = \check{h}_0(\hat{h}) + \check{h}_1. \tag{6.60}$$

For a given $\hat{z} \in \mathcal{B}_\Upsilon$, introduce

$$G_0(\hat{z}) = \Xi_2\big(\check{h}_0(\hat{z})\big)$$

and consider the equations

$$h'_{0,k} + \lambda_k h_{0,k} + \sum_{l=J+1}^{M} \mathcal{R}_{kl} h_{0,l} + \Gamma_k(\check{h}_0) = -\sum_{l=1}^{J} \mathcal{R}_{kl} z_l - \gamma_k(\hat{z}) \tag{6.61}$$

for $k = J+1,\dots,M$. By comparing (6.61) with (6.44), we see that Corollary 6.2 can be used with $\psi = 0$ to prove that the system (6.61), (6.57) has a unique solution $\check{h}_0(\hat{z})$ in $\mathcal{A}_\Upsilon$ (and hence $\check{h}_0(\hat{h}) \in \mathcal{A}_{a_0}$). We also obtain the estimate

$$\big(G_0(\hat{z})\big)(t) \le C\left(\int_{-1}^{t} e^{-\mu(t-s)}\kappa(s)(\Xi_1\hat{z})(s)\,ds + \kappa(t)(\Xi_1\hat{z})(t)\right) \tag{6.62}$$

for $t \ge -1$.

Let us now consider the system (6.58), (6.59) and introduce

$$G_1 = \Xi_2 \check{h}_1.$$

Corollary 6.2 can be used with $\hat{z} = 0$ to prove uniqueness of the solution. We also obtain the estimate

$$G_1(t) \le C\|\psi\|_{L^2(\Omega)} e^{-\mu t}. \tag{6.63}$$

7 Functions $h_1, \dots, h_J$

7.1 Equation for $\hat{h}$; existence and uniqueness results

We now consider the system of equations (3.6), (3.7) for $k = 1,\dots,J$. Equation (3.6) obviously becomes

$$h'_k + \sum_{l=1}^{M} \mathcal{R}_{kl} h_l + g_k(v) = 0$$

and using the decompositions (6.4) and (6.60) gives the system

$$h'_k + \sum_{l=1}^{J} \mathcal{R}_{kl} h_l + \mathcal{M}_k(\hat{h}) = \beta_k, \tag{7.1}$$

$$h_k(0) = a_k \tag{7.2}$$

for $k = 1, \dots, J$, where

$$\mathcal{M}_k(\hat{h}) = \sum_{l=J+1}^{M} \mathcal{R}_{kl} h_{0,l}(\hat{h}) + \Gamma_k\big(\check{h}_0(\hat{h})\big) + \gamma_k(\hat{h}), \tag{7.3}$$

$$\beta_k = -\sum_{l=J+1}^{M} \mathcal{R}_{kl} h_{1,l} - \Gamma_k(\check{h}_1) - g_k(v_2) \tag{7.4}$$

and

$$a_k = \int_\Omega (\psi, \phi_k)\, dx. \tag{7.5}$$

We concluded in Section 6 that the functions $\check{h}_0$ and γ_k depend linearly on $\hat{h}$ while Γ_k depends linearly on $\check{h}$ and the functions $\check{h}_1$ and $g_k(v_2)$ are independent of $\hat{h}$. It follows that $\mathcal{M}_k$ depends linearly on $\hat{h}$ while β_k is independent of $\hat{h}$. Let us prove the following estimates for the operators $\mathcal{M}_k$ and functions β_k, $k = 1, \dots, J$.

Lemma 7.1 (i) *For every $\Upsilon > 0$, the operator $\mathcal{M}_k$ is defined on $\mathcal{B}_\Upsilon$ and satisfies*

$$\begin{aligned} \|\mathcal{M}_k(\hat{z})\|_{L^1(t,t+1)} \le C\kappa(t)\bigg(\int_{-1}^{t} & e^{-\mu(t-s)}\kappa(s)(\Xi_1\hat{z})(s)\, ds \\ & + \kappa(t)(\Xi_1\hat{z})(t)\bigg), \quad t \ge -1. \end{aligned} \tag{7.6}$$

The space $\mathcal{B}_\Upsilon$ and the operator Ξ_1 were introduced on page 96.

(ii) *The function β_k, given by* (7.4)*, satisfies*

$$\|\beta_k\|_{L^1(t,t+1)} \le C\|\psi\|_{L^2(\Omega)}\kappa(t)e^{-\mu t}, \quad t \ge -1. \tag{7.7}$$

The proofs of (i) and (ii) are similar, compare the right-hand sides in (7.3) and (7.4).

Proof. (i): Let $\hat{z} \in \mathcal{B}_\Upsilon$. From (5.1) and (6.62) it follows that

$$\begin{aligned} \|\mathcal{R}_{kl} h_{0,l}(\hat{z})\|_{L^1(t,t+1)} &\le c_1\kappa(t)\big(G_0(\hat{z})\big)(t) \\ &\le C\kappa(t)\left(\int_{-1}^{t} e^{-\mu(t-s)}\kappa(s)(\Xi_1\hat{z})(s)\, ds + \kappa(t)(\Xi_1\hat{z})(t)\right). \end{aligned} \tag{7.8}$$

We get an estimate for $\Gamma_k\big(\check{h}_0(\hat{z})\big)$ from (6.26). After using (6.62) again and changing the order of integration, we obtain

$$\begin{aligned}
&\|\Gamma_k\big(\check{h}_0(\hat{z})\big)\|_{L^1(t,t+1)} \\
&\qquad \le C\kappa(t)\Big(\int_{-1}^{t} e^{-2\lambda_{J+1}(t-s)}\kappa(s)\big(G_0(\hat{z})\big)(s)\,ds + \kappa(t)\big(G_0(\hat{z})\big)(t)\Big) \\
&\qquad \le C\kappa(t)\left(\int_{-1}^{t} e^{-\mu(t-s)}\kappa(s)(\Xi_1\hat{z})(s)\,ds + \kappa(t)(\Xi_1\hat{z})(t)\right).
\end{aligned} \tag{7.9}$$

As in (6.50) it finally follows that

$$\|\gamma_k(\hat{z})\|_{L^1(t,t+1)} \le C\kappa(t)\left(\int_{-1}^{t} e^{-2\lambda_{J+1}(t-s)}\kappa(s)(\Xi_1\hat{z})(s)\,ds + \kappa(t)(\Xi_1\hat{z})(t)\right) \tag{7.10}$$

and by combining (7.8), (7.9) and (7.10), inequality (7.6) follows.

(ii): The proof is similar to that of (i) but instead of (6.50) and (6.62), we use (6.52) and (6.63). □

We are going to study problem (7.1), (7.2) with a general right-hand side $\beta = (\beta_1, \dots, \beta_J)$ from $\mathcal{B}_\Upsilon$ for some appropriate Υ and arbitrary complex numbers a_k, $k = 1, \dots, J$. We will need a two-sided estimate for the number μ introduced in (6.15). In the remaining part of the paper we assume that $\varkappa$ fulfills the inequality

$$\varkappa \le \frac{\lambda_{J+1}}{8c_0}. \tag{7.11}$$

This, together with (6.15), gives

$$\frac{7\lambda_{J+1}}{8} \le \mu \le \lambda_{J+1}. \tag{7.12}$$

In the following lemma we hence assume that β_k, $k = 1, \dots, J$, are arbitrary functions, not necessarily the same as in (7.4).

Lemma 7.2 *Assume that there exist positive constants C_1 and Υ such that*

$$\|\beta_k\|_{L^1(t,t+1)} \le C_1 e^{\Upsilon t}, \quad k = 1, \dots, J \tag{7.13}$$

and let $a_1, \dots, a_J$ be arbitrary complex numbers. Provided that $\varkappa$ is small enough, there exists a unique solution of (7.1), (7.2) *in $\mathcal{B}_\Upsilon$.*

Proof. The proof is similar to the corresponding part of Proposition 6.1. It follows that (7.1), (7.2) is equivalent to

$$h_k = -(S\hat{h})_k + \alpha_k, \quad k = 1, \dots, J,$$

where

$$(S\hat{h})_k(t) = \int_0^t \left(\sum_{l=1}^{J} \mathcal{R}_{kl} h_l + \mathcal{M}_k(\hat{h}) \right) ds \tag{7.14}$$

for $t \geq 0$ and

$$\alpha_k(t) = a_k + \int_0^t \beta_k \, ds.$$

We set $(S\hat{h})(t)$ to 0 for $t < 0$. The aim is to prove that S is a bounded, linear operator from $\mathcal{B}_\Upsilon$ to $\mathcal{B}_\Upsilon$ with norm less than 1 if $\varkappa$ is small enough. So suppose that $\hat{z} \in \mathcal{B}_\Upsilon$. We estimate $\|S\hat{z}\|_{L^\infty(t,t+1)}$ for $t \geq -1$ by using (7.14). This yields that

$$\begin{aligned}\operatorname*{ess\,sup}_{t<s<t+1} |(S\hat{z})_k(s)| \leq \int_{-1}^t \left(\sum_{l=1}^{J} \|\mathcal{R}_{kl} z_l\|_{L^1(s,s+1)} + \|\mathcal{M}_k(\hat{z})\|_{L^1(s,s+1)} \right) ds \\ + \sum_{l=1}^{J} \|\mathcal{R}_{kl} z_l\|_{L^1(t,t+1)} + \|\mathcal{M}_k(\hat{z})\|_{L^1(t,t+1)}.\end{aligned}$$

Here we use Lemma 7.1 to estimate the terms containing $\mathcal{M}_k$ and (5.1) to estimate the terms containing $\mathcal{R}_{kl}$. Furthermore, it follows from (7.12) that

$$\begin{aligned}\int_{-1}^t \kappa(s) \int_{-1}^s e^{-\mu(s-\tau)} \kappa(\tau)(\Xi_1 \hat{z})(\tau) \, d\tau \, ds &\leq \frac{\kappa_0}{\mu} \int_{-1}^t \kappa(\tau)(\Xi_1 \hat{z})(\tau) \, d\tau \\ &\leq \frac{8\kappa_0}{7\lambda_{J+1}} \int_{-1}^t \kappa(\tau)(\Xi_1 \hat{z})(\tau) \, d\tau.\end{aligned}$$

It is now easy to derive the inequality

$$\operatorname*{ess\,sup}_{t<s<t+1} \sum_{k=1}^{J} |(S\hat{z})_k(s)| \leq C_0 \left(\int_{-1}^t \kappa(s)(\Xi_1 \hat{z})(s) \, ds + \kappa(t)(\Xi_1 \hat{z})(t) \right),$$

from which it follows that

$$\begin{aligned}e^{-\Upsilon t} \operatorname*{ess\,sup}_{t<s<t+1} \sum_{k=1}^{J} |(S\hat{z})_k(s)| &\leq C_0 \|\hat{z}\|_{\mathcal{B}_\Upsilon} \left(e^{-\Upsilon t} \int_{-1}^t \kappa(s) e^{\Upsilon s} \, ds + \kappa(t) \right) \\ &\leq C_0 \left(\frac{1}{\Upsilon} + 1 \right) \varkappa \|\hat{z}\|_{\mathcal{B}_\Upsilon},\end{aligned}$$

i.e.

$$\|S\hat{z}\|_{\mathcal{B}_\Upsilon} \leq C_0 \left(\frac{1}{\Upsilon} + 1 \right) \varkappa \|\hat{z}\|_{\mathcal{B}_\Upsilon}.$$

The condition $\|S\| < 1$ is thus met if

$$\varkappa < \frac{1}{C_0(1 + 1/\Upsilon)}.$$

In order to complete the proof, it suffices to prove that $\alpha = (\alpha_1, \ldots, \alpha_J) \in \mathcal{B}_\Upsilon$. Since a_k is constant, it is sufficient to show that $f = (f_1, \ldots, f_J)$, where

$$f_k(t) = \int_0^t \beta_k \, ds,$$

belongs to $\mathcal{B}_\Upsilon$. It follows from (7.13) that

$$\operatorname*{ess\,sup}_{t<s<t+1} \sum_{k=1}^{J} |f_k(s)| \le Ce^{\Upsilon t}\left(1 + \frac{1}{\Upsilon}\right),$$

so $f \in \mathcal{B}_\Upsilon$ and thus $\alpha \in \mathcal{B}_\Upsilon$. □

We write problem (7.1), (7.2) in matrix form

$$\hat{h}' + \mathcal{R}\hat{h} + \mathcal{M}(\hat{h}) = \beta, \tag{7.15}$$

$$\hat{h}(0) = a, \tag{7.16}$$

where $\mathcal{R}$ is the $J \times J$-matrix with entry (k, l) equal to $\mathcal{R}_{kl}$ and $\mathcal{M}(\hat{h})$, β and a are vectors of dimension J with k:th component equal to $\mathcal{M}_k(\hat{h})$, β_k and a_k, respectively, as given in (7.3)–(7.5). Taking $\Upsilon = a_0$ in Lemma 7.2, where a_0 was defined in Lemma 2.1, we arrive at the following corollary.

Corollary 7.3 *With β_k and a_k given in* (7.4), (7.5)*, there exists a unique solution $\hat{h}$ of* (7.15), (7.16) *in $\mathcal{B}_{a_0}$.*

7.2 The homogeneous equation

We now study the homogeneous version of (7.15), (7.16), i.e. the system

$$\hat{h}' + \mathcal{R}\hat{h} + \mathcal{M}(\hat{h}) = 0, \tag{7.17}$$

$$\hat{h}(0) = a. \tag{7.18}$$

We use here an approach for studying the above problem suggested in Section 16, 17, Kozlov, Maz'ya [10] and Section 4, Kozlov [7]. See also Kozlov, Langer [8].

Suppose that $a \neq 0$. The question is whether it is possible to have $\hat{h}(t_0) = 0$ for some $t_0 \ge 0$. The answer is no, as follows from the following lemma.

Lemma 7.4 *Assume that $\hat{h}$ is a solution of* (7.17)*. If $\hat{h}(t_0) = 0$ for some $t_0 \ge 0$, then $\hat{h}$ is identically* 0*.*

Proof. By translating the t-variable, it follows immediately from Corollary 7.3 that $\hat{h}(t) = 0$ for $t \ge t_0$. Set

$$f(t) = e^{\frac{\mu t}{2}} H_1(t). \tag{7.19}$$

Obviously, $f(t) = 0$ for $t < -1$, and, by continuity, there exists a $t_1 \in [-1, t_0]$ such that

$$A := f(t_1) = \max_{t \in [-1, t_0]} f(t). \tag{7.20}$$

Since f is increasing on $[-1, 0]$, it follows that $t_1 \geq 0$.

From (7.17) we write

$$\hat{h}(t) = \int_t^{t_0} \Big(\mathcal{R}\hat{h} + \mathcal{M}(\hat{h}) \Big)\, ds$$

and obtain the estimate

$$|\hat{h}(t)| \leq C\kappa_0 \int_{t-1}^{t_0} \left(H_1(s) + \kappa_0 \int_{-1}^{s} e^{-\mu(s-\tau)} H_1(\tau)\, d\tau \right) ds. \tag{7.21}$$

Here we estimate

$$\int_{t-1}^{t_0} H_1(s)\, ds \leq A \int_{t-1}^{t_0} e^{-\frac{\mu s}{2}}\, ds \leq \frac{2A}{\mu} e^{\frac{\mu}{2}} e^{-\frac{\mu t}{2}}$$

and

$$\int_{t-1}^{t_0} \int_{-1}^{s} e^{-\mu(s-\tau)} H_1(\tau)\, d\tau\, ds \leq A \int_{t-1}^{t_0} e^{-\mu s} \int_{-1}^{s} e^{\frac{\mu\tau}{2}}\, d\tau\, ds \leq \frac{4Ae^{\frac{\mu}{2}}}{\mu^2} e^{-\frac{\mu t}{2}}.$$

Using these estimates in (7.21) and the fact that $\mu \geq 7\lambda_{J+1}/8$ from (7.12), we obtain

$$H_1(t) \leq C_1 \kappa_0 A e^{-\frac{\mu t}{2}}$$

and hence

$$A \leq C_1 \varkappa A.$$

By requiring $\varkappa < C_1^{-1}$, it follows that $A = 0$. But then (7.19) and (7.20) imply that $\hat{h}(t) = 0$ for $0 \leq t \leq t_0$ so h is identically 0. □

We assume that $\hat{h}(t)$ is not identically 0. From Lemma 7.4 it then follows that $|\hat{h}(t)| > 0$ for all $t \geq 0$. This means that we can represent $\hat{h}$ as

$$\hat{h}(t) = \rho(t)\Theta(t), \tag{7.22}$$

where

$$\rho(t) = |\hat{h}(t)|, \qquad |\Theta(t)| = 1$$

and ρ and Θ are both absolutely continuous. We insert (7.22) in (7.17) and multiply by Θ. This yields

$$\big((\rho\Theta)', \Theta\big) + \rho\big(\mathcal{R}\Theta, \Theta\big) + \big(\mathcal{M}(\rho\Theta), \Theta\big) = 0. \tag{7.23}$$

Obviously,

$$\big((\rho\Theta)', \Theta\big) = \rho'\big(\Theta, \Theta\big) + \rho\big(\Theta', \Theta\big)$$

and $(\Theta, \Theta) = |\Theta|^2 = 1$. This implies that

$$0 = \frac{d}{dt}(\Theta, \Theta) = (\Theta', \Theta) + (\Theta, \Theta') = 2\Re(\Theta', \Theta),$$

so (Θ', Θ) is purely imaginary. This shows that

$$\Re\big((\rho\Theta)', \Theta\big) = \rho'.$$

The ellipticity and symmetry properties of the matrices $A_{ij}^{(1)}$ and $A^{(1)}$ imply that $\mathcal{R}$ is hermitian, which in particular means that $(\mathcal{R}\Theta, \Theta)$ is real. After taking the real part of (7.23), we obtain

$$\rho' + f\rho + \mathcal{M}_0(\rho) = 0 \tag{7.24}$$

on $(0, \infty)$, where

$$f = (\mathcal{R}\Theta, \Theta) \tag{7.25}$$

and

$$\mathcal{M}_0(\rho) = \Re\big(\mathcal{M}(\rho\Theta), \Theta\big).$$

Here we temporarily consider Θ as a given function and the reason is that we want an equation for ρ. We also have the initial condition

$$\rho(0) = \rho_0, \tag{7.26}$$

where

$$\rho_0 = |a| > 0,$$

with a from (7.18). Since $\mathcal{M}$ is linear, it also follows that $\mathcal{M}_0$ is linear so equation (7.24) is linear.

Suppose that Υ is a given positive number. If $\xi \in L^\infty_{\text{loc}}(0, \infty)$, we set

$$(\Xi_3\xi)(t) = \begin{cases} \|\xi\|_{L^\infty(t,t+1)} & \text{if } t \geq 0 \\ \|\xi\|_{L^\infty(0,t+1)} & \text{if } -1 < t < 0 \\ 0 & \text{if } t \leq -1. \end{cases}$$

and say that $\xi \in \mathcal{D}_\Upsilon$ if

$$\|\xi\|_{\mathcal{D}_\Upsilon} = \sup_{t \geq -1} e^{-\Upsilon t}(\Xi_3\xi)(t) < \infty.$$

We also introduce the operator $\mathcal{Z}$ on functions defined on $[0, \infty)$ as

$$(\mathcal{Z}y)(t) = y(t)\exp\bigg(\int_0^t f(s)\,ds\bigg)$$

and set

$$z = \mathcal{Z}\rho.$$

From (7.24), (7.26) we obtain the problem

$$z' + \mathcal{N}(z) = 0 \quad \text{on } (0,\infty), \tag{7.27}$$

$$z(0) = z_0, \tag{7.28}$$

where

$$\mathcal{N}(z) = \mathcal{Z}\mathcal{M}_0(\mathcal{Z}^{-1}z) \tag{7.29}$$

and $z_0 = \rho_0$. We will need the following estimate for the operator $\mathcal{N}$.

Lemma 7.5 *$\mathcal{N}$ is a linear operator on $\mathcal{D}_\Upsilon$ with the estimate*

$$\|\mathcal{N}(g)\|_{L^1(t,t+1)} \le C\kappa(t)\left(\int_{-1}^{t} e^{-\mu_1(t-s)}\kappa(s)(\Xi_3 g)(s)\,ds + \kappa(t)(\Xi_3 g)(t)\right), \tag{7.30}$$

where

$$\mu_1 = \mu - c_3\kappa_0 \tag{7.31}$$

for some constant c_3.

Proof. The linearity of $\mathcal{N}$ follows from (7.29) since every operator there is linear. Furthermore, it follows that

$$\|\mathcal{N}(g)\|_{L^1(t,t+1)} \le \operatorname*{ess\,sup}_{t<s<t+1}\left\{\exp\left(\int_0^s f(\tau)\,d\tau\right)\right\}\|\mathcal{M}_0(\mathcal{Z}^{-1}g)\|_{L^1(t,t+1)}. \tag{7.32}$$

As before, we carry out this analysis for $t \ge -1$. This means that if $-1 \le t < 0$, we take supremum over $(0,t+1)$ instead of $(t,t+1)$ in (7.32).

From Lemma 5.1 it follows that $f \in L^1_{\text{loc}}(0,\infty)$ with the estimate

$$\|f\|_{L^1(t,t+1)} \le c_3\kappa(t) \tag{7.33}$$

for some constant c_3. Hence, the function

$$s \mapsto \exp\left(\int_0^s f(\tau)\,d\tau\right)$$

is continuous and the essential supremum in (7.32) is attained for some $t_0 \in [t,t+1]$. By writing $h = \mathcal{Z}^{-1}g$, it follows that

$$\|\mathcal{M}_0(\mathcal{Z}^{-1}g)\|_{L^1(t,t+1)} \le \|\mathcal{M}(h\Theta)\|_{L^1(t,t+1)}$$

and since $|\Theta| = 1$, we get an estimate for the last norm from (7.6). Using all this in (7.32), we obtain

$$\|\mathcal{N}(g)\|_{L^1(t,t+1)} \le C\kappa(t)\exp\left(\int_0^{t_0} f(\tau)\,d\tau\right) \times\left(\int_{-1}^{t} e^{-\mu(t-s)}\kappa(s)(\Xi_3 h)(s)\,ds + \kappa(t)(\Xi_3 h)(t)\right). \tag{7.34}$$

We furthermore have that

$$\begin{aligned}(\Xi_3 h)(s) &= \operatorname*{ess\,sup}_{s<u<s+1} \left\{ \exp\left(- \int_0^u f(\tau)\,d\tau \right) |g(u)| \right\} \\ &\le (\Xi_3 g)(s) \sup_{s<u<s+1} \left\{ \exp\left(- \int_0^u f(\tau)\,d\tau \right) \right\} \qquad (7.35) \\ &= (\Xi_3 g)(s) \exp\left(- \int_0^{u_0(s)} f(\tau)\,d\tau \right),\end{aligned}$$

for some $u_0(s) \in [s, s+1]$.

We consider the term

$$\exp\left(\int_0^{t_0} f(\tau)\,d\tau \right) \int_{-1}^{t} e^{-\mu(t-s)} \kappa(s)(\Xi_3 h)(s)\,ds \qquad (7.36)$$

appearing in (7.34). According to (7.35), the factor

$$\exp\left(\int_0^{t_0} f(\tau)\,d\tau \right) (\Xi_3 h)(s)$$

can be majorized by

$$(\Xi_3 g)(s) \exp\left(\int_{u_0(s)}^{t_0} f(\tau)\,d\tau \right).$$

In turn,

$$\int_{u_0(s)}^{t_0} f(\tau)\,d\tau \;\le \int_a^b \|f\|_{L^1(\tau,\tau+1)}\,d\tau, \qquad (7.37)$$

where

$$a = \min\{t_0, u_0(s)\} - 1, \qquad b = \max\{t_0, u_0(s)\}.$$

Since $t_0 \in [t, t+1]$ as well as $u_0(s) \in [s, s+1]$ and $s \le t$, we see that $b - a \le t - s + 2$. Hence, it follows from (7.33) that

$$\int_a^b \|f\|_{L^1(\tau,\tau+1)}\,d\tau \le (t - s + 2) c_3 \kappa_0. \qquad (7.38)$$

Summing this up, we get the estimate

$$\exp\left(\int_0^{t_0} f(\tau)\,d\tau \right) (\Xi_3 h)(s) \le e^{2c_3\kappa_0} (\Xi_3 g)(s) e^{c_3\kappa_0(t-s)}. \qquad (7.39)$$

For the term in (7.36), it then yields the inequality

$$\begin{aligned}&\exp\left(\int_0^{t_0} f(\tau)\,d\tau \right) \int_{-1}^{t} e^{-\mu(t-s)} \kappa(s)(\Xi_3 h)(s)\,ds \\ &\qquad\qquad \le C \int_{-1}^{t} e^{-\mu_1(t-s)} \kappa(s)(\Xi_3 g)(s)\,ds, \qquad (7.40)\end{aligned}$$

with $\mu_1 = \mu - c_3\kappa_0$. By setting $s = t$ in (7.39), we get the inequality

$$\exp\biggl(\int_0^{t_0} f(\tau)\,d\tau\biggr)(\Xi_3 h)(t) \le C(\Xi_3 g)(t). \tag{7.41}$$

By combining (7.40) and (7.41), estimate (7.30) follows from (7.34). □

As was the case for μ, we will also need a two-sided estimate for μ_1. Assuming that $\varkappa \le \lambda_{J+1}/(8c_3)$ and using (7.31) and (7.12), we obtain

$$\frac{3\lambda_{J+1}}{4} \le \mu_1 \le \lambda_{J+1}. \tag{7.42}$$

We now prove uniqueness of solutions of at most exponential increase of problem (7.27), (7.28).

Lemma 7.6 *Let Υ be an arbitrary, positive number. If $z_0 = 0$ and $\varkappa$ is small enough, then $z = 0$ is the only solution in $\mathcal{D}_\Upsilon$ of* (7.27), (7.28).

Proof. We assume that z is a solution of problem (7.27), (7.28) with $z_0 = 0$ and prove that $z = 0$. The problem is equivalent to the integral equation

$$z(t) = -\int_0^t \mathcal{N}(z)\,ds. \tag{7.43}$$

Equation (7.43) and estimate (7.30) imply that

$$\begin{aligned}(\Xi_3 z)(t) &\le \int_{-1}^t \|\mathcal{N}(z)\|_{L^1(s,s+1)}\,ds + \|\mathcal{N}(z)\|_{L^1(t,t+1)}\\ &\le C_1\kappa_0\biggl(\int_{-1}^t\int_{-1}^s e^{-\mu_1(s-\tau)}\kappa(\tau)(\Xi_3 z)(\tau)\,d\tau\,ds\\ &\quad + \int_{-1}^t \kappa(s)(\Xi_3 z)(s)\,ds + \kappa(t)(\Xi_3 z)(t)\biggr).\end{aligned}$$

After changing the order of integration and using (7.42), we see that the inequalities

$$\begin{aligned}\int_{-1}^t\int_{-1}^s e^{-\mu_1(s-\tau)}\kappa(\tau)(\Xi_3 z)(\tau)\,d\tau\,ds &\le \frac{1}{\mu_1}\int_{-1}^t \kappa(\tau)(\Xi_3 z)(\tau)\,d\tau\\ &\le \frac{4}{3\lambda_{J+1}}\int_{-1}^t \kappa(\tau)(\Xi_3 z)(\tau)\,d\tau\end{aligned}$$

hold, so

$$(\Xi_3 z)(t) \le C_1\kappa_0\left(\left(1 + \frac{4}{3\lambda_{J+1}}\right)\int_{-1}^t \kappa(s)(\Xi_3 z)(s)\,ds + \kappa(t)(\Xi_3 z)(t)\right). \tag{7.44}$$

We now assume that $\varkappa$ is so small that the inequality $1 - C_1\varkappa^2 \geq 1/2$ is fulfilled. It then follows from (7.44) that

$$\frac{(\Xi_3 z)(t)}{2} \leq C\kappa_0 \int_{-1}^{t} \kappa(s)(\Xi_3 z)(s)\,ds$$

and Grönwall's inequality gives that $(\Xi_3 z)(t) = 0$ for every $t \geq -1$, i.e. $z = 0$. □

Lemma 7.6 deals with the uniqueness of solutions of (7.27), (7.28). The existence of solutions follows from the following lemma.

Lemma 7.7 *For $z_0 \neq 0$ and $\varkappa$ small enough, there exists a solution of (7.27), (7.28) of the form*

$$z(t) = z_0 \exp\left(\int_0^t \Lambda(s)\,ds\right). \tag{7.45}$$

Here, the real-valued function Λ satisfies the inequality

$$\|\Lambda\|_{L^1(t,t+1)} \leq C\kappa(t)\left(\int_{-1}^{t} e^{-\mu_2(t-s)}\kappa(s)\,ds + \kappa(t)\right), \quad t \geq 1, \tag{7.46}$$

where

$$\mu_2 = \mu - c_4\kappa_0 \tag{7.47}$$

for some positive constant c_4.

Proof. It follows from (7.45) and the fact that z is real-valued that the same is true for Λ. Because of the linearity, we can without loss of generality assume that $z_0 = 1$. Inserting

$$z(t) = \exp\left(\int_0^t \Lambda(s)\,ds\right)$$

in (7.27), we obtain the equation

$$\mathcal{T}\Lambda = \Lambda, \tag{7.48}$$

where

$$(\mathcal{T}\Lambda)(t) = -\mathcal{N}\left[s \mapsto \exp\left(-\int_s^t \Lambda(\tau)\,d\tau\right)\right](t). \tag{7.49}$$

The existence of a solution of the given form is proved if we can find a fixed point of the operator $\mathcal{T}$. We introduce

$$\|\Lambda\|_{\mathcal{B}} = \operatorname*{ess\,sup}_{t \geq -1} \|\Lambda\|_{L^1(t,t+1)}$$

and define the Banach space

$$\mathcal{B} = \left\{\Lambda \in L^1_{\text{loc}}(\mathbf{R}) : \Lambda(t) = 0 \text{ for } t < 0 \text{ and } \|\Lambda\|_{\mathcal{B}} < \infty\right\}.$$

The quantity $\|\cdot\|_{\mathcal{B}}$ is a norm in this space. We will prove, provided $\varkappa$ is small, that $\mathcal{T}$ maps the set

$$B = \{\Lambda \in \mathcal{B} : \|\Lambda\|_{\mathcal{B}} \leq \kappa_0\},$$

where κ_0 is the same number as in (1.15), into itself and is a contraction on B.

We estimate the expression given in (7.49) by using (7.30). We first have to estimate

$$G(s) = \operatorname*{ess\,sup}_{s<\tau<s+1} \left\{ \exp\left(-\int_\tau^t \Lambda(u)\,du \right) \right\}$$

for $s \leq t$. Note that in the function which $\mathcal{N}$ is acting on in (7.49), the symbol t is considered a fixed quantity. Similarly to (7.37), we obtain

$$G(s) \leq \exp\left(\int_a^b \|\Lambda\|_{L^1(u,u+1)}\,du \right),$$

for some a, b such that $0 \leq b - a \leq t - s + 2$. Since $\Lambda \in B$, we can make the estimate $G(s) \leq e^{2\kappa_0} e^{\kappa_0 (t-s)}$. When using this in (7.30), we see from (7.49) that

$$\|\mathcal{T}\Lambda\|_{L^1(t,t+1)} \leq C\kappa(t) \left(\int_{-1}^t e^{-\mu_2(t-s)}\kappa(s)\,ds + \kappa(t) \right), \tag{7.50}$$

where $\mu_2 = \mu_1 - \kappa_0 = \mu - (c_3+1)\kappa_0$ with c_3 as in (7.31). Here we also see that the constant c_4 in (7.47) is equal to $c_3 + 1$ and by requiring that $\varkappa \leq \lambda_{J+1}/4$, it follows that $\mu_1 - \mu_2 \leq \lambda_{J+1}/4$. Together with (7.42), it gives the bounds

$$\frac{\lambda_{J+1}}{2} \leq \mu_2 \leq \lambda_{J+1}. \tag{7.51}$$

By estimating κ in (7.50) by κ_0 and computing the resulting integral, we obtain

$$\|\mathcal{T}\Lambda\|_{L^1(t,t+1)} \leq C\kappa_0^2 \left(\frac{1}{\mu_2} + 1 \right) \leq C \left(\frac{2}{\lambda_{J+1}} + 1 \right) \varkappa\kappa_0 = C'\varkappa\kappa_0.$$

We see that

$$\|\mathcal{T}\Lambda\|_{L^1(t,t+1)} \leq \kappa_0$$

if $\varkappa \leq 1/C'$, so $\mathcal{T}$ maps B into itself, as desired.

We go on and prove that $\mathcal{T}$ is a contradiction on B, i.e. that there exists a constant $C < 1$ such that

$$\|\mathcal{T}\Lambda_1 - \mathcal{T}\Lambda_2\|_{\mathcal{B}} \leq C\|\Lambda_1 - \Lambda_2\|_{\mathcal{B}}$$

if Λ_1, $\Lambda_2 \in B$. Because of the linearity of $\mathcal{N}$, it follows from (7.49) that

$$(\mathcal{T}\Lambda_1 - \mathcal{T}\Lambda_2)(t) = \mathcal{N}(g_2 - g_1)(t), \tag{7.52}$$

where

$$g_k(s) = \exp\biggl(-\int_s^t \Lambda_k(u)\,du\biggr), \quad k = 1, 2.$$

We also set

$$\zeta_k(s) = -\int_s^t \Lambda_k(u)\,du, \quad k = 1, 2$$

and introduce

$$G(s) = \operatorname*{ess\,sup}_{s<\tau<s+1} |g_2(\tau) - g_1(\tau)|.$$

Since $|e^a - e^b| \le |a - b| e^{\max\{a,b\}}$, we obtain

$$G(s) \le \operatorname*{ess\,sup}_{s<\tau<s+1} \left(|\zeta_2(\tau) - \zeta_1(\tau)|\, e^{\max\{\zeta_1(\tau), \zeta_2(\tau)\}} \right). \tag{7.53}$$

As in (7.37) and (7.38) we get, with $\tau \in [s, s+1]$ and $s \le t$, the estimates

$$\int_\tau^t \bigl|\Lambda_1(u) - \Lambda_2(u)\bigr|\, du \le \|\Lambda_1 - \Lambda_2\|_{\mathcal{B}} (t - s + 2)$$

and

$$-\int_\tau^t \Lambda_k(u)\,du \le \|\Lambda_k\|_{\mathcal{B}} (t - s + 2).$$

Since Λ_1, $\Lambda_2 \in B$, it follows from (7.53) that

$$G(s) \le \|\Lambda_1 - \Lambda_2\|_{\mathcal{B}} (t - s + 2)\, e^{\kappa_0 (t-s+2)}.$$

We use this in (7.30) with $g = g_2 - g_1$ and obtain from (7.52) the inequality

$$\|\mathcal{T}\Lambda_1 - \mathcal{T}\Lambda_2\|_{L^1(t,t+1)} \le C_1 \kappa_0^2 \|\Lambda_1 - \Lambda_2\|_{\mathcal{B}} \left(\int_{-1}^t (t - s + 2) e^{-\mu_2 (t-s)}\, ds + 1 \right). \tag{7.54}$$

Since

$$\int_{-\infty}^t (t - s + 2) e^{-\mu_2(t-s)}\, ds = \frac{1}{\mu_2^2} + \frac{2}{\mu_2},$$

it follows from (7.54) that

$$\begin{aligned} \|\mathcal{T}\Lambda_1 - \mathcal{T}\Lambda_2\|_{L^1(t,t+1)} &\le C_1 \kappa_0^2 \left(\frac{1}{\mu_2^2} + \frac{2}{\mu_2} + 1 \right) \|\Lambda_1 - \Lambda_2\|_{\mathcal{B}} \\ &= C_1 \kappa_0^2 \left(\frac{1}{\mu_2} + 1 \right)^2 \|\Lambda_1 - \Lambda_2\|_{\mathcal{B}} \\ &\le C_2 \|\Lambda_1 - \Lambda_2\|_{\mathcal{B}}, \end{aligned}$$

where (7.51) is used to calculate

$$C_2 = C_1 \varkappa^2 \left(\frac{2}{\lambda_{J+1}} + 1 \right)^2.$$

We assume that $\varkappa$ so chosen so small that this value is less than 1. Hence, $\mathcal{T}$ is a contraction on B and, since B is closed in $\mathcal{B}$, it follows from the Banach fixed point theorem that there exists a unique function Λ in B satisfying relation (7.48). As a consequence, we see that estimate (7.50) is valid also for Λ, so (7.46) follows.

□

We now summarize what has been done and get the following result.

Lemma 7.8 *Provided that $\varkappa$ is small enough, there exists a unique solution $\hat{h}$ of* (7.17), (7.18) *in $\mathcal{B}_{a_0}$. This solution can be represented in the form*

$$\hat{h}(t) = |a| \exp\left(\int_0^t \big(-f(s) + \Lambda(s)\big)\, ds\right)\Theta(t), \tag{7.55}$$

where

$$f = (\mathcal{R}\Theta, \Theta), \tag{7.56}$$

$$|\Theta(t)| = 1, \quad t \geq 0, \tag{7.57}$$

$$\|\Lambda\|_{L^1(t,t+1)} \leq C\kappa(t)\left(\int_{-1}^t e^{-\mu_2(t-s)}\kappa(s)\, ds + \kappa(t)\right), \quad t \geq -1, \tag{7.58}$$

and

$$\|\Theta'\|_{L^1(t,t+1)} \leq C\kappa(t), \quad t \geq -1. \tag{7.59}$$

The functions Λ and Θ' are here extended by 0 *for $t < 0$.*

Proof. According to Lemma 7.2, there exists a unique solution of (7.17), (7.18) in $\mathcal{B}_{a_0}$. If $a = 0$, the solution is obviously 0 and we are done. Suppose that $a \neq 0$. Then it follows from Lemma 7.4 that $\hat{h}$ is non-vanishing. Hence we can write $\hat{h} = \rho\Theta$ as in (7.22) and obtain the problem (7.24), (7.26). Here, the function f is introduced in (7.25), which is identical to (7.56).

We make one more transformation, namely $z = \mathcal{Z}\rho$, and see that z solves the problem (7.27), (7.28) with $z_0 = |a|$. It is easy to realize that this is equivalent to (7.24), (7.26).

It follows that $z \in \mathcal{D}_\Upsilon$ for some Υ with $\mathcal{D}_\Upsilon$ defined on page 106. Namely, since

$$z(t) = \exp\left(\int_0^t f(s)\, ds\right)|\hat{h}(t)|$$

and estimate (7.33) is valid, we see that

$$(\Xi_3 z)(t) \leq C_1 e^{C_2\kappa_0 t} H_1(t),$$

where H_1 was defined in (6.55). By using (6.56), this implies that

$$(\Xi_3 z)(t) \leq C_1\|\psi\|_{L^2(\Omega)} e^{(C_2\kappa_0 + a_0)t},$$

so $z \in \mathcal{D}_\Upsilon$ for $\Upsilon = C_2\kappa_0 + a_0$. According to Lemma 7.6, this is the only solution of problem (7.27), (7.28) in $\mathcal{D}_\Upsilon$. By Lemma 7.7, it can be written in the form (7.45) with Λ satisfying (7.46). The conclusion is that $\hat{h} = (\mathcal{Z}^{-1}z)\Theta$ and (7.55) follows.

It remains to prove (7.59). We write $\hat{h}(t) = |a| \exp\big(\int_0^t g(s)\,ds\big)\Theta(t)$, where $g = -f + \Lambda$. Differentiating this formula with respect to t, we obtain

$$|a|\Theta'(t) = \exp\Big(-\int_0^t g(s)\,ds\Big)\hat{h}'(t) - |a|g(t)\Theta(t).$$

From this, the norm inequality

$$\begin{aligned} |a|\|\Theta'\|_{L^1(t,t+1)} \leq & \|\hat{h}'\|_{L^1(t,t+1)} \sup_{t\leq s\leq t+1} \Big\{\exp\Big(-\int_0^s g(\tau)\,d\tau\Big)\Big\} \\ & + |a|\|g\|_{L^1(t,t+1)} \end{aligned} \tag{7.60}$$

follows.

We proceed with estimating $\|g\|_{L^1(t,t+1)}$. Using estimate (7.33) for f and the estimate $\|\Lambda\|_{L^1(t,t+1)} \leq C\kappa_0\kappa(t)$ for Λ, which follows from (7.46), we obtain

$$\|g\|_{L^1(t,t+1)} \leq \|\Lambda\|_{L^1(t,t+1)} + \|f\|_{L^1(t,t+1)} \leq c_6\kappa(t), \tag{7.61}$$

where c_6 is a constant.

We next estimate $\|\hat{h}'\|_{L^1(t,t+1)}$. From (7.17) we obtain the equality

$$\hat{h}' = -\big(\mathcal{R}\hat{h} + \mathcal{M}(\hat{h})\big).$$

Using (5.1) and (7.6), it follows that

$$\|\hat{h}'\|_{L^1(t,t+1)} \leq C\kappa(t)\left(H_1(t) + \int_{-1}^t e^{-\mu(t-s)}\kappa(s)H_1(s)\,ds\right). \tag{7.62}$$

We search for a convenient expression for $H_1(t)$. Because of continuity, the following is valid for some $t_0 \in [t, t+1]$:

$$\begin{aligned} \sup_{t\leq s\leq t+1} |h_k(s)| \leq & \sup_{t\leq s\leq t+1} |a| \exp\Big(\int_0^s g(\tau)\,d\tau\Big) \\ & = |a| \exp\Big(\int_0^{t_0} g(\tau)\,d\tau\Big) \\ & = |a| \exp\Big(\int_0^t g(\tau)\,d\tau\Big) \exp\Big(\int_t^{t_0} g(\tau)\,d\tau\Big). \end{aligned} \tag{7.63}$$

From (7.61) it follows that

$$\exp\Big(\int_t^{t_0} g(\tau)\,d\tau\Big) \leq \exp\Big(\int_{t-1}^{t+1} \|g\|_{L^1(\tau,\tau+1)}\,d\tau\Big) \leq e^{2c_6\kappa_0},$$

i.e. a constant, so from (7.63) we get

$$H_1(t) \le C|a| \exp\left(\int_0^t g(s)\,ds\right).$$

We use this in (7.62) to obtain

$$\|\hat{h}'\|_{L^1(t,t+1)} \le C|a|\kappa(t)\Bigg(\exp\left(\int_0^t g(\tau)\,d\tau\right) + \int_{-1}^t e^{-\mu(t-s)}\kappa(s)\exp\left(\int_0^s g(\tau)\,d\tau\right)ds\Bigg).$$

Since

$$\exp\left(\int_0^s g(\tau)\,d\tau\right) \le Ce^{c_6\kappa_0(t-s)}\exp\left(\int_0^t g(\tau)\,d\tau\right),$$

where c_6 is the same constant as in (7.61), we get, with $\mu_4 = \mu - c_6\kappa_0$, the estimate

$$\|\hat{h}'\|_{L^1(t,t+1)} \le C|a|\kappa(t)\exp\left(\int_0^t g(\tau)\,d\tau\right)\left(1 + \int_{-1}^t e^{-\mu_4(t-s)}\kappa(s)\,ds\right). \tag{7.64}$$

When assuming that $\varkappa \le \lambda_{J+1}/(4c_6)$ and using (7.12), we conclude that $\mu_4 \ge 5\lambda_{J+1}/8$. Therefore, the last integral in (7.64) is estimated by a constant depending on n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2. Hence

$$\|\hat{h}'\|_{L^1(t,t+1)} \le C|a|\kappa(t)\exp\left(\int_0^t g(\tau)\,d\tau\right). \tag{7.65}$$

A use of (7.61) and (7.65) in (7.60) yields

$$|a|\|\Theta'\|_{L^1(t,t+1)} \le C|a|\kappa(t),$$

which implies (7.59). □

7.3 A particular solution of (7.15)

We start the subsection with the following lemma.

Lemma 7.9 *Let*

$$\mu_3 = \lambda_{J+1} - 2c_0\kappa_0, \tag{7.66}$$

where c_0 *is the same constant as in* (6.15)*. Suppose that the right-hand side* β *in* (7.15) *is a measurable vector function satisfying*

$$\|\beta\|_{L^1(t,t+1)} \le b\kappa_0 e^{-\mu_3 t}, \quad t \ge 0 \tag{7.67}$$

for some constant b*. Then equation* (7.15) *has a solution* $\hat{h}$ *subject to the estimate*

$$|\hat{h}(t)| \le Cb\kappa_0 e^{-\mu_3 t}, \quad t > 0, \tag{7.68}$$

provided that $\varkappa$ *is small enough.*

Proof. We introduce the Banach space $\mathcal{E}$ consisting of functions

$$z = (z_1, \dots, z_J) \in L^\infty_{\text{loc}}(0, \infty)$$

such that

$$\|z\|_{\mathcal{E}} = \sup_{t \ge -1} e^{\mu_3 t}(\Xi_1 z)(t) < \infty,$$

where Ξ_1, was introduced in (6.42). If $\hat{h}$ is a solution from $\mathcal{E}$ of (7.15), then obviously

$$\hat{h} = \mathcal{K}\hat{h}, \tag{7.69}$$

where

$$(\mathcal{K}\hat{h})(t) = \int_t^\infty \big(\mathcal{R}\hat{h} + \mathcal{M}(\hat{h}) - \beta\big)\, ds,$$

since the integrand belongs to $L^1(0,\infty)$. We prove the lemma by showing that $\mathcal{K}$ has a fixed point in $\mathcal{E}$. Let us first prove that $\mathcal{K}$ maps the set

$$E = \{z \in \mathcal{E} : \|z\|_{\mathcal{E}} \le b\}$$

into itself so assume that $\hat{h} \in E$. By using estimates (5.1), (7.6) and (7.67), we obtain, after having extended β by 0 for $t < 0$, the estimate

$$|(\mathcal{K}\hat{h})(t)| \le C\kappa_0 \int_{t-1}^\infty \left(H_1(s) + \kappa_0 \int_{-1}^s e^{-\mu(s-\tau)} H_1(\tau)\, d\tau + b e^{-\mu_3 s}\right) ds. \tag{7.70}$$

We estimate the terms in the right-hand side. Since $\hat{h} \in E$, it follows that

$$H_1(t) \le b e^{-\mu_3 t}. \tag{7.71}$$

By using the inequality

$$\frac{3\lambda_{J+1}}{4} \le \mu_3 \le \lambda_{J+1}, \tag{7.72}$$

obtained when combining (7.11) and (7.66), we get

$$\int_{t-1}^{\infty} be^{-\mu_3 s}\,ds \le \frac{be^{\mu_3}}{\mu_3}e^{-\mu_3 t} \le \frac{4be^{\lambda_{J+1}}}{3\lambda_{J+1}}e^{-\mu_3 t}, \tag{7.73}$$

Furthermore,

$$\int_{t-1}^{\infty} \kappa_0 \int_{-1}^{s} e^{-\mu(s-\tau)}H_1(\tau)\,d\tau\,ds \le \frac{be^{\mu_3}\kappa_0}{\mu_3(\mu-\mu_3)}e^{-\mu_3 t} \le \frac{4be^{\lambda_{J+1}}}{3\lambda_{J+1}c_0}e^{-\mu_3 t}, \tag{7.74}$$

where we have used (7.71) in the first inequality and (6.15), (7.66) and (7.72) in the second. Combining (7.70), (7.71), (7.73) and (7.74), we finally obtain

$$|(\mathcal{K}\hat{h})(t)| \le Cb\kappa_0 e^{-\mu_3 t}. \tag{7.75}$$

From this it follows that $\|\mathcal{K}\hat{h}\|_{\mathcal{E}} \le b$ if $\varkappa$ is small, so $\mathcal{K}$ maps E into itself, as desired.

In the same way it follows that

$$\|\mathcal{K}\hat{h}_1 - \mathcal{K}\hat{h}_2\|_{\mathcal{E}} \le \alpha\|\hat{h}_1 - \hat{h}_2\|_{\mathcal{E}}$$

with $\alpha < 1$ if $\hat{h}_1$, $\hat{h}_2 \in E$ and $\varkappa$ being small enough. Hence $\mathcal{K}$ is a contraction on the closed set E and has a unique fixed point there. This function is a solution of (7.15) and the estimate (7.68) is obtained from (7.69) and (7.75). □

Corollary 7.10 *With $\beta = (\beta_1, \dots, \beta_J)$, where β_k is given by* (7.4)*, there exists a solution $\hat{h}_p$ of* (7.15) *such that*

$$|\hat{h}_p(t)| \le C\|\psi\|_{L^2(\Omega)}e^{-\mu_3 t}. \tag{7.76}$$

Proof. From (7.7) it follows that (7.67) is fulfilled with $b = C\|\psi\|_{L^2(\Omega)}$ so (7.76) follows from (7.68). □

We combine what has been proved in Section 7 to obtain the following splitting of the solution of problem (7.15), (7.16).

Lemma 7.11 *The unique solution $\hat{h}$ in $\mathcal{B}_{a_0}$ of* (7.15), (7.16) *from Corollary* 7.3*, can be written as*

$$\hat{h} = \hat{h}_h + \hat{h}_p, \tag{7.77}$$

where $\hat{h}_p$ is the same as in Corollary 7.10*. The vector function $\hat{h}_h$ is a solution of* (7.15) *with $\beta = 0$ and can be represented as*

$$\hat{h}_h(t) = |\hat{h}_h(0)|\exp\Big(\int_0^t \big(-f(s) + \Lambda(s)\big)\,ds\Big)\Theta(t), \tag{7.78}$$

where the function f is given by (7.56), Θ *is satisfying* (7.57), *the inequality*

$$|\hat{h}_h(0)| \leq C\|\psi\|_{L^2(\Omega)} \tag{7.79}$$

is valid and estimates for Λ *and* Θ' *are found in* (7.58) *and* (7.59).

Proof. The function $\hat{h}_h$ satisfies the problem

$$\hat{h}_h' + \mathcal{R}\hat{h}_h + \mathcal{M}(\hat{h}_h) = 0,$$

$$\hat{h}_h(0) = a - \hat{h}_p(0),$$

where a is the same vector as in (7.16). The representation (7.78) follows from Lemma 7.8. Inequality (7.76) implies that $|\hat{h}_p(0)| \leq C\|\psi\|_{L^2(\Omega)}$ and from (7.5) it follows that $|a_k| \leq \|\psi\|_{L^2(\Omega)}$. This proves (7.79). □

8 Proof of Theorem 1.1

As mentioned in Section 2.2, it is sufficient to prove the theorem under the assumption that $\lambda_1 = 0$. Using the representation (7.77), we can rewrite (3.1) in the form

$$u(x,t) = \sum_{k=1}^{J} h_{h,k}(t)\phi_k(x) + \sum_{k=1}^{J} h_{p,k}(t)\phi_k(x) + \sum_{k=J+1}^{M} h_k(t)\phi_k(x) + v(x,t), \tag{8.1}$$

where $h_{h,k}$ and $h_{p,k}$ denote the k:th component of $\hat{h}_h$ and $\hat{h}_p$, respectively. We set

$$V(x,t) = \exp\bigg(\int_0^t \big(f(s) - \Lambda(s)\big)\,ds\bigg) \times \bigg(\sum_{k=1}^{J} h_{p,k}(t)\phi_k(x) + \sum_{k=J+1}^{M} h_k(t)\phi_k(x) + v(x,t)\bigg).$$

By using the representation (7.78), it follows that (8.1) implies (1.17) with $w_0 = |\hat{h}_h(0)|$. The estimates (1.20), (1.21) and (1.22) follow from (7.79), (7.58) and (7.59), respectively, with $b_0 \leq \mu_2$ in (1.21). We will return to the question how b_0 should be chosen.

It remains to prove (1.23). We start with estimating $H_2(t)$, where H_2 is given by (6.55). Set $g = -f + \Lambda$ and $H_{1,h} = \Xi_1\hat{h}_h$, where Ξ_1 is defined in (6.42). From (7.78) and (7.79), it follows that

$$H_{1,h}(t) \leq C\|\psi\|_{L^2(\Omega)} \exp\bigg(\int_0^t g(s)\,ds\bigg). \tag{8.2}$$

An estimate for $H_{1,p}(t)$, defined analogously to $H_{1,h}(t)$, follows from (7.76) as

$$H_{1,p}(t) \leq C\|\psi\|_{L^2(\Omega)}e^{-\mu_3 t},$$

where μ_3 is given by (7.66). Since $\|g\|_{L^1(t,t+1)} \leq c_6\kappa_0$, where c_6 is the same as in (7.61), we can rewrite this estimate as

$$H_{1,p}(t) \leq C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right)e^{-\mu_5 t}, \tag{8.3}$$

where $\mu_5 = \mu_3 - c_6\kappa_0$. When combining (8.2) and (8.3), we get

$$H_1(t) \leq C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right), \tag{8.4}$$

where H_1 is given by (6.55). Using (8.4) and Corollary 6.3, we arrive at the inequality

$$H_2(t) \leq C\|\psi\|_{L^2(\Omega)}\Bigg(e^{-\mu t} + \int_{-1}^t e^{-\mu(t-s)}\kappa(s)\exp\left(\int_0^s g(\tau)\,d\tau\right)ds + \kappa(t)\exp\left(\int_0^t g(s)\,ds\right)\Bigg).$$

This implies that

$$H_2(t) \leq C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right) \times\left(e^{-\mu_4 t} + \int_{-1}^t e^{-\mu_4(t-s)}\kappa(s)\,ds + \kappa(t)\right), \tag{8.5}$$

where $\mu_4 = \mu - c_6\kappa_0$.

We now turn to estimating the function H given by (4.41). Since $H(t) \leq H_1(t) + H_2(t)$, it follows from (8.4) and (8.5) that

$$H(t) \leq C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right). \tag{8.6}$$

In order to obtain (1.23), we also need an estimate for the function v. The inequalities (4.42) and (8.6) imply that

$$|v|_{\mathcal{C}_t} \leq C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right) \times\left(e^{-\lambda_{J+1}t} + \int_{-1}^t e^{-\lambda_{J+1}(t-s)}\kappa(s)\,ds + \kappa(t)\right). \tag{8.7}$$

We are now in position to complete the derivation of estimate (1.23) but let us first find an estimate for $|W|_{\mathcal{C}_t}$, where

$$W(x,t) = \sum_{k=1}^J h_{p,k}(t)\phi_k(x) + \sum_{k=J+1}^M h_k(t)\phi_k(x) + v(x,t).$$

From (7.76) it follows that

$$|h_{p,k}\phi_k|_{\mathcal{C}_t} \le C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right) e^{-\mu_5 t} \tag{8.8}$$

and (8.5) implies that

$$|h_k\phi_k|_{\mathcal{C}_t} \le C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right) \times \left(e^{-\mu_4 t} + \int_{-1}^t e^{-\mu_4(t-s)}\kappa(s)\,ds + \kappa(t)\right) \tag{8.9}$$

for $k = J+1, \ldots, M$. By combining (8.7), (8.8) and (8.9), we obtain

$$|W|_{\mathcal{C}_t} \le C\|\psi\|_{L^2(\Omega)} \exp\left(\int_0^t g(s)\,ds\right) \times \left(e^{-\mu_5 t} + \int_{-1}^t e^{-\mu_4(t-s)}\kappa(s)\,ds + \kappa(t)\right)$$

and, since

$$V(x,t) = \exp\left(-\int_0^t g(s)\,ds\right) W(x,t),$$

it follows that

$$|V|_{\mathcal{C}_t} \le C\|\psi\|_{L^2(\Omega)} \left(e^{-\mu_5 t} + \int_{-1}^t e^{-\mu_4(t-s)}\kappa(s)\,ds + \kappa(t)\right).$$

The estimate (1.23) follows if we choose $b_0 = \min\{\mu_2, \mu_4, \mu_5\}$. Using the definitions of μ_2 and μ given by (7.47) and (6.15), respectively, and the definitions of μ_4 and μ_5, we see that $b_0 = \lambda_{J+1} - C_1\kappa_0$ for some constant C_1 depending only on n, N, Ω, $A_{ij}^{(0)}$, $A^{(0)}$, p, s_2', ν, ν_1 and ν_2. This completes the proof of Theorem 1.1.

9 Corollaries of Theorem 1.1

In the next corollary we see that if $\kappa \in L^1(0,\infty)$, then the leading term in the asymptotics is the same as in the case when the perturbation is zero.

Corollary 9.1 *If $\kappa \in L^1(0,\infty)$, then*

$$u(x,t) = e^{-\lambda_1 t}\left(\sum_{k=1}^J b_k\phi_k(x) + \omega(x,t)\right), \tag{9.1}$$

where b_k, $k = 1, \ldots, J$, are constants and $|\omega|_{\mathcal{C}_t} \to 0$ as $t \to \infty$. The constants b_k may depend on $A_{ij}^{(1)}$, $i, j = 1, \ldots, n$, $A^{(1)}$ and ψ.

Proof. We begin by proving that

$$|V|_{\mathcal{C}_t} \to 0 \quad \text{as } t \to \infty \tag{9.2}$$

for the function V in (1.17). We first show that

$$\kappa(t) \to 0 \quad \text{as } t \to \infty. \tag{9.3}$$

Similar to (5.43), Kozlov, Maz'ya [9], it can be proved that

$$\kappa(t) \le 2 \int_{t-1/2}^{t+1/2} \kappa(s)\, ds. \tag{9.4}$$

Since $\kappa \in L^1(0,\infty)$, this implies (9.3). Using (9.3), we check that

$$\int_{-1}^{t} e^{-b(t-s)} \kappa(s)\, ds \to 0 \quad \text{as } t \to \infty$$

for any positive number b. This, together with (1.23) and (9.3), proves (9.2).

It follows from (7.59) that

$$\int_{-1}^{\infty} \|\Theta'\|_{L^1(t,t+1)}\, dt < \infty.$$

This implies that the vector function $\Theta(t)$ has a limit Θ_0 as $t \to \infty$ and

$$|\Theta(t) - \Theta_0| \le \int_{t-1}^{\infty} \|\Theta'\|_{L^1(s,s+1)}\, ds. \tag{9.5}$$

Analogously, $\|g\|_{L^1(t,t+1)}$, where $g(t) = -f(t) + \Lambda(t)$, is estimated by $C\kappa(t)$. Therefore,

$$\int_0^t g(s)\, ds = C_1 - \int_t^\infty g(s) ds$$

and

$$\int_t^\infty g(s)\, ds \le C \int_{t-1}^\infty \kappa(s)\, ds.$$

This implies that

$$\exp\left(\int_0^t g(s)\, ds \right) - e^{C_1} \to 0 \qquad \text{as } t \to \infty. \tag{9.6}$$

We apply (1.17), (9.2), (9.5) and (9.6) and arrive at (9.1). □

We give an estimate for the function u which is of particular interest if $\kappa \in L^2(0,\infty)$.

Corollary 9.2 *The estimate*

$$|u|_{\mathcal{C}_t} \le C_1 \|\psi\|_{L^2(\Omega)} e^{-\lambda_1 t + \int_0^t (-f(s) + C_2 \kappa(s)^2)\, ds} \tag{9.7}$$

is valid. If, in particular, $\kappa \in L^2(0,\infty)$*, then we have*

$$|u|_{\mathcal{C}_t} \le C \|\psi\|_{L^2(\Omega)} e^{-\lambda_1 t - \int_0^t f(s)\, ds}.$$

Proof. We begin by proving the inequality

$$\int_0^t |\Lambda(s)|\, ds \le C \int_{-1}^t \kappa(s)^2\, ds. \tag{9.8}$$

From (7.58), we obtain

$$\int_0^t |\Lambda(s)|\, ds \le C \left(\int_{-1}^t \kappa(s) \int_{-1}^s e^{-b_0(s-\tau)} \kappa(\tau)\, d\tau\, ds + \int_{-1}^t \kappa(s)^2\, ds \right). \tag{9.9}$$

Using Hölder's inequality to estimate the double integral in the right-hand side and boundedness of convolution in L^2-norm, it follows that

$$\int_{-1}^t \kappa(s) \int_{-1}^s e^{-b_0(s-\tau)} \kappa(\tau)\, d\tau\, ds \le C \int_{-1}^t \kappa(s)^2\, ds.$$

This, together with (9.9), gives (9.8).

From (1.23), it follows that

$$|V|_{\mathcal{C}_t} \le C \|\psi\|_{L^2(\Omega)}. \tag{9.10}$$

Estimate (9.7) follows from (1.17), (1.20), (9.8) and (9.10). □

Corollary 9.3 *Assume, in addition to the conditions in Theorem* 1.1*, that the matrix* $A^{(1)} = 0$*,* $A^{(1)}_{ij} \in L^1(Q)$ *for* $i,j = 1,2,\dots,n$ *and that* p*, as given by* (2.6)*, is equal to* ∞*. Then it follows that*

$$u(x,t) = e^{-\lambda_1 t} \left(b \sum_{k=1}^{J} \theta_k(t) \phi_k(x) + \omega(x,t) \right), \tag{9.11}$$

where $|\omega|_{\mathcal{C}_t} \to 0$ *as* $t \to \infty$ *and* b *is a constant which may depend on* $A^{(1)}_{ij}$*,* $i,j = 1,\dots,n$*, and* ψ*.*

Proof. Since $p = \infty$, the number s_1 given by (1.13) is equal to 2 and the function κ is in this case defined by (1.12) as

$$\kappa(t) = \sum_{i,j=1}^{n} \|A^{(1)}_{ij}\|_{L^2(\mathcal{C}_t)}.$$

By changing the order of integration, it follows for $i, j = 1, \dots, n$ that

$$\int_0^\infty \|A_{ij}^{(1)}\|^2_{L^2(\mathcal{C}_t)}\,dt = \int_0^\infty \int_t^{t+1} \int_\Omega |A_{ij}^{(1)}(x,s)|^2\,dx\,ds\,dt \\ \le \int_0^\infty \int_\Omega |A_{ij}^{(1)}(x,s)|^2\,dx\,ds.$$

Since $|A_{ij}^{(1)}|$ is bounded and $A_{ij}^{(1)} \in L^1(Q)$, we have

$$\int_0^\infty \kappa(t)^2\,dt < \infty. \tag{9.12}$$

Since

$$\kappa(t)^2 \le 4 \int_{t-1/2}^{t+1/2} \kappa(s)^2\,ds,$$

compare with (9.4), it follows that $\kappa(t) \to 0$ as $t \to \infty$. Using that $\nabla\phi_k \in \big(L^\infty(\Omega)\big)^N$, $k = 1, \dots, J$, we derive from (1.18) and (1.19) that

$$\int_0^\infty |f(t)|\,dt \le C \sum_{i,j=1}^n \int_0^\infty \int_\Omega |A_{ij}^{(1)}(x,t)|\,dx\,dt < \infty. \tag{9.13}$$

In the same way as in Corollary 9.1, we prove that

$$|V|_{\mathcal{C}_t} \to 0 \text{ as } t \to \infty \tag{9.14}$$

and that

$$\exp\left(\int_0^t \big(-f(s) + \Lambda(s)\big)\,ds\right) \to C \quad \text{as } t \to \infty. \tag{9.15}$$

In the derivation of (9.15), we use the estimates (9.8), (9.12) and (9.13). The formulae (1.17), (9.14) and (9.15) give (9.11). □

The asymptotic formula (1.17) contains two unknown functions, Θ and Λ. If the first eigenvalue is simple, i.e. $J = 1$, and the coefficients are real, then Θ can be chosen as 1. Thus, we arrive at the following theorem.

Theorem 9.4 *If $J = 1$ and the matrices A_{ij}, i, $j = 1, \dots, n$ and A are real-valued, then the solution u in $\big(V^2_{0,\mathrm{loc}}(Q)\big)^N$ of problem* (1.1)–(1.3) *can be written as*

$$u(x,t) = e^{-\lambda_1 t + \int_0^t (-\mathcal{R}(s) + \Lambda(s))\,ds}\big(w_0\phi_1(x) + V(x,t)\big),$$

where

$$\mathcal{R} = \int_\Omega \left[\sum_{i,j=1}^n (A_{ij}^{(1)}\phi_{1x_j}, \phi_{1x_i}) + (A^{(1)}\phi_1, \phi_1)\right] dx,$$

w_0 is a constant and the estimates (1.20), (1.21) *and* (1.23) *are valid.*

A Eigenfunctions of a time-independent operator

In this section we consider the boundary value problem

$$\begin{cases} L^{(0)}u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where

$$L^{(0)}u(x) = -\sum_{i,j=1}^{n} \left(A_{ij}^{(0)}(x)u_{x_j}(x)\right)_{x_i} + A^{(0)}(x)u(x)$$

and $f \in \left(L^2(\Omega)\right)^N$. The main result is that there exists an ON-basis of $\left(L^2(\Omega)\right)^N$ consisting of eigenfunctions of $L^{(0)}$.

We define the sesquilinear form $B : \left(H_0^1(\Omega)\right)^N \times \left(H_0^1(\Omega)\right)^N \to \mathbf{C}$ as

$$B[u,\eta] = \int_\Omega \left[\sum_{i,j=1}^{n} \left(A_{ij}^{(0)}u_{x_j}, \eta_{x_i}\right) + \left(A^{(0)}u, \eta\right)\right] dx \tag{A.2}$$

and say that $u \in \left(H_0^1(\Omega)\right)^N$ is a weak solution of the problem (A.1) if

$$B[u,\eta] = \int_\Omega (f,\eta)\, dx, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^N.$$

Before studying the eigenfunctions of $L^{(0)}$, let us first prove the inequality

$$\int_\Omega \left(A^{(0)}u, \eta\right) dx \le C\|A^{(0)}\|_{L^q(\Omega)}\|u\|_{H^1(\Omega)}\|\eta\|_{H^1(\Omega)}, \tag{A.3}$$

with q given in (1.8). In the case when $n \ge 3$, Hölder's inequality with $s = 2n/(n-2)$ implies that

$$\int_\Omega \left(A^{(0)}u, \eta\right) dx \le C\|A^{(0)}\|_{L^{n/2}(\Omega)}\|u\|_{L^s(\Omega)}\|\eta\|_{L^s(\Omega)}.$$

Since $n/2 < q$ and $\left(H^1(\Omega)\right)^N$ is compactly embedded in $\left(L^s(\Omega)\right)^N$, inequality (A.3) follows.

If $n \le 2$, we set $s = 2q/(q-1)$ and observe that $s \in [2,4]$. From Hölder's inequality it follows that

$$\int_\Omega \left(A^{(0)}u, \eta\right) dx \le C\|A^{(0)}\|_{L^q(\Omega)}\|u\|_{L^s(\Omega)}\|\eta\|_{L^s(\Omega)}.$$

In this case, the space $\left(H^1(\Omega)\right)^N$ is compactly embedded in every $\left(L^{r_1}(\Omega)\right)^N$ for $1 \le r_1 < \infty$ so (A.3) follows.

For convenience, we will use the notation $\|\cdot\|_H$ instead of $\|\cdot\|_{H^1(\Omega)}$. We have the following result.

Theorem A.1 *Let Ω be an open, bounded region in $\mathbf{R}^n$ with Lipschitz boundary. Then there exists an ON-basis $\{\phi_k\}_{k=1}^{\infty}$ of $\big(L^2(\Omega)\big)^N$ consisting of eigenfunctions of the operator $L^{(0)}$, i.e.*

$$B[\phi_k, \eta] = \lambda_k \int_\Omega (\phi_k, \eta)\, dx, \quad \forall \eta \in \big(H_0^1(\Omega)\big)^N, \tag{A.4}$$

where the eigenvalues $\{\lambda_k\}$ satisfy

$$\lambda_1 \le \lambda_2 \le \ldots, \quad \lambda_k \to \infty \text{ as } k \to \infty.$$

Furthermore, $\phi_k \in \big(H_0^1(\Omega)\big)^N$ and

$$B[u,u] \ge \lambda_1 \|u\|^2_{L^2(\Omega)}. \tag{A.5}$$

Proof. We first prove the theorem under the extra assumption that $A_-^{(0)} = 0$, i.e. when the matrix $A^{(0)}$ is positive. We note that $B[\cdot,\cdot]$ defines a scalar product on $\big(H_0^1(\Omega)\big)^N$. In fact, condition (1.7) implies that all elements in the matrices $A_{ij}^{(0)}$ are bounded. From this and (A.3) we conclude that there exists a constant C not depending on u or η such that

$$\big|B[u,\eta]\big| \le C\|u\|_H \|\eta\|_H. \tag{A.6}$$

Hence $B[u,\eta]$ is finite. The linearity in the first argument of B is straightforward. The relation $B[\eta,u] = \overline{B[u,\eta]}$ follows from (1.6) and that $A^{(0)}$ is hermitian. By using (1.7) and the extra assumption that $A^{(0)}$ is positive we see that

$$B[u,u] \ge \nu \|\nabla u\|^2_{L^2(\Omega)}. \tag{A.7}$$

Thus $B[u,u] \ge 0$ with equality only for $u = 0$, showing that B is a scalar product on $\big(H_0^1(\Omega)\big)^N$.

Our next step is to show that the norms $\|\cdot\|_H$ and $\|\cdot\|_B$, defined as

$$\|u\|_B = \big(B[u,u]\big)^{1/2},$$

are equivalent. It follows immediately from (A.6) that there exists a constant C independent of u such that $\|u\|_B \le C\|u\|_H$. Conversely, we can use Poincaré's inequality and (A.7) to obtain

$$\|u\|_H^2 = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \le C_1 \|\nabla u\|^2_{L^2(\Omega)} \le \frac{C_1}{\nu}\|u\|_B^2,$$

where C_1 does not depend on u. Together this gives that $\|\cdot\|_H$ and $\|\cdot\|_B$ are equivalent on $\big(H_0^1(\Omega)\big)^N$.

For u fixed in $\big(H_0^1(\Omega)\big)^N$, we introduce the linear and bounded functional $\eta \mapsto \int_\Omega (\eta, u)\, dx$ on $\big(H_0^1(\Omega)\big)^N$. It follows from Riesz representation theorem that there exists a unique element $Ku \in \big(H_0^1(\Omega)\big)^N$ such that

$$\int_\Omega (\eta, u)\, dx = B[\eta, Ku], \quad \forall \eta \in \big(H_0^1(\Omega)\big)^N.$$

In the same way as in the proof of Theorem A.8, Section A.1 in Part 1 of this thesis, we can show that K is linear, bounded, symmetric and compact but this time all computations are made with respect to the scalar product B. Hence there exists an, with respect to B, orthogonal basis $\{\phi_k\}_{k=1}^{\infty}$ of $\left(H_0^1(\Omega)\right)^N$ consisting of eigenvectors of K with corresponding eigenvalues $\{\mu_k\}_{k=1}^{\infty}$. We can assume that

$$\|\phi_k\|_{L^2(\Omega)} = 1, \quad k = 1, 2, \ldots$$

and we will soon see that this set is orthogonal even in $\left(L^2(\Omega)\right)^N$.

We introduce

$$m = \inf\{B[Ku, u] : u \in \left(H_0^1(\Omega)\right)^N, \|u\|_B = 1\}$$

and

$$M = \sup\{B[Ku, u] : u \in \left(H_0^1(\Omega)\right)^N, \|u\|_B = 1\} \tag{A.8}$$

and get from Lemma A.5 in Section A.1 in Part 1 of the thesis that $m, M \in \sigma(K) \subset [m, M]$ where $\sigma(K)$ is the spectrum of K. We observe that

$$B[Ku, u] = B[u, Ku] = \int_{\Omega} (u, u)\, dx = \|u\|_{L^2(\Omega)}^2$$

and, from the equivalence of norms, that there exists a constant c_0 such that

$$\|u\|_{L^2(\Omega)}^2 \leq \|u\|_H^2 \leq c_0 \|u\|_B^2,$$

so $M \leq c_0$. On the other hand, it follows from Lemma A.4(i), Section A.1, Part 1, that $0 \in \sigma(K)$. Thus $m = 0$.

From Lemma A.4(ii), Part 1, we see that the point spectrum $\sigma_p(K)$ is equal to $\sigma(K)$, except possibly for 0. But 0 can not be an eigenvalue of K since $Ku = 0$ would imply that

$$\int_{\Omega} (\eta, u)\, dx = B[\eta, 0] = 0, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^N,$$

which with $\eta = u$ shows that $u = 0$. As in the proof of Theorem A.8, Part 1, we conclude that

$$M = \mu_1 \geq \mu_2 \geq \ldots > 0, \quad \mu_k \to 0 \text{ as } k \to \infty.$$

The relation $K\phi_k = \mu_k \phi_k$ implies that

$$B[\phi_k, \eta] = \lambda_k \int_{\Omega} (\phi_k, \eta)\, dx, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^N,$$

where $\lambda_k = \mu_k^{-1}$ is an eigenvalue of $L^{(0)}$. Since $\lambda_k \neq 0$, it follows, by setting $\eta = \phi_l$, that the sequence $\{\phi_k\}_{k=1}^{\infty}$ is orthogonal also in $\left(L^2(\Omega)\right)^N$. Furthermore, it follows from (A.8) that $B[Ku, u] \leq M\|u\|_B^2$ and, since

$M = \mu_1 = \lambda_1^{-1}$ and $B[Ku,u] = \|u\|^2_{L^2(\Omega)}$, the inequality (A.5) follows. This concludes the proof in the case $A_-^{(0)} = 0$.

Suppose that $A_-^{(0)} \neq 0$. Since $A_-^{(0)}$ is bounded, there is a constant c such that the matrix $-A_-^{(0)} + cI$ is positive. Then the matrix $G = A_+^{(0)} - A_-^{(0)} + cI$ is positive, so we can apply the proof above on the form

$$B'[u,\eta] = \int_\Omega \left[\sum_{i,j=1}^{n} \left(A_{ij}^{(0)} u_{x_j}, \eta_{x_i}\right) + (Gu, \eta) \right] dx$$

to obtain an ON-basis $\{\phi_k\}_{k=1}^\infty$ of $\left(H_0^1(\Omega)\right)^N$ with corresponding scalars $\{\lambda'_k\}_{k=1}^\infty$ satisfying

$$B'[\phi_k, \eta] = \lambda'_k \int_\Omega (\phi_k, \eta)\, dx, \quad \forall \eta \in \left(H_0^1(\Omega)\right)^N$$

and $\lambda'_k \to \infty$ as $k \to \infty$. But

$$B[\phi_k, \eta] = B'[\phi_k, \eta] - c \int_\Omega (\phi_k, \eta)\, dx = (\lambda'_k - c) \int_\Omega (\phi_k, \eta)\, dx,$$

so ϕ_k is an eigenfunction of $L^{(0)}$ with eigenvalue $\lambda_k = \lambda'_k - c$. Also (A.5) follows from the last line and the proof is complete. □

Actually, more can be said about the eigenfunctions.

Theorem A.2 *There exists a* $p \in (2, \infty]$ *such that* $\phi_k \in \left(W_0^{1,p}(\Omega)\right)^N$ *for* $k = 1, 2, \ldots$.

Proof. If $a > -\lambda_1$, then the operator

$$L^{(0)} + a : \left(W_0^{1,2}(\Omega)\right)^N \to \left(W^{-1,2}(\Omega)\right)^N$$

is isomorphic. Furthermore, it can be checked, using the assumptions on $A_{ij}^{(0)}$ and $A^{(0)}$, that the operator

$$L^{(0)} + a : \left(W_0^{1,p}(\Omega)\right)^N \to \left(W^{-1,p}(\Omega)\right)^N \tag{A.9}$$

is continuous for any $p \in [1, \infty]$. By Theorem 3.5, Triebel [18], $\left(W_0^{1,p}(\Omega)\right)^N$ and $\left(W^{-1,p}(\Omega)\right)^N$ are interpolation spaces. Therefore, using Shneiberg [17] or Krugljak, Milman [11], we derive the existence of $\epsilon > 0$ such that the operator in (A.9) is isomorphic for $p \in [2-\epsilon, 2+\epsilon]$. This implies in particular that $\phi_k \in \left(W^{1,2+\epsilon}(\Omega)\right)^N$ for all k, completing the proof. □

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