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Coupled azimuthal and radial flows and oscillations in a rotating plasma

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Nonlinear coupling between the radial, axial, and azimuthal flows in a cold rotating plasma is considered nonperturbatively by first constructing a basis solution for a rotating flow. Simple but exact solutions that describe an expanding plasma with oscillatory flow fields are then obtained. These solutions show that the energy in the radial and axial flow components can be transferred to the azimuthal component but not the vice versa. Nonlinear electron velocity oscillations in the absence of electron density oscillations at the same frequency are shown to exist. © 2009 American Institute of Physics. [DOI: 10.1063/1.3158596]

I. INTRODUCTION

The rotating plasma is one of the simple physical systems exhibiting unusual nonlinear wave and pattern phenomena (see e.g., Refs. 1–9 and the references therein). Finite rotating cold plasmas are of special interest for understanding plasma crystal formation and other phenomena.^{4,6–8} In one of the simple models, the electrons and ions are treated as cold fluids and the interaction is described by a set of nonlinear differential equations representing coupled radial, axial, and azimuthal oscillators.

Most investigations of nonlinear wave phenomena consider finite but small perturbations of an equilibrium or steady state.^{10–12} The evolution of the perturbations is mainly governed by the dynamic properties (such as dispersion, phase relations, dissipation, etc.) of the corresponding linear normal modes but with inclusion of weakly nonlinear wave-particle and/or wave-wave interactions. Thus, the resulting nonlinear waves behave quite similarly to the corresponding linear modes, and the problem is usually studied by carrying out small-amplitude expansion of the relevant physical quantities. Depending on the characteristics of the corresponding normal modes, weakly nonlinear evolution equations can be derived. Since many waves have similar dispersion, propagation, and nonlinear properties, their nonlinear behavior can usually be described by one of the paradigm evolution equations^{10–12} whose derivations are now routine and whose solutions are well understood.

However, for evolving systems far from equilibrium, the perturbative approach is not applicable. One must then use a fully nonlinear treatment to investigate the possibility of steady states, waves, and patterns. One can attempt to find time-dependent wavelike dynamic-equilibrium states using nonperturbative methods.¹³ Such states, in the form of highly nonlinear plasma oscillations and/or patterns, are little understood but have indeed been shown to exist.^{14–17} Here we consider a class of exact solutions of the fully nonlinear

equations describing a cold plasma by first constructing a basis solution for a free flow. Solutions describing an expanding plasma with oscillatory flow fields are then obtained. The solutions show that the energy in the radial and axial flow components can be transferred to the rotating component but not the vice versa. Furthermore, in contrast to linear electrostatic waves, there can exist electron velocity oscillations in the absence of density oscillations at similar frequencies.

II. FORMULATION

We shall consider highly nonlinear flow behavior, which may exist in cold plasmas containing electrons and ions. The displacement current is assumed to be fully compensated by the conduction current, so that the temporal dependence of the magnetic field can be represented by an azimuthal electric field. Accordingly, the dimensionless cold-fluid equations are

$$\partial_t n_j + \nabla \cdot (n_j \mathbf{v}_j) = 0, \quad (1)$$

$$\partial_t \mathbf{v}_j + \mathbf{v}_j \cdot \nabla \mathbf{v}_j = \mu_j (\mathbf{E} + \mathbf{v}_j \times \mathbf{B}), \quad (2)$$

$$\nabla \cdot \mathbf{E} = n_i - n_e, \quad (3)$$

$$\partial_t \mathbf{E} = n_e \mathbf{v}_e - n_i \mathbf{v}_i, \quad (4)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (5)$$

$$\nabla \times \mathbf{E} = -\partial_t \mathbf{B}, \quad (6)$$

where the time and space have been normalized by the inverse plasma frequency $\omega_{pe} = (4\pi n_0 e^2 / m_e)^{1/2}$ and R_0 , an arbitrary space scale, say the initial size of the plasma, respectively, e and m_e are the charge and mass of the electrons, n_j and \mathbf{v}_j are the densities and velocities of the particle species $j=e, i$ (for electrons and ions, respectively) normalized by a reference density n_0 and $R_0 \omega_{pe}$, respectively, $\mu_e = -1$, and $\mu_i = m_e / m_i$. The electric and magnetic fields \mathbf{E} and \mathbf{B} are

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normalized by $4\pi en_0 R_0$ and $4\pi en_0 R_0 c / R_0 \omega_{pe} = (4\pi n_0 m_e)^{1/2} c$, respectively, and c is the light speed.

We are interested in rotating flows. In cylindrical coordinates, the fluid velocity can be expressed as $\mathbf{v} = v_r(r, \varphi, t)\mathbf{e}_r + v_\varphi(r, \varphi, t)\mathbf{e}_\varphi$, which satisfies the incompressibility condition $\nabla \cdot \mathbf{v} = 0$. Here \mathbf{e}_s denotes the unit vector in the s direction. One can thus define a stream function Ψ satisfying $\mathbf{v} = \mathbf{e}_z \times \nabla \Psi$ or $\nabla^2 \Psi = \omega$, where \mathbf{e}_z is the unit vector and $\omega = \nabla \times \mathbf{v}$ is the vorticity. Thus, for irrotational flow ($\omega = 0$) one can write,^{14–17}

$$v_\varphi(r, \varphi, t) = A(t)r \sin(2\varphi) + B(t)r \cos(2\varphi), \quad (7)$$

$$v_r(r, \varphi, t) = B(t)r \sin(2\varphi) - A(t)r \cos(2\varphi),$$

which can be used as the starting point for constructing time-dependent exact solutions of Eqs. (1)–(6).

III. BASIS FLOW STRUCTURE

We shall first consider the basis velocity structure,

$$\mathbf{V} = V_r(t, r, z)\mathbf{e}_r + V_\varphi(t, r, z)\mathbf{e}_\varphi + V_z(t, r, z)\mathbf{e}_z, \quad (8)$$

which satisfies $\omega_\varphi = \mathbf{e}_\varphi \cdot \nabla \times \mathbf{V} = 0$ or

$$\partial_r V_z - \partial_z V_r = 0, \quad (9)$$

so that there exists a potential $\psi(t, r, z)$ satisfying $V_r = \partial_z \psi$ and $V_z = \partial_r \psi$, such that Eq. (14) for the radial and axial velocity components can be written as $\partial_r \Pi = V_\varphi^2 / r$, $\partial_z \Pi = 0$, and $\Pi = \partial_t \psi + (V_r^2 + V_z^2) / 2$. Thus, the evolution of the velocity field is determined by the potential ψ . Since Π does not depend on z , the azimuthal velocity V_φ also does not depend on z . Accordingly,

$$\partial_t V_\varphi + (V_r / r) \partial_r (r V_\varphi) = 0, \quad (10)$$

which is satisfied only if $\partial_z V_r = 0$, or V_r is also not a function of z . It follows from Eq. (9) that the axial velocity component V_z can depend only on t and z . We can then assume

$$V_r = A(t)r, \quad V_z = B(t)z, \quad \text{and} \quad V_\varphi = C(t)r, \quad (11)$$

where $A(t)$, $B(t)$, and $C(t)$ are associated with the radial, axial, and azimuthal velocity components and are still to be determined. The corresponding vorticity is $\omega_z = 2C(t)$. Clearly, the simple basis structure proposed in Eq. (11) is not unique, and its validity still has to be verified by the existence of solutions.

Substitution of Eq. (11) into the corresponding continuity equation leads to a first-order linear partial differential equation for $N(t, r, z)$,

$$\partial_t N + Ar \partial_r N + Bz \partial_z N = -(2A + B)N, \quad (12)$$

whose characteristics are given by

$$d_t N = -(2A + B)N, \quad d_r = Ar, \quad \text{and} \quad d_z = Bz. \quad (13)$$

A nontrivial simple case is when the functions A , B , and C do not depend on space. Substituting the Ansatz (11) into the force-free momentum equation,

$$\partial_t \mathbf{V} + \mathbf{V} \cdot \nabla \mathbf{V} = 0, \quad (14)$$

we obtain

$$d_t A + A^2 - C^2 = 0, \quad (15)$$

$$d_t B + B^2 = 0, \quad (16)$$

$$d_t C + 2AC = 0, \quad (17)$$

which is a set of nonlinear ordinary differential equations (ODEs) giving the basis solutions that can be used to construct multidimensional exact solutions of the cold-plasma equations. We note that the axial and radial flow components are not coupled, and the density, given by Eq. (13), does not affect the flow field.

The set (15)–(17) of simple but nonlinear ODEs can be integrated numerically. As an illustration, we present here two cases of rotating flow ($C(0) \neq 0$). Figure 1(a) shows the evolution of a system, which is initially free from radial flow [$A(0) = 0$]. Figure 1(b) shows the corresponding phase space. Here one can clearly see the close relationship among the fields. For comparison, Fig. 2 shows the evolution of a flow, which does have an initial radial component. The difference between the two initial conditions can be seen by comparing the phase spaces of the resulting flows shown in Figs. 1(b) and 2(b). In both cases the dynamics of the free-flow involves only inertial acceleration, resulting in radial expansion of the fluid. However, when the radial velocity [given by $A(t)r$] is initially zero, the corresponding rotational component C , as well as the density $N(t)$, decreases much slower than the case with a nonvanishing initial radial flow. From Eq. (16), we see that the axial flow [given by $B(t)z$] is not affected by the radial and azimuthal flows, nor the density.

It may be of interest to briefly examine the asymptotic dynamics of the simple basis flow. Assuming that at large times the flow fields are small (say, $A \ll 1$), we obtain from Eqs. (15)–(17),

$$d_t (d_t A + A^2)^{1/2} + 2A(d_t A + A^2)^{1/2} = 0, \quad (18)$$

which can be formally integrated to quadrature. An approximate solution is

$$A(t) \sim \kappa \text{th}(\kappa t), \quad B(t) \sim (\theta + t)^{-1}, \quad \text{and} \quad C(t) \sim \kappa, \quad (19)$$

where κ and $\theta \geq 0$ are arbitrary integration constants, and we have made use of the relation $\exp(-4 \int A dt) \rightarrow 1$. It follows that for $t \rightarrow \infty$ we have the asymptotic solutions $A(t) = \kappa$, $B(t) = 0$, $C(t) = \kappa$, and $N \sim \exp(-\kappa t)$. That is, the flow field in this special case remains finite for $t \rightarrow \infty$, with the vorticity given by $2\kappa \mathbf{e}_z$.

IV. DYNAMICS OF A ROTATING PLASMA

We now use the basis solution given by Eqs. (13) and (15)–(17) to investigate the set (1)–(6) for a cold plasma. Following the basis solution, the velocity fields may be represented by

$$v_{jr} = rA_j(t), \quad v_{j\varphi} = rC_j(t), \quad \text{and} \quad v_{jz} = zB_j(t), \quad (20)$$

where $j = e, i$. We shall consider a spatially uniform plasma and define $n_e = n(t)$. The electric field can be written as

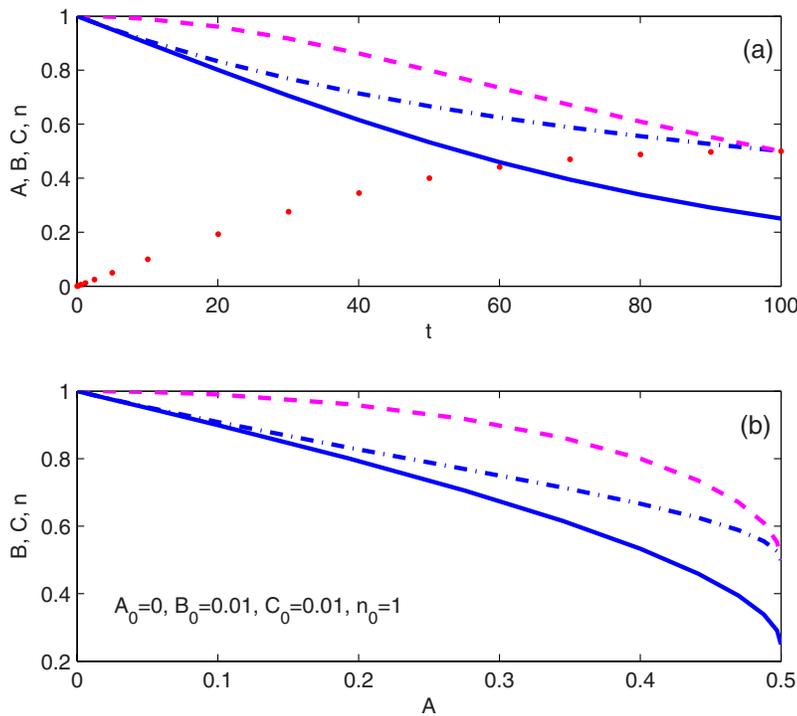


FIG. 1. (Color online) (a) The evolution of A (dotted curve), B (dash-dotted curve), C (dashed curve), and n (solid curve) for $A_0=0$. (b) The corresponding phase space. Here and in Fig. 2, the plotted values of A , B , and C are renormalized by B_0 for visual clarity. The initial values given in the figures are values before the renormalization.

$$\mathbf{E} = r\varepsilon_r(t)\mathbf{e}_r + r\varepsilon_\varphi(t)\mathbf{e}_\varphi + z\varepsilon_z(t)\mathbf{e}_z, \quad (21)$$

$$n_i = n + 2\varepsilon_r + \varepsilon_z, \quad (23)$$

such that Eq. (6) gives

$$\begin{aligned} \partial_t B_r &= 0, & \partial_t B_\varphi &= 0, \\ \partial_t B_z &= -2\varepsilon_\varphi. \end{aligned} \quad (22)$$

so that the ion continuity equation is satisfied.

Inserting Eqs. (20), (21), and (23) into Eqs. (1), (2), and (4), and equating the terms with similar spatial dependence, we obtain

$$d_t n + (2A_e + B_e)n = 0, \quad (24)$$

$$d_t A_e + A_e^2 - C_e^2 + \varepsilon_r + C_e B_z = 0, \quad (25)$$

$$d_t B_e + B_e^2 + \varepsilon_z = 0, \quad (26)$$

It is clear that with the initial conditions $B_r(t=0)=B_\varphi(t=0)=B_z(t=0)=0$, only B_z can differ from zero inside the plasma and it can only be a function of time. Accordingly, $\nabla \times \mathbf{B} = 0$. Moreover, from the Poisson's Eq. (3) we obtain

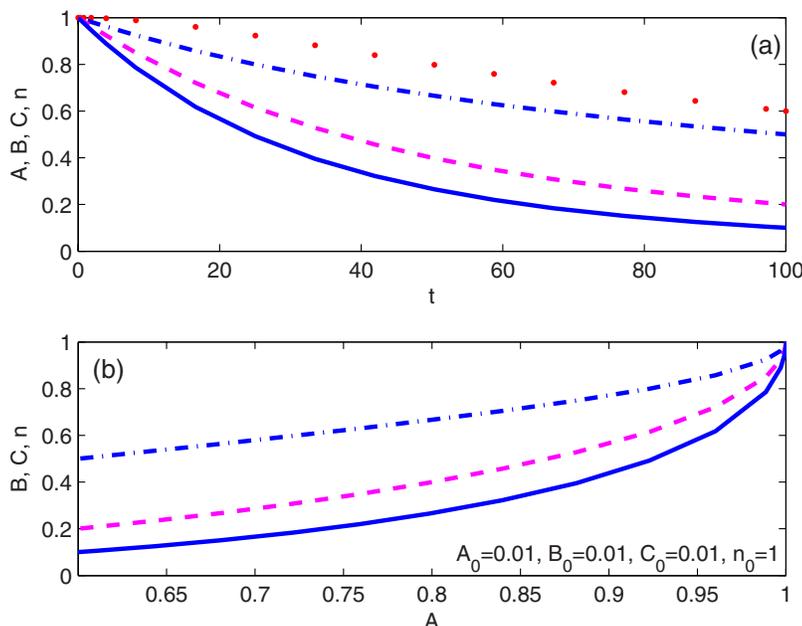


FIG. 2. (Color online) (a) The evolution of A (dotted curve), B (dash-dotted curve), C (dashed curve), and n (solid curve) for $A_0=0.01$. (b) The corresponding phase space.

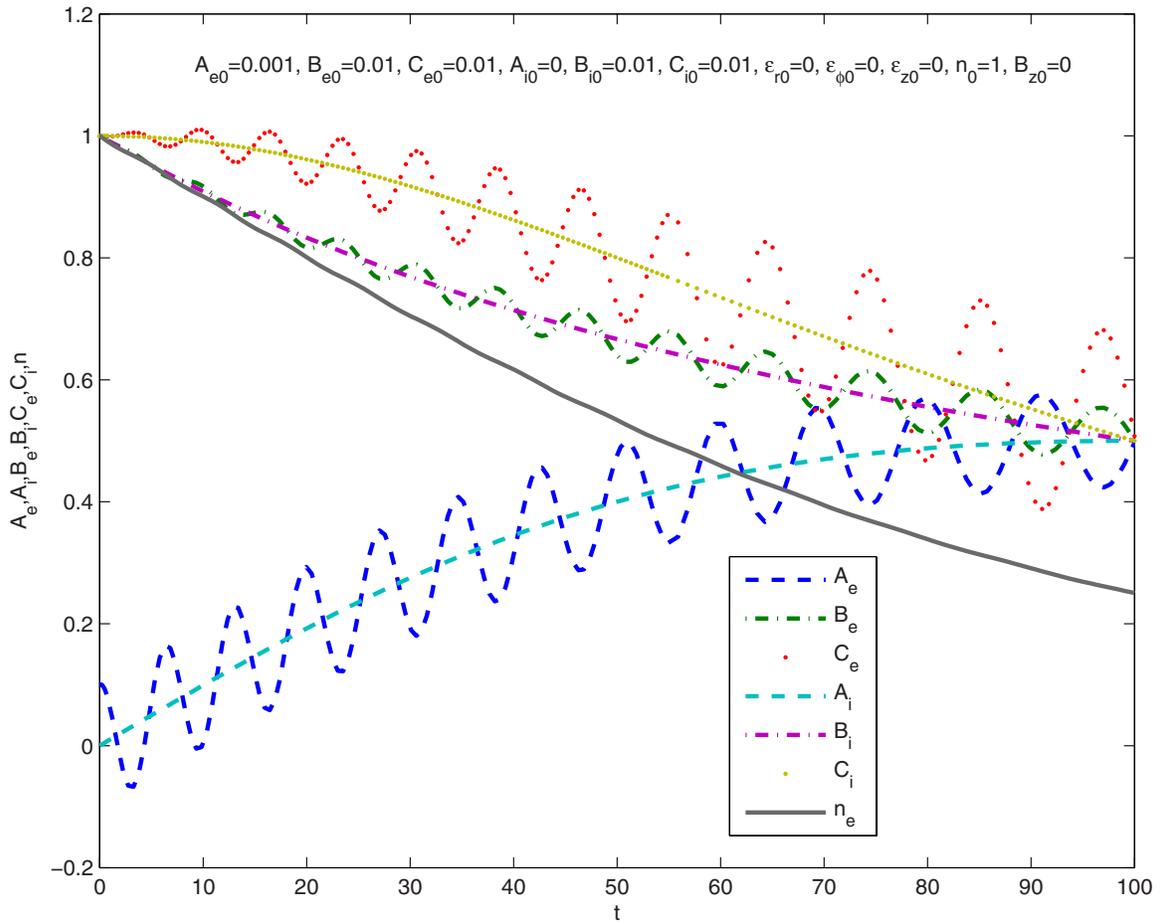


FIG. 3. (Color online) The evolution of A_j , B_j , and C_j (where $j=i, e$), and n , for $\mu=10^{-5}$ and $A_{e0}=0.001$. Oscillations occur in A_e , B_e , and C_e . Here and in Fig. 4, the curves for A_j , B_j , and C_j have been renormalized by B_{e0} for visual clarity. The initial values given in the figures are values before the renormalization.

$$d_t C_e + 2A_e C_e + \varepsilon_\varphi - A_e B_z = 0, \quad (27)$$

$$d_t A_i + A_i^2 - C_i^2 - \mu_i \varepsilon_r - \mu_i C_i B_z = 0, \quad (28)$$

$$d_t B_i + B_i^2 - \mu_i \varepsilon_z = 0, \quad (29)$$

$$d_t C_i + 2A_i C_i - \mu_i \varepsilon_\varphi + \mu_i A_i B_z = 0, \quad (30)$$

$$d_t \varepsilon_r = n(A_e - A_i) - (2\varepsilon_r + \varepsilon_z)A_i, \quad (31)$$

$$d_t \varepsilon_\varphi = n(C_e - C_i) - (2\varepsilon_r + \varepsilon_z)C_i, \quad (32)$$

$$d_t \varepsilon_z = n(B_e - B_i) - (2\varepsilon_r + \varepsilon_z)B_i, \quad (33)$$

and we recall that the ion continuity equation is already satisfied. Thus, the spatial and temporal variations of the physical quantities are separated, and the ODEs (24)–(33) together with Eqs. (22) and (23) completely determine the dynamics of the vortical flow with the electron and ion vorticities given by $\boldsymbol{\omega}_j = \nabla \times \mathbf{v}_j = 2C_j(t)\mathbf{e}_z$.

To consider the evolution of the plasma, we shall assume initial states close to that of the basis flow, which we emphasize is purely inertial and not dependent on the mass and charge. To avoid singular and other solutions (such as that involving too large gradients) that may invalidate our starting equations, we shall restrict our numerical solutions to the

case of an initially expanding plasma, namely, $A_j(0) \geq 0$ and $B_r(0) \geq 0$, and no initial magnetic field [$B_z(0)=0$]. Furthermore, we assume a realistic mass ratio $\mu_i=10^{-5}$, and take $\varepsilon_r(0)=0$ and $\varepsilon_\varphi(0)=0$. Depending on the initial values, the system can evolve in many ways, i.e., there are many different solutions of the set of nonlinear ODEs. As an example, we look for solutions by setting the initial radial and azimuthal electron flow components slightly different from that of the basis flows.

Figures 3 and 4 show that highly nonlinear oscillations of the electron velocity can occur without any noticeable change of the electron density. This is because the temporal dependence of the density given by Eq. (24) depends only on time integrals of A_e and B_e , so that the oscillations in the latter are smoothed out. Such solutions do not exist in the linear limit since higher harmonics do not appear. Comparing the two figures, we see also that a small difference in the initial flow can lead to very different oscillation patterns. In particular, it is possible to have oscillations occurring only in the azimuthal flow component. We note that the initial disturbance of the basis radial flow has a strong effect on the azimuthal and axial flows. However, initial disturbance of the basis azimuthal flow, even when strong, does not affect the radial and axial flow components. That is, the energy in the azimuthal flow oscillations cannot be converted into the

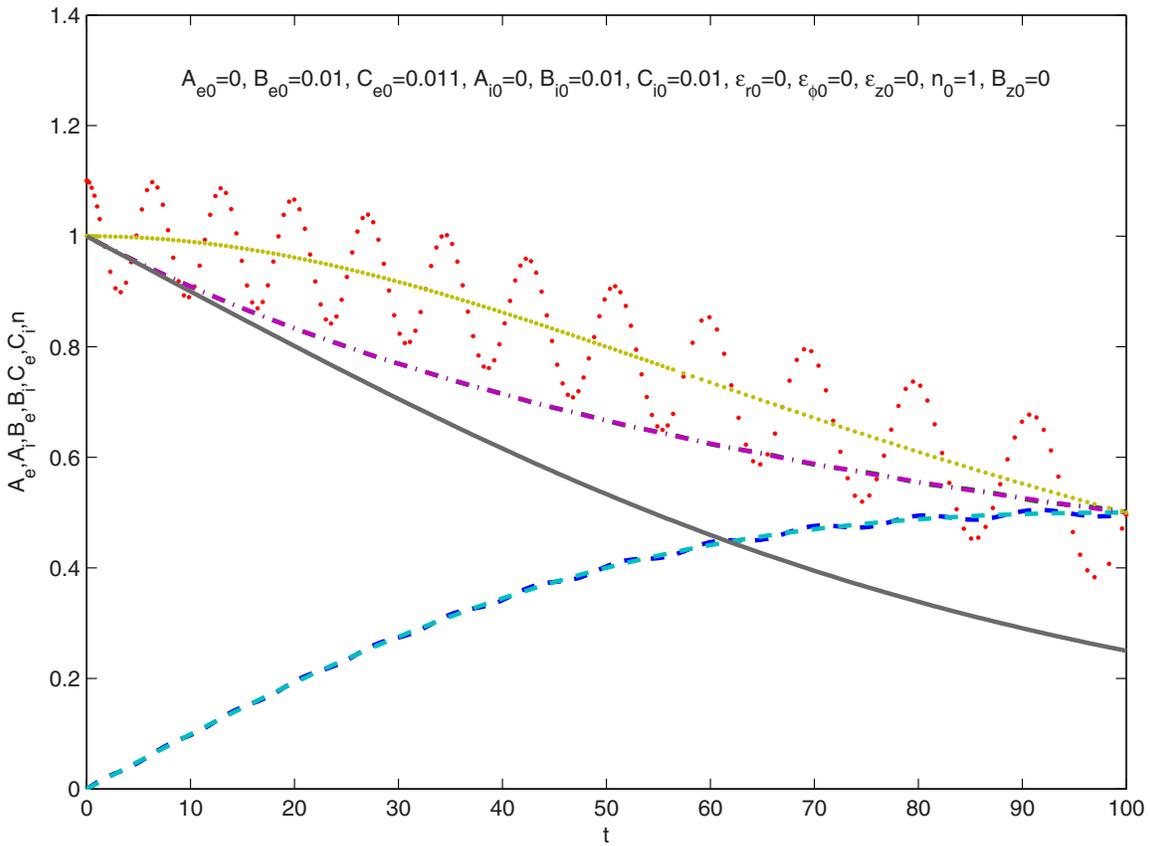


FIG. 4. (Color online) The evolution of $A_j, B_j,$ and C_j (where $j=i, e$), and n , for $\mu=10^{-5}$ and $A_{e0}=0$. The legend and renormalization are the same as that in Fig. 3. Note that the curves for A_e and B_e nearly overlap that for A_i and B_i , respectively. Oscillations occur only in the azimuthal electron flow component C_e , demonstrating the sensitivity of the solution on the initial conditions and the dominance of the rotating flow component.

other degrees of freedom, but the energy in the latter can be converted into the azimuthal component. This conclusion has obvious implications in applications involving rotating plasmas, such as for magnetically confined plasmas and solar dynamos.

We can easily estimate the evolution of the spatial extent of the electrons and ions.⁷ If initially the plasma is located in $\{0 \leq r \leq R_0, 0 \leq z \leq L_0\}$ and satisfies the neutrality condition $n_e(t=0)=n_i(t=0)=n_0$, the total number in each species is $N_j=2\pi \int_0^{L_j} \int_0^{R_j} n_j(t) r dr dz$. From the particle conservation condition $d_t N_j=0$ and the continuity equations, we obtain

$$\begin{aligned}
 d_t N_j &= 2\pi \int_0^{R_j} [d_t L_j - v_{jz}(t, r, L_j)] r n_j dr \\
 &+ \int_0^{L_j} [d_t R_j - v_{jr}(t, R_j, z)] n_j R_j dz \\
 &= 0,
 \end{aligned}
 \tag{34}$$

where $R_j(t)$ and $L_j(t)$ are the spatial extents of the electrons and ions at the time t in the radial and axial directions, respectively. In general, this relation is satisfied if the integrals vanish, or $d_t L_j = v_{jz}(t, r, L_j)$ and $d_t R_j = v_{jr}(t, R_j, z)$. Equation (20) then becomes

$$d_t L_j = B_j L_j, \quad d_t R_j = A_j R_j,
 \tag{35}$$

which may be expressed as

$$\begin{aligned}
 L_j &= L_0 \exp \left[\int_0^t B_j(t') dt' \right], \\
 R_j &= R_0 \exp \left[\int_0^t A_j(t') dt' \right],
 \end{aligned}
 \tag{36}$$

thus giving R_j and L_j in terms of $A_j, B_j,$ and N_j via the continuity equations. Equation (36) implies that the nonlinear oscillations of the electron velocity shown in Figs. 3 and 4 take place without modifying the expanding plasma boundaries $L_j(t)$ and $R_j(t)$, which like the densities depend only on time integrals of A_j and B_j .

V. CONCLUSION

We have investigated three-dimensional nonlinear flows and oscillations in a rotating and expanding plasma in the cylindrical geometry. To obtain a mathematical description of the nonlinear dynamics, we have introduced a basis solution for the rotating flow. Using the basis solution, we consider the evolution of a simple plasma system. The evolution of each flow and density component of the plasma system is determined by a set of coupled nonlinear ODEs, given by Eqs. (13) and (15)–(17). Depending on the initial state, there exists a rich variety of solutions of the nonlinear ODEs (22)–(33) even with the very simple basis flow structure. For

the cases presented, where the amplitude of the oscillations remains not too large, all the physical variables inherit the general characteristics of the basis solution.

More general basis flow structures can also be constructed, and plasma dynamics that deviate greatly from that shown in Figs. 3 and 4 can be found. For example, from the results of Sec. III one can see that an initial external electrostatic field can be included and the same analysis can be performed to obtain new solutions. Also, by removing the condition (9), one can modify the Ansatz in Eqs. (20) and (21) by introducing additional radial dependence into the axial component of the velocity and electric fields, and obtain other nonlinear oscillations and patterns. Further generalizations of the model can be realized by carefully modifying the structure of the basis solution.

It is well known that to obtain exact nonlinear solutions even for a simple plasma system with arbitrary initial/boundary conditions is generally difficult. The basis flow and the other solutions considered here are special cases. However, these solutions are analytically exact and thus inherently stable within the limits of the starting equations.^{14–16} It is also unlikely that they can be obtained by other available methods. Physically interesting and useful conclusions can also be drawn. For example, the result that the energy in the azimuthal flow is not convertible into the other degrees of freedom, but that in the latter can be converted into the azimuthal flow, and that (depending on the initial conditions) oscillations can be limited to only the azimuthal flow component, can have important implications in applications involving rotating plasmas and fluids. Another example is that there can exist flow oscillations that are not accompanied by density oscillations on the same time scale. However, as the evolution is sensitive to the initial conditions before the nonlinearity of the motion becomes predominant, in real applications one will still have to take into consideration effects such as strong background inhomogeneity, dissipation, heating, magnetic viscosity, etc. On the other hand, the exact solutions describing possible highly nonlinear final states should be useful as a bench test for new analytical and nu-

merical schemes for solving nonlinear partial differential equations.^{10–12} They can also be used as a starting point for numerical investigation of more complex and more realistic problems.^{4,6–8} The analysis here can also be readily extended to investigating electron-position plasmas, non-neutral plasmas, and large scale motion of complex fluids. From the physics point of view, several interesting problems also remain: What are conditions under which a basis flow structure is robust and not destroyed by the oscillations? How exactly does the basis flow affect the subsequent evolution of the system? Can there be basis flow structures that eventually contract, collapse, or reach finite equilibrium states?

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