

Topology Optimization of Structures with Contact Constraints by using a Smooth Formulation and a Nested Approach

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1. Abstract

In this paper a method for topology optimization of structures in unilateral contact is developed. A linear elastic structure that is unilateral constrained by rigid supports is considered. The supports are modeled by Signorini's contact conditions which in turn are treated by a smooth approximation. This approximation must not be confused with the well-known penalty approach. The state of the system, which is defined by the equilibrium equations and the smooth approximation, is solved by a Newton method. The design parametrization is obtained by using the SIMP-model. The minimization of compliance for a limited value of volume is considered. The optimization problem is solved by a nested approach where the equilibrium equations are linearized and sensitivities are calculated by the adjoint method. The problem is then solved by SLP. The LP-problem is in turn solved by an interior point method that is available in Matlab. In order to avoid mesh-dependency the sensitivities are filtered by Sigmund's filter. The method is implemented by using Matlab and Visual Fortran, where the Fortran code is linked to Matlab as mex-files. The implementation is done for a general design domain in 2D by using fully integrated isoparametric elements. The implementation seems to be very efficient and robust.

2. Keywords: SIMP, Contact, Newtons'method, SLP

3. Introduction

A proper modeling of boundary conditions is crucial when performing topology optimization of machine components. Improper modeling will result in a poor concepts. Often it is not sufficient to use the same boundary conditions that are used in a direct finite element analysis of the state problem. For instance, a boundary which can appropriately be considered fixed in a finite element analysis of the state might be too stiff in a topology optimization analysis, resulting in a too weak concept. In many situation one must also include the contact conditions in order to set up a proper model.

In this work a method for topology optimization that includes frictionless contact is proposed. Recently, in Strömberg and Klarbring [1], we suggested a similar method where the contact conditions were treated by the augmented Lagrangian approach. In this work, inspired by Hilding [2], we utilize instead a smooth approximation which should not be confused with the more familiar penalty approach. The approximation is based on the fact that the following smooth equation

$$\xi + \eta - \sqrt{(\xi - \eta)^2 + 4\mu^2} = 0$$

is equivalent to

$$\xi \geq 0, \quad \eta \geq 0, \quad \xi\eta = \mu^2,$$

which in turn approximates $\xi \geq 0, \eta \geq 0, \xi\eta = 0$ for small values on μ . This smoothing approach was suggested by Facchinei et al. [3].

A brief review of papers concerning topology optimization of structures in unilateral contact gives the conclusion that the number of papers on this subject is surprisingly few. The list is longer, however, if one consider structural optimization for contact problems in general. A review on the subject can be found in the paper by Hilding et al. [4]. A pioneering work on topology optimization of structures in

unilateral contact is found in the paper by Petersson and Patriksson [5]. A more recent work on this subject is by Fancello [6]. Another recent work is by Mankame and Ananthasuresh [7], where compliant mechanisms were generated by including contact conditions.

In this work the minimization of compliance for a linear elastic structure in unilateral contact with a rigid support that is modeled by Signorini's contact conditions is the subject. Signorini's contact conditions are numerically treated by the smoothing procedure discussed above. The state equations are then solved by the Newton method that was used in Strömberg [8] for solving an augmented Lagrangian formulation of contact. The design parametrization is performed by utilizing the SIMP-model. The nested approach presented in Klarbring and Rönqvist [9] is adopted, and sensitivities are treated by the adjoint method. The non-linear optimization problem is solved by SLP using an interior point method for the LP-problem. The stiffness is calculated by using fully integrated isoparametric elements and Sigmund's filter [10] is utilized in order to avoid mesh dependency. The method is implemented in Topo2D by using Matlab and Fortran and it is most efficient and robust.

The outline of the paper is as follows: in section 4 the governing equations of the state problem are given, in section 5 the optimization problem is defined, in section 6 the numerical treatment is discussed, in section 7 two numerical examples are given and, finally, some concluding remarks are presented.

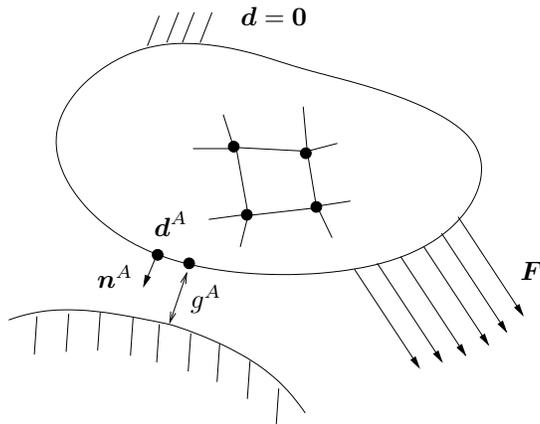


Figure 1: A linear elastic body unilaterally constrained by a rigid support.

4. The state equations

Let us consider a linear elastic body which is represented by a stiffness matrix \mathbf{K} obtained by using finite elements. The body is constrained to a rigid support as shown in Figure 1. The support is considered to be rigid and frictionless. The potential contact surface is defined by a set of potential contact nodes. The normal of the contact surface at each contact node is denoted \mathbf{n}^A . The distance between the contact surface and the support in the normal direction is given by g^A . The kinematic constraint between the contact surface and the support can now be formulated as

$$\mathbf{d}^A \cdot \mathbf{n}^A - g^A \leq 0. \quad (1)$$

For all contact nodes these conditions can be summarized as the following inequality:

$$\mathbf{C}_N \mathbf{d} - \mathbf{g} \leq \mathbf{0}, \quad (2)$$

where \mathbf{C}_N is a transformation matrix containing the normal directions \mathbf{n}^A and the column vector \mathbf{g} contains all gaps g^A . We will also use \mathbf{C}_N^A which represents a row of \mathbf{C}_N such that $\mathbf{C}_N^A \mathbf{d} = \mathbf{d}^A \cdot \mathbf{n}^A$. The equilibrium equations of the system reads

$$\mathbf{K} \mathbf{d} + \mathbf{C}_N^T \mathbf{P}_N = \mathbf{F}, \quad (3)$$

where \mathbf{d} is the displacement vector, \mathbf{F} is the external forces and \mathbf{P}_N is a column vector containing normal contact forces P_N^A which are governed by Signorini's contact conditions.

Signorini's contact conditions are formulated as

$$P_N^A \geq 0, \quad C_N^A \mathbf{d} - g^A \leq 0, \quad P_N^A (C_N^A \mathbf{d} - g^A) = 0. \quad (4)$$

The state of our system is defined by the solution of (3) and (4), which obviously are the Karush-Kuhn-Tucker conditions for

$$\begin{aligned} & \min_{\mathbf{d}} \Pi(\mathbf{d}) \\ & \text{s.t. } \mathbf{C}_N \mathbf{d} \leq \mathbf{g}, \end{aligned} \quad (5)$$

where

$$\Pi(\mathbf{d}) = \frac{1}{2} \mathbf{d}^T \mathbf{K} \mathbf{d} - \mathbf{F}^T \mathbf{d} \quad (6)$$

is the potential energy of our system.

5. The optimization problem

We are interested in minimizing the compliance

$$c = c(\mathbf{d}) = \mathbf{F}^T \mathbf{d} \quad (7)$$

for the system defined in the previous section. The design parametrization is made by using the SIMP-model. Thus, the stiffness matrix $\mathbf{K} = \mathbf{K}(\boldsymbol{\rho})$ is generated by the following assembly procedure:

$$\mathbf{K}(\boldsymbol{\rho}) = \prod_e \rho_e^n \mathbf{k}_e, \quad (8)$$

where \prod is an assembly operator, $\boldsymbol{\rho}$ contains density variables $\epsilon \leq \rho_e \leq 1$ for each element e . Here ϵ is a small number that is set to 0.001 in the calculations. When $\rho_e = 1$ there is material and $\rho_e = \epsilon$ represents no material. \mathbf{k}_e is the stiffness matrix for element e . In order to obtain a good "0-1"-design experience has shown $n = 3$ to be a good choice. The total volume of the design $V = V(\boldsymbol{\rho})$ is obtained as

$$V = \sum_e \rho_e V_e, \quad (9)$$

where V_e represent the volume of element e . This total volume is constrained by

$$V(\boldsymbol{\rho}) - V_0 \leq 0, \quad (10)$$

where V_0 is the amount of material that can be distributed over the design domain.

Summarized, the following optimization problem is considered:

$$\begin{aligned} & \min_{(\boldsymbol{\rho}, \mathbf{d})} \mathbf{F}^T \mathbf{d} \\ & \text{s.t. } \begin{cases} \min_{\mathbf{d}} \frac{1}{2} \mathbf{d}^T \mathbf{K}(\boldsymbol{\rho}) \mathbf{d} - \mathbf{F}^T \mathbf{d} \\ \text{s.t. } \mathbf{C}_N \mathbf{d} \leq \mathbf{g} \\ V(\boldsymbol{\rho}) - V_0 \leq 0 \\ \epsilon \leq \boldsymbol{\rho} \leq \mathbf{1}. \end{cases} \end{aligned} \quad (11)$$

6. The numerical treatment

The optimization problem in (11) is solved by SLP. This is done by using a nested approach, so the problem is solved in the density variable. The contact conditions in (4) are treated by introducing a smoothed approximation. At the solution defining the state, the cost function and the state equations are linearized. Finally, the derived LP-problem is solved by an interior point method. More details about the numerical approach are presented below.

Signorini's contact conditions are replaced by the following smooth approximation:

$$\Phi^A = \Phi(P_N^A, r(g^A - C_N^A \mathbf{d})) = 0, \quad (12)$$

where $r > 0$ is any constant,

$$\Phi = \Phi(\xi, \eta) = \xi + \eta - \sqrt{(\xi - \eta)^2 + 4\mu^2} \quad (13)$$

and μ is small value which is set to be constant or is decreasing during the optimization. $\Phi = 0$ is equivalent to

$$\xi \geq 0, \quad \eta \geq 0, \quad \xi\eta = \mu^2. \quad (14)$$

Thus, by setting μ sufficiently small a proper approximation of (4) is obtained. We have also found that a proper value of r is important for the performance. This is a modification of the smoothing procedure which was used in Hilding [2]. All Φ^A is collected in the column vector $\Phi = \Phi(\mathbf{x})$, where $\mathbf{x} = (\mathbf{d}, \mathbf{P}_N)$. In conclusion, the state is obtained by solving

$$\mathbf{H} = \mathbf{H}(\boldsymbol{\rho}, \mathbf{x}) = \left\{ \begin{array}{c} \mathbf{K}(\boldsymbol{\rho})\mathbf{d} + \mathbf{C}_N^T \mathbf{P}_N - \mathbf{F} \\ \Phi(\mathbf{x}) \end{array} \right\} = \mathbf{0}. \quad (15)$$

By utilizing (15) we rewrite (11) as

$$\begin{aligned} & \min_{(\boldsymbol{\rho}, \mathbf{x})} \mathbf{F}^T \mathbf{d} \\ & \text{s.t.} \left\{ \begin{array}{l} \mathbf{H}(\boldsymbol{\rho}, \mathbf{x}) = \mathbf{0} \\ V(\boldsymbol{\rho}) - V_0 \leq 0 \\ \mathbf{0} \leq \boldsymbol{\rho} \leq \mathbf{1}. \end{array} \right. \end{aligned} \quad (16)$$

For a given density distribution $\boldsymbol{\rho} = \hat{\boldsymbol{\rho}}$, the state problem in (15) is solved by a Newton algorithm with an Armijo line-search. The search direction is given by

$$\mathbf{z} = -(\nabla_x \mathbf{H})^{-1} \mathbf{H}(\hat{\boldsymbol{\rho}}, \mathbf{x}), \quad (17)$$

where $\nabla_x \mathbf{H} = \nabla_x \mathbf{H}(\hat{\boldsymbol{\rho}}, \mathbf{x})$. In particular,

$$\begin{aligned} \frac{\partial \Phi}{\partial \xi} &= 1 - \frac{(\xi - \eta)}{\sqrt{(\xi - \eta)^2 + 4\mu^2}}, \\ \frac{\partial \Phi}{\partial \eta} &= 1 + \frac{(\xi - \eta)}{\sqrt{(\xi - \eta)^2 + 4\mu^2}}. \end{aligned} \quad (18)$$

At a state $(\hat{\boldsymbol{\rho}}, \hat{\mathbf{x}})$, where $(\hat{\mathbf{x}})$ is the solution from the Newton algorithm, we make a sensitivity analysis of (16). By using a nested approach, we consider \mathbf{x} to be a function of $\boldsymbol{\rho}$, i.e. $\mathbf{x} = \mathbf{x}(\boldsymbol{\rho})$. Sensitivity analysis of (15) yields

$$\nabla_x \mathbf{H} \frac{\partial \mathbf{x}}{\partial \rho_e} = -\frac{\partial \mathbf{H}}{\partial \rho_e}. \quad (19)$$

The compliance can be represented by

$$\mathbf{c} = \mathbf{R}^T \mathbf{x}, \quad (20)$$

where $\mathbf{R} = (\mathbf{F}; \mathbf{0})$. The sensitivity of the compliance is then found by first solving the following adjoint equation:

$$\nabla_x \mathbf{H}^T \boldsymbol{\Gamma} = \mathbf{R} \quad (21)$$

for the vector $\boldsymbol{\Gamma}$. Together with (19), this yields

$$s_e = \frac{\partial \mathbf{c}}{\partial \rho_e} = \boldsymbol{\Gamma}^T \nabla_x \mathbf{H} \frac{\partial \mathbf{x}}{\partial \rho_e} = -\boldsymbol{\Gamma}^T \frac{\partial \mathbf{H}}{\partial \rho_e}. \quad (22)$$

The sensitivity s_e is treated by Sigmund's filter in order to avoid mesh dependency. This procedure reads

$$\hat{s}_e = \sum_{f=1}^{n_{el}} \delta_f \rho_f s_f \left/ \rho_e \sum_{f=1}^{n_{el}} \delta_f \right., \quad (23)$$

where

$$\delta_f = (r_{\min} - \text{dist}(e, f))_+. \quad (24)$$

Here $\text{dist}(e, f)$ denotes the distance between the centers of element e and f , and r_{\min} is a parameter. Finally, by putting (22) into the linearization of the nested formulation of (16), we obtain the following LP-problem at a state $(\hat{\mathbf{x}})$:

$$\begin{aligned} & \min_{\boldsymbol{\rho}} \hat{s}_e \rho_e \\ & \text{s.t.} \left\{ \begin{array}{l} V(\boldsymbol{\rho}) - V_0 \leq 0 \\ \hat{\boldsymbol{\rho}} + \boldsymbol{\rho}_l \leq \boldsymbol{\rho} \leq \hat{\boldsymbol{\rho}} + \boldsymbol{\rho}_u, \end{array} \right. \end{aligned} \quad (25)$$

where ρ_l and ρ_u define lower and upper moving limits, respectively. In this work we let $\rho_{le} = -0.025$ and $\rho_{ue} = 0.025$. Furthermore, we also check that the global limits are satisfied. For instance, if $\hat{\rho}_e + 0.025 > 1$, then $\rho_{ue} = 1 - \hat{\rho}_e$ instead of $\rho_{ue} = 0.025$.

The problem in (25) is solved by the interior point method that is available in Matlab. The optimal solution to the problem in (25) defines a new design point $\hat{\rho}$ around which we derive a new LP-problem by following the procedure above. In such manner a sequence of LP-problems are generated and the sequence continues until a solution of (25) is also believed to solve the problem in (11).

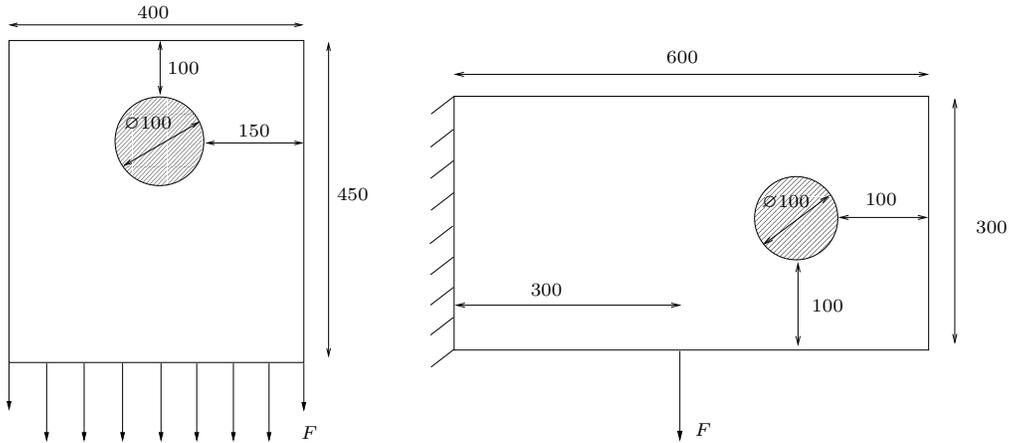


Figure 2: *The design domain, loads and boundary conditions for two numerical examples.*

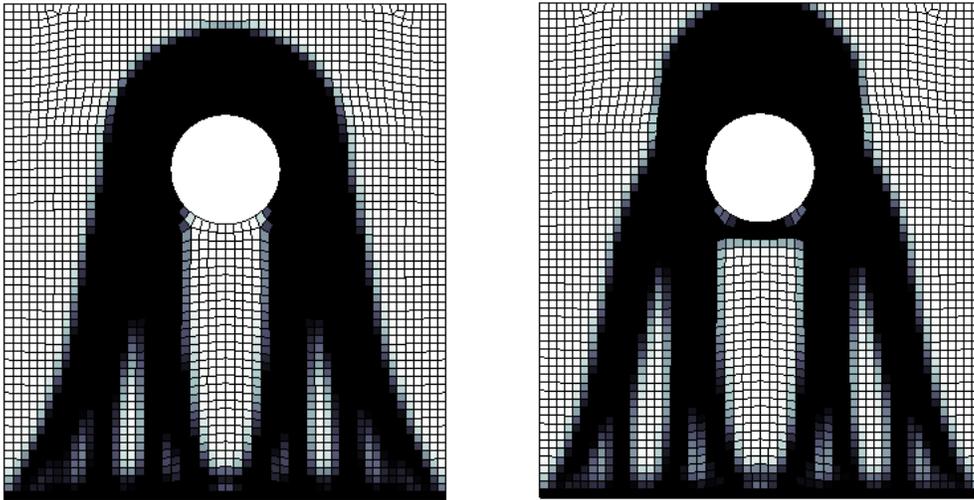


Figure 3: *Different optimal designs for different initial contact gaps: the initial gap is zero for the design to the left and $g = 0.01$ for the result to the right.*

7. Numerical examples

The method presented above is implemented in Topo2D which is a toolbox developed by using Matlab and Fortran. The method is most efficient and robust. This is demonstrated here by presenting the solutions for a lug connected to a rigid pin and a clamped beam connected to a rigid pin as shown in Figure 2. The figure shows the design domain, loads and boundary conditions of the examples. All units are SI-units. The geometry is discretized by using fully integrated isoparametric elements, Young's modulus is $2.1E11$ and Poisson's ratio is 0.3. The number of elements for the lug is 2956 and r_{\min} is set to 12. The beam is discretized by using 3011 elements and $r_{\min} = 12$. For both problems it is assumed that 50 percent of the design domain can be filled by material. The lug is subjected to a line load at the bottom corresponding to a force $F = 54E6$ and for the beam three load cases are considered: first

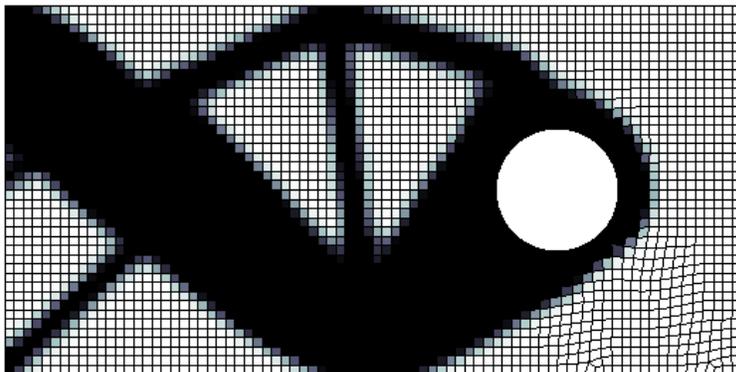
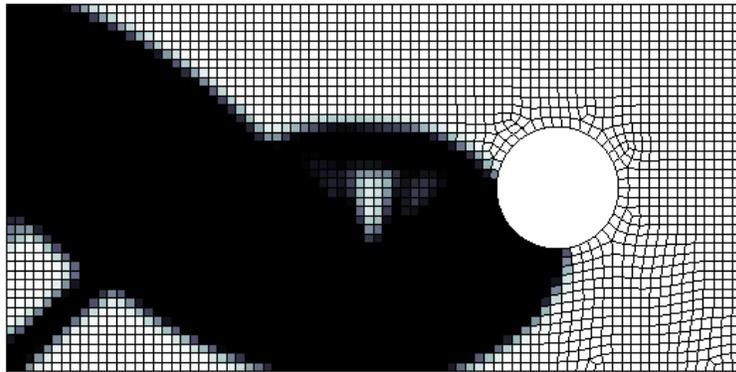
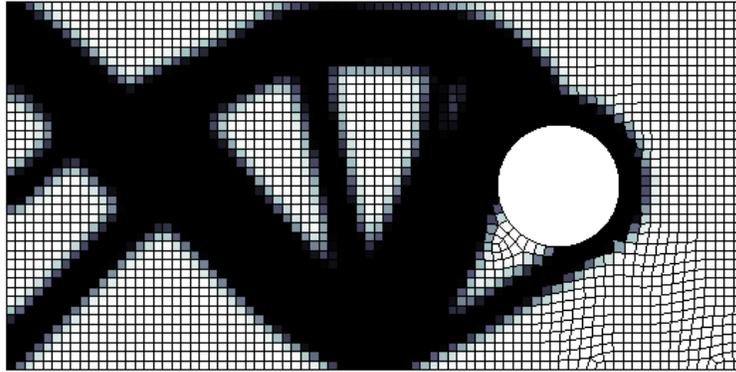


Figure 4: *Different optimal designs for different load cases.*

$F = F_0 = 100E6$ is applied, secondly $F = -F_0$ and, finally the weighted compliance of these two load cases is optimized (equal weights).

The optimal designs are presented in Figure 3 and Figure 4. The first figure shows how the initial gap influence the optimal design. It should be noted that the symmetry is not utilized in order to check the method. The next figure shows how the load influence the optimal design. It is obvious that the optimal design is very sensitive to the sign of the load. This is of course explained by unilateral character of the state equations. If the contact constraints are switched to bilateral constraints, then all these load cases will of course yield the same solution. The solutions are obtained by using a laptop with 2.00 GHz Intel dual core processor and 1.96 GB of RAM. The CPU-time for the single load problems is approximately 25 s for each problem (0.84 s/iteration). However, for the weighted compliance problem, the CPU-time is almost twice that time depending on more Newton iterations as well as more SLP iterations. In Figure 5 numerical convergence in compliance for the weighted compliance problem is shown.

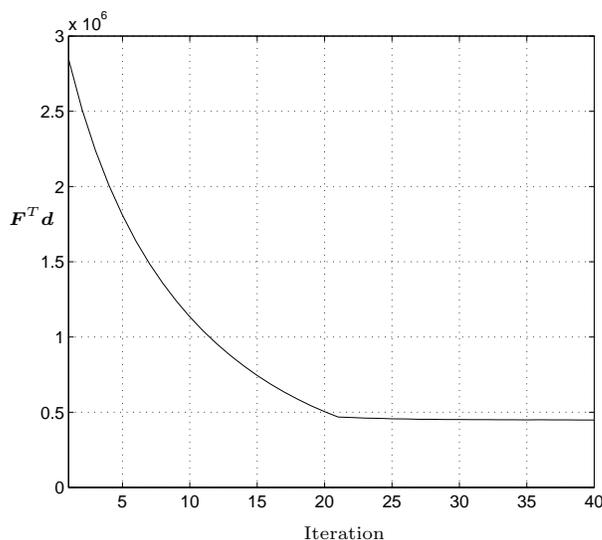


Figure 5: *Convergence in compliance for the weighted compliance problem.*

8. Concluding remarks

In this paper a method for topology optimization of structures in unilateral contact with rigid supports is presented. The method is developed for a linear elastic structure in unilateral contact with rigid supports. The contact is frictionless and it is formulated by using a smooth approximation of Signorini's contact conditions. For this structure the compliance is minimized. The design parametrization is performed by utilizing the SIMP-model. The state problem is solved by a Newton method. The optimization problem is treated by SLP where the LP-problem is solved by an interior point method. Sigmund's filter is also utilized in this procedure in order to avoid mesh dependency and patterns of checker-boards. The method is implemented in Topo2D and it is most efficient and robust. The convergence is very smooth and never show the noisy behavior which sometimes is present in the non-smooth augmented Lagrangian approach. In a forthcoming paper [11], this smooth approach is compared to the augmented Lagrangian approach when applied to topology optimization.

9. References

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