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Efficient detection of second-degree variations in 2D and 3D images

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Abstract

Estimation of local second-degree variation should be a natural first step in computerized image analysis, just as it seems to be in human vision. A prevailing obstacle is that the second derivatives entangle the three features signal strength (i.e. magnitude or energy), orientation and shape. To disentangle these features we propose a technique where the orientation of an arbitrary pattern f is identified with the rotation required to align the pattern with its prototype p . This is more strictly formulated as solving the derotating equation. The set of all possible prototypes spans the shape-space of second degree variation. This space is one-dimensional for 2D-images, two-dimensional for 3D-images. The derotation decreases the original dimensionality of the response vector from three to two in the 2D-case and from six to three in the 3D-case, in both cases leaving room only for magnitude and shape in the prototype. The solution to the derotation and a full understanding of the result requires i) mapping the derivatives of the pattern f onto the orthonormal basis of spherical harmonics, and ii) identifying the eigenvalues of the Hessian with the derivatives of the prototype p . But once the shape-space is established the possibilities to put together independent discriminators for magnitude, orientation, and shape are easy and almost limitless.

Keywords: Second derivatives, spherical harmonics, rotation invariance, line detection, 3D-volume analysis, Hessian, eigenvalues, shape, derotation.

1. Introduction

1.1 Why second derivatives?

In this introduction we want convey two things. We want to underpin our general conviction that derivatives, especially the second derivatives, are of fundamental importance to image analysis. We also want to outline a general model for local feature detection, which utilizes these derivatives.

Commonly, the first steps in image analysis consist of a few neighborhood operations with the intention to detect, or rather to enhance some local features in the presence of noise and other unwanted features. And since established descriptions of local behavior of functions employ derivatives (cf. the Taylor expansion), nothing could be more natural than to estimate the derivatives in these first steps. Traditionally, only the gradient, i.e. the first derivatives, is computed. Edge detection is the first chapter in image processing textbooks. However, we find it just as natural and necessary to estimate the second derivatives. At any point, like any other function, the variation of the image has one odd and one even component. The gradient picks up much of the odd part of this variation, but is blind to the even components of the signal. And in most applications these even components can be captured quite well by the second derivatives.

We can think of two reasons why the 2nd derivatives have been shunned in the past. One is their reputation of being very noise sensitive. The noise sensitivity is easily curbed and controlled, however, by using 2nd derivative estimating convolution kernels (derivators) in the form of differentiated Gaussians. The other reason why 2nd derivatives may have been avoided is that it has not always been understood how to retrieve their embedded directional information. Like the 1st derivatives, the 2nd derivatives carry information on magnitude and orientation in the neighborhood of a certain point. However, in addition, there is one degree of freedom representing shape in the 2D second derivatives, in the 3D-case there are two. Unfortunately, this shape information is entangled with the orientation information. In 3D-functions there are no less than six second-derivatives, and to disentangle these six primary measurements into magnitude, orientation and shape is certainly not trivial.

Having advocated for the employment of second degree derivatives, why should we stop there? Are the third degree variations less interesting? Yes, we believe so. Including second derivatives has the huge benefit of capturing most of the otherwise missing even signal component. Compared to this, the contributions from the third derivatives should be small. Also, it might be worth noticing that many, if not all, physical processes seem to be driven by partial differential equations of second order. Therefore, it seems likely that local second degree variations will dominate the stochastic processes in Nature that produce the signals we call images.

The following observation supports this conjecture. Assume that we initially have severely defocused (blurred) a given 2D-image so that nothing is left but the average gray-level of the original. The effect is the same as if we were very near-sighted and looking at the image from afar. Then we sharpen the image again (moving closer to the image), but very slowly and gradually. What do the first observable features look like? Do we see borders between different areas, which segment the image into different parts? Of course we don't. Instead, standing out from the background, we see faint border-less objects in the form of blobs, more or less elongated. The reason we perceive these structures is NOT that we see edges with opposite gradient directions running parallel to each other at a close distance. Instead we see these objects directly with the help of operators in our own vision system when these operators hit the line or blob right on, i.e. when these objects are seen as even functions. Scale-space behavior of blobs have been investigated by Lindeberg [37], [38].

Koenderink and van Doorn [31] observe that the first order variations are highly dependent on "intereospecific factors", i.e. on illumination and camera design rather than the physical environment we intend to analyze. They also note that the human visual system seems to discard the first order variation at a very early stage. Our conclusion is that edge detectors, which traditionally have been considered to be the most basic tools in low level image analysis, are not natural at all. The basic components of an image are not edges, but events that appear locally as an even function when they are encountered head-on. Therefore, it is more important to apply low and high-resolution convolution kernels that are able to detect blobs, and probably more importantly, lines. The cross-sections of these operators are likely to have the shape of a Mexican hat, which means second derivatives of Gaussians.

1.2 Orientation. The compass and the phase methods

The quest for orientation distinguishes multi-dimensional from one-dimensional signal processing. Local orientation can be estimated by two principally different methods: The compass method and the phase method. The **compass method** is a genuine matched filter technique, where the matching kernel is placed over the point of interest and its neighborhood in a number of rotated versions. The response, i.e. the correlation value is recorded for all these rotation angles and the embedded feature is assumed to have an orientation, which is somewhere between the two orientations where the matching filter gave the two largest responses. Interpolation may be used to increase the precision in this estimation. The matching pattern can be made highly discriminative and specific so that only a certain type (shape) of pattern is detected. However, the more specific the pattern is that we want to discriminate for, the larger is the number of rotated versions that have to be employed. Estimation of specific shape and precise orientation becomes rather expensive.

In general, the **phase** measurement technique is less expensive than the compass method. In the 2D-case the orientation angle of the pattern is retrieved from a pair of responses from two convolution kernels, say (b_{nx}, b_{ny}) which should be **rotation-invariant**. A pair of operators (b_{nx}, b_{ny}) is rotation invariant if for any pattern f the detected local response energy

$$f_{nx}^2 + f_{ny}^2 \equiv (f \otimes b_{nx})^2 + (f \otimes b_{ny})^2 \quad (1)$$

is invariant to rotations of f (convolution is denoted by $\otimes$). It can be shown [8], [36] that in two dimensions all rotation-invariant operator pairs have the form

$$\begin{aligned} b_{nx} &= h_n(r) \cos n(\phi + \phi_n(r)) \\ b_{ny} &= h_n(r) \sin n(\phi + \phi_n(r)) \end{aligned} \quad (2)$$

where $h_n(r)$ and $\phi_n(r)$ are one-dimensional functions for $0 \leq r < \infty$. Under these circumstances the orientation angle β_n of the pattern f is retrieved as

$$\beta_n = \frac{1}{n} \arg(f_{nx}, f_{ny}) \quad (3)$$

In principal the radial variation $h_n(r)$ is free to use for application dependent purposes. Figure 1 shows symbolic icons for two pairs of rotation-invariant operators of second order. Lines and curves indicate zero-crossings. The pair to the left has constant $\phi_n(r) = 0$, while the pair to the right is of a more general type with a phase angle $\phi_n(r)$, which varies with the radius r .

Figure 1. Two rotation invariant operator pairs. The general type (right) is not used in this paper

In practice we have to use operators which are not perfectly rotation invariant. Imperfect rotation invariance creates systematic errors in orientation estimation. For an investigation in these matters, see [9]. For most applications, the common Sobel operators are sufficiently rotation-invariant.

It was shown by Hummel [26] and Zucker [50] that rotation-invariant operators (2) with $\phi_n(r) = 0$ are optimal in the following sense for detection of step edges. Assume that the orientations of step edge patterns appear with isotropic probability. For a given number N , the set of $N/2$ rotation-invariant operator pairs are able to approximate the step function better than any other set of N basis functions. Clearly, since a step edge is an odd function, only basis sets of odd order should be included. This result was generalized by Lenz [34] to encompass all patterns, which are either odd or even. The actual pattern prototype may be used to optimize the radial function $h_n(r)$ for best match, a design problem that will not be dealt with in this treatise.

1.3 The derotation equation

Our general model for interpretation, or rather, translation of a local neighborhood response is given by the **derotation equation** (4).

$$A f \otimes \mathbf{g} \equiv A \mathbf{f}_g = f \otimes A \mathbf{g} = f \otimes \mathbf{b} \equiv \mathbf{f}_b = \|\mathbf{f}_b\| R(\boldsymbol{\beta}) \mathbf{p}(\boldsymbol{\kappa}) \quad (4)$$

Here, on the far left the image f is convolved with a vector of convolution kernels (operators, filters, basis functions) $\mathbf{g}$ to obtain a response vector $\mathbf{f}_g$, which is an ordered set of images. See Figure 2. Two operators

(g_1, g_2) are said to be orthogonal if their inner product $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (g_1 \cdot g_2)(x, y) dx dy = 0$. In this paper, these

operators are derivative estimators, **derivators**, and in general, these are not all mutually orthogonal. The matrix A produces a linear combination at each point of these images so that we obtain the equivalent responses $\mathbf{f}_b$ from an orthonormal set of operators $\mathbf{b}$, which have harmonic variation in the angular direction. Clearly, from now on it does not matter if we regard (4) as an equation for the set of all points and neighborhoods in the input image or an equation for the responses from a single local neighborhood. See Figure 2.

Figure 2. The derivative responses seen as a vector of images **and** as a response vector at each sample point of the original image

One might ask why we do not use (the hitherto undefined) orthonormal set $\mathbf{b}$ of operators to begin with, which would eliminate the mapping operation A . The reason is speed. The derivators are perfectly separable and therefore very computationally efficient. Thus, to get the harmonic responses $\mathbf{f}_b$ as linear combinations of the derivatives is actually not a detour but a short-cut. In any case, given these response vectors we want to extract **magnitude** and **orientation** for each point and its associated neighborhood in the image. These quantities are found to the right in (4). Here, the magnitude $\|\mathbf{f}_b\|$ is the L_2 -norm of $\mathbf{f}_b$, $R(\boldsymbol{\beta})$ is a rotation operator representing orientation, and $\mathbf{p}(\boldsymbol{\kappa})$ is a **prototype** pattern, a normalized linear combination of the basis functions, a unit vector in the function space $\mathbf{b}$. In the sequel, $R(\boldsymbol{\beta})$ will be defined by one, two or three

parameters, $\mathbf{p}(\boldsymbol{\kappa})$ by zero, one or two parameters. The rotation parameters $\boldsymbol{\beta}$ are of course angles, and, as we will see, the parameters in the vector $\boldsymbol{\kappa}$ are also most conveniently seen as angles, representing **shape** rather than rotation and orientation.

In some cases we might use the following variation of the derotation equation (4).

$$\mathbf{f}_b = R(\boldsymbol{\beta}) \mathbf{p} \quad (5)$$

Here, the prototype response vector $\mathbf{p}$ is still a vector in the basis function space $\mathbf{b}$ but it is no longer normalized but equal in magnitude to the response vector $\mathbf{f}_b$. Therefore, this vector $\mathbf{p}$ has one more degree of freedom than the vector $\mathbf{p}(\boldsymbol{\kappa})$ in (4).

Clearly, the orientation of $\mathbf{f}_b$ embedded in $R(\boldsymbol{\beta})$ will be given relative to the predefined prototype orientation, which is to say that we interpret the response vector of the local pattern f as a rotated and amplified version of a given prototype response. We may also rewrite (4) as

$$R^{-1}(\boldsymbol{\beta}) \|\mathbf{f}_b\|^{-1} \mathbf{f}_b = \mathbf{p}(\boldsymbol{\kappa}) \quad (6)$$

which more appropriately would deserve the name derotation (and demagnification) of the feature response to yield the prototype. However, since we are going to use (4) extensively rather than (5) or (6), we reserve the name derotation equation for (4).

In our first derotation example the convolution basis set consists of the three first derivative estimators $\mathbf{g}_1^T = (g_x, g_y, g_z)$, which are convolved with a 3D-signal $f(x, y, z)$. Since these basis functions are orthonormal, the mapping A then reduces to a unity matrix that can be discarded from the derotation equation. As our prototype we choose $\mathbf{p}_b^T = (0, 0, 1)$, a gradient of unit magnitude oriented exactly in the positive z-direction. This prototype has zero degree of freedom, meaning that there is no shape parameter $\boldsymbol{\kappa}$. A rotator $R(\beta, \gamma)$ for the 3D-gradient can be defined by two angles (β, γ) , where β (β is now a single angle) rotates the gradient vector around the y-axis followed by rotation with γ around the z-axis. Since this is a first derivative response we call the vector $\mathbf{f}_1$. Eq. (4) then takes the form

$$\mathbf{f}_1 = f \otimes \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} = \|\mathbf{f}_1\| \begin{bmatrix} \cos \beta \cos \gamma & -\sin \gamma & \sin \beta \cos \gamma \\ \cos \beta \sin \gamma & \cos \gamma & \sin \beta \sin \gamma \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (7)$$

which yields

$$\begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix} = \|\mathbf{f}_1\| \begin{bmatrix} \sin \beta \cos \gamma \\ \sin \beta \sin \gamma \\ \cos \beta \end{bmatrix} \quad (8)$$

As in most of the following derotation cases, the equation system (7)-(8) is neither under- nor over-determined having the same degree of freedom on both sides. The three unknowns, the magnification $\|\mathbf{f}_1\|$, and the two parameters (β, γ) are obtained as follows. Instead of solving for $\|\mathbf{f}_1\|$, we may just as well solve for $\|\mathbf{f}_1\|^2$.

$$\begin{aligned}
\|f_1\|^2 &= f_x^2 + f_y^2 + f_z^2 \\
\gamma &= \arg(f_x, f_y) & -\pi < \gamma \leq \pi \\
\beta &= \arctan \frac{f_x \cos \gamma + f_y \sin \gamma}{f_z} & -\frac{\pi}{2} < \beta \leq \frac{\pi}{2}
\end{aligned} \tag{9}$$

1.4 How this paper is organized

The main body of the paper is organized as follows. In section 2 we develop a line detection technique for 2D-images based on second derivatives and circular harmonic operators, the latter being mandatory to obtain orthogonality and rotation invariance. We develop energy measures, as well as shape sensitive factors and combinations of the basic responses and we apply these things to image enhancement of fingerprints. The latter was published in [11] but for the sake of completeness and for understanding the more complex 3D-case, we feel that both theory and application of line detection technique is a necessary prelude.

Section 3 gives the theoretical background and the basic equations for detecting second degree structures in 3D-volumes. In principle, the theory and the technique are generalizations of the 2D-case where the circular harmonic basis functions are replaced by basis functions with spherical harmonic variation. But because the number of 3D-second-derivatives are now six instead of three, and because 3D-orientation-rotation is defined with three degrees of freedom instead of one, the complexity of the problem increases considerably. Concepts and algorithms, which might be understood and developed intuitively and implicitly in 2D, have to be tackled with more care in the 3D-case. In the 2D-case, given the response in an orthonormal feature space, we are able to solve the derotation equation with rather heuristic methods. Converting the six-component response vector to magnitude, shape, and orientation in the 3D-case calls for a more systematic approach. In the general case we have to employ diagonalization of the Hessian to solve the derotation problem.

The eigen values of the Hessian can be identified with the prototype derivatives. However, since there are six possible permutations of the eigen values the diagonalization makes room for six different permutations. Rules for selecting one of these is necessary to establish a uniquely defined position for each prototypic shape. We then develop discriminative shape factors, which in section 4 we are employed in an algorithm for detection of the curvilinear 3D-shapes we call strings. The validity of the algorithm is demonstrated by an application where we enhance the blood vessels in a 3D-volume obtained from angiographic magnetic resonance imaging.

References to earlier work are mostly deferred to section 5. Since the content of this paper concerns rather fundamental concepts in image analysis, an exhaustive list of references is almost unfeasible. We apologize to the undoubtedly numerous authors who rightfully should have been referred to. For the interested reader we also want to refer to the report [12] and the master thesis [36]. Our own interest in these matters goes back in time more than twenty years [8].

2. Line detection in two-dimensional images

2.1 The circular harmonic orthonormal operator set. The shape space.

Assume that $h_0(r)$ is a rotationally symmetric and differentiable function. Typically, but not necessarily, this function is a 2D Gaussian,

$$h_0(r) = e^{-r^2} = e^{-(x^2+y^2)} \quad (10)$$

or a variation thereof. The choice of function in (10) determines the exact profile matching characteristics of the final line detection procedure, but in practice, this choice is not critical as long as the procedure is embedded in a scale-space. With the help of $h_0(r)$ we construct three second degree **derivators** (= derivative operators or estimators) as follows. Three Fourier domain representations are given to the right of the $\Leftrightarrow$ sign.

$$\begin{aligned} g_{xx} &= \frac{\partial^2}{\partial x^2} h_0(r) \Leftrightarrow G_{xx} = -4\pi^2 u^2 H_0(\rho) = -4\pi^2 \rho^2 \cos^2 \phi H_0(\rho) \equiv H_2(\rho) \cos^2 \phi \\ g_{yy} &= \frac{\partial^2}{\partial y^2} h_0(r) \Leftrightarrow G_{yy} = -4\pi^2 v^2 H_0(\rho) = -4\pi^2 \rho^2 \sin^2 \phi H_0(\rho) \equiv H_2(\rho) \sin^2 \phi \\ g_{xy} &= \frac{\partial^2}{\partial x \partial y} h_0(r) \Leftrightarrow G_{xy} = -4\pi^2 uv H_0(\rho) = -4\pi^2 \rho^2 \cos \phi \sin \phi H_0(\rho) \equiv H_2(\rho) \cos \phi \sin \phi \end{aligned} \quad (11)$$

Because of the theorem below, the angular variation in the variable ϕ is the same in both domains, while the radial variations correspond over the Hankel transform. The perfectly rotationally symmetric Gaussian and the perfect derivation are ideals that have to be compromised in practice. In our implementations we have used generalized Sobel operators as reasonable approximations of ideal first derivators [10]. Second derivators are then obtained by convolving two first derivators.

We observe in (11) that although these three operators measure independent features when applied as convolution kernels, they are not fully orthogonal. Instead, an orthonormal basis functions consisting of one zero order and two second order circular harmonics (B_{20}, B_{21}, B_{22}) are obtained with the following linear combinations of (G_{xx}, G_{yy}, G_{xy}) , which defines the mapping A_2 employed later in (17) and (20).

$$\begin{aligned} \mathbf{B}_2 &= \begin{bmatrix} B_{20} \\ B_{21} \\ B_{22} \end{bmatrix} = A_2 \begin{bmatrix} G_{xx} \\ G_{yy} \\ G_{xy} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{8}{3}} \end{bmatrix} \begin{bmatrix} G_{xx} \\ G_{yy} \\ G_{xy} \end{bmatrix} = H_2(\rho) \begin{bmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \cos 2\phi \\ \sqrt{\frac{2}{3}} \sin 2\phi \end{bmatrix} = \frac{H_2(\rho)}{\rho^2} \begin{bmatrix} \sqrt{\frac{1}{3}}(u^2 + v^2) \\ \sqrt{\frac{2}{3}}(u^2 - v^2) \\ \sqrt{\frac{8}{3}}uv \end{bmatrix} \Leftrightarrow \\ \mathbf{b}_2 &= \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} h_{20}(r)\sqrt{\frac{1}{3}} \\ h_2(r)\sqrt{\frac{2}{3}} \cos 2\phi \\ h_2(r)\sqrt{\frac{2}{3}} \sin 2\phi \end{bmatrix} = A_2 \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{xy} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{3}} & \sqrt{\frac{1}{3}} & 0 \\ \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & \sqrt{\frac{8}{3}} \end{bmatrix} \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{xy} \end{bmatrix} = \frac{1}{r^2} \begin{bmatrix} h_{20}(r)\sqrt{\frac{1}{3}}(x^2 + y^2) \\ h_2(r)\sqrt{\frac{2}{3}}(x^2 - y^2) \\ h_2(r)\sqrt{\frac{8}{3}}xy \end{bmatrix} \quad (12) \end{aligned}$$

Note that we establish these linear combinations first in the Fourier domain where the radial function $H_2(\rho)$ is common to (G_{xx}, G_{yy}, G_{xy}) . The corresponding orthonormal basis functions (b_{20}, b_{21}, b_{22}) in the signal domain follow from the following theorem, the proof of which is given in *Appendix A*.

Theorem. *If a function $B(u, v)$ is separable into radial variation $H(\rho)$ and angular variation $a(\phi)$ and the angular variation is harmonic, then the Fourier transform preserves the angular variation while the radial functions in the two domains are Hankel transforms of each other.*

The theorem implies the existence of Fourier pairs of the type

$$b(x, y) = h(r) a(\phi) \Leftrightarrow B(u, v) = H(\rho) a(\phi) \quad (13)$$

which was applied in (12). The zero order Hankel transform is used to produce the Laplacian operator $b_{20} = h_{20}(r)$, while the second order Hankel transform produces (b_{21}, b_{22}) with the radial variation $h_2(r) \neq h_{20}(r)$. In the special case that $h_0(r)$ is the Gaussian (10) we may see how the two functions $h_{20}(r)$, $h_2(r)$ are related to $h_0(r)$. We use the following two expressions for g_{xx}

$$\begin{aligned} g_{xx} &= \frac{1}{2} \left(\sqrt{3} b_{20} + \sqrt{\frac{3}{2}} b_{21} \right) = \frac{1}{2} (h_{20}(r) + h_2(r) \cos 2\phi) \\ g_{xx} &= \frac{\partial^2}{\partial x^2} h_0(r) = 2(-1 + 2x^2) h_0(r) = 2(r^2 - 1 + r^2 \cos 2\phi) h_0(r) \end{aligned} \quad (14)$$

from which we get

$$\begin{aligned} h_{20}(r) &= 4(r^2 - 1) h_0(r) \\ h_2(r) &= 4r^2 h_0(r) \end{aligned} \quad (15)$$

The mapping (12) is designed to yield the following normalizing and orthogonality conditions.

$$\begin{aligned} \int_0^{2\pi} [B_i(\rho, \phi)]^2 d\phi &= \frac{2\pi}{3} [H_2(\rho)]^2 \quad \text{for } i = 20, 21, 22 \\ \sum B_i^2 &= 2\pi H_2^2 \\ \int_0^{2\pi} B_i B_j(\rho, \phi) d\phi &= 0 \quad \text{for } i \neq j \end{aligned} \quad (16)$$

Let us make a minor digression at this point. We note in (11) that in the Fourier domain the derivatives are represented by polynomials, i.e. second degree **moment functions**. Suppose we had decided to use moment functions to begin with as basis functions in the in the signal domain, which then could be mapped into orthogonal functions with harmonic angular variation. These functions appear as derivators in the Fourier domain, and using the above theorem, these can be translated into orthogonal basis functions of type circular harmonics the matrix A_2 . And from (12) we note that the basis functions generated from derivators also can be given a polynomial form. Therefore, several results in this treatise should be applicable also for operator sets based on moment functions. Further comments are found in section 5.

The three-dimensional function space, spanned by the three circular harmonics in (12), is shown in Figure 3. The second derivative response f_2 for the image neighborhood $f(x, y)$ at $(x, y) = (0, 0)$ maps into this space as follows. Convolution is denoted as $\otimes$.

$$\mathbf{f}_2 \equiv \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \end{bmatrix} \equiv f \otimes \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} = f \otimes A_2 \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{xy} \end{bmatrix} = \sqrt{\frac{1}{3}} \begin{bmatrix} 1 & 1 & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} f_{xx} \\ f_{yy} \\ f_{xy} \end{bmatrix} \quad (17)$$

Figure 3. The three-dimensional space of second degree variation in two-dimensional images

From (17) we also have the inverse relations

$$\begin{bmatrix} f_{xx} \\ f_{yy} \\ f_{xy} \end{bmatrix} = A_2^{-1} \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \end{bmatrix} = \sqrt{\frac{3}{8}} \begin{bmatrix} \sqrt{2} & 1 & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \end{bmatrix} \quad (18)$$

As mentioned above, the two derivators g_{xx} and g_{yy} in Figure 3 are not orthogonal, but form an angle $\theta \neq \pi/2$. From the analytical expressions of the derivators in (11) and (12) we retrieve θ as

$$\theta = \arccos \frac{\int_0^{2\pi} g_{xx} g_{yy} d\theta}{\left(\int_0^{2\pi} g_{xx}^2 d\theta \right)^{1/2} \left(\int_0^{2\pi} g_{yy}^2 d\theta \right)^{1/2}} = \arccos \frac{1}{3} = 70.53^\circ \quad (19)$$

We are now ready to formulate the derotation equation (20) for the 2D second derivative case.

$$\begin{aligned} \mathbf{f}_2 = \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \end{bmatrix} &= f \otimes A_2 \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{xy} \end{bmatrix} = f \otimes \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} = \\ &= \|\mathbf{f}_2\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\beta & -\sin 2\beta \\ 0 & \sin 2\beta & \cos 2\beta \end{bmatrix} \begin{bmatrix} \sin \kappa \\ \cos \kappa \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\beta & -\sin 2\beta \\ 0 & \sin 2\beta & \cos 2\beta \end{bmatrix} \begin{bmatrix} p_{20} \\ p_{21} \\ 0 \end{bmatrix} \end{aligned} \quad (20)$$

The response vector is called $\mathbf{f}_2$, since this is a second order variation. The matrix A_2 was defined in (12). In (20) there is one parameter β representing orientation, and one free parameter κ representing shape. The prototype response is a vector in the $[b_{20}, b_{21}, b_{22}]$ -space. Unlike the situation in (7), magnitude and orientation only cannot describe the three degrees of freedom in the response vector. Even so, the wanted parameters $\|\mathbf{f}_2\|$, β , and κ in (20) are found similarly to $\|\mathbf{f}_1\|$, β , and γ in (8).

$$\begin{aligned} \|\mathbf{f}_2\|^2 &= f_{20}^2 + f_{21}^2 + f_{22}^2 = f_{xx}^2 + f_{yy}^2 - \frac{2}{3} f_{xx} f_{yy} + \frac{8}{3} f_{xy}^2 \\ \beta &= \frac{1}{2} \arg(f_{21}, f_{22}) \\ \kappa &= \arctan \frac{f_{20}}{f_{21} \cos 2\beta + f_{22} \sin 2\beta} = \arctan \frac{f_{20}}{\sqrt{f_{21}^2 + f_{22}^2}} \end{aligned} \quad (21)$$

From (20) and (21) we find the following more explicit expression for the prototype vector.

$$\mathbf{p}_2 \equiv \begin{bmatrix} p_{20} \\ p_{21} \\ p_{22} \end{bmatrix} = \begin{bmatrix} \frac{f_{20}}{\sqrt{f_{21}^2 + f_{22}^2}} \\ 0 \end{bmatrix} = \|\mathbf{f}_2\| \begin{bmatrix} \sin \kappa \\ \cos \kappa \\ 0 \end{bmatrix} \quad (22)$$

The normalized prototype vector is residing on a semi-circle, a meridian, of the unit sphere in Figure 3, which goes from the top to bottom of the sphere while passing b_{21} . This is the **shape space** for second-degree variations and shown separately in Figure 4. Here we find all possible shapes (prototype patterns) as linear combinations of (b_{20}, b_{21}) . An actual pattern f and its prototype p have the very same shape; only the orientation is different. Following the semicircle from top to bottom we encounter the following specific shapes: The positive **blob** b_{20} , the **derivator** g_{xx} , the optimal **valley detector** q_{valley} , the **saddle** b_{21} , the optimal **ridge detector** q_{ridge} , the **derivator** $-g_{yy}$, and finally the negative **blob** $-b_{20}$. The optimal valley and ridge detectors will be explained shortly.

Figure 4. The shape and prototype space for 2D second degree variation.

2.2 Optimal valley and ridge detectors. Quadratic combinations

We have defined our derotated responses, i.e. the prototypical second degree variations, as those for which the response to b_{22} is zero. From (18) we then get the **prototype derivatives** as

$$\mathbf{p}_g = \begin{bmatrix} p_{xx} \\ p_{yy} \\ p_{xy} \end{bmatrix} = \sqrt{\frac{3}{8}} \begin{bmatrix} \sqrt{2} & 1 & 0 \\ \sqrt{2} & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} p_{20} \\ p_{21} \\ 0 \end{bmatrix} \quad (23)$$

A special prototype response, a linear combination between b_{20} and b_{21} , is the derivator

$$g_{xx} = \sqrt{\frac{9}{8}} \left(\sqrt{\frac{2}{3}} b_{20} + \sqrt{\frac{1}{3}} b_{21} \right) \quad (24)$$

and it would seem that this convolution kernel should be the perfect match for an ideal vertical line of type valley. Assuming identical profiles in the center, the only difference between g_{xx} and the ideal vertically oriented valley is that while g_{xx} tapers off with increasing radius, the ideal valley keeps its profile along the y -axis to infinity. Somewhat counter-intuitively, the **optimal line detectors**, the normalized operator for best matching of such perfectly linear structures as valleys and ridges are instead

$$q_{valley} = \sqrt{\frac{1}{3}}b_{20} + \sqrt{\frac{2}{3}}b_{21} \quad \text{and} \quad q_{ridge} = -\sqrt{\frac{1}{3}}b_{20} + \sqrt{\frac{2}{3}}b_{21} \quad (25)$$

Figure 5. The line response from the two basis functions portrayed in the Fourier domain

The optimality can be shown in several ways, one of which is found in [10]. Here we will use the Fourier domain for an alternative deduction. As illustrated in Figure 5, the ideal vertical line has no variation along the y -axis. Therefore its energy is concentrated along the u -axis in the Fourier domain. Along the u -axis the functions B_{20} and B_{21} have a common variation $H_2(u)$ except for the two normalization factors $\sqrt{\frac{1}{3}}$ and $\sqrt{\frac{2}{3}}$, respectively.

Thus, the response f_{valley} at a central point of the perfect valley is the weighted sum of the two products between the Fourier transform $VALLEY(u)$ and a normalized linear combination the two Fourier transforms $\sqrt{\frac{1}{3}}H_2(u)$ and $\sqrt{\frac{2}{3}}H_2(u)$ respectively. We use sine and cosine factors of the parameter $\mathcal{G}$ to express this linear combination as $[\sin \mathcal{G} B_{20} + \cos \mathcal{G} B_{21}] = \left[\frac{1}{\sqrt{3}} \sin \mathcal{G} + \sqrt{\frac{2}{3}} \cos \mathcal{G} \right] H_2(u)$ so that

$$f_{valley}(\mathcal{G}) = \left[\frac{1}{\sqrt{3}} \sin \mathcal{G} + \sqrt{\frac{2}{3}} \cos \mathcal{G} \right] \int_{-\infty}^{\infty} VALLEY(u) H_2(u) du \quad (26)$$

This response has a maximum and a minimum for $\mathcal{G} = \pm \arctan \frac{1}{\sqrt{2}}$, respectively, which proofs (25) and explains the valley and ridge indications in Figures 3 and 4. Gray-scale versions of the four normalized functions b_{20} , $\frac{\sqrt{8}}{3}g_{xx}$, q_{valley} , and b_{21} are shown in Figure 6. We note that q_{valley} , in spite of its curved zero-crossings, indeed seems to be the best match to a straight line.

Figure 6. The four operators b_{20} , $\frac{\sqrt{8}}{3}g_{xx}$, l_{valley} , and b_{21} . Black <0 , Gray $=0$, White >0 .

The rotation-invariant valley and ridge responses p_{valley} and p_{ridge} expressed in derivatives yield

$$\begin{aligned} p_{\text{valley}} &= \sqrt{\frac{1}{3}}p_{20} + \sqrt{\frac{2}{3}}p_{21} = p_{xx} - \frac{1}{3}p_{yy} \\ p_{\text{ridge}} &= -\sqrt{\frac{1}{3}}p_{20} + \sqrt{\frac{2}{3}}p_{21} = \frac{1}{3}p_{xx} - p_{yy} \end{aligned} \quad (27)$$

Call the normalized responses $q_{\text{valley}} = \|\mathbf{f}_2\|^{-1} p_{\text{valley}}$ and $q_{\text{ridge}} = \|\mathbf{f}_2\|^{-1} p_{\text{ridge}}$, respectively as in (24).

Figure 7 (left) shows how these quantities vary with the shape angle $\kappa = \arctan \frac{p_{20}}{p_{21}}$.

Figure 7. The quantities q_{valley} , q_{ridge} , q_{line} , and q'_{line} as a function of the shape angle κ

The quantities p_{valley} and p_{ridge} are just one of many possibilities to define rotation invariant combinations of the available responses. Any function using the arguments $(p_{20}, p_{21}) = \left(f_{20}, \sqrt{f_{21}^2 + f_{22}^2}\right)$ is of course

rotation invariant, including $\kappa = \arctan \frac{p_{20}}{p_{21}}$. We may define a rotation-invariant and energy independent shape

factor $Q_s(\mathcal{G}, \kappa)$ with the parameter $\mathcal{G}$ and the quadratic expression

$$Q_s(\mathcal{G}, \kappa) = \|\mathbf{p}_2\|^{-2} \left((\cos \mathcal{G})^2 (\sin \kappa)^2 - (\sin \mathcal{G})^2 (\cos \kappa)^2 \right) \quad (28)$$

In fact, for $\kappa = \vartheta$ and $\kappa = \pm\vartheta$ we find $Q_s(\vartheta, \kappa) = 0$. The angle ϑ corresponds to the shape for total response *suppression*. With $\vartheta = \arctan \sqrt{\frac{1}{2}}$, eq. (28) yields

$$Q_s(\vartheta, \kappa) = \|\mathbf{p}_2\|^{-2} \left(\frac{2}{3} (\cos \kappa)^2 - \frac{1}{3} (\sin \kappa)^2 \right) = \|\mathbf{p}_2\|^{-2} \frac{8}{9} (f_{xx}f_{yy} - f_{xy}^2)$$

According to Horn [24] $f_{xx}f_{yy} - f_{xy}^2$ is called *The quadratic variation*. However, from (28) we understand that this is just one of several rotation-invariant but shape-dependent energy measures.

The following energy independent shape factor $Q(\vartheta, \kappa)$ is also quadratic but instead of suppressing the response it attains its maximum and minimum values of ± 1 for $\kappa = \pm\vartheta$.

$$Q(\vartheta, \kappa) = \frac{2 \cos \vartheta \sin \kappa \sin \vartheta \cos \kappa}{(\cos \vartheta \sin \kappa)^2 + (\sin \vartheta \cos \kappa)^2} = \frac{2 \cos \vartheta p_{20} \sin \vartheta p_{21}}{(\cos \vartheta)^2 p_{20}^2 + (\sin \vartheta)^2 p_{21}^2} \quad (29)$$

For line detection we chose $\vartheta = \arctan \sqrt{\frac{1}{2}}$ in (29) which yields the dimensionless line-ness factor

$$q_{line}(\kappa) = \frac{2\sqrt{2} p_{20} p_{21}}{2p_{20}^2 + p_{21}^2} = \frac{2\sqrt{2} f_{20} \sqrt{f_{21}^2 + f_{22}^2}}{2f_{20}^2 + f_{21}^2 + f_{22}^2} = \frac{(f_{xx} + f_{yy}) \sqrt{(f_{xx} - f_{yy})^2 + (2f_{xy})^2}}{f_{xx}^2 + f_{yy}^2 + 2f_{xy}^2} \quad (30)$$

q_{line} attains its maximum +1 for valleys and its minimum -1 for ridges, equals 0 for blobs and saddles as illustrated in Figure 7. It is rather discriminative against shapes that deviate from the ideal line. A slightly simpler liness factor q'_{line} is obtained from (29) as follows with $\vartheta = 45^\circ$. It behaves almost identically to the factor q'_{line} for valleys and ridges.

$$q'_{line}(\kappa) = \frac{2p_{20}p_{21}}{p_{20}^2 + p_{21}^2} = \frac{2f_{20}\sqrt{f_{21}^2 + f_{22}^2}}{f_{20}^2 + f_{21}^2 + f_{22}^2} = \frac{(f_{xx} + f_{yy}) \sqrt{(f_{xx} - f_{yy})^2 + (2f_{xy})^2}}{f_{xx}^2 + f_{yy}^2 - \frac{2}{3} f_{xx}f_{yy} + \frac{8}{3} f_{xy}^2} \quad (31)$$

Using (30) or (31) we may define and compute a **line response vector** $\mathbf{f}_{line}$ for rotated valleys and ridges.

Since (31) leads to somewhat simpler formulas we use q'_{line} to obtain

$$\mathbf{f}_{line}(\kappa, \beta) = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \|\mathbf{f}_2\| q'_{line} \begin{bmatrix} \cos 2\beta \\ \sin 2\beta \end{bmatrix} = \frac{\|\mathbf{f}_2\| q'_{line}}{p_{21}} \begin{bmatrix} f_{21} \\ f_{22} \end{bmatrix} = \frac{2f_{20}}{\|\mathbf{f}_2\|} \begin{bmatrix} f_{21} \\ f_{22} \end{bmatrix} \quad (32)$$

and the signed scalar **line measure** f_{line} as

$$f_{line} = \|\mathbf{f}_2\| q'_{line} = \frac{2p_{20}p_{21}}{\|\mathbf{f}_2\|} = \frac{2f_{20}\sqrt{f_{21}^2 + f_{22}^2}}{\|\mathbf{f}_2\|} \quad (33)$$

It should be noted that a ridge and a valley having the same strength, shape, and orientation give two values for β in (21), which differ by 90° . From (32), on the other hand, we get $\beta = \frac{1}{2} \arg(f_{20}f_{21}, f_{20}f_{22})$, which becomes identical for ridge and valley because of the Laplacian f_{20} . The sign of f_{20} flips the (f_{21}, f_{22}) -

vector 180° in Figure 3. Although derived in a different manner, it is interesting to note that the quantity $p_{20}p_{21} = \frac{1}{2} \|\mathbf{f}_2\|^2 q'_{line}$ is one of three ridge measures in [39] and [40].

2.3 An application: Fingerprint enhancement

We have applied the above line detection principle to **fingerprint enhancement** [11]. Although we use rather different tools, the general strategy is borrowed from Knutsson et al [29]. According to this strategy, local orientation features that are consistent with the more global orientation should be enhanced, the non-consistent ones suppressed. Using the harmonic responses (17), we compute the line vector response $\mathbf{f}_{line}^T = [l_1, l_2]$ as in (32). See Figure 8. Next, this vector field is smoothed by averaging the images l_1 and l_2 over a suitable larger neighborhood yielding two other images L_1 and L_2 . The direction cosines

$\left(\cos 2\beta = \frac{L_1}{\sqrt{L_1^2 + L_2^2}}, \sin 2\beta = \frac{L_2}{\sqrt{L_1^2 + L_2^2}} \right)$ for the smoothed field are computed. A valley of unit strength with this orientation would have the response vector

$$\mathbf{f}_{2\beta} = (f_{20\beta}, f_{21\beta}, f_{22\beta}) = \left(\frac{1}{\sqrt{3}}, \sqrt{\frac{2}{3}} \cos 2\beta, \sqrt{\frac{2}{3}} \sin 2\beta \right) \quad (34)$$

Therefore, if we match (compute the inner product) between the actual local response $\mathbf{f}_2 = (f_{20}, f_{21}, f_{22})$ and the more global unit vector response $\mathbf{f}_{2\beta}$ in (34), the local response will be preserved in proportion to the “correctness” of both orientation and shape. Specifically, as can be observed from Figure 3, a valley or a ridge that is off by 90° in orientation is orthogonal to $\mathbf{f}_{2\beta}$ and will be suppressed completely. On the other hand, for a valley that matches the dominating orientation in the neighborhood, the inner product will yield the full value $\|\mathbf{f}_2\|$, while a ridge with matching orientation will yield $-\|\mathbf{f}_2\|$. As a consequence, the enhanced image $g(x, y)$ is bipolar in valley-ridge strength.

Figure 8. Fingerprint enhancement procedure

Figure 9a) shows the original fingerprint image $f(x, y)$, which is of poor quality and has very low contrast. Overlaid to the original image, the smoothed local orientation map is shown in b). The lengths of the small line segments are proportional to local energy. The enhanced image $g(x, y)$ and the thresholded result $t(x, y)$ are presented in c) and d), respectively. Note that the procedure yields quite reasonable results at the center part of the fingerprint. It seems to form a fairly intelligent hypothesis of the ridge/valley structure. Farther out from the center, the responses are rather random as could be expected from thresholding a very noisy result. To avoid false interpretations of these results, a rather obvious augmentation of the procedure would be to suppress the output altogether on the basis of a low line response $\|L_1, L_2\|$.

Figure 9. a) The original fingerprint. b) The orientation map. c) The enhanced image $g(x, y)$. d) The thresholded image $t(x, y)$.

2.4 Conclusions of the 2D-case. The Hessian, non-orthogonal bases, and steerable filters

Let us conclude our treatment of two-dimensional line detection by introducing the **Hessian** H , its eigen values and eigenvectors.

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} \quad (35)$$

Many authors have made use of the Hessian for second derivatives analysis (see section 5 below). Here, we note that finding the eigenvalues means diagonalization of the Hessian matrix. In this process the cross derivative is brought to zero just as in the procedure we have called derotation. Furthermore, except for scaling and the arbitrary permutation, the eigenvalues are obviously identical to the derivatives of the prototype in (23).

$$\begin{aligned}
p_{xx} &= \frac{\sqrt{3}}{2} p_{20} + \sqrt{\frac{3}{8}} p_{21} = \frac{1}{2} (f_{xx} + f_{yy}) + \frac{1}{2} \sqrt{(f_{xx} - f_{yy})^2 + 4f_{xy}^2} = \lambda_1 \text{ (or } \lambda_2) \\
p_{yy} &= \frac{\sqrt{3}}{2} p_{20} - \sqrt{\frac{3}{8}} p_{21} = \frac{1}{2} (f_{xx} + f_{yy}) - \frac{1}{2} \sqrt{(f_{xx} - f_{yy})^2 + 4f_{xy}^2} = \lambda_2 \text{ (or } \lambda_1)
\end{aligned} \tag{36}$$

Thus, the eigenvalue computation does indeed deliver the prototype derivatives. We also note that the orientation can then be obtained from the eigenvector matrix. It may then be argued that the eigenvalue/eigenvector information is an equivalent or even advantageous representation of the second-degree variation. We don't think so. We think it is important to understand and observe the computational process of derotation in the orthonormal function space of Figure 3. Such a space lends itself to intuitive and geometric reasoning and to fully grasp the concept of rotation invariance. It helps us to avoid the embarrassing mistake to assume that the two derotated responses $\lambda_1 = p_{xx}$ (or p_{yy}) and $\lambda_2 = p_{yy}$ (or p_{xx}) represent orthogonal quantities. From the theory of statistics and probability prevails the habit of portraying the covariance as an ellipse and label the two orthogonal axes λ_1 and λ_2 , even if the eigenvalues only determine the eccentricity of this ellipse. As a matter of fact, the *eigenvectors* corresponding to the eigenvalues are clearly orthogonal, since they constitute the rotator, the orientation in the signal space. But the *operators*, the derivators that are used to compute the Hessian are not orthogonal. They deliver their results in the three-dimensional second order variation feature space shown in Figure 3. This is where the eigenvalues reside in the form of prototype derivatives, as response vectors that are clearly non-orthogonal.

In the 2D-case we seem to have been able to do well without explicitly introduce the Hessian and its eigenvalues. But this is only because the derotation equation is still rather simple. As we will see in section three of this treatise, the Hessian is quite useful, in fact almost indispensable in the 3D-case.

Let us summarize the 2D-derotation and shape detection procedure in the following mapping sequence, a sequence which in practice offers many variations and short-cuts.

$$f(x, y) \Rightarrow (f_{xx}, f_{yy}, f_{xy}) \Rightarrow (f_{20}, f_{21}, f_{22}) \Rightarrow (p_{20}, p_{21}, \beta) \Rightarrow (\|f_2\|, \kappa, \beta) \tag{37}$$

The first step is to map the local neighborhood variation into derivative estimation (three convolutions), the second is the linear mapping onto circular harmonic responses, the third is derotation to prototype and orientation angle, and the fourth is to compute signal strength, shape and orientation. As we also have shown, the shape angle information can be used for computing dimensionless shape factors, e.g. q_{line} , which yields a highly discriminative liness measure $\|f_{line}\|$.

So far we have solved the derotation equation (4) after having converted the measured vector f_g into the vector f_b . The advantage of using the non-orthogonal basis set g for measurements is that these convolution kernels are separable and possible to compute with great efficiency. But to find the rotator $R(\beta)$ we used the vector f_b and a prototype vector p , expressed as coefficients of an orthogonal basis set in (5).

$$A f_g = f_b = R(\beta) p \tag{38}$$

However, this derotation equation is readily transformed into

$$f \otimes g = f_g = A^{-1} R(\beta) p = A^{-1} R(\beta) A A^{-1} p \tag{39}$$

With the definitions $R_g(\beta) = A^{-1} R(\beta) A$ and $p_g = A^{-1} p$ we write (39) as

$$f_g = R_g(\beta) p_g \tag{40}$$

Here, $\mathbf{f}_g$ contains the non-orthogonal filter responses, $R_g(\beta) = A^{-1}R(\beta)A$ is a rotator for the non-orthogonal function space and $\mathbf{p}_g$ is the prototype vector (32) given as components in this very same space.

First-degree derivators are orthogonal basis functions, which are rotated versions of each other. Inspired by this, and by compass method thinking, the following **non-orthogonal basis** was proposed in [15] for 2D second derivative estimation. In fact, this basis set can be seen as an archetypal case of **steerable filters**. It consists of three 60° -rotated versions of g_{xx} , namely $(g_{xx}, g_{60}, g_{120})$. In the function space of Figure 3, all three are located on the ideal valley latitude, 120° apart in longitude. The mapping A of this basis set onto the previous orthogonal basis set (b_{20}, b_{21}, b_{22}) is obtained from the geometry in Figure 3, so that for this case the matrix A and its inverse A^{-1} in (40) are as follows.

$$\begin{aligned} \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} &= A \begin{bmatrix} g_{xx} \\ g_{60} \\ g_{120} \end{bmatrix} = \begin{bmatrix} \frac{2}{3\sqrt{3}} & \frac{2}{3\sqrt{3}} & \frac{2}{3\sqrt{3}} \\ \frac{2\sqrt{8}}{3\sqrt{3}} & -\frac{\sqrt{8}}{3\sqrt{3}} & -\frac{\sqrt{8}}{3\sqrt{3}} \\ 0 & \frac{\sqrt{8}}{3} & -\frac{\sqrt{8}}{3} \end{bmatrix} \begin{bmatrix} g_{xx} \\ g_{60} \\ g_{120} \end{bmatrix} \\ \begin{bmatrix} g_{xx} \\ g_{60} \\ g_{120} \end{bmatrix} &= A^{-1} \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} & \sqrt{\frac{3}{8}} & 0 \\ \frac{\sqrt{3}}{2} & -\sqrt{\frac{3}{32}} & \frac{3}{\sqrt{32}} \\ \frac{\sqrt{3}}{2} & -\sqrt{\frac{3}{32}} & \frac{-3}{\sqrt{32}} \end{bmatrix} \begin{bmatrix} b_{20} \\ b_{21} \\ b_{22} \end{bmatrix} \end{aligned} \quad (41)$$

From (41) we find that the response energy expressed as

$$f_{xx}^2 + f_{60}^2 + f_{120}^2 = \frac{9}{8} \left(2f_{20}^2 + \frac{1}{2}(f_{21}^2 + f_{22}^2) \right) = \frac{9}{8} \left(2p_{20}^2 + \frac{1}{2}p_{21}^2 \right) \quad (42)$$

is not particularly appropriate as a line energy measure. It maximizes its response for the shape angle $\kappa = \arctan 2 = 63.4^\circ$ while the optimal line detector has a shape angle $\kappa = \arctan \frac{1}{\sqrt{2}} = 35.3^\circ$. Nevertheless, with the two matrices A and A^{-1} at hand we are also able to compute the matrix $R_g(\beta) = A^{-1}R(\beta)A$ in eq. (40). Using $\mathbf{g}_{xx}^T = (\frac{\sqrt{3}}{2}, \sqrt{\frac{3}{8}}, 0)$ as prototype $\mathbf{p}$ in the orthonormal space, in the non-orthogonal space this prototype becomes $\mathbf{p}_g = A^{-1}\mathbf{p} = (1, 0, 0)^T$. Given a response vector $(f_{xx}, f_{60}, f_{120})^T$ the derotation equation (40) then takes the form

$$\begin{aligned} \begin{bmatrix} f_{xx} \\ f_{60} \\ f_{120} \end{bmatrix} &= \frac{1}{3} \|f_2\| \begin{bmatrix} 1 + 2\cos 2\beta & 1 - \cos 2\beta + \sqrt{3}\sin 2\beta & 1 - \cos 2\beta - \sqrt{3}\sin 2\beta \\ 1 - \cos 2\beta + \sqrt{3}\sin 2\beta & 1 + 2\cos 2\beta & 1 - \cos 2\beta - \sqrt{3}\sin 2\beta \\ 1 - \cos 2\beta - \sqrt{3}\sin 2\beta & 1 - \cos 2\beta + \sqrt{3}\sin 2\beta & 1 + 2\cos 2\beta \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \\ &= \frac{1}{3} \|f_2\| \begin{bmatrix} 1 + 2\cos 2\beta \\ 1 - \cos 2\beta + \sqrt{3}\sin 2\beta \\ 1 - \cos 2\beta - \sqrt{3}\sin 2\beta \end{bmatrix} \end{aligned} \quad (43)$$

Hence, we are now perfectly able to estimate the orientation β for the second derivative. It may then seem that equations (40) and (43) are a general recipe for replacing the derotation equations (4), (5), (38) with an

alternative for non-orthogonal bases. However, equation (43) is over-determined and the three measurements are not specifically exploited to detect any shape information. Any attempt to interpret the shape content in the steerable filter responses is bound to land in the orthogonal space of Figure 3. If the purpose is to separate magnitude, orientation, and shape without destroying information in the original response vector, there seems to be no reasonable alternative to the orthogonal basis representation.

3. Second derivatives and spherical harmonics in three-dimensional images

3.1 Diagonalizing the Hessian. Mapping derivatives to harmonics

For 3D-functions, the Hessian matrix

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} \quad (44)$$

harbors three cross-derivatives, all of which should be brought to zero in a derotation/diagonalizing procedure. With the cross-derivatives set to zero, the six degrees of freedom in the Hessian are reduced to three, from which we infer that there are three degrees of freedom for the orientation. This is in full accordance with the fact that a 3x3 rotator with three degrees of freedom describes all rotations in 3D. Therefore, already now we may conclude that the counterpart to the 2D-case in (37) will be the following mapping sequence.

$$\begin{aligned} f(x, y, z) &\Rightarrow (f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, f_{yz}) \Rightarrow (f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25}) \Rightarrow \\ &(p_{20}, p_{21}, p_{22}, \alpha, \beta, \gamma) \Rightarrow (\|f_2\|, \kappa, \eta, \alpha, \beta, \gamma) \end{aligned} \quad (45)$$

With a slight abuse of notation, we do not change the name of the responses from the 2D-case, although the derivators are quite different in the 3D-case. Disregarding this, in (45) the first step is akin to the one in (37), namely to compute the derivatives, followed by a mapping onto spherical harmonic operator responses, which are then to be translated into prototype responses and three rotation-orientation parameters (α, β, γ) . In the last step, the prototype data are converted to signal strength $\|f_2\|$ and two parameters (κ, η) defining shape. It should be noted, however, that in an actual implementation/application considerable shape information could be retrieved much earlier than indicated by (45).

The derotation equation (4) takes the following form for three-dimensional second derivatives.

$$A_3 f \otimes \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{zz} \\ g_{xy} \\ g_{xz} \\ g_{yz} \end{bmatrix} \equiv A_3 \begin{bmatrix} f_{xx} \\ f_{yy} \\ f_{zz} \\ f_{xy} \\ f_{xz} \\ f_{yz} \end{bmatrix} = f \otimes A_3 \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{zz} \\ g_{xy} \\ g_{xz} \\ g_{yz} \end{bmatrix} = f \otimes \begin{bmatrix} c_{20} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{24} \\ c_{25} \end{bmatrix} \equiv \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{24} \\ f_{25} \end{bmatrix} \equiv \mathbf{f}_2 = \|\mathbf{f}_2\| R(\alpha, \beta, \gamma) \begin{bmatrix} \sin \kappa \\ \cos \kappa \cos \eta \\ \cos \kappa \sin \eta \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (46)$$

The 6 x 6 matrix A_3 will be defined shortly in eq. (50) below, and the 6 x 6 rotator $R(\alpha, \beta, \gamma)$ is defined in Appendix C. A_3 is used to map the derivatives onto the six responses $(f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25})$ which is equivalent to convolve f with the six orthogonal basis functions $(c_{20}, c_{21}, c_{22}, c_{23}, c_{24}, c_{25})$. We will not use the rotator $R(\alpha, \beta, \gamma)$ explicitly to solve the derotation equation in the manner we did for the 2D-case. Instead we will exploit the fact that the chosen prototype in (46) has zeroes in the positions which correspond to those basis functions (c_{23}, c_{24}, c_{25}) which estimate the cross-derivatives. Therefore, as already mentioned, solving the derotation equation becomes indeed equivalent to diagonalizing the Hessian, which is also equivalent to computing the eigenvalues of H . For this task the cyclic Jacobi technique is recommended in [44]. This procedure is also possible to cast in terms of derotating the spherical harmonic responses [13]. In the latter case the aim is explicitly set to recover the rotator $R(\alpha, \beta, \gamma)$, rather than the eigenvalues and eigenvectors.

To find the matrix A_3 we have to use the Fourier domain in the same way as in the 2D-case (12). In the Fourier domain there is an orthogonal set of six second degree 3D-derivators $(G_{xx}, G_{yy}, G_{zz}, G_{xy}, G_{xz}, G_{yz})$, while the basis set consists of six spherical harmonics operators $(C_{20}, C_{21}, C_{22}, C_{23}, C_{24}, C_{25})$. One of these is of zero degree (the Laplacian operator C_{20}); the other five are of second degree. Although much more difficult than in the 2D-case, it has been shown by L.-E. Andersson [1] that for functions of any dimension, the harmonic angular variations are preserved over the Fourier transform while the radial variation undergoes a Hankel transform. The proof is not included here for the sake of brevity. However, it follows from this result that for 3D-functions, the zero order operator corresponds bi-directionally over the Fourier transform and its inverse with a Hankel transform of order $1/2$, while the second order operators correspond over the Hankel transform of order $2\frac{1}{2}$. Therefore, and in accordance with the 2D-case (17), the local second derivatives $(f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{xz}, f_{yz})$ can be linearly combined into responses $\mathbf{f}_2^T = (f_{20}, f_{21}, f_{22}, f_{23}, f_{24}, f_{25})$. These responses are of course identical to the direct responses from the operators $(c_{20}, c_{21}, c_{22}, c_{23}, c_{24}, c_{25})$, obtainable from the six second-degree derivators using the same linear combinations A_3 . Thanks to this important theorem we know that the basis consists of spherical harmonics also in the signal domain and that in this domain, the derivators are separable in a radial variation multiplied with a polynomial.

The Cartesian coordinates (u, v, w) and the polar coordinates (ρ, θ, ϕ) are connected by

$$\begin{aligned} u &= \rho \sin \theta \cos \phi \\ v &= \rho \sin \theta \sin \phi \\ w &= \rho \cos \theta \end{aligned} \quad (47)$$

As in the 2D-case (10) we obtain the derivators by differentiating $h_0(r)$, which is now a rotationally symmetric 3D-function. The Fourier transforms appear as follows.

$$\begin{aligned} g_{xx} &= \frac{\partial^2}{\partial x^2} h_0(r) \Leftrightarrow G_{xx} = -4\pi^2 H_0(\rho) u^2 = -4\pi^2 \rho^2 H_0(\rho) \sin^2 \theta \cos^2 \phi \\ g_{yy} &= \frac{\partial^2}{\partial y^2} h_0(r) \Leftrightarrow G_{yy} = -4\pi^2 H_0(\rho) v^2 = -4\pi^2 \rho^2 H_0(\rho) \sin^2 \theta \sin^2 \phi \\ g_{zz} &= \frac{\partial^2}{\partial z^2} h_0(r) \Leftrightarrow G_{zz} = -4\pi^2 H_0(\rho) w^2 = -4\pi^2 \rho^2 H_0(\rho) \cos^2 \theta \\ g_{xy} &= \frac{\partial^2}{\partial x \partial y} h_0(r) \Leftrightarrow G_{xy} = -4\pi^2 H_0(\rho) uv = -4\pi^2 \rho^2 H_0(\rho) \sin^2 \theta \cos \phi \sin \phi \\ g_{xz} &= \frac{\partial^2}{\partial x \partial z} h_0(r) \Leftrightarrow G_{xz} = -4\pi^2 H_0(\rho) uw = -4\pi^2 \rho^2 H_0(\rho) \sin \theta \cos \theta \cos \phi \\ g_{yz} &= \frac{\partial^2}{\partial y \partial z} h_0(r) \Leftrightarrow G_{yz} = -4\pi^2 H_0(\rho) vw = -4\pi^2 \rho^2 H_0(\rho) \sin \theta \cos \theta \sin \phi \end{aligned} \quad (48)$$

In the Fourier domain we obtain the orthogonal and normalized spherical harmonic operators with the linear combinations in (50). The normalization factors are chosen so that the signal energy integrated over the unit sphere equals

$$\int_0^{2\pi} \int_0^\pi [C_i(\rho)]^2 \sin \theta d\theta d\phi = \frac{4\pi}{6} [H_2(\rho)]^2 \quad \text{for } i = 20, 21, 22, 23, 24, 25 \quad (49)$$

which means that the sum of the energies of the six basis functions is 4π , the area of the unit sphere. Note that since $\frac{4\pi}{6} = \frac{2\pi}{3}$, the same quantity as in the 2D-case (16) appears on the right hand side of (49). The orthonormal spherical harmonic functions in (49) are often defined with Legendre polynomials and complex valued functions [47]. For our purposes, however, it is important to normalize the zero order and second order harmonics into one single set. Also, we see no advantage of bringing in complex-valued functions at this point. Note again that we may express the basis functions with polynomials in *both* domains.

$$\begin{aligned}
 \begin{bmatrix} C_{20} \\ C_{21} \\ C_{22} \\ C_{23} \\ C_{24} \\ C_{25} \end{bmatrix} &= A_3 \begin{bmatrix} G_{xx} \\ G_{yy} \\ G_{zz} \\ G_{xy} \\ G_{yz} \\ G_{xz} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & 0 & 0 & 0 \\ -\sqrt{\frac{5}{24}} & -\sqrt{\frac{5}{24}} & \sqrt{\frac{5}{6}} & 0 & 0 & 0 \\ \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} \end{bmatrix} \begin{bmatrix} G_{xx} \\ G_{yy} \\ G_{zz} \\ G_{xy} \\ G_{yz} \\ G_{xz} \end{bmatrix} = \\
 &-4\pi^2 \rho^2 H_0(\rho) \begin{bmatrix} \sqrt{\frac{1}{6}} \\ \sqrt{\frac{5}{24}}(3\cos^2\theta - 1) \\ \sqrt{\frac{5}{8}}\sin^2\theta\cos 2\phi \\ \sqrt{\frac{5}{8}}\sin^2\theta\sin 2\phi \\ \sqrt{\frac{5}{8}}\sin 2\theta\cos\phi \\ \sqrt{\frac{5}{8}}\sin 2\theta\sin\phi \end{bmatrix} = -4\pi^2 H_0(\rho) \begin{bmatrix} \sqrt{\frac{1}{6}}(u^2 + v^2 + w^2) \\ \sqrt{\frac{5}{24}}(2w^2 - u^2 - v^2) \\ \sqrt{\frac{5}{8}}(u^2 - v^2) \\ \sqrt{\frac{5}{2}}uv \\ \sqrt{\frac{5}{2}}uw \\ \sqrt{\frac{5}{2}}vw \end{bmatrix} \Leftrightarrow \\
 \Leftrightarrow \begin{bmatrix} c_{20} \\ c_{21} \\ c_{22} \\ c_{23} \\ c_{24} \\ c_{25} \end{bmatrix} &= \begin{bmatrix} \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & 0 & 0 & 0 \\ -\sqrt{\frac{5}{24}} & -\sqrt{\frac{5}{24}} & \sqrt{\frac{5}{6}} & 0 & 0 & 0 \\ \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} \end{bmatrix} \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{zz} \\ g_{xy} \\ g_{yz} \\ g_{xz} \end{bmatrix} = \\
 &\begin{bmatrix} h_{20}(r)\sqrt{\frac{1}{6}} \\ h_2(r)\sqrt{\frac{5}{24}}(3\cos^2\theta - 1) \\ h_2(r)\sqrt{\frac{5}{8}}\sin^2\theta\cos 2\phi \\ h_2(r)\sqrt{\frac{5}{8}}\sin^2\theta\sin 2\phi \\ h_2(r)\sqrt{\frac{5}{8}}\sin 2\theta\cos\phi \\ h_2(r)\sqrt{\frac{5}{8}}\sin 2\theta\sin\phi \end{bmatrix} = \frac{1}{r^2} \begin{bmatrix} h_{20}(r)\sqrt{\frac{1}{6}}(x^2 + y^2 + z^2) \\ h_2(r)\sqrt{\frac{5}{24}}(2z^2 - x^2 - y^2) \\ h_2(r)\sqrt{\frac{5}{8}}(x^2 - y^2) \\ h_2(r)\sqrt{\frac{5}{2}}xy \\ h_2(r)\sqrt{\frac{5}{2}}xz \\ h_2(r)\sqrt{\frac{5}{2}}yz \end{bmatrix} \quad (50)
 \end{aligned}$$

The same matrix A_3 is used to map the derivatives to spherical harmonic responses in (51).

$$\mathbf{f}_2 = \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{24} \\ f_{25} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & 0 & 0 & 0 \\ -\sqrt{\frac{5}{24}} & -\sqrt{\frac{5}{24}} & \sqrt{\frac{5}{6}} & 0 & 0 & 0 \\ \sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}} \end{bmatrix} \begin{bmatrix} f_{xx} \\ f_{yy} \\ f_{zz} \\ f_{xy} \\ f_{xz} \\ f_{yz} \end{bmatrix} \quad (51)$$

The inverse of (51) yields

$$\begin{bmatrix} f_{xx} \\ f_{yy} \\ f_{zz} \\ f_{xy} \\ f_{xz} \\ f_{yz} \end{bmatrix} = A_3^{-1} \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{24} \\ f_{25} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{15}} & \sqrt{\frac{2}{5}} & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}} & -\sqrt{\frac{2}{15}} & -\sqrt{\frac{2}{5}} & 0 & 0 & 0 \\ \sqrt{\frac{2}{3}} & \sqrt{\frac{8}{15}} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{\frac{2}{5}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{2}{5}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{2}{5}} \end{bmatrix} \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{24} \\ f_{25} \end{bmatrix} \quad (52)$$

Presume that the prototype for the response $\mathbf{f}_2$ has been found by solving the derotation equation (46), which means that we now should know the actual prototype $\mathbf{p}_2 = (p_{20}, p_{21}, p_{22})$. This vector can then be mapped into the three-dimensional function space (c_{20}, c_{21}, c_{22}) as shown in Figure 10. This is of course a subspace of the six-dimensional function space $(c_{20}, c_{21}, c_{22}, c_{23}, c_{24}, c_{25})$, and in the latter space we find the original response $\mathbf{f}_2$. The derotation procedure should rotate this six-dimensional vector into the three-dimensional prototype space of Figure 10. The shape is represented by $\frac{\mathbf{p}_2}{\|\mathbf{p}_2\|}$ and this quantity resides on the surface of the unit sphere of Figure 10. A position on this unit sphere may be defined by the two angles κ and η_λ . However, as we will see shortly, this position is not unique.

3.2 Mapping eigenvalues to prototype derivatives and absolute orientation

As in the 2D-case, we define the **orientation of a pattern** $f(x, y, z)$ as its rotation relative to its prototype. And by fixing the orientation of the prototype space of Figure 10 to the coordinate system (x, y, z) , the orientation of a rotated pattern $f(x, y, z)$ also becomes defined in the (x, y, z) – space, the world coordinates for the original 3D signal. Unfortunately, the prototype space of Figure 10 is redundant in the sense that a certain shape, say, the double cone, will appear in this space not once but six times. The reason is that there are six permutations of the eigenvalues $(\lambda_1, \lambda_2, \lambda_3)$, all of which are valid solutions to the Hessian diagonalization. To get a unique orientation of the prototype in the signal space, one of these should be selected, which is to eliminate most of the unit sphere in Figure 10 from the viable shape space. Of the six eigenvalue permutations $(\lambda_1, \lambda_2, \lambda_3)$, $(\lambda_2, \lambda_1, \lambda_3)$, only one should be identified with the prototype derivatives (p_{xx}, p_{yy}, p_{zz}) .

Let us first observe how the two basis functions (c_{21}, c_{22}) span the equatorial plane of Figure 10. From (52) we have the expressions

Figure 10. The shape and prototype space (c_{20}, c_{21}, c_{22}) for 3D second-degree variation

$$\begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = h_2(r) \sqrt{\frac{5}{8}} \begin{bmatrix} \sqrt{\frac{1}{3}}(2z^2 - x^2 - y^2) \\ x^2 - y^2 \end{bmatrix} \quad (53)$$

By linear combinations we can construct the following pairs of alternative basis functions, e.g.

$$\begin{bmatrix} c_{31} \\ c_{32} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = h_2(r) \sqrt{\frac{5}{8}} \begin{bmatrix} \sqrt{\frac{1}{3}}(2x^2 - y^2 - z^2) \\ z^2 - y^2 \end{bmatrix}$$

$$\begin{bmatrix} c_{41} \\ c_{42} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} c_{21} \\ c_{22} \end{bmatrix} = h_2(r) \sqrt{\frac{5}{8}} \begin{bmatrix} \sqrt{\frac{1}{3}}(2y^2 - z^2 - x^2) \\ z^2 - x^2 \end{bmatrix}$$

Because the polynomials for (c_{21}, c_{22}) and (c_{31}, c_{32}) have the same form except for the cyclic permutation $(x, y, z) \rightarrow (y, z, x)$ of the variables it follows that they also have the same shape. This cyclic permutation is to rotate the signal space coordinate system (x, y, z) 60° around the cube diagonal, which in Figure 10 is to rotate the prototype coordinate system 120° around the c_{20} -axis. The third cyclic permutation (z, x, y) is to rotate

the prototype system 240° around the same axis. A non-cyclic permutation such as $(x, y, z) \rightarrow (y, x, z)$ is equivalent to rotate the signal space coordinates $+90^\circ$ or -90° around the z -axis, which in Figure 10 corresponds to a 180° rotation around the c_{21} -axis so that the two orange shapes c_{22} and $-c_{22}$ are swapped. Clearly, these two shapes are the same and should not both appear in a non-redundant shape space. The two remaining non-cyclic permutations (z, y, x) and (x, z, y) correspond to a 180° rotation around the c_{31} and c_{41} -axes, respectively. Thus, an arbitrary assignment of the three eigenvalues onto the three prototype derivatives could make the vector $\mathbf{p}'_2$ appear in any of six positions in the shape space of Figure 10. To eliminate this six-fold redundancy we select the 60° -sector around the positive c_{22} -axis of the unit sphere in Figure 10 as the “chosen one”. Naturally, we could have chosen another 60° sector of the original space, or even two disjunct 30° sectors. The chosen sector of Figure 10 is shown in Figure 11 with several archetypal shapes. From this choice follows that a response vector, which falls outside this sector in Figure 11, is not a prototype but a response vector that is not fully derotated.

The 60° sector in Figure 11 has the characteristic feature that for the shape angle $\eta = \arg(p_{21}, p_{22})$ we have

$$\frac{\pi}{3} \leq \eta < \frac{2\pi}{3}$$

We use the polynomial expressions for (c_{21}, c_{22}) to obtain

$$\begin{aligned} \text{For } [p_{21} \geq 0] \text{ and } \left[\frac{\pi}{3} \leq \arg(p_{21}, p_{22})\right] &\Rightarrow \sqrt{3} \leq \frac{\sqrt{\frac{5}{8}}(p_{xx} - p_{yy})}{\sqrt{\frac{5}{24}}(2p_{zz} - p_{xx} - p_{yy})} \Rightarrow p_{zz} \leq p_{xx} \\ \text{For } [p_{21} < 0] \text{ and } \left[\arg(p_{21}, p_{22}) < \frac{2\pi}{3}\right] &\Rightarrow \sqrt{3} < \frac{-\sqrt{\frac{5}{8}}(p_{xx} - p_{yy})}{\sqrt{\frac{5}{24}}(2p_{zz} - p_{xx} - p_{yy})} \Rightarrow p_{yy} < p_{zz} \end{aligned}$$

Hence, we should make the eigenvalue/prototype assignment as

$$p_{xx} \Leftarrow \lambda_1, \quad p_{yy} \Leftarrow \lambda_3, \quad p_{zz} \Leftarrow \lambda_2 \quad \text{where} \quad \lambda_1 > \lambda_2 > \lambda_3$$

Having assigned a value to the three prototype derivatives, the response vector $\mathbf{p}_2$ is obtained from (52) as

$$\mathbf{p}_2 \equiv \begin{bmatrix} p_{20} \\ p_{21} \\ p_{22} \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} & \sqrt{\frac{1}{6}} \\ -\sqrt{\frac{5}{24}} & -\sqrt{\frac{5}{24}} & \sqrt{\frac{5}{6}} \\ -\sqrt{\frac{5}{8}} & -\sqrt{\frac{5}{8}} & 0 \end{bmatrix} \begin{bmatrix} p_{xx} \\ p_{yy} \\ p_{zz} \end{bmatrix} \quad (54)$$

Figure 11. The shape space for 3D second degree variation

3.3. Eigenvector computation. The 3x3 signal space rotator

In the general case, the orientation is defined by the rotator

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \quad (55)$$

the columns of which consist of three eigenvectors corresponding to the three eigenvalues identified with the prototype derivatives (p_{xx}, p_{yy}, p_{zz}) as described in section 3.2. The complete rotator may then be obtained from the equation system

$$\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{bmatrix} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} p_{xx} & 0 & 0 \\ 0 & p_{yy} & 0 \\ 0 & 0 & p_{zz} \end{bmatrix} \quad (56)$$

This rotator R should be distinguished from the 6x6 basis function rotator $R(\alpha, \beta, \gamma)$ in (46) and Appendix C. In spite of its nine components, R has only three degrees of freedom. To see the dependencies among the nine components, we may first note that by definition and construction, the three eigenvectors computed from (56) will be orthogonal to each other. This fact yields three equations of the type

$$r_{11}r_{12} + r_{21}r_{22} + r_{31}r_{32} = 0 \quad (57)$$

which reduces the degrees of freedom from nine to six. Then, each eigenvector column can be arbitrary scaled, and in this case we should of course normalize so that

$$r_{1i}^2 + r_{2i}^2 + r_{3i}^2 = 1, \quad i = 1, 2, 3 \quad (58)$$

which reduces the degrees of freedom from six to three. It is a common fallacy to believe that because the eigenvectors (e_x, e_y, e_z) corresponding to the eigenvalues (p_{xx}, p_{yy}, p_{zz}) are orthogonal the estimators (g_{xx}, g_{yy}, g_{zz}) also form an orthogonal basis set. The fact that this is not true was mentioned already in connection with two-dimensional second derivatives in the previous section. Neither the functions (g_{xx}, g_{yy}) in 2D set nor the (g_{xx}, g_{yy}, g_{zz}) in the 3D set are orthogonal.

Commonly, a rotation in three dimensions is defined as a sequence of three planar rotations around the coordinate axes. For instance, a rotation of an object with the angles (α, β, γ) around z -, y -, and x -axes, respectively, is obtained by the rotator

$$\begin{bmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{bmatrix} \begin{bmatrix} \cos \alpha & -\sin \alpha & 0 \\ \sin \alpha & \cos \alpha & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \gamma & -\sin \alpha \cos \beta \cos \gamma - \cos \alpha \sin \beta & \sin \beta \cos \gamma \\ \cos \alpha \cos \beta \sin \gamma + \sin \alpha \cos \gamma & -\sin \alpha \cos \beta \sin \gamma + \cos \alpha \cos \gamma & \sin \beta \sin \gamma \\ -\cos \alpha \sin \beta & \sin \alpha \sin \beta & \cos \beta \end{bmatrix} \quad (59)$$

Identification between this rotator and the rotator R in (55), will give us a host of possibilities to compute the angles (α, β, γ) . If $r_{13}^2 + r_{23}^2 \geq r_{33}^2$, then $|\sin \beta| \geq |\cos \beta|$, in which case we may suggest

$$\begin{aligned}
\alpha &= \arctan \frac{r_{32}}{-r_{31}} \\
\gamma &= \arctan \frac{r_{23}}{r_{13}} \\
\beta &= \arctan \frac{r_{13} \cos \gamma + r_{23} \sin \gamma}{r_{33}}
\end{aligned} \tag{60}$$

For $\beta \approx 0$, (60) reduces to

$$R = \begin{bmatrix} \cos(\alpha + \gamma) & -\sin(\alpha + \gamma) & 0 \\ \sin(\alpha + \gamma) & \cos(\alpha + \gamma) & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{61}$$

which is natural since the middle planar rotation disappears and the total rotation is reduced to two successive rotations around the z-axis with the angle

$$\alpha + \gamma = \arctan \frac{r_{21}}{r_{11}} \tag{62}$$

3.4 Rotation-invariants. Shape factors

Whatever pose we give a certain prototype, the derotation procedure described above will deliver the same coefficients (p_{20}, p_{21}, p_{22}) in the (c_{20}, c_{21}, c_{22}) -space. Hence, any function using the arguments

(p_{20}, p_{21}, p_{22}) is rotation invariant. One such combination that have been used by some authors is the energy measure formed by the sum of the squared eigenvalues as

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = p_{xx}^2 + p_{yy}^2 + p_{zz}^2 = 2p_{20}^2 + \frac{4}{5}(p_{21}^2 + p_{22}^2)$$

where the last part of this expression is obtained from (52). It follows that the sum of the energies of the eigenvalues is not shape-invariant, although this was by all likelihood the intention. It over-emphasizes the p_{20} -component and under-emphasizes the p_{21} and p_{22} -components, which means that it gives too high a value for blob-like shapes, too low a value for second order variations, which in the shape space of Figure 11 are located on the equator. The only energy measure that is indifferent to both rotation and shape is of course the straight sum of the squared *orthogonal* components, which yields

$$\begin{aligned}
\|p_2\|^2 &= p_{20}^2 + p_{21}^2 + p_{22}^2 = p_{xx}^2 + p_{yy}^2 + p_{zz}^2 - \frac{1}{2}(p_{xx}p_{yy} + p_{xx}p_{zz} + p_{yy}p_{zz}) = \\
\|f_2\|^2 &= f_{20}^2 + f_{21}^2 + f_{22}^2 + f_{23}^2 + f_{24}^2 + f_{25}^2 = \\
&f_{xx}^2 + f_{yy}^2 + f_{zz}^2 - \frac{1}{2}(f_{xx}f_{yy} + f_{xx}f_{zz} + f_{yy}f_{zz}) + \frac{5}{2}(f_{xy}^2 + f_{xz}^2 + f_{yz}^2)
\end{aligned} \tag{63}$$

Rotation-invariant functions with arguments $(\|p_2\|, p_{20})$ or, equivalently, $\left(\|p_2\|, \sqrt{\|p_2\|^2 - p_{20}^2}\right)$ or

$\left(p_{20}, \sqrt{p_{21}^2 + p_{22}^2}\right)$ can be computed *without derotation*. One such quantity is the **shape angle** κ in Figure 10, which is defined as

$$\kappa \equiv \arg\left(p_{20}, \sqrt{p_{21}^2 + p_{22}^2}\right) \quad (64)$$

A rather discriminative **shape factor** $Q(\vartheta, \kappa)$, constructed in the same manner as (29) with a preferred angle ϑ as parameter can be defined as

$$Q(\vartheta, \kappa) = \frac{2 \cos \vartheta \sin \kappa \sin \vartheta \cos \kappa}{(\cos \vartheta \sin \kappa)^2 + (\sin \vartheta \cos \kappa)^2} = \frac{2 \cos \vartheta p_{20} \sin \vartheta \sqrt{p_{21}^2 + p_{22}^2}}{(\cos \vartheta)^2 p_{20}^2 + (\sin \vartheta)^2 (p_{21}^2 + p_{22}^2)} \quad (65)$$

which has a maximum $Q(\vartheta, \kappa) = 1$ for $\kappa = \vartheta$ and a minimum $Q(\vartheta, \kappa) = -1$ for $\kappa = -\vartheta$. Zero-crossings occur for both for fully symmetric, purely Laplacian blob shapes where $\|\mathbf{p}_2\| = p_{20}$, $\kappa = \pm \frac{\pi}{2}$, and for purely Laplacian-free shapes where $p_{20} = 0$, $\kappa = 0$. By setting the parameter $\vartheta = \frac{\pi}{4}$ in (65) this shape factor takes the following simple form, which we name simply Q_κ .

$$Q\left(\frac{\pi}{4}, \kappa\right) \equiv Q_\kappa = \frac{2 p_{20} \sqrt{p_{20}^2 + p_{21}^2}}{p_{20}^2 + (p_{20}^2 + p_{21}^2)}$$

Maximum and minimum values +1 and -1 occur for $\kappa = \pm \frac{\pi}{4}$, respectively, i.e. for shapes, which are in between the two extremes. This is, approximately, where we find second-degree variations of type planes and strings to be dealt with below.

Also the angle η in Figure 11 is shape specific. From this Figure we find that all shapes at the two edges of the shape space are rotationally symmetric around the x-axis and the y-axis, respectively. In general **axially symmetric** prototypes have the characteristic feature

$$\sqrt{3}|p_{21}| = p_{22} \Rightarrow \sin \eta = \frac{\sqrt{3}}{2}$$

Objects (signals) $f(x, y, z)$ which produce such prototype responses can be defined by a 2D-function $f_{\beta, \lambda}(s, t) = f(x, y, z)$ with a symmetry axis in the (β, λ) -direction and the coordinates (s, t) defined as

$$\begin{aligned} \text{For } p_{21} < 0 \quad s &= [\cos \beta, \sin \beta \cos \gamma, \sin \beta \sin \gamma]^T \cdot [x, y, z]^T \\ t &= \left\| [\cos \beta, \sin \beta \cos \gamma, \sin \beta \sin \gamma]^T \times [x, y, z]^T \right\| \\ \text{For } p_{21} \geq 0 \quad s &= [\sin \beta \sin \gamma, \cos \beta, \sin \beta \cos \gamma]^T \cdot [x, y, z]^T \\ t &= \left\| [\sin \beta \sin \gamma, \cos \beta, \sin \beta \cos \gamma]^T \times [x, y, z]^T \right\| \end{aligned} \quad (66)$$

Derotation with $(-\beta, -\gamma)$ transforms the function $f_{\beta, \gamma}(s, t)$ into $f_{0,0}(x, \sqrt{y^2 + z^2})$ or $f_{0,0}(y, \sqrt{x^2 + z^2})$ depending on the polarity of if p_{21} .

It is interesting to note that the inner loop of the Jacobi method for diagonalization [44] should terminate rather quickly for axially symmetric and nearly axially symmetric shapes. The details of this argument are left out here for the sake of brevity. In case we are only interested in axially symmetric shapes, lack of fast termination indicates an uninteresting shape, which means that we may terminate quickly anyway. The net result can be dramatic savings in computation time.

With the full prototype vector computed the following **shape factor** $Q(\vartheta, \eta)$ **for axial symmetry** can be derived in the manner of (29) and the above derivation of $Q(\vartheta, \kappa)$ and Q_κ .

$$Q(\vartheta, \eta) = \frac{2 \sin \vartheta \cos \eta \cos \vartheta \sin \eta}{(\sin \vartheta \cos \eta)^2 + (\cos \vartheta \sin \eta)^2} = \frac{2 \sin \vartheta p_{21} \cos \vartheta p_{22}}{(\sin \vartheta)^2 p_{21}^2 + (\cos \vartheta)^2 p_{22}^2} =$$

$$\left[\vartheta = \arctan \sqrt{3} \right] \equiv Q_\eta = \frac{2 \frac{\sqrt{3}}{2} p_{21} \frac{1}{2} p_{22}}{\frac{3}{4} p_{21}^2 + \frac{1}{4} p_{22}^2} = \frac{2\sqrt{3} p_{21} p_{22}}{3 p_{21}^2 + p_{22}^2}$$

which is $+1$ and -1 for $\eta = \pm \vartheta$, i.e. shapes with a symmetry axis along y and x respectively. A somewhat simpler shape factor, which still take the values ± 1 in the very same positions as Q_η is the following.

$$Q'_\eta = \frac{4}{\sqrt{3}} \frac{p_{21} p_{22}}{p_{21}^2 + p_{22}^2}$$

In Figure 11 we see that many of the axially symmetric shapes can be grouped as either strings and planes. However, the eradication of redundant shapes has forced the positive and negative versions of these archetypes to become tilted 90° with respect to each other. It is not difficult to see that the factor $p_{20} p_{21}$ takes on positive and negative values for strings and planes, respectively. The sign of the products

$$Q_\kappa Q_\eta = \frac{2 p_{20} \sqrt{p_{21}^2 + p_{22}^2}}{\|p_2\|^2} \frac{2\sqrt{3} p_{21} p_{22}}{3 p_{21}^2 + p_{22}^2} \quad \text{and} \quad Q_\kappa Q'_\eta = \frac{8}{\sqrt{3}} \frac{p_{20} p_{21} p_{22}}{\|p_2\|^2 \sqrt{p_{21}^2 + p_{22}^2}}$$

is likewise taking its sign from the factor $p_{20} p_{21}$, since all other factors including p_{22} are positive. But from the properties of the two shape factors Q_κ and Q_η also follows that the maximum value $+1$ and the minimum value -1 are obtained when the prototype vector is on one of the two outer meridians in Figure 11 halfway between the equator and one of the poles. Furthermore, the zero values for the combined rather discriminative shape factor $Q_\kappa Q_\eta$ are found on the equator and along the c_{22} -meridian. A qualitative illustration of how the $Q_\kappa Q_\eta$ -factor varies over the shape space is given by Figure 12.

We will now make some observations in the 3D Fourier domain shown in Figure 13, which is akin to the 2D-case, Figure 5. In the Fourier domain, the ideal positive plane has all its energy concentrated along the u -axis.

Let $H_2(\rho) = -4\pi^2 H_0(\rho)$ in eq. (50). Then we find that in the u -direction we obtain

$C_{20}(u) = H_2(u) \sqrt{\frac{1}{6}} u^2$ and $C_{31}(u) = H_2(u) \sqrt{\frac{5}{6}} u^2$. The second-derivative response for such a plane from a unit vector $c_{20} \sin \kappa + c_{31} \cos \kappa$ can then be expressed as follows.

Figure 12. The shape space with approximate isocurves for the shape factor $Q_\kappa Q_\eta$

Figure 13. The ideal plane and the ideal string (upper left) are detected optimally by certain combinations of c_{20} , c_{31} , and c_{41} , which are derived in the Fourier domain (lower half)

$$f_{plane}(\kappa) = \int_{-\infty}^{\infty} PLANE(u) [\sin \kappa C_{20} + \cos \kappa C_{31}](u) du =$$

$$\left[\sqrt{\frac{1}{6}} \sin \kappa + \sqrt{\frac{5}{6}} \cos \kappa \right] \int_{-\infty}^{\infty} PLANE(u) H_2(u) u^2 du \quad (67)$$

which takes its maximum for $\kappa = \arctan \frac{1}{\sqrt{5}}$. This is the angle where we find the operator (with normalized energy) that gives us maximum response for an ideal plane. Using the same technique as in (29) we can *optimize the plane sensitivity* of the previous shape factor $Q(\vartheta, \kappa)$ in (65) for planes with the formula

$$Q(\vartheta, \kappa) = \frac{2 \cos \vartheta p_{20} \sin \vartheta \sqrt{p_{21}^2 + p_{22}^2}}{\cos^2 \vartheta p_{20} \sin^2 \vartheta (p_{21}^2 + p_{22}^2)} = \left[\vartheta = \arctan \frac{1}{\sqrt{5}} \right] \Rightarrow$$

$$Q_{plane}(\kappa) = \frac{2 \sqrt{\frac{5}{6}} p_{20} \sqrt{\frac{1}{6}} \sqrt{\|\mathbf{p}_2\|^2 - p_{20}^2}}{\frac{5}{6} p_{20}^2 + \frac{1}{6} (\|\mathbf{p}_2\|^2 - p_{20}^2)} = \frac{2 \sqrt{5} p_{20} \sqrt{\|\mathbf{p}_2\|^2 - p_{20}^2}}{\|\mathbf{p}_2\|^2 + 4 p_{20}^2}$$

The positive ideal string is running in the y -direction In Figure 13. Therefore, its energy in the Fourier domain is concentrated with perfect rotationally symmetry in the (u, w) -plane. From (50) we find that in this plane

$C_{20}(u, w) = H_2(u, w) \sqrt{\frac{1}{6}} (u^2 + w^2)$ and $-C_{41}(u, w) = H_2(u, w) \sqrt{\frac{5}{24}} (u^2 + w^2)$. The second derivative response from a unit vector $c_{20} \sin \kappa - c_{41} \cos \kappa$ can then be expressed as

$$f_{string}(\kappa) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} STRING(u, w) [\sin \kappa C_{20} - \cos \kappa C_{41}](u, w) du dw =$$

$$= \left[\sqrt{\frac{1}{6}} \sin \kappa + \sqrt{\frac{5}{24}} \cos \kappa \right] \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} STRING(u, w) H_2(u^2 + w^2) du dw \quad (68)$$

which takes its maximum for $\kappa = \arctan \frac{2}{\sqrt{5}}$. Hence this is the shape angle where we find the operator (with normalized energy) that gives us maximum response for an ideal string. *Optimizing the shape factor $Q(\vartheta, \kappa)$ for string detection* then gives us

$$Q(\vartheta, \kappa) = \frac{2 \cos \vartheta p_{20} \sin \vartheta \sqrt{p_{21}^2 + p_{22}^2}}{(\cos \vartheta)^2 p_{20}^2 + (\sin \vartheta)^2 (p_{21}^2 + p_{22}^2)} = \left[\vartheta = \arctan \frac{2}{\sqrt{5}} \right] \Rightarrow$$

$$Q_{string}(\kappa) = \frac{2 \frac{\sqrt{5}}{3} p_{20} \frac{2}{3} \sqrt{\|\mathbf{p}_2\|^2 - p_{20}^2}}{\frac{5}{9} p_{20}^2 + \frac{4}{9} (\|\mathbf{p}_2\|^2 - p_{20}^2)} = \frac{4 \sqrt{5} p_{20} \sqrt{\|\mathbf{p}_2\|^2 - p_{20}^2}}{4 \|\mathbf{p}_2\|^2 + p_{20}^2} \approx \frac{2 p_{20} \sqrt{\|\mathbf{p}_2\|^2 - p_{20}^2}}{\|\mathbf{p}_2\|^2}$$

Hence, because the optimal string detector is so close to the angle $\kappa = 45^\circ$ in Figure 11, the complete string detector with shape discrimination in η as well as in κ becomes nearly identical to the previous factors $Q_\kappa Q_\eta$, or $Q_\kappa Q'_\eta$. Multiplying these factors with the magnitude $\|\mathbf{f}_2\| = \|\mathbf{p}_2\|$ we get the rotation invariant *string responses*

$$p_{string} = \|\mathbf{p}_2\| Q_\kappa Q_\eta = \frac{2p_{20}\sqrt{p_{21}^2 + p_{22}^2}}{\|\mathbf{p}_2\|} \frac{2\sqrt{3}p_{21}p_{22}}{3p_{21}^2 + p_{22}^2}$$

$$p'_{string} = \|\mathbf{p}_2\| Q_\kappa Q'_\eta = \frac{8}{\sqrt{3}} \frac{p_{20}}{\|\mathbf{p}_2\|} \frac{p_{21}p_{22}}{\sqrt{p_{21}^2 + p_{22}^2}}$$

Let us return again to the shape factor Q_η , which completely suppresses prototypes of type four-wedged orange together with all other prototypes along the c_{22} -meridian in Figure 11-12. Such suppression might be quite reasonable in the equatorial plane where the shape really is c_{22} , a four-wedged Orange, the antithesis of anything that is rotationally symmetric around an axis. But if we move away from the equatorial plane, say, halfway to the pole, following the c_{22} or $\eta = 90^\circ$ meridian, the 3D shape in Figure 11 has changed to something we call an **Oval** and which is a flattened string with elliptic cross-section, a compromise between a plane and a string. In applications where we want to measure the stringness, it is not evident that we should set the stringness value to zero for this shape, which is what the previously deducted measure Q_η will deliver.

Fortunately, if we want to consider such a half-breed as having a non-zero string-ness or non-zero plane-ness value we can simply bias the original $Q_\kappa Q_\eta$ -value with any constant a provided that $-1 < a < 1$.

The non-redundant shape space of Figure 11 is not the common way to present second order variation in three-dimensional functions. Many authors feel content to describe this variation with ellipsoids, the shapes of which are fitted in a triangle [28]. In this triangle, the following three archetypes occupy the three corners: The sphere, the flat plane-like ellipsoid, and the elongated string-like ellipsoid. This shape space is a subset of the complete shape space of Figure 11, located at the top and bottom close to the blob and the negative blob, respectively. These ellipsoidal shapes appear when all eigenvalues (all prototype derivatives) have the same polarity. At the very boundary of these areas g_{yy} is zero in the upper half, g_{xx} in the lower half in Figure 11. The upper

boundary is a part of a great circle which runs in the plane normal to the unit vector $-\sqrt{\frac{5}{6}}g_{yy}$ (the Ideal negative Plane) and which encompasses the two unit vectors Optimal string detector and Optimal string detector. The lower boundary is a great circle which runs in a plane normal to the unit vector $\sqrt{\frac{5}{6}}g_{xx}$ (the Ideal plane) and which encompasses the two unit vectors Optimal negative string detector and Optimal negative plane detector. As could be expected, the two areas where all eigenvalues have the same sign cover only a part of the total shape space. Therefore, it seems important to underscore that the ellipsoidal model for second order variation is valid only for cases where the tensor is known to have eigenvalues of the same polarity. Among the cases where this condition is upheld are diffusion coefficients [28], covariance matrices, and quadrature filter responses [30]. It is not upheld for the Hessian, the second order variation of a multidimensional function.

4. Application example: Detection of blood vessels in MRI volume data

Based on the insights provided in the previous section, we have implemented and applied the following **string detection** algorithm. The input volume is $f(x, y, z)$, the output volume is $String(x, y, z)$, both of which are embedded in a scale-space pyramid described in Appendix A. See also Figure 14. The relative steps in scale are (approximately) $\frac{1}{\sqrt{2}}$ and the number of data points to be computed is $6MNL(1 + 1 + \frac{1}{8} + \frac{1}{8} + \frac{1}{64} + \frac{1}{64} + \dots)$. Here, the factor six stems from the six derivatives, MNL is the number of grid points in the 3D input volume, and the terms inside the parenthesis reflect the decreasing voxel numbers for every second level of the pyramid. Using separable generalized Sobel operators as in [9] the total number of operations in Step 1 is not more than $29MNL(1 + 1 + \frac{1}{8} + \frac{1}{8} + \frac{1}{64} + \frac{1}{64} + \dots)$, all of which are simple additions and subtractions.

- Step 1. Convolution.** Convolve the input data volume with highly separable convolution kernels of type generalized Sobel to obtain the six second derivatives in (x, y, z) .
- Step 2. Energy discrimination.** Compute the total second degree energy $\|f_2\|^2(x, y, z)$ as in (63). For energies below a certain threshold of interest $String(x, y, z) \Leftarrow 0$ and quit all computations for (x, y, z) . Else
- Step 3. Shape discrimination I.** Compute $f_{20}(x, y, z)$ and a preliminary, still incomplete quadratic shape factor Q_κ as in (65). For Q_κ below a certain threshold of interest, $String(x, y, z) \Leftarrow 0$ and quit all computations for (x, y, z) . Else
- Step 4. Eigen values.** Compute the eigenvalues of the Hessian with early termination ($String(x, y, z) \Leftarrow 0$) for non-axially symmetric shapes as described in section 3.4.
- Step 5. Prototype derivatives and prototype spherical harmonic responses.** Map the three eigen values onto the second derivatives (p_{xx}, p_{yy}, p_{zz}) and compute the harmonic responses with (54).
- Step 6. Shape discrimination II.** Compute the shape factor $Q_\kappa Q'_\eta$ as in section 3.4. For $Q_\kappa Q'_\eta$ smaller than a certain threshold, let $String(x, y, z) \Leftarrow 0$ and quit all computations for (x, y, z) . Else
- $$String(x, y, z) \Leftarrow p'_{string} = \|p_2\| Q_\kappa Q'_\eta$$
- Step 7. Orientation.** If orientation is required, compute the orientation $(\beta, \gamma)(x, y, z)$ of the symmetry axis as in (60) – (62).
- Step 8. Scale-space merge.** Merge the scale-space result by saving max responses from all levels of scale

As illustrated in Figure 14, the above algorithm exploits a number of shortcuts, which will improve the processing speed dramatically. The first such shortcut is after *Step 2*. If the total second-degree signal energy is below a certain threshold we might decide not to spend more time on the measurements in this specific grid point of the volume. Having passed this test, we might find after *Step 3* that the computed shape factor is below another, rather liberal threshold and further investigation of this grid point and its responses can be abandoned. Only then are we taking on the more cumbersome task in *Step 4* of computing the eigenvalues. However already here we will have a chance to detect and discriminate against non-axially symmetric cases, finish early and quit the subsequent steps. Hence, this shape discrimination in the η -direction is of the same nature as, and a possible replacement for, the subsequent *Step 6*. From the prototype responses (p_{20}, p_{21}, p_{22}) computed in *Step 5* and the more precise shape factor $Q_\kappa Q'_\eta$ we may find after all in *Step 6* that the specific shape factors were not reaching the required threshold. For this reason or because orientation is not an issue for the application in question, the final *Step 7* and *Step 8* might not be necessary.

Figure 14. A string detection procedure

We have implemented and tested the above algorithm for image enhancement of blood vessels in an MRI-volume using only shape discrimination (Step 7 not included). The scale-space result is merged into a single final output volume by going from coarse-to-fine scale while saving maximum string response. The input volume comprises 25 transversal slices, each slice consisting of 236×176 voxels. In Figure 15, a) and b), we see the 8th slice before and after enhancement. Although the larger vessels are bright and clear, smaller vessels are not so easy detectable in the input data. Furthermore, many other structures besides the vessels have delivered strong signals to this MR-image. Most of these are suppressed in the enhanced image and the beneficial effect of this is illustrated clearly in Figure 15 c) and d), which show a Maximum Intensity Profiles (MIP) before and after enhancement. Figure 16 shows density profiles along two pixel rows of the 9th slice. The powerful shape discrimination is clearly visible in these diagrams. The enhanced image profile contains almost nothing but blood vessel information while the original signal has a multitude of peaks of the same magnitude as the blood

vessels themselves. Figure 17 shows a volume rendered image with three different parameter settings for the input (left) and the enhanced data (right). Slight remnants of the background structure is seen to the right and is often wanted for geometric reference.

Compared to the fingerprint experiment the 3D-image enhancement procedure presented here has fewer levels of processing and does not include local averaging of orientation information. In the 2D-case, we found that such averaging had to take into account that the symmetry axes for derotated ridges and valleys were 90° apart. In the 3D-case, the same precautions are necessary. A positive string or plane will be aligned with a negative counterpart only if their components $(f_{21}, f_{22}, f_{23}, f_{24}, f_{25})$ are all multiplied with $Sgn(f_{20})$.

Figure 15. MR angiography volume data from the neck, $236 \times 176 \times 25$ voxels. Upper: Slice 8 before and after processing. Bottom: Maximum Intensity Projection (MIP) before and after processing

Figure 16. Intensity profiles slice, at rows $y = 103$ (left) and $y = 114$ (right). Dotted curves: input-signal, solid curves: after enhancement. 16. Input and output data subjected to volume rendering with three different parameter settings

Figure 17. Input and output data subjected to volume rendering with three different parameter settings

5. Comparisons and references to other methods

The concepts of **prototype** and **derotation** can be traced back to Lenz [34], who also describes the decomposition of an arbitrary 3D-function into **spherical harmonics**. However, the connection to the Hessian and eigen values are missing in this early work as well as a derotation algorithm for the second degree variation in 3D. In a later contribution [35] Lenz molds the rotation invariance problem in group-theoretical terms and proposes a derotation algorithm which is similar to the following.

Burel and Hénocq are using **spherical harmonics** of different degrees (mainly degree 1 and 2) to determine with high precision the difference in orientation between a rotated 3D-object and the same object in a standard position [5], [6]. The link between derivatives and spherical harmonics is not established. Also, the scope is somewhat different from the local feature detection and low-level image analysis problem of the present paper. Typically, the rotated object in the intended applications is macroscopic and rich in structural information. Assuming that the unknown rotation is to be described as a sequence of three planar rotation steps the spherical harmonic responses of the test object can be described by a 5x5 matrix multiplied with the object in a standard position (cf. the prototype concept in this paper). In various combinations the arguments of the three angles appear as parameters in the elements of this matrix. The matrix is in principal identical to the rotator in Appendix C of this paper. The angles can be solved analytically via a higher degree equation system or iteratively as in [13]. However, as mentioned above in the present paper, a closer look at this latter scheme reveals that it is identical to the Jacobi method for diagonalizing a symmetric matrix [Press88], which in this case is the Hessian. In [5], [6] it is also demonstrated that the spherical harmonics method has a link to the “inertia method” and moment functions. However, these moment functions are not the second order polynomials that we find in the Fourier as well as in the signal domain in eq. (12) and (50). In any case it is shown that inertia matrices are lacking much information of the object, information which is indeed taken into account by expanding the object into spherical harmonics.

Koller et al [32], [33] have proposed a method of 3-D feature discrimination using the **Hessian**. After computing and sorting the eigenvalues according to magnitude, a curvilinear structure called *line* (in this paper: string) is detected when the two largest ones have (approximately) the same magnitude and sign (negative for bright lines, positive for dark lines) whereas the third one is close to zero. The geometrical mean $\sqrt{\lambda_1 \lambda_2}$ of the two largest eigenvalues are then used as a measure of “string-ness”. Standing by itself, this measure seems to be lacking much of the discriminative power of the $Q_\kappa Q_\eta$ -factor in section 3.4. In [33] detection of both strings and planes are discussed. Considerable attention is given to the problem of “false” second derivative responses at edges. A non-linear combination of a pair of edge responses is suggested as an alternative to the second derivatives.

Lorenz et al [41], [42] are using the **Hessian** of 3D second derivatives in a similar manner to what has been presented in this paper. After deriving the eigen values, their relative signs and magnitudes are exploited to distinguish between strings (called lines) and planes, although the given measure for “line-ness”

$$\lambda_{\text{magnitude}} = \frac{|\lambda_1| + |\lambda_2|}{2|\lambda_3|} \quad (69)$$

seems to discriminate poorly against “plane-ness”. Hence, the specificity of our measure $Q_\kappa Q'_\eta$ should be superior. Furthermore, the fact that in our algorithm, the eigenvalue computation is avoided for uninteresting data points should make it considerably faster. In [17] Frangi et al distinguish clearly between strings and planes using more refined eigenvalue analysis, although the dual-cone shape is not identified.

Squared first derivatives for line detection was first proposed by Bigun in [2]. In 2D, the local first derivatives (the gradient) expressed as $f_x + if_y$ are used to produce $(f_x + if_y)^2 = (f_x^2 - f_y^2, if_x f_y)$. The magnitude of this complex quantity is the gradient energy while the argument equals the double gradient orientation angle. The

quantities are then weighted and summed (a second convolution step) over a larger neighborhood. Since a line profile constitutes gradients in its neighborhood this operator also responds (indirectly) to second derivatives. This idea is further developed and generalized in [3] where the squared first derivatives over the larger neighborhood are weighted in a matrix $\bar{J}$ called inertia matrix. We note that the diagonal element consist of sums several terms. In the 3D-case, this matrix in [3] yields

$$\bar{J} = \begin{bmatrix} \overline{f_y^2 + f_z^2} & \overline{-f_x f_y} & \overline{-f_x f_z} \\ \overline{-f_x f_y} & \overline{f_x^2 + f_z^2} & \overline{-f_y f_z} \\ \overline{-f_x f_y} & \overline{-f_y f_z} & \overline{f_x^2 + f_y^2} \end{bmatrix} \quad (70)$$

where the $\overline{\text{bar}}$ stands for weighted averaging. A large value for $\overline{f_y^2 + f_z^2}$ may be interpreted as a measure of variation, the variation in the x -direction excluded. In the local Fourier domain of the very same larger neighborhood, the corresponding component of $\hat{J}$ is therefore rightfully interpreted as the energy content in a plane, the normal of which is the u -axis. The components of $\bar{J}$ are also interpreted as moments. The moment of inertia $\bar{V}_f^2$ with respect to the axis of orientation, the **symmetry axis** is given by the unit vector k as follows.

$$\bar{V}_f^2 = k^T \bar{J} k \quad (71)$$

Finding the *least* eigenvalue λ_0 of $\bar{J}$ and inserting the corresponding eigenvector k_0 in (71) minimizes $\bar{V}_f^2$. Therefore, this vector is the best fit for a linear symmetry axis through the neighborhood along which we find a minimum of variation.

There are certain similarities with this method and the string/plane detector technique presented in this paper, although the tensor $\bar{J}$ is not so easily understood and analyzed, primarily because of the *non-linearity* introduced by squaring operation before the smoothing. Since there are only three degrees of freedom in the original measurements (f_x, f_y, f_z) the “derotation” of an unsmoothed tensor J will result in one energy response V_f^2 and one axis orientation (β, γ) . Clearly, so far there is no room for any shape information. Hence, we will not know whether a large response V_f^2 is caused by an unusually high general signal level and a poor symmetry shape or by a moderate signal level and perfect linear symmetry. The smoothed tensor $\bar{J}$, however, has more degrees of freedom than three, which allows for certain embedded shape information.

In Haglund’s thesis [23] the same squared first derivatives are also smoothly weighted but used to form a somewhat different tensor. In the 2D-case we get the **Smoothed Squared Gradient Tensor** $\bar{G}$ as

$$\bar{G} = \begin{bmatrix} \overline{f_x^2} & \overline{f_x f_y} \\ \overline{f_x f_y} & \overline{f_y^2} \end{bmatrix} \quad (72)$$

The unsmoothed tensor is $G = \mathbf{f}_1 \cdot \mathbf{f}_1^T = \|\mathbf{f}_1\|^2 \begin{bmatrix} g_x \\ g_y \end{bmatrix} \begin{bmatrix} g_x & g_y \end{bmatrix}$ is a tensor, which represents the gradient with squared magnitude and double angle orientation. It is shown in [49] by van Vliet and Verbeek that the eigenvector of the largest eigenvalue λ_1 of the smoothed tensor $\bar{G}$ equals the average gradient orientation. Furthermore, the eigenvalues of $\bar{G}$ are used in an elliptic shape model preassuming that both eigenvalues are

positive. The relation λ_2 / λ_1 is taken as a measure of asymmetry. As a bonus, the operation it is said to respond to both 1st and 2nd degree variations, i.e. edges and lines, which is somewhat doubtful since on the first detection level, all even components of the signal are neglected. Even so, the smoothed squared gradient tensor is claimed to be especially well suited to detect and segment objects and patterns characterized by one-dimensional textures. In a paper by van Kempen et al [27] the smoothed 3D gradient tensor is applied to confocal microscopy 3D-volumes and used to distinguish between what in this paper is called planes and strings.

Detection of one-dimensionality is the aim and goal of the much older **quadrature filter** approach by Granlund and Knutsson [20], [29]. Here, both odd and even components of the local signal variation is detected and taken into account on the lowest level. Because of the quadrature, however, in the 2D-case the orientation ϕ of the odd squared component is limited to the same interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$ as the even squared component. The filter response

f_ϕ is computed as $\sqrt{(f \otimes a_\phi)^2 + (f \otimes b_\phi)^2}$ where the filters a_ϕ and b_ϕ are akin to first and second degree derivators. We notice that this filter is linear in the sense that the response grows linearly with the signal level of f . By careful matching of the frequency content in the odd and even filter kernels a and b, respectively, it is possible to design a quadrature filter that occupies a well-defined sector of the Fourier domain. The technique is basically a compass method employing a set of identical but rotated filters for at least three orientations (ϕ_1, ϕ_2, ϕ_3) in the 2D-case. Thus the works [20] and [29] predates the steerable filter publications [15] with approximately ten years. Interpolation is required to obtain orientation and each filter is carefully designed in the Fourier domain to get the correct angular overlap. The filters are not orthogonal.

Originally developed for 2D-images, this technique was later transformed by Knutsson to the 3D-case in [30] and [21] with the orientation interval limited to a unit hemisphere. Hence it is understood that these filters only detect the axially rotationally symmetric variation of the signal. To properly cover the 3D Fourier domain a minimum of six filters are needed. The filter responses are transformed into elements of a 3x3 matrix, which is called the *orientation tensor*. Like the Hessian, this tensor contains information on both shape and orientation, which are retrievable by computing eigenvalues and eigenvectors. However, since all eigenvalues are positive the shapes are limited to the upper part of the shape space in Figure 11. It is not clear what the operator response will be for a second-degree variations of type double-cone. A non-orthogonal operator set based on polynomial functions of order 1 and 2 is presented in [16]. Without mapping onto an orthogonal function space, these responses are used in a quadrature style to derive an orientation tensor along the lines of [39], [21].

A specific feature of the quadrature filters is there ability to give a *phase independent response* to a frequency component, a sinusoidal wave in the image. At the crest of this wave, the filter kernel b_ϕ will give maximum response while the kernel a_ϕ will give maximum response where the gradient of the sinusoid has its maximum. Evidently, the given kernel size should match the wavelength of the sinusoid to give a perfectly phase independent response.

It might now be interesting to see if we may extend the orientation and derotation concept of this paper to achieve such a phase-independent response. To this end we first convolve the 2D-image function f with all derivators of first and second degree to establish the following derotation equations.

$$\begin{aligned} f_1 &= f \otimes \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} g_x \\ g_y \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \end{bmatrix} = \|f_1\| \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ f_2 &= f \otimes \frac{1}{\sqrt{3}} \begin{bmatrix} 1 & 1 & 0 \\ \sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & \sqrt{8} \end{bmatrix} \begin{bmatrix} g_{xx} \\ g_{yy} \\ g_{xy} \end{bmatrix} = \begin{bmatrix} f_{20} \\ f_{21} \\ f_{22} \end{bmatrix} = \|f_2\| \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos 2\beta & -\sin 2\beta \\ 0 & \sin 2\beta & \cos 2\beta \end{bmatrix} \begin{bmatrix} \sin \kappa \\ \cos \kappa \\ 0 \end{bmatrix} \end{aligned} \quad (73)$$

These two equation systems are fully determined; none is over-determined. The derotation solution consists of two magnitudes, two rotation angles and one shape angle. We assume that the two sets of derivators are designed

so that their frequency responses are as similar as possible. These responses of first and second degree may then be utilized as follows in a quadrature approach. The definition of $\mathbf{f}_{line}$ is the same as in (32).

$$\begin{aligned}
 \mathbf{f}_{edge} &\equiv \frac{1}{\|\mathbf{f}_1\|} \begin{bmatrix} f_x^2 - f_y^2 \\ 2f_x f_y \end{bmatrix} = \|\mathbf{f}_1\| \begin{bmatrix} \cos 2\alpha \\ \sin 2\alpha \end{bmatrix} \\
 \mathbf{f}_{line} &\equiv \frac{2f_{20}}{\|\mathbf{f}_2\|} \begin{bmatrix} f_{21} \\ f_{22} \end{bmatrix} = \frac{2f_{20}\sqrt{f_{21}^2 + f_{22}^2}}{\|\mathbf{f}_2\|} \begin{bmatrix} \cos 2\beta \\ \sin 2\beta \end{bmatrix} \\
 \mathbf{f}_{quad} = \mathbf{f}_{edge} + \mathbf{f}_{line} &= \begin{bmatrix} \|\mathbf{f}_1\|^{-1}(f_x^2 - f_y^2) + \|\mathbf{f}_2\|^{-1}2f_{20}f_{21} \\ \|\mathbf{f}_1\|^{-1}2f_x f_y + \|\mathbf{f}_2\|^{-1}2f_{20}f_{22} \end{bmatrix}
 \end{aligned} \tag{74}$$

The magnitude response $\|\mathbf{f}_{quad}\|$ depends on the double-angle difference $2(\alpha - \beta)$ between the line and the edge components, so that just as for the filters in [20], [29]

$$\begin{aligned}
 \alpha - \beta = 0 \text{ or } \pi &\Rightarrow \|\mathbf{f}_{quad}\| = \|\mathbf{f}_{edge}\| + \|\mathbf{f}_{line}\| \\
 \alpha - \beta = \frac{\pi}{2} \text{ or } -\frac{\pi}{2} &\Rightarrow \|\mathbf{f}_{quad}\| = \|\mathbf{f}_{edge}\| - \|\mathbf{f}_{line}\|
 \end{aligned} \tag{75}$$

Pros and cons for this alternative method to compute quadrature responses in 2D- and 3D-images remain to be investigated.

We mentioned earlier in section 2 that the derivative theorem for Fourier transforms, which represent derivators as polynomials and the theorem for preservation of harmonics, creates a strong link between derivatives and polynomials, i.e. moment functions. Since **moment invariants** have generated a substantial literature, e.g. [25], [45], [46], [7], [18], [4], [19] it seems appropriate to comment on the relation between the present treatise and these concepts. We notice again that the polynomial functions appear in both domains in (12) and (50) and that the linear combinations $\mathcal{A}$ brings about very similar orthogonal basis sets. Hence, it is only natural that some of the invariants brought forward in this paper have appeared previously in the literature on moment functions. In fact, for the 2D-case, the so-called complex moments, correspond to the orthogonal (but not normalized) basis set of circular harmonics given in complex form. Likewise, in the 3D-case, the complex moments correspond to orthogonal spherical harmonic functions in complex form as given in [47]. However, the two domains in (12) and (50) are not quite symmetric. The zero-order harmonic components correspond over a different Hankel transform than the other five. Therefore, it is only in the Fourier domain that all six basis function have the same radial variation so that the six moment functions in the Fourier domains span the six-dimensional orthogonal space of spherical harmonic functions via $\mathcal{A}$. The six polynomials in the signal domain cannot be used in the same manner.

6. Conclusions

In this paper we have made two initial claims. One of these is that second derivatives have not yet been properly exploited in low level image processing. The other one is that second order variations, for several reasons, might be inherently significant in two-dimensional image formation and vision. Leaving the first claim aside, we would like to bring forward further arguments for the second claim.

Physiological experiments show that the retina as well as the visual cortex in mammal vision systems contains neurons which act as Laplacian operators. Orientation specific cells were found in the visual cortex area. As shown by Hubel and Wiesel [22], these cells react preferably to *line-shaped* stimuli, and they seem to tile the visual cortex area into regions of about 0.5 mm^2 . Each such region is responsible for line detection in a small region of the visual field. Thus, via the retinas the visual field is topologically mapped patch by patch onto the visual cortex. Two regions, serving the left and right eye, respectively, which receive stimuli from the same point of the visual field are located side by side on the cortex area forming interlaced bands.

Hence, the cells in a certain region are triggered only if the line-shaped object is at a certain position in the visual field. However, a response from the microelectrode inserted in the cortex will be detected only if the specific electrode position within the region matches a specific *orientation* of the line. The traditional interpretation of this effect is that each region contains a set of cells a “super-cell”, which form a compass detector set, each cell being sensitive to one specific direction. We think it would be worthwhile to test another hypothesis, namely that the observed line detection effect could arise from an architecture such as the one we have proven useful in Section 2 of this paper. For every function, Nature tends to produce the most economic hardware, which would be to determine orientation by phase rather than by the compass principle. Following this simplicity rule, one should replace the expensive compass super-cell model by the three cells with excitation and inhibition inputs distributed as the operator kernels (b_{20}, b_{21}, b_{22}) in Figure 3, possibly followed by a line detection cell producing a signal akin to the one in eq. (33).

From a practical point of view the main goal of this treatise is to deliver measures of magnitude, shape, and orientation for the local second order variation in images. To achieve this purpose and to design actual algorithms we have used some unconventional models and concepts, e.g. the prototype and the derotation equation. The latter seems to be a novelty, at least in the present form. Although it would be possible to deal with the 2D-problem without these two vehicles, we feel that they are almost indispensable for the 3D-case.

The virtue of orthogonal basis functions has been clearly demonstrated. Fortunately, the lack of orthogonality of second order derivators is not a problem. In the Fourier domain these operators appear as polynomials, alternatively as trigonometric functions. Well-known linear mappings onto the orthogonal function spaces of circular and spherical harmonics can then be applied. Correct normalization is important. It might be a novelty to many readers of this article that the two operators g_{xx} and g_{yy} are not orthogonal and that the total rotation and shape invariant second derivative signal energy is the one given in (21) and (63). With the local response vector firmly embedded in an orthonormal function space all kinds of rotation-invariant measures are easy to obtain. Moreover, they can be given an intuitive geometrical meaning. In this sense we believe that Figures 3, 4, 5, 6, 9, 10, 11 are important contributions to the art of image processing.

Speed is the main reason why we in the first step of our algorithms estimate derivatives and not directly compute the circular or spherical harmonic responses. The derivators are perfectly separable into surprisingly simple kernels which all require nothing but additions and subtractions. As illustrated in Appendix 1 to retrieve the derivatives in a multilevel scale space becomes highly efficient which is especially important for the 3D-case, where six derivatives are to be computed. Full resolution at all levels would have become rather costly, while a straight-forward pyramid, halving the resolution at each step, would produce kernels of type triangular functions, which is a poor approximation of Gaussians. In our proposal, two images are generated at each resolution level with the relative scale factor of $\sqrt{2}$, which helps to preserve the Gaussian shape of the kernels reasonably well. The number of operations is still not more than doubled compared to the simplest pyramid scheme. Another feature of the pyramid in Appendix 1 is the half resolution unit offset between the grid-points at one resolution

level relative to the grid-points at all other levels. Hereby, all father-son relation sets become non-intersecting, which simplifies algorithms of type split-and merge in the scale space.

Speed is also the hallmark of our string enhancement procedure in Section 4. Just as the fingerprint experiment in Section 2, this is just an example to illustrate and validate the more theoretical parts of the treatise. Nevertheless, we feel that the blood vessel example shows how even seemingly overwhelmingly heavy computations can be reduced by fully exploiting a thorough theoretical understanding. As soon as we have found the rotation-invariant response energy to be too low at a certain grid-point, we avoid further processing. Furthermore, since the Laplacian response is so easy to retrieve we may also do a shape specific discrimination before we decide to enter the derotation procedure. Also the more demanding eigenvalue computation can be finished early with a proper test condition. The full computation scheme is carried out for just a small portion of the total number of grid-points.

As should be obvious from any of the derotation equations in this treatise, shape and orientation retrieval goes together in second derivative analysis. The authors of this paper think of orientation as synonymous with rotation, rotational position, or pose. In the 2D-case where orientation is defined by one single free parameter, normally an angle, the synonymy is unquestionable and trivially true. However, in the 3D-case this synonymy might be less clear. To define rotation of a general arbitrary 3D object, or a local 3D-signal variation, we need three free parameters. These can be defined by a 3x3 matrix with three degrees of freedom or with three Euler angles. But should we really name any of these representations *orientation*? For objects with various forms of symmetries the case is more obvious, since the rotation/orientation can be described with fewer than three parameters. By limiting the shape space to what we have called axially symmetric objects, both the shape space and the orientation space loses one dimension, which means that the second derivative response vector has not six but four degrees of freedom for these shapes. Therefore, when the derotation has aligned an axis (the symmetry axis) of the object with one of the coordinate axes, the result is akin to the two angles for latitude and longitude on the globe. And surely, we all agree that these two angles constitute the orientation of the object.

For the general 3D-object an obvious symmetry axis is missing and this is probably the reason why the pose of such an object is not so naturally taken to be the same as orientation. Fortunately, as we see from Figure 11, for all the prototypes of second derivative responses there is a unique pose. Therefore, we think it is quite natural to define the orientation of a local second order variation as the rotator that transforms the actual neighborhood response to its prototype response. If we are forced to name a common axis for all the prototypic shapes in Figure 11, it should be the z -axis along which the basis function c_{22} has no variation. In reality it is enough to realize that all the other shapes have an orientation, which is fixed with respect to the signal space coordinates. Thereby, the rotated version of each prototype indeed has a unique orientation. This fact notwithstanding we may use more practical orientation definitions for certain applications. For instance, if we have to trace a positive string structure more globally, it may be practical to define the orientation with two polar angles with the y -axis as symmetry axis rather than the z -axis since the prototype for these shapes is oriented along the y -axis in Figure 11.

This treatise has not been presenting any full-fledged applications that would satisfy an end user such as a radiologist. We will deal in depth with problems such as segmentation and visualization in future papers but we hope and expect the methods and algorithms presented in this treatise to be a cornerstone in these efforts. Various applications will raise the need for various secondary procedures to be applied immediately following the low-level neighborhood analysis presented in this paper. These things are clearly outside the scope of the present paper.

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Appendix A

The Fourier transform of two-dimensional functions preserves harmonic angular variation

Only the 2D-case is treated here, where functions with harmonic angular variation of order n can be described with the complex-valued expression

$$h_n(r, \phi) = h_n(r) [\cos n\phi + i \sin n\phi] = h_n(r) e^{in\phi}$$

which also means that we are going to consider the two operators $h_n(r) \cos n\phi$ and $h_n(r) \sin n\phi$ in simultaneously as one single complex-valued function. For $n = 0$ we have $h_0(r)$, the 2D-Fourier transform F_2 of which yields

$$\begin{aligned} F_2[h_0(r)] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_0(r) e^{-i2\pi(ux+vy)} du dv \quad [x - iy = r^{-i}\phi, u + iv = e^{i\psi}] \\ &= \int_0^{\infty} \int_0^{2\pi} h_0(r) e^{-i2\pi r \rho \cos(\psi-\phi)} d\phi r dr = \int_0^{\infty} h_0(r) \int_0^{2\pi} e^{-i2\pi r \rho \cos\beta} d\beta r dr \\ &= 2\pi \int_0^{\infty} h_0(r) J_0(2\pi r \rho) r dr = \mathbf{H}_0[h_0(r)](\rho) \stackrel{\Delta}{=} H_0(\rho) \end{aligned}$$

where J_0 is the Bessel function of zero order and $\mathbf{H}_0$ is the Hankel transform of zero order, which transforms the one-dimensional radial function $h_0(r)$ in the signal domain to $H_0(\rho)$ in the Fourier domain. For $n \neq 0$ we get

$$\begin{aligned} F_2[h_n(r) e^{in\phi}] &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h_n(r) e^{in\phi} e^{-i2\pi(ux+vy)} du dv = [x - iy = r^{-i}\phi, u + iv = e^{i\psi}] \\ &= \int_0^{\infty} \int_0^{2\pi} h_n(r) e^{in\phi} e^{-i2\pi r \rho \cos(\psi-\phi)} d\phi r dr = \int_0^{\infty} h_n(r) e^{-in\psi} \int_0^{2\pi} e^{-i2\pi r \rho \cos(\psi-\phi)} e^{in(\psi-\phi)} d\phi r dr \\ &= e^{in\psi} e^{-in\frac{\pi}{2}} \int_0^{\infty} h_n(r) \int_0^{2\pi} e^{-i2\pi r \rho \cos(\psi-\phi)} e^{-in(\psi-\phi-\frac{\pi}{2})} d\phi r dr \quad \left[e^{-in\frac{\pi}{2}} = (-i)^n, [\psi - \phi - \frac{\pi}{2} = \alpha] \right] \\ &= e^{in\psi} (-i)^n \int_0^{\infty} h_n(r) \int_0^{2\pi} e^{-i2\pi r \rho \sin\alpha} e^{-in\alpha} d\alpha r dr = \\ &= e^{in\psi} (-i)^n \int_0^{\infty} h_n(r) J_n(2\pi r \rho) r dr = e^{in\psi} (-i)^n \mathbf{H}_n[h_n(r)](\rho) \stackrel{\Delta}{=} H_n(\rho) e^{in\phi} \end{aligned}$$

where J_n is the Bessel function of order n and $\mathbf{H}_n$ is the Hankel transform of order n , which transforms the one-dimensional function $h_n(r)$ in the signal domain to $H_n(\rho)$ in the Fourier domain. Obviously, the harmonic variation is preserved over the Fourier transform. Clearly, the proof will hold also for the general form (2) for rotation-invariant operators with radius-dependent phase $\phi_n(r)$.

Appendix B. A computationally efficient scale-space pyramid

We describe only the one-dimensional case. The 2D- and 3D-operators are perfectly separable and the generalizations to these and higher dimensions are rather straightforward. The input signal is smoothed in a series of convolutions. See Figure 1. The first five employ the simple smoothing kernel $(1, 1)$ which at the next resolution level is followed by *four* convolutions with the kernel $(1 \ 0 \ 1)$, at the next level by *four* convolutions with the kernel $(1 \ 0 \ 0 \ 0 \ 1)$, followed at the next resolution level by another *four* convolution kernels etc. Each arrow is a convolution operation. At each new level the effective length of the kernels is doubled. These kernels allow us to decrease the resolution at each level with a factor of two, while we attain a succession of smoothing kernels that only moderately deviate from optimally sampled Gaussians. Therefore, at each level the smoothing steps all consist of summing two neighboring samples. The output pyramid is receiving data from two taps at each level.

Figure 1. Appendix B. A one-dimensional scale-space pyramid

The smoothing preceding the different taps is best described by the variance of the effective kernels, which amount to what is shown in the following table.

Table1. Appendix B. Variance of the Gaussian kernels at different level of the scale-space pyramid

Tap No	1	2	3	4	5	6	7	8	9
σ^2	0.5	1.0	2.25	4.25	9.25	17.25	37.25	69.25	149.25
σ	0.707	1.0	1.5	2.06	3.04	4.15	6.10	8.32	12.21

Note that the standard deviation (measured in sampling distances for the input signal) increases with a factor which is rather constant and $\approx \sqrt{2}$. On each level there are two taps where the second derivative f_{xx} is made available via differentiating kernels of type $(1 \ -1) \otimes (1 \ -1) = (1 \ -2 \ 1)$. Effectively, including the preceding smoothing, the first four second-degree derivator kernels have the following weight coefficients in the resolution of the original input signal. All of them seem to deviate only marginally from the ideal “Mexican hat”.

Tap 1 5-point kernel $(1 \ 0 \ -2 \ 0 \ 1)$

Tap 2 7-point kernel $(1 \ 2 \ -1 \ -4 \ -1 \ 2 \ 1)$

Tap 3 12-point kernel, half of which is $(1 \ 5 \ 9 \ 5 \ -6 \ -14 \ \dots)$

Tap 4 16-point kernel, half of which is $(1 \ 5 \ 11 \ 15 \ 13 \ 1 \ -17 \ -29 \ \dots)$

In the 3D-case, each basic smoothing step utilizes a $2 \times 2 \times 2$ kernel, (Figure 2, left) the convolution by which is executed by summing two neighbors in the three directions. The derivators at each tap position are now six $3 \times 3 \times 3$ kernels (the g_{xx} kernel is shown below, right). Thanks to their exceptional separability (factorization) and common factors, they can be executed using only 23 operations of type addition or subtraction between neighboring samples.

Figure 2. Appendix B. The three-dimensional kernels for smoothing and double differentiating

The **pyramid** can be seen as an **oct-tree**, where one voxel at a certain level is a father of eight sons. For the one-dimensional signal $f(x)$ above the oct-tree is reduced to a “bi-tree” where each father has two sons and this is illustrated in the Figure 3. It should be noted that the unambiguous farther-son hierarchy is due to the extra smoothing step in at the first level. Hereby, the data-points in all of the coarser levels 2, 3, 4, 5 are forced to occupy sampling positions that fall in-between the original samples at Level 1 and also to avoid occupying overlapping positions between themselves.

Figure 3, Appendix B. A one-dimensional scale-space tree with noninterseting father-son relations

Appendix C*Table 1, Appendix C. The 6 x 6 element rotator $[R(\alpha, \beta, \gamma)]$ for 3D second order harmonics. For axially symmetric shapes, $\cos \alpha \Leftarrow 1$, $\sin \alpha \Leftarrow 0$*

1	0	0	0	0	0
0	$\frac{1}{4}(3 \cos 2\beta + 1)$	$\frac{\sqrt{3}}{4} \cos 2\alpha (1 - \cos 2\beta)$	$-\frac{\sqrt{3}}{4} \sin 2\alpha (1 - \cos 2\beta)$	$-\frac{\sqrt{3}}{2} \cos \alpha \sin 2\beta$	$\frac{\sqrt{3}}{2} \sin \alpha \sin 2\beta$
0	$\frac{\sqrt{3}}{4} (1 - \cos 2\beta) \cos 2\gamma$	$\frac{1}{4} \cos 2\alpha (\cos 2\beta + 3) \cos 2\gamma$ $-\sin 2\alpha \cos \beta \sin 2\gamma$	$-\frac{1}{4} \sin 2\alpha (\cos 2\beta + 3) \cos 2\gamma$ $-\cos 2\alpha \cos \beta \sin 2\gamma$	$\frac{1}{2} \cos \alpha \sin 2\beta \cos 2\gamma$ $-\sin \alpha \sin \beta \sin 2\gamma$	$-\frac{1}{2} \sin \alpha \sin 2\beta \cos 2\gamma$ $-\cos \alpha \sin \beta \sin 2\gamma$
0	$\frac{\sqrt{3}}{4} (1 - \cos 2\beta) \sin 2\gamma$	$\frac{1}{4} \cos 2\alpha (\cos 2\beta + 3) \sin 2\gamma$ $+\sin 2\alpha \cos \beta \cos 2\gamma$	$\frac{1}{4} \sin 2\alpha (\cos 2\beta + 3) \sin 2\gamma$ $-\cos 2\alpha \cos \beta \cos 2\gamma$	$\frac{1}{2} \cos \alpha \sin 2\beta \sin 2\gamma$ $+\sin \alpha \sin \beta \cos 2\gamma$	$-\frac{1}{2} \sin \alpha \sin 2\beta \sin 2\gamma$ $+\cos \alpha \sin \beta \cos 2\gamma$
0	$\frac{\sqrt{3}}{2} \sin 2\beta \cos \gamma$	$-\frac{1}{2} \cos 2\alpha \sin 2\beta \cos \gamma$ $+\sin 2\alpha \sin \beta \sin \gamma$	$\frac{1}{2} \sin 2\alpha \sin 2\beta \cos \gamma$ $+\cos 2\alpha \sin \beta \sin \gamma$	$\cos \alpha \cos 2\beta \cos \gamma$ $-\sin \alpha \cos \beta \sin \gamma$	$-\sin \alpha \cos 2\beta \cos \gamma$ $-\cos \alpha \cos \beta \sin \gamma$
0	$\frac{\sqrt{3}}{2} \sin 2\beta \sin \gamma$	$-\frac{1}{2} \cos 2\alpha \sin 2\beta \sin \gamma$ $+\sin 2\alpha \sin \beta \cos \gamma$	$\frac{1}{2} \sin 2\alpha \sin 2\beta \sin \gamma$ $-\cos 2\alpha \sin \beta \cos \gamma$	$\cos \alpha \cos 2\beta \sin \gamma$ $+\sin \alpha \cos \beta \cos \gamma$	$\cos \alpha \cos 2\beta \cos \gamma$ $-\sin \alpha \cos \beta \sin \gamma$