Rampfilter implementation on truncated projection data. Application to 3D linear tomography for logs.

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Abstract

A common wish in non-destructive testing is to investigate a large object with a small interesting detail inside. Due to practical circumstances, the projections may sometimes be truncated. According to the theory on tomography, it is then impossible to reconstruct the object. However, sometimes it is possible to receive an approximate result. It turns out that the key-point is how to implement the ramp-filter. The quality of the result depends on the object itself. We show one good experiment on real data, linear cone-beam tomography for logs. We also show experiments on the Shepp-Logan phantom, well-known from medical CT, and discuss the varying reconstruction quality.

Key words

truncated projections, local tomography, linear conebeam tomography, log scanning

1 Introduction

According to the theory on tomography, see for example [1], the following theorem can be stated.

Theorem 1 An image can be reconstructed from its projections provided that

a The projections span the whole, \(180^\circ\), angular interval around the object.

b The projections cover the whole object, i.e. the projections are not truncated.

However, sometimes the only accessible projection data lacks projection angles and/or is truncated. Then it is theoretically impossible to get an exact reconstructed image, but it might still be possible to get useful information. The goal is then to diminish the artifacts as much as possible. Our concern here is how to handle truncated projection data.

2 Truncated projections in computed tomography

At first, we recall the whole process of computed tomography (CT), which is illustrated in Figure 1 and contains the following steps, projection generation, ramp-filtering and back-projection.

![Figure 1: Computed tomography (CT) illustrated.](image)

For the sake of simplicity, we have assumed parallel projection data. Denote the object \(f(x,y)\), the projection \(p(r,\theta)\), the ramp-filter \(h(r)\), and the reconstructed object \(\hat{f}(x,y)\). A couple of equations relates these functions to each other. The equation for projection generation is

\[
p(r, \theta) = \int_{-\infty}^{\infty} f(x,y) \, ds, \tag{1}
\]

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} r \\ s \end{bmatrix}. \tag{2}
\]

The equation for application of the ramp-filter is

\[
q(r, \theta) = \mathcal{F}_r^{-1}\left[\mathcal{F}_r[p(r,\theta)] \cdot \mathcal{F}_r[h(r)]\right], \tag{3}
\]

where \(\mathcal{F}_r\) denotes the Fourier transform in the \(r\)-direction, \(\mathcal{F}_r^{-1}\) denotes the inverse Fourier transform, and

\[
\mathcal{F}[h(r)] = H(\rho) = \begin{cases} |ho|, & |\rho| \leq \rho_{\text{max}}, \\ 0, & \text{elsewhere}. \end{cases} \tag{4}
\]
Back-projection means smearing of filtered projection data over the image plane and the equation is

$$\hat{f}(x, y) = \int_0^\pi q(x \cos \theta + y \sin \theta, \theta) \, d\theta. \quad (5)$$

A more thorough description of the reconstruction process can be found in, for example, [1].

A common wish in non-destructive testing is to investigate a large object with a small interesting detail inside. Let us divide the object according to

$$f_{\text{tot}}(x, y) = f(x, y) + f_d(x, y),$$

where \(f(x, y)\) is the large object, \(f_d(x, y)\) the detail, and \(f_{\text{tot}}(x, y)\) the complete object. Since CT reconstruction is a linear operation, the analysis of the reconstruction of the large object to \(\hat{f}(x, y)\) can be treated separately. The final reconstruction result is the sum of the two reconstructions,

$$\hat{f}_{\text{tot}}(x, y) = \hat{f}(x, y) + \hat{f}_d(x, y). \quad (6)$$

Suppose that the projection rays do not cover the large object, i.e. the projection data is truncated. Further on, suppose that the detail is completely covered by projection rays. Therefore, the detail can be reconstructed to \(\hat{f}_d(x, y) \approx f_d(x, y)\). However, what about the reconstruction of the large object?

Let the large object be a centrally located circular disc with radius \(R\) and density \(1\) according to

$$f(x, y) = \begin{cases} 1, & x^2 + y^2 \leq R^2, \\ 0, & \text{elsewhere}, \end{cases} \quad (7)$$

that is, it looks as the object in Figure 1. At first, regard a case when the projections are non-truncated. The equation for the projections is then

$$p(r, \phi) = p(r) = \begin{cases} 2 \cdot \sqrt{R^2 - r^2}, & |r| \leq R, \\ 0, & \text{elsewhere}. \end{cases} \quad (8)$$

see Figure 1, left. As indicated in Figure 1, right, the filtered projections have a constant value equal to the the density value \(1\) divided by \(\pi\) for values \(|r| < R\). For values \(|r| \geq R\), the filtered projections have negative values. During back-projection, see Figure 1, right, values equal to \(1/\pi\) are smeared back over the area \(x^2 + y^2 \leq R^2\) for each projection \(i = 1, ..., M\). Then the whole result is multiplied with \(d\theta = \pi/M\) according to equation (5). Therefore, the interior of reconstructed object gets the desired value \(1\). In a more complicated way, the positive and negative values in the filtered projections cancel each other outside the area \(x^2 + y^2 < R^2\). A cross-section through the reconstructed object at \(y = 0\) gives the correctly reconstructed result, i.e. \(1\) for \(y < R\) and \(0\) for \(y = 0\).

What is happening during the reconstruction process if the projections are truncated as illustrated in Figure 2a-c. The projection in Figure 2a has been truncated at \(r = \pm k = \pm 64\). Therefore, the area \(x^2 + y^2 \leq k^2\) is the region of interest, where we want a reconstruction result equal to \(1\). The borders of the region of interest at \(\pm k\) are marked with dotted lines in Figure 2. Note that the filtered projection in Figure 2b no longer has a constant value in the region of interest. Instead, the shape is heavily concave or bowl-shaped, which results in the incorrect bowl-shaped reconstruction in Figure 2c. According to equation (6), the incorrectly reconstructed large object \(\hat{f}(x, y)\) will be added to the correctly reconstructed detail \(\hat{f}_d(x, y)\), which results in a bowl-shaped artifact.

The experiment shows that the large object cannot be reconstructed with the common reconstruction method. This is in line with [1] which states that it is theoretically impossible to reconstruct an object from truncated projections correctly. Therefore, our strategy will be as follows.

**Theorem 2** Suppose that an object consists of several sub-objects and that the projections of some of them are truncated. Regard the sub-objects with truncated projections as lost, impossible to reconstruct. Then try to diminish the influence of the truncated projections as much as possible, readily so that they are reconstructed to a constant value close to \(0\).

When comparing Figure 2a and b it seems that the ramp-filter gives heavy response at the edges in projection data. One possibility would then be to soften the edges of the truncated projection. In Figure 2d, we have extrapolated the projection outwards by using a smooth \(\cos^2()\)-function, that is

$$p_2(r) = \begin{cases} p(r), & -k \leq r \leq k, \\ p(k) \cdot \cos^2 \left(\frac{\pi(r-k)}{2k}\right), & k < r < 2k, \\ p(-k) \cdot \cos^2 \left(\frac{\pi(r+k)}{2k}\right), & -2k < r < -k, \\ 0, & \text{elsewhere}. \end{cases} \quad (9)$$
The result shown in Figure 2e and f after ramp-filtering and reconstruction is not much better than in Figure 2b and c. It seems that it is the step from 0 to \( p(k) \) that should be eliminated instead of the shape of the edge. Suppose that \( p(k) = p(-k) \). Then the step can be eliminated in two ways, either by subtracting all values in \( p(r), |r| < k \) with \( p(k) \), or by padding the truncated projection with \( p(k) \) to the right and to the left. This gives two functions \( p3(k) \) and \( p3b(k) \),

\[
p3(r) = \begin{cases} 
  p(r) - p(k), & |r| \leq k, \\
  0, & |r| > k,
\end{cases}
\]

\[
p3b(r) = \begin{cases} 
  p(r), & |r| \leq k, \\
  p(k), & |r| > k,
\end{cases}
\]

(10)

Applying the ramp-filter on these two variants gives exactly the same result. The reason is as follows. The only difference between \( p3(r) \) and \( p3b(r) \) is a constant DC-level, \( p(k) \). The ramp-filter has zero response for frequency \( \rho = 0 \) according to equation (4) and therefore any DC-level in the projection data will be eliminated.

The projections \( p3(r) \) and \( p3b(r) \), the filtered version and reconstruction is shown in Figure 2g-i. We observe that the reconstruction in Figure 2i is close to 0 and much less bowl-shaped than the reconstruction results in Figure 2c and f. When the incorrectly reconstructed large object \( f(x, y) \) is added to the correctly reconstructed detail \( \hat{f}(x, y) \), it will give a bowl-shaped artifact, however much less severe than previously. It is clear that the more flat projection data of the large object, the less artifacts in the reconstruction.

In the case when \( p(k) \neq p(-k) \), we suggest the following combined procedure of subtraction and smoothing,

\[
p4(r) = \begin{cases} 
  p(r) - \frac{p(k) + p(-k)}{2}, & -k < r < k, \\
  \frac{p(k) - p(-k)}{2} \cdot \cos^2 \left( \frac{\pi(r-k)}{2k} \right), & k < r < 2k, \\
  \frac{p(-k) - p(k)}{2} \cdot \cos^2 \left( \frac{\pi(r+k)}{2k} \right), & -2k < r < -k,
\end{cases}
\]

(11)

i.e. the projection data is first subtracted by \( (p(k) - p(-k))/2 \) and then extrapolated outwards by using a smooth \( \cos^2(.) \)-function.

3 Truncated projections in linear computed tomography

Linear cone-beam tomography for logs is a non-destructive method to get information about the interior of a log. The log is translated with high speed (approximately 2m/s) through an arrangement of two 90° oriented X-ray source-detector systems. The detectors are two-dimensional and consist of a number of sparsely placed line detectors. The reconstruction algorithm consists of the following steps

1. Projection generation and rebinning.
2. Individual reconstruction of data from both source-detector systems.
3. Merging of the results from both systems.

The final reconstruction result is a volume showing the interior of the log. The whole algorithm is carefully described in [2]. It can be shown that the first task in theorem 1 is not satisfied. Despite this, it is possible to receive useful information about the sub-objects (knots) inside the main object (the log itself).

If the whole log is scanned before reconstruction, the second task in theorem 1 is satisfied, but not if piecewise processing is desired. With such processing the memory requirement would decrease and the computations could be applied in a pipe-line manner, so that the reconstruction results from the first part of the log could be received before the last part of the log is scanned.

Step 1 and 3 is carefully described in [2]. Let us instead concentrate on step 2 which is very much alike common computed tomography. Figure 3 describes the reconstruction of a longitudinal cross-section of the log with data from one source-detector system. The projection data span the interval \(-45° \leq \theta \leq 45°\). The ramp-filter is applied along every line in the y-direction. Finally, back-projection or some equivalent Fourier method is applied.

The X-ray attenuation coefficient is approximately proportional to the density. Typical densities for green-wood are 0.6 kg/dm³ for heart-wood, 1.0 kg/dm³ for sap-wood, and 1.1 kg/dm³ for knots. Therefore, the projection data looks approximately as indicated in the Figure 3. Heart- and sap-wood give the main contribution to projection data, whereas the knots only give small contributions. The projection data from heart- and sap-wood is truncated, whereas the projection data from the knots is non-truncated. The strategy described in theorem 2 should therefore be applicable here and the procedure described in equation (11) should be applied before ramp-filtering. Then the heart- and sap-wood will be reconstructed to a value close to zero, whereas the knots will come forward.
Note that since reconstruction is a linear space invariant operation, the ramp-filter can equally well be applied after back-projection. Then the non-filtered reconstruction result is exposed to the procedure in equation (11). This course of action has been chosen in the experiments in the next section.

4 Experimental results

The first experiment reported here is a simulation performed with tomography data from the stem bank at the Dept of Wood Technology, Skellefteå Campus, Luleå University of Technology. It is a $127 \times 127 \times 127$ data volume with voxel size $2 \times 2 \times 2 \text{ mm}^3$. This volume has then served as a reference volume as well as input for the simulations. One slice through this reference volume along with two reconstruction results are shown in Figure 4.

Linear cone-beam tomography as described in the previous section was applied together with the subtraction-padding procedure in equation (11). The reconstruction result is is shown in Figure 4, right. Note that the knots are clearly visible, whereas the heart- and sap-wood have disappeared. If the area outside the truncated data is assumed to be zero, the result looks as shown in Figure 4, middle. Note the heavy bowl-shaped grey-scale artifact that is similar to the artifact in Figure 2c. The contrast between the knots and the back-ground are much less that the dynamics of the bowl-shaped artifact.

Figure 4: From left: reference slice, reconstruction, reconstruction with subtraction-padding.

The second experiment was performed on the Shepp-Logan phantom, well-known in medical tomographic literature and constructed to reflect the situation in the human brain, a bone-ring of density 2, brain tissue of 1.02 and brain objects of 1.00 and 1.03. We call this phantom the low-contrast Shepp-Logan phantom. Another Shepp-Logan phantom (high-contrast) contains a bone-ring of density 0.04, brain tissue of 0.02, and brain objects of 0.00 and 0.03. The phantom along with three experiments are shown in Figure 5. The contrast has been adjusted so that the brain objects looks as clear as possible. At first, note that plain reconstruction from truncated projections does not work. Secondly, the subtraction-padding in equation (11), works much better on the high-contrast phantom, than on the low-contrast phantom.

Figure 5: Reconstruction from truncated projection data of the Shepp-Logan phantom. 60% of projection data is truncated. From left: phantom, high-contrast reconstruction without padding, high-contrast reconstruction, low-contrast reconstruction.

5 Conclusions and discussion

According to the theory on tomography, it is impossible to reconstruct the object from truncated projections. However, we have shown that it is sometimes possible to receive an approximate reconstruction result by a careful implementation of the ramp-filter. Experiments show that good image quality is possible to receive for some applications. Linear cone-beam tomography for logs is one such application. Further on, it is more likely to receive a good result for high-contrast objects than for low-contrast objects. According to this, we wonder if the successful tomography results with wavelets on truncated projection data reported recently, are not due to the wavelets themselves, but to successful ramp-filter implementation.

6 Acknowledgment

The support by VINNOVA (previously NUTEK, project number P11608) and CENIIT (Center for Industrial Information Technology), Linköping University are gratefully acknowledged.

References
