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Complete Characterization of the Pareto Boundary for the MISO Interference Channel

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Abstract—In this correspondence, we study the achievable rate region of the multiple-input single-output (MISO) interference channel, under the assumption that all receivers treat the interference as additive Gaussian noise. Our main result is an explicit parametrization of the Pareto boundary for an arbitrary number of users and antennas. The parametrization describes the boundary in terms of a low-dimensional manifold. For the two-user case we show that a single real-valued parameter per user is sufficient to achieve all points on the Pareto boundary and that any point on the Pareto boundary corresponds to beamforming vectors that are linear combinations of the zero-forcing (ZF) and maximum-ratio transmission (MRT) beamformers. We further specialize the results to the MISO broadcast channel (BC). A numerical example illustrates the result.

Index Terms—Beamforming, interference channel, multiple antenna, Pareto optimal, performance region.

I. INTRODUCTION

Interference channels (IFC) consist of at least two transmitters and two receivers. The first transmitter wants to transfer information to the first receiver and the second transmitter to the second receiver, respectively. This happens at the same time on the same frequency causing interference at the receivers. Information-theoretic studies of the IFC have a long history [1]–[4]. These references have provided various achievable rate regions, which are generally larger in the more recent papers than in the earlier ones. However, the capacity region of the general IFC remains an open problem. For certain limiting cases, for example when the interference is weak or very strong, respectively, the sum capacity is known [5]. If the interference is weak, it can simply be treated as additional noise. For very strong interference, successive interference cancellation (SIC) can be applied at one or more of the receivers. Multiple antenna IFCs are studied in [10]. Multiple-input multiple-output (MIMO) IFCs have also recently been studied in [6], from the perspective of spatial multiplexing gains. In [7], the rate region of the single-input single-output (SISO) IFC was characterized in terms of convexity and concavity.

The IFC is a building block in many communication systems, for example, for ad hoc networks and cognitive radio. It also specializes to scenarios with cooperation either at the transmitter or at the receiver

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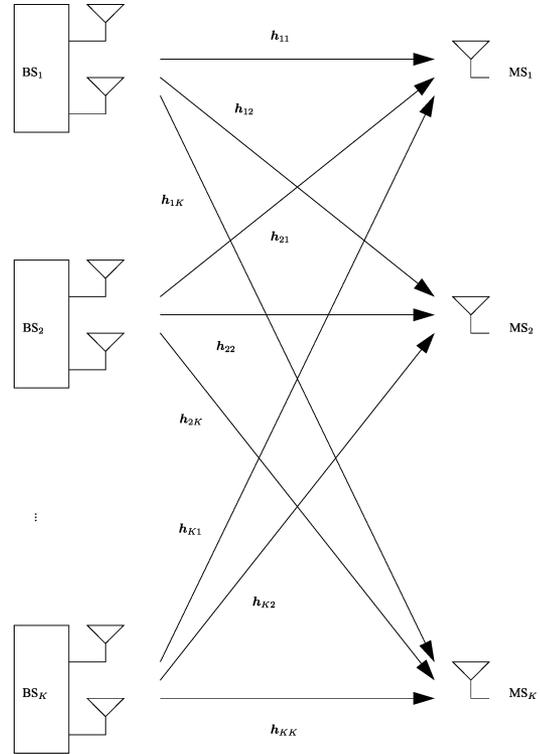
Fig. 1. K-user MISO interference channel under study (illustrated for $N = 2$ transmit antennas).

side, leading to, for instance, the multiple-access channel (MAC) and the broadcast channel (BC). For system design it is important to analyze the achievable rate region of the general Gaussian IFC (as will be defined in Section II) and to design transmit strategies that operate on the Pareto boundary of that region. (The Pareto boundary is the set of rate points at which it is impossible to improve any of the rates without simultaneously decreasing at least one of the others.)

In this correspondence, we study the multiple-input single-output (MISO) Gaussian IFC and completely characterize the rate region achievable by treating interference as additive Gaussian noise. Our main contribution is an explicit parametrization of the Pareto boundary for the K-user Gaussian MISO IFC (see Section III, especially Proposition 1). For the special case of $(K = 2)$ users we show that any point in the rate region can be achieved by choosing beamforming vectors that are linear combinations of the zero-forcing (ZF) and the maximum-ratio transmission (MRT) beamformers (see Section IV-A, especially Corollary 2). We further specialize the results to the MISO BC (the BC is a special case of the IFC), see Section IV-B. The special cases for $K = 2$ were presented partly in conference papers [9] and [13].

Notation: The notation for this paper is as follows: $(\cdot)^*$: complex conjugate; $(\cdot)^T$: transpose; $(\cdot)^H$: Hermitian transpose; $\boldsymbol{I}$: the identity matrix; $\Pi_{\boldsymbol{X}} \triangleq \boldsymbol{X}(\boldsymbol{X}^H \boldsymbol{X})^{-1} \boldsymbol{X}^H$: orthogonal projection onto the column space of $\boldsymbol{X}$; and $\Pi_{\boldsymbol{X}}^{\perp} \triangleq \boldsymbol{I} - \Pi_{\boldsymbol{X}}$: orthogonal projection onto the orthogonal complement of the column space of $\boldsymbol{X}$.

II. SYSTEM MODEL AND PRELIMINARIES

We consider the MISO interference channel with K transmitters and K receivers, as illustrated in Fig. 1. All base stations BS_k have N transmit antennas each, that can be used with full phase coherency. The mobiles MS_k, however, have a single receive antenna each. We shall

assume that transmission consists of scalar coding followed by beamforming and that all propagation channels are frequency-flat. This leads to the following basic model for the matched-filtered, symbol-sampled complex baseband data received at MS_k:

$$y_k = \boldsymbol{h}_{kk}^T \boldsymbol{w}_k s_k + \sum_{l=1, l \neq k}^{K} \boldsymbol{h}_{lk}^T \boldsymbol{w}_l s_l + e_k \tag{1}$$

where s_l, $1 \leq l \leq K$ is the symbol transmitted by BS_l, $\boldsymbol{h}_{ij}$ is the (complex-valued) $N \times 1$ channel-vector between BS_i and MS_j, and $\boldsymbol{w}_l$ is the beamforming vector used by BS_l. The variables e_k are noise terms which we model as independent and identically distributed (i.i.d.) complex Gaussian with zero mean and variance σ^2.

We assume that each base station can use the transmit power P, but that power cannot be traded between the base stations. Without loss of generality, we shall take $P = 1$. This gives the power constraints

$$\|\boldsymbol{w}_k\|^2 \leq 1, \quad 1 \leq k \leq K. \tag{2}$$

Throughout, we define the signal-to-noise ratio (SNR) as $1/\sigma^2$. The precoding scheme that we will discuss requires that the transmitters (BS_k) have access to channel state information (CSI) for some of the links. However, at no point we will require phase coherency between the base stations.

In what follows we will assume that all receivers treat co-channel interference as noise, i.e., they make no attempt to decode and subtract the interference. The main justification for this assumption is that in most envisioned applications, MS_i would use receivers with a simple structure. Additionally, interference cancellation is difficult in an environment where the receivers do not know the coding and modulation schemes used by the interfering transmitters. For a general, interference-free $N \times 1$ MISO channel with zero-mean Gaussian noise at the receiver, scalar coding with beamforming is uniformly optimal with respect to the variance of the Gaussian noise. (A more detailed discussion of this can be found in [8].)

For a given set of beamforming vectors $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_K\}$, the following rate is then achievable for the link $\mathrm{BS}_k \rightarrow \mathrm{MS}_k$, by using codebooks approaching Gaussian ones:

$$R_k(\boldsymbol{w}_1, \dots, \boldsymbol{w}_K) = \log_2 \left(1 + \frac{\left| \boldsymbol{w}_k^T \boldsymbol{h}_{kk} \right|^2}{\sum_{l \neq k} \left| \boldsymbol{w}_l^T \boldsymbol{h}_{lk} \right|^2 + \sigma^2} \right). \tag{3}$$

We define the *achievable rate region* to be the set of all rates that can be achieved using beamforming vectors that satisfy the power constraint:

$$\mathcal{R} \triangleq \bigcup_{\{\boldsymbol{w}_k : \boldsymbol{w}_k \in \mathbb{C}^N, \|\boldsymbol{w}_k\|^2 \leq 1, 1 \leq k \leq K\}} \{R_1(\boldsymbol{w}_1, \dots, \boldsymbol{w}_K), \dots,$$
$$\dots, R_K(\boldsymbol{w}_1, \dots, \boldsymbol{w}_K)\} \subset \mathbb{R}_+^K. \tag{4}$$

The outer boundary of this region is called the *Pareto boundary*, because it consists of operating points $(R_1, \dots, R_K)$ for which it is impossible to improve one of the rates, without simultaneously decreasing at least one of the other rates. More precisely we define the *Pareto optimality* of an operating point as follows.

Definition 1: A rate tuple $(R_1, \dots, R_K)$ is Pareto optimal if there is no other tuple $(Q_1, \dots, Q_K)$ with $(Q_1, \dots, Q_K) \geq (R_1, \dots, R_K)$ and $(Q_1, \dots, Q_K) \neq (R_1, \dots, R_K)$ (the inequality is component-wise). $\square$

III. EXPLICIT PARAMETRIZATION OF THE PARETO BOUNDARY

The description of the rate region in (4) is not suitable for evaluation of the Pareto boundary in practice. In this section we present a general, more useful representation of the boundary.

Proposition 1: Let i be given and fixed. Suppose that $\boldsymbol{h}_{ij}$ are linearly independent for $j = 1, \dots, K$ and that $\boldsymbol{h}_{ij}^H \boldsymbol{h}_{ij'} \neq 0$ for all j, j', $j' \neq j$.[1]

Then if $\boldsymbol{w}_i$ is a beamforming vector that corresponds to a rate point on the Pareto boundary, there exist complex numbers $\{\xi_{ij}\}_{j=1}^K$ such that

$$\boldsymbol{w}_i = \sum_{j=1}^{K} \xi_{ij} \boldsymbol{h}_{ij}^* \tag{5}$$

and

$$\|\boldsymbol{w}_i\|^2 = 1. \tag{6}$$

$\square$

Before we give the proof of Proposition 1, note that by setting $\boldsymbol{H}_i \triangleq [\boldsymbol{h}_{i1}, \boldsymbol{h}_{i2}, \dots, \boldsymbol{h}_{iK}]$ and $\boldsymbol{\xi}_i \triangleq [\xi_{i1}, \dots, \xi_{iK}]^T$, condition (6) is equivalent to the following quadratic constraint:

$$\boldsymbol{\xi}_i^H \left(\boldsymbol{H}_i^H \boldsymbol{H}_i \right)^* \boldsymbol{\xi}_i = 1. \tag{7}$$

Therefore, all $\{\xi_{ij}\}$ in (5) are bounded by the inverse of the smallest singular value of $\boldsymbol{H}_i$. Note also that in the signal-to-interference plus noise ratio (SINR) expressions only terms of the form $\left| \boldsymbol{w}_i^T \boldsymbol{h}_{ii} \right|^2$ or $\left| \boldsymbol{w}_i^T \boldsymbol{h}_{ij} \right|^2$ occur. Hence, the complex angle of $\boldsymbol{w}_i$ can be shifted by an arbitrary amount. This means that without loss of generality, at least one of the parameters $\xi_{i1}, \dots, \xi_{iK}$ can be chosen real-valued.

Note also that each transmitter k needs to know only its own channels $\boldsymbol{h}_{k1}, \dots, \boldsymbol{h}_{kK}$ to compute the beamforming vectors that achieve rates on the Pareto boundary.

Proof: Let $\{\boldsymbol{u}_{im}\}$ be an orthonormal basis for the orthogonal complement of the space spanned by $\{\boldsymbol{h}_{i1}^*, \dots, \boldsymbol{h}_{iK}^*\}$ (under the assumptions made, this space has dimension $N - K$, thhus if $K = N$, there is nothing to prove). Then let $\boldsymbol{w}_i$ be an arbitrary beamforming vector that corresponds to a rate point on the Pareto boundary, and that satisfies the power constraint $\|\boldsymbol{w}_i\|^2 \leq 1$. Since the set of vectors $\{\boldsymbol{h}_{i1}^*, \dots, \boldsymbol{h}_{iK}^*, \boldsymbol{u}_{i1}, \dots, \boldsymbol{u}_{i(N-K)}\}$ spans $\mathbb{C}^N$, we can write

$$\boldsymbol{w}_i = \sum_{j=1}^{K} \xi_{ij} \boldsymbol{h}_{ij}^* + \sum_{m=1}^{N-K} \gamma_{im} \boldsymbol{u}_{im} \tag{8}$$

for some set of complex-valued ξ_{ij}, γ_{im}.

To verify the proposition we need to show that if $\boldsymbol{w}_i$ corresponds to a rate point at the boundary, then we have $\gamma_{im} = 0$ for $m = 1, \dots, N - K$. The proof goes by contradiction. Suppose $\gamma_{im} \neq 0$ for some m, say $m = m'$. Then with $\boldsymbol{w}_i' \triangleq \boldsymbol{w}_i - \gamma_{im'} \boldsymbol{u}_{im'}$ we have that

$$R_p(\boldsymbol{w}_1, \dots, \boldsymbol{w}_i', \dots, \boldsymbol{w}_K) = R_p(\boldsymbol{w}_1, \dots, \boldsymbol{w}_i, \dots, \boldsymbol{w}_K)$$

for all $p = 1, \dots, K$ because $\boldsymbol{u}_{im'}^T \boldsymbol{h}_{ij} = 0$ for all $j = 1, \dots, K$. At the same time

$$\|\boldsymbol{w}_i'\|^2 = \left\| \sum_{j=1}^{K} \xi_{ij} \boldsymbol{h}_{ij}^* \right\|^2 + \sum_{m=1, m \neq m'}^{N-K} |\gamma_{im}|^2$$
$$< \left\| \sum_{j=1}^{K} \xi_{ij} \boldsymbol{h}_{ij}^* \right\|^2 + \sum_{m=1}^{N-K} |\gamma_{im}|^2 = \|\boldsymbol{w}_i\|^2.$$

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[1]In a fading environment, this will be the case with probability one as long as $K \leq N$, i.e., the number of antennas at the base station must be larger than or equal to the number of users.

In particular, we have that $\|\boldsymbol{w}_i'\|^2 < 1$. Now define

$$\phi \triangleq \arg\left(\boldsymbol{w}_i'^T \boldsymbol{h}_{ii}\right)$$

and

$$\epsilon \triangleq \frac{1 - \|\boldsymbol{w}_i'\|}{\|\boldsymbol{\nu}_i\|} \tag{9}$$

where

$$\boldsymbol{\nu}_i \triangleq \Pi^{\perp}_{[\boldsymbol{h}_{i1}^*,\ldots,\boldsymbol{h}_{ii-1}^*,\boldsymbol{h}_{ii+1}^*,\ldots\boldsymbol{h}_{iK}^*]} \boldsymbol{h}_{ii}^*.$$

Note that $\boldsymbol{\nu}_i$ is a nonzero vector since $\boldsymbol{h}_{i1}, \ldots, \boldsymbol{h}_{iK}$ are linearly independent. Thus, ϵ is well-defined and we have $\epsilon > 0$. Also note that $\boldsymbol{h}_{ii}^T \boldsymbol{\nu}_i \in \mathbb{R}$, $\boldsymbol{h}_{ii}^T \boldsymbol{\nu}_i > 0$ and that $\boldsymbol{h}_{ij}^T \boldsymbol{\nu}_i = 0$ for $i \neq j$. Define

$$\boldsymbol{w}_i'' \triangleq \boldsymbol{w}_i' + \epsilon e^{j\phi} \boldsymbol{\nu}_i = \boldsymbol{w}_i - \gamma_{im'}\boldsymbol{u}_{im'} + \epsilon e^{j\phi}\boldsymbol{\nu}_i.$$

Then we have

$$\left|\boldsymbol{h}_{ij}^T \boldsymbol{w}_i''\right| = \left|\boldsymbol{h}_{ij}^T \boldsymbol{w}_i'\right|, \quad j \neq i \tag{10}$$

$$\left|\boldsymbol{h}_{ii}^T \boldsymbol{w}_i''\right| = \left|\boldsymbol{h}_{ii}^T \boldsymbol{w}_i' + \epsilon e^{j\phi}\boldsymbol{h}_{ii}^T\boldsymbol{\nu}_i\right| = \left|\boldsymbol{h}_{ii}^T \boldsymbol{w}_i'\right| + \epsilon\boldsymbol{h}_{ii}^T\boldsymbol{\nu}_i > \left|\boldsymbol{h}_{ii}^T \boldsymbol{w}_i'\right| \tag{11}$$

$$\|\boldsymbol{w}_i''\| = \left\|\boldsymbol{w}_i' + \epsilon e^{j\phi}\boldsymbol{\nu}_i\right\| \leq \|\boldsymbol{w}_i'\| + \epsilon\|\boldsymbol{\nu}_i\| = 1. \tag{12}$$

Hence, $\boldsymbol{w}_i''$ satisfies the power constraint, while for the rates we have

$$R_i(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_i'', \ldots, \boldsymbol{w}_K) > R_i(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_i', \ldots \boldsymbol{w}_K)$$
$$= R_i(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_i, \ldots \boldsymbol{w}_K)$$

and

$$R_p(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_i'', \ldots, \boldsymbol{w}_K) = R_p(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_i, \ldots, \boldsymbol{w}_K)$$

for $p \neq i$. It follows that $\boldsymbol{w}_i$ cannot correspond to a rate point on the boundary. Thus, we must have $\gamma_{im'} = 0$, so $\boldsymbol{w}_i$ has the form (5). ∎

IV. SPECIAL CASES

A. The Two-User $(K = 2)$ MISO IFC

In this subsection we consider the special case of two users $(K = 2)$. The main results presented here can be found in [9] as well, but the proofs there did not exploit Proposition 1 and therefore were somewhat lengthy.

For $K = 2$, Proposition 1 specializes to the following corollary.

Corollary 1: All vectors $\boldsymbol{w}_1$ that correspond to points on the Pareto boundary have the form

$$\boldsymbol{w}_1 = \alpha_1 \frac{\Pi_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^*}{\left\|\Pi_{\boldsymbol{h}_{12}}\boldsymbol{h}_{11}\right\|} + \beta_1 \frac{\Pi^{\perp}_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^*}{\left\|\Pi^{\perp}_{\boldsymbol{h}_{12}}\boldsymbol{h}_{11}\right\|} \tag{13}$$

where α_1, β_1 are non-negative real-valued scalars that satisfy $\alpha_1^2 + \beta_1^2 = 1$.

The beamforming vector $\boldsymbol{w}_2$ of the second user can be parametrized similarly. □

Proof: We need to show that any vector described by the parametrization in (5) can also be described via the parametrization in (13). But this is clear since $\{\boldsymbol{h}_{11}^*, \boldsymbol{h}_{12}^*\}$ and $\left\{\Pi_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^*, \Pi^{\perp}_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^*\right\}$ span the same space. To see this, note that $\boldsymbol{h}_{11}^* = \Pi_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^* + \Pi^{\perp}_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^*$ and that $\boldsymbol{h}_{12}^* = \gamma\Pi_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^*$ for some complex-valued scalar γ. Next, note that the condition in (6) is equivalent to $|\alpha_1|^2 + |\beta_1|^2 = 1$. This

follows because $\Pi_{\boldsymbol{h}_{12}^*}\boldsymbol{h}_{11}^*$ and $\Pi^{\perp}_{\bar{\boldsymbol{h}}_{12}^*}\boldsymbol{h}_{11}^*$ are orthogonal by construction so $\|\boldsymbol{w}_1\|^2 = |\alpha_1|^2 + |\beta_1|^2$.

It remains to show that α_1 and β_1 can be chosen non-negative and real-valued. Due to the power constraint they satisfy $|\alpha_1|^2 + |\beta_1|^2 = 1$. Consider the desired-signal part in the rate expression for user 1:

$$|\boldsymbol{w}_1^T \boldsymbol{h}_{11}|^2 = \left|\alpha_1\left\|\Pi_{\boldsymbol{h}_{12}}\boldsymbol{h}_{11}\right\| + \beta_1\left\|\Pi^{\perp}_{\bar{\boldsymbol{h}}_{12}}\boldsymbol{h}_{11}\right\|\right|^2 \tag{14}$$

and the interference term in the rate expression for user 2:

$$\left|\boldsymbol{w}_1^T \boldsymbol{h}_{12}\right|^2 = |\alpha_1|^2 \frac{\left|\boldsymbol{h}_{11}^H \boldsymbol{h}_{12}\right|^2}{\left\|\Pi_{\boldsymbol{h}_{12}}\boldsymbol{h}_{11}\right\|^2}. \tag{15}$$

The expression in (15) depends only on $|\alpha_1|^2$. From the triangle inequality we have that

$$\left|\alpha_1\left\|\Pi_{\boldsymbol{h}_{12}}\boldsymbol{h}_{11}\right\| + \beta_1\left\|\Pi^{\perp}_{\bar{\boldsymbol{h}}_{12}}\boldsymbol{h}_{11}\right\|\right|$$
$$\leq |\alpha_1|\left\|\Pi_{\boldsymbol{h}_{12}}\boldsymbol{h}_{11}\right\| + |\beta_1|\left\|\Pi^{\perp}_{\bar{\boldsymbol{h}}_{12}}\boldsymbol{h}_{11}\right\|.$$

with equality only if $\arg(\alpha_1) = \arg(\beta_1)$. Hence, on the boundary, α_1 and β_1 have the same phase. Since the complex angle of $\boldsymbol{w}_1$ can be shifted by an arbitrary amount, it follows that all points on the boundary can be achieved by taking α_1, β_1 non-negative real-valued. ∎

In the remainder of this section, we consider two specific choices of beamformers, namely maximum-ratio transmission (MRT) and zero-forcing (ZF). Starting from Corollary 1, we shall show that any beamforming vector that corresponds to a rate tuple on the boundary must be a linear combination of the MRT and ZF beamformers, *with real-valued coefficients*.

The ZF point $\left(R_1^{\mathrm{ZF}}, R_2^{\mathrm{ZF}}\right)$ is the set of rates which are achieved if the two BS choose beamforming vectors such that no interference is created for the other point-to-point link at all. If we assume that both BS use their maximum permitted power, then BS_1 should choose a unit-norm beamforming vector $\boldsymbol{w}_1$ that is orthogonal to the channel of the second user, and which at the same time maximizes $\left|\boldsymbol{w}_1^T \boldsymbol{h}_{11}\right|$. This beamformer is given by (see proof in [12])

$$\boldsymbol{w}_1^{\mathrm{ZF}} = \frac{\Pi^{\perp}_{\bar{\boldsymbol{h}}_{12}^*}\boldsymbol{h}_{11}^*}{\left\|\Pi^{\perp}_{\bar{\boldsymbol{h}}_{12}}\boldsymbol{h}_{11}\right\|}. \tag{16}$$

(A similar result holds for $\boldsymbol{w}_2^{\mathrm{ZF}}$; interchange the indexes $(\cdot)_1$ and $(\cdot)_2$.)

The MRT beamforming vector for user k, $1 \leq k \leq 2$ is the vector that maximizes the transmission rate in the absence of interference. This is given by [11]

$$\boldsymbol{w}_k^{\mathrm{MRT}} = \frac{\boldsymbol{h}_{kk}^*}{\|\boldsymbol{h}_{kk}\|}, \quad k = 1, 2.$$

From a game theoretic point of view, one can show that for a one-shot noncooperative beamforming game on the MISO interference channel, the MRT beamforming is a unique Nash equilibrium (NE) [12]. For this reason one could call it the "selfish beamforming strategy."

We now present a parametrization of the Pareto boundary expressed in terms of the ZF and MRT beamformers defined above.

Corollary 2: Any point on the Pareto boundary is achievable with the beamforming strategy

$$\boldsymbol{w}_k(\lambda_k) = \frac{\lambda_k \boldsymbol{w}_k^{\mathrm{MRT}} + (1 - \lambda_k)\boldsymbol{w}_k^{\mathrm{ZF}}}{\|\lambda_k \boldsymbol{w}_k^{\mathrm{MRT}} + (1 - \lambda_k)\boldsymbol{w}_k^{\mathrm{ZF}}\|} \tag{17}$$

for $k = 1, 2$, and for some set of *real-valued* parameters λ_k, $0 \leq \lambda_k \leq 1$, $k = 1, 2$. □

Proof: Consider $\boldsymbol{w}_1$ ($\boldsymbol{w}_2$ is handled similarly). Define $\ell_1 \triangleq \left\| \Pi_{\boldsymbol{h}_{12}} \boldsymbol{h}_{11} \right\|^2$ and $\ell_2 \triangleq \left\| \Pi_{\boldsymbol{h}_{12}}^\perp \boldsymbol{h}_{11} \right\|^2$. Note that $\ell_1 + \ell_2 = \|\boldsymbol{h}_{11}\|^2$. Then

$$\frac{\Pi_{\boldsymbol{h}_{12}^*} \boldsymbol{h}_{11}^*}{\left\| \Pi_{\boldsymbol{h}_{12}} \boldsymbol{h}_{11} \right\|} = \sqrt{\frac{\ell_1 + \ell_2}{\ell_1}} \boldsymbol{w}_1^{\mathrm{MRT}} - \sqrt{\frac{\ell_2}{\ell_1}} \boldsymbol{w}_1^{\mathrm{ZF}}.$$

Also note that $\boldsymbol{w}_1^{\mathrm{ZF}}$ is identical to the second basis vector in (13). From Corollary 1 it then follows that any point on the Pareto boundary is achievable by taking

$$\begin{aligned} \boldsymbol{w}_k = {}& \alpha_k \left(\sqrt{\frac{\ell_1 + \ell_2}{\ell_1}} \boldsymbol{w}_k^{\mathrm{MRT}} - \sqrt{\frac{\ell_2}{\ell_1}} \boldsymbol{w}_k^{\mathrm{ZF}} \right) \\ & + \sqrt{1 - \alpha_k^2} \, \boldsymbol{w}_k^{\mathrm{ZF}} \\ = {}& \alpha_k \sqrt{\frac{\ell_1 + \ell_2}{\ell_1}} \boldsymbol{w}_k^{\mathrm{MRT}} \\ & + \left(\sqrt{1 - \alpha_k^2} - \alpha_k \sqrt{\frac{\ell_2}{\ell_1}} \right) \boldsymbol{w}_k^{\mathrm{ZF}} \end{aligned} \tag{18}$$

where $0 \le \alpha_k \le 1$. By construction, the vectors $\boldsymbol{w}_k$ given by (18) have unit norm and clearly, any vector given by (18) for some $\alpha_k, 0 \le \alpha_k \le 1$ is also given by (17) for some $\lambda_k, 0 \le \lambda_k \le 1$. ∎

Corollary 2 shows that we only need to vary the scalar, *real-valued* parameters λ_1, λ_2 in order to reach any point on the Pareto boundary. This is much simpler than varying the beamforming vectors, or using the parametrization in [12].

A consequence of Corollary 2 is that each transmitter needs to know only its MRT and ZF beamformers to achieve points on the Pareto boundary. In order to compute these beamformers, knowledge of the transmitters' own channel to all other users is sufficient. In a game-theoretic framework [9], the parameter $\lambda_k, 0 \le \lambda_k \le 1$ can be interpreted as the "selfishness" of user k. For $\lambda_k = 1$ the transmitter falls back to the selfish NE (MRT) solution. For $\lambda_k = 0$ the transmitter acts in a completely altruistic way and applies the ZF beamformer. Note that the converse of Corollary 2 does not hold, i.e., many rate tuples that correspond to beamformers of the form (17) do not lie on the Pareto boundary. For example, the choice $\lambda_k = 1$ for all k (i.e., all users do pure MRT) was shown in [12] to be far from the boundary for high SNR.

The achievable rates in (4) can be expressed as functions of λ_k as follows:

$$\begin{aligned} R_1(\lambda_1, \lambda_2) &= \log \left(1 + \frac{\left| \boldsymbol{w}_1^T(\lambda_1) \boldsymbol{h}_{11} \right|^2}{\sigma^2 + \left| \boldsymbol{w}_2^T(\lambda_2) \boldsymbol{h}_{21} \right|^2} \right), \\ & \qquad 0 \le \lambda_k \le 1, \quad k = 1, 2. \\ R_2(\lambda_1, \lambda_2) &= \log \left(1 + \frac{\left| \boldsymbol{w}_2^T(\lambda_2) \boldsymbol{h}_{22} \right|^2}{\sigma^2 + \left| \boldsymbol{w}_1^T(\lambda_1) \boldsymbol{h}_{12} \right|^2} \right), \\ & \qquad 0 \le \lambda_k \le 1, \quad k = 1, 2. \end{aligned}$$

B. The MISO Broadcast Channel

The broadcast channel (BC) is a special case of the IFC, where the transmitters (BS_k here) are collocated and allowed to cooperate. This section treats this special case for the general K-user case, as well as for the case of $K = 2$ users. The channel model simplifies since in the BC there are only K channel vectors, from the BS to each mobile. We denote these by $\boldsymbol{h}_1, \ldots, \boldsymbol{h}_K$, where

$$\boldsymbol{h}_{ij} = \boldsymbol{h}_j \quad \text{for all } 1 \le i, \ j \le K. \tag{19}$$

For the MISO BC case, a sum-power constraint is applied at the transmitter rather than individual constraints. Denote the transmit power for user k by $P_k \ge 0$. Then the power constraint is $\sum_{k=1}^K P_k \le P$. We have the following counterpart to Proposition 1.

Proposition 2: If $\boldsymbol{w}_i$ is a beamforming vector that corresponds to a rate point on the Pareto boundary, there exist complex numbers $\{\xi_{ij}\}_{j=1}^K$ such that

$$\boldsymbol{w}_i = \sum_{j=1}^K \xi_{ij} \boldsymbol{h}_j^*, \quad \|\boldsymbol{w}_i\|^2 = P_i \text{ and } \sum_{i=1}^K P_i = P. \tag{20}$$

∎

Proof: The result is a variation of Proposition 1. As an intermediate step, one must first show that to achieve points on the Pareto boundary one must use all available transmit power, i.e., that $\sum_{i=1}^K \|\boldsymbol{w}_i\|^2 = P$ holds on the boundary. This fact was shown for $K = 2$ in [13, Lemma 1], and can be easily generalized to arbitrary K. ∎

Next, we consider the two-user ($K = 2$) MISO BC. The channels from the two transmitters to the two receivers in the IFC simplify as follows:

$$\boldsymbol{h}_{11} = \boldsymbol{h}_1, \quad \boldsymbol{h}_{12} = \boldsymbol{h}_2, \quad \boldsymbol{h}_{21} = \boldsymbol{h}_1, \quad \boldsymbol{h}_{22} = \boldsymbol{h}_2. \tag{21}$$

Since a sum-power constraint is applied at the transmitter rather than individual constraints, the available power P is split between the two users according to P_1, P_2, where $P_1 + P_2 = P$. We can show that the characterization of the boundary (Proposition 2) simplifies to the following.

Corollary 3: Any point on the Pareto boundary of the MISO BC rate region is achievable with the power allocation $0 \le P_1, P_2 \le P$, $P_1 + P_2 = P$ and the beamforming vectors

$$\boldsymbol{w}_1(\lambda_1) = \frac{\lambda_1 \boldsymbol{w}_1^{\mathrm{MRT}} + (1 - \lambda_1) \boldsymbol{w}_1^{\mathrm{ZF}}}{\| \lambda_1 \boldsymbol{w}_1^{\mathrm{MRT}} + (1 - \lambda_1) \boldsymbol{w}_1^{\mathrm{ZF}} \|} \tag{22}$$

$$\boldsymbol{w}_2(\lambda_2) = \frac{\lambda_2 \boldsymbol{w}_2^{\mathrm{MRT}} + (1 - \lambda_2) \boldsymbol{w}_2^{\mathrm{ZF}}}{\| \lambda_2 \boldsymbol{w}_2^{\mathrm{MRT}} + (1 - \lambda_2) \boldsymbol{w}_2^{\mathrm{ZF}} \|} \tag{23}$$

for some set of real-valued parameters $\lambda_1, \lambda_2, 0 \le \lambda_1, \lambda_2 \le 1$. ∎

Proof: The proof follows from Proposition 2 in a similar way as Corollary 2 follows from Proposition 1. A direct proof is given in [13]. ∎

For the MISO BC, other parametrizations (alternative to Proposition 2 and Corollary 3) exist. By duality theory [14] the optimal beamformers for the MISO BC are known to be MMSE beamformers and take the form

$$\boldsymbol{w}_1 = \sqrt{P_1} \frac{\left(\sigma_n^2 \boldsymbol{I} + Q_2 \boldsymbol{h}_2 \boldsymbol{h}_2^H \right)^{-1} \boldsymbol{h}_{11}}{\left\| \left(\sigma_n^2 \boldsymbol{I} + Q_2 \boldsymbol{h}_2 \boldsymbol{h}_2^H \right)^{-1} \right\|} \tag{24}$$

$$\boldsymbol{w}_2 = \sqrt{P_2} \frac{\left(\sigma_n^2 \boldsymbol{I} + Q_1 \boldsymbol{h}_1 \boldsymbol{h}_1^H \right)^{-1} \boldsymbol{h}_{22}}{\left\| \left(\sigma_n^2 \boldsymbol{I} + Q_1 \boldsymbol{h}_1 \boldsymbol{h}_1^H \right)^{-1} \right\|} \tag{25}$$

where Q_1 and Q_2 are the powers in the dual model (see [14] for details). Comparing the two parametrizations in Corollary 3 and in (24)–(25), we see that both parametrizations require one non-negative real-valued parameter for the power allocation and two non-negative real-valued parameters for the beamforming vectors.

V. ILLUSTRATION

Fig. 2 illustrates the achievable rate region for a two-user two-antenna Gaussian MISO IFC. The points are generated from Corollary 2 by varying λ_1 and λ_2 over a grid where $0 \le \lambda_1 \le 1$ and $0 \le \lambda_2 \le 1$.
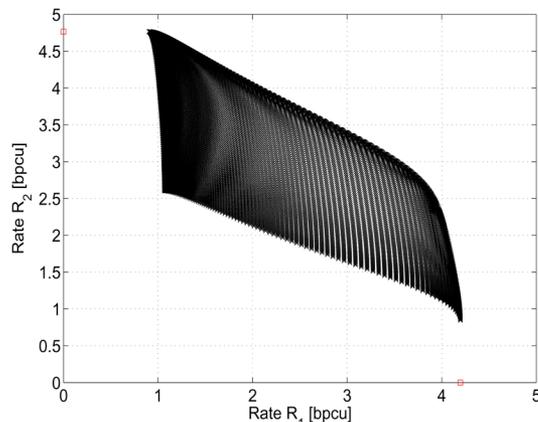
Fig. 2. Pareto boundary for a sample channel realization with $N = 2$ two transmit antennas at high SNR 30 dB.

VI. CONCLUDING REMARKS

The motivation for this correspondence has been the recent, huge interest in IFCs as a model for spectrum resource conflicts (see, e.g., [5], [7], [9], [10], and [12], and the references therein). Our main contribution has been a characterization of the MISO IFC for arbitrary SNR, and specifically a parametrization of the Pareto boundary of the rate region. Our hope is that the results will be useful for future research on resource allocation and spectrum sharing for situations that are well modeled via the MISO IFC.

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Multimode Precoding for MIMO Systems: Performance Bounds and Limited Feedback Codebook Design

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Abstract—This correspondence investigates the problem of designing the precoding codebook for limited feedback multiple-input multiple-output (MIMO) systems. We first analyze the asymptotic capacity loss of a suboptimal multimode precoding scheme as compared to optimal waterfilling and show that the suboptimal scheme is sufficient when negligible capacity loss is allowed. This knowledge is then applied to the design of the limited feedback codebook. In the design, the generalized Lloyd algorithm is employed, where the computation of the centroid is formulated as an optimization problem and solved optimally. Numerical results show that the proposed codebook design outperforms the comparable algorithms reported in the literature.

Index Terms—Given's rotation, limited feedback codebook design, Lloyd algorithm, waterfilling.

I. INTRODUCTION

A well-known result of information theory establishes that feedback does not improve the capacity of a discrete memoryless channel [1]. Nonetheless, for the cases where the channel is selective in either time, frequency, or space, feedback of the channel state to the transmitter can bring substantial benefits to the forward communications system in terms of either capacity, performance, or complexity. The theoretical study of capacity and coding with channel state information at the transmitter (CSIT) can be traced back as early as to Shannon [2]. More recently, information-theoretic capacity on channels with both perfect [3]–[5] and imperfect [6] CSIT and practical coding schemes using CSIT [7], [8] have been studied.

With the advent of multiple-input and multiple-output (MIMO) antenna systems, investigation on the potential benefits of CSIT for MIMO systems has been intensified and design of a practical scheme to achieve the potential benefits as closely as possible becomes very important. The channel estimation done at the receiver needs to be sent back to the transmitter to provide the potential CSIT benefit. Thus, the study of MIMO system with limited feedback is of practical interests. In the past, various options in MIMO transmit beamforming