Controllable 3-D Filters for Low Level Computer Vision

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Abstract

Three-dimensional data processing is becoming more and more common. Typical operations are for example estimation of optical flow in video sequences and orientation estimation in 3-D MR images. This paper proposes an efficient approach to robust low level feature extraction for 3-D image analysis. In contrast to many earlier algorithms the methods proposed in this paper support the use of relatively complex models at the initial processing steps. The aim of this approach is to provide the means to handle complex events at the initial processing steps and to enable reliable estimates in the presence of noise. A limited basis filter set is proposed which forms a basis on the unit sphere and is related to spherical harmonics. From these basis filters, different types of orientation selective filters are synthesized. An interpolation scheme that provides a rotation as well as a translation of the synthesized filter is presented. The purpose is to obtain a robust and invariant feature extraction at a manageable computational cost.

1 Introduction

Multidimensional signal processing is with few exceptions performed on discrete and quantified data, while on the other hand local low-level descriptors are interpreted as continuous functions in the feature space. A line or a plane can for example appear at arbitrary orientation or position in an image. In order to achieve a general and invariant processing it is essential that the feature extraction supports a continuous representation of the specific model. In practice only a limited number of filters can be applied at each neighbourhood of the image to obtain a reasonable computational speed. In many earlier approaches this problem is solved by using a coarse and incomplete partitioning of the feature space, where both the filters used in the feature extraction and the feature space partitioning are chosen to suit the application. This approach reduces the computational demands but restricts the possibilities to solve more complicated tasks due to the limited number of features and the inaccuracy in the feature estimates. Recently there has been an increasing interest among several researchers to develop methods that enable a general feature extraction from a limited set of *basis filters*, since this is a way to obtain a more complete feature extraction at a manageable computational cost, [10, 19, 20, 21]. The proposed feature extraction method uses the following steps:

- Convolve the image with the *basis filter set*.
- Use the basis filter responses to interpolate (synthesize) different types of filters in large number of orientations.
- Analyze the output from the synthesized filters to produce pixel-wise descriptions in terms of:
  - the number of events present in the neighbourhood.
  - the type, e.g. a line or a plane.
  - the orientation.
  - the size.

The selection of the basis filters is based on a number of observations and design decisions:

1. What image features are useful to describe from the proposed methods, e.g. orientation, position and scale.
2. What type of filters are feasible for the detection and estimation these image features?
3. How shall the basis filter set be chosen to make the interpolation procedure both computational efficient and precise.

This presentation is focused on control of orientation and to a certain extent on the position of the synthesized filter. The scale, i.e. the size of the synthesized filters, can be controlled by the proposed method, but unfortunately this extension requires a vast increase of the number of basis filters [20]. A more efficient method to obtain estimates in scale space is to apply the same basis filters at subsampled versions of the original image [1, 12]. Step 2 and 3 above are of course depending on each other. The requirement on the basis filters are, however, fairly general and the presentation of the synthesized filters is postponed to section 4.

2 Basis Filters

A 3-D basis filter set is required to support a uniform approximation of functions defined on the unit sphere. The Weierstrass theorem [14] states that a continuous function:

$$F(u_1, u_2, u_3) \qquad u_1^2 + u_2^2 + u_3^2 \leq 1 \tag{1}$$

can be uniformly approximated by the polynomial

$$F(u_1, u_2, u_3) = \sum_{\eta,\mu,\nu=0}^{N} a_{\eta,\mu,\nu}\, u_1^{\eta}\, u_2^{\mu}\, u_3^{\nu} \qquad (2)$$

The polynomials $u_1^{\eta}\, u_2^{\mu}\, u_3^{\nu}$ form a complete set of functions in the unit sphere. If these polynomials are grouped together as homogeneous polynomials of degree $l = \eta + \mu + \nu$, there will for each l be:

$$\frac{1}{2}\,(l+1)\,(l+2) \qquad (3)$$

linearly independent polynomials. On the unit sphere these polynomials are subject to the constraint $u_1^2 + u_2^2 + u_3^2 = 1$ which removes the linearly independent property. If this constraint is used to eliminate for example u_1^2, each term $u_1^{\eta}\, u_2^{\mu}\, u_3^{\nu}$ can be reduced to $u_2^{\mu}\, u_3^{\nu}$, where $\mu + \nu = l$ or $u_1\, u_2^{\mu}\, u_3^{\nu}$ where $\mu + \nu + 1 = l$, plus lower order polynomials. There are $l + 1$ possible combinations in the first case and l in the second which gives a total of $2l+1$ independent polynomials of degree l. To obtain an intuitive feeling of the directional properties of these functions spherical coordinates are introduced.

$$u_1 = \sin(\theta)\cos(\varphi) \quad u_2 = \sin(\theta)\sin(\varphi) \quad u_3 = \cos(\theta)$$

In table 1 the orthonormal homogeneous polynomials of order $l = 0, 1, 2$ and 3 are listed in spherical coordinates.

These functions are identified as spherical harmonics Y_l^m which are frequently used in physics, where they constitute eigenvectors to the angular momentum operator. Consequently the spherical harmonics constitute an orthonormal base on the unit sphere and are generally interpreted as a natural 3-D generalization of the circular harmonics, $e^{il\varphi} = \cos(l\varphi) + i\sin(l\varphi)$.

To simplify the interpolation procedure and to provide an efficient computation by sequential and/or recursive filtering, it is desired that the basis filters (of the same order) obtain a uniform shape and a well defined orientation. This property is unfortunately not fulfilled for spherical harmonics of order $l \geq 2$. An alternative choice that meets these conditions would be

$$B_{li}(\bar{u}) = G(\rho)\,(\hat{n}_{li} \cdot \hat{u})^l \qquad (4)$$

where

$\bar{u} = (u_1, u_2, u_3)$ is an arbitrary coordinate vector in the Fourier domain and $\hat{u}$ is the normalized coordinate vector.

$G(\rho)$ defines the radial frequency response, $\rho^2 = u_1^2 + u_2^2 + u_2^2$.

$\hat{n}_{li}$ defines the orientation of the i-th basis filter of order l.

Here all basis filters of the same order can be expressed as rotated versions of a single filter. The number of basis filters required for each order l is according to eq. (3) $(l+1)\,(l+2)/2$, as opposed to $2l+1$ for the spherical harmonics. It is, however, straightforward to show that the proposed basis filters of order l span the corresponding basis filters of order $(l-2, l-4, \ldots)$ [1]. For a basis filter *set* of order $l = (0, 1, 2, \ldots, N)$ it is sufficient to compute the filter responses of order $l = N$ and $l = N - 1$. The lower order basis filter responses are then obtained by a simple projection scheme. This gives a total of:

$$\frac{1}{2}\,(N+1)\,(N+2) + \frac{1}{2}\,N\,(N+1) = (N+1)^2$$

basis filters. For the spherical harmonics the corresponding number of basis filters is calculated as:

$$\sum_{l=0}^{N} 2l + 1 = (N+1)^2$$

The spherical harmonics and the basis filters of eq. (4) are consequently equally efficient in terms of the required number of filters.

It may be argued that the proposed basis filters are not orthogonal as opposed to the spherical harmonics. We are convinced that this feature is not critical for the performance of the filter set, especially in relation to the computational benefits. It is, however, possible to obtain orthogonal basis filters with uniform shape according to the above requirements. An alternative orthonormal basis filter

l	$\lvert m \rvert$	Angular function
0	0	$1/\sqrt{4\pi}$
1	0	$\sqrt{3/4\pi}\,\cos(\theta)$
	1	$\sqrt{3/4\pi}\,\sin(\theta)\,\cos(\varphi)$ $\sqrt{3/4\pi}\,\sin(\theta)\,\sin(\varphi)$
2	0	$\sqrt{5/16\pi}\,(3\cos^2(\theta) - 1)$
	1	$\sqrt{15/4\pi}\,\sin(\theta)\,\cos(\theta)\,\cos(\varphi)$ $\sqrt{15/4\pi}\,\sin(\theta)\,\cos(\theta)\,\sin(\varphi)$
	2	$\sqrt{15/16\pi}\,\sin^2(\theta)\,\cos(2\varphi)$ $\sqrt{15/16\pi}\,\sin^2(\theta)\,\sin(2\varphi)$
3	0	$\sqrt{7/16\pi}\,(5\cos^3(\theta) - 3\cos(\theta))$
	1	$\sqrt{21/32\pi}\,\sin(\theta)\,(5\cos^2(\theta) - 1)\,\cos(\varphi)$ $\sqrt{21/32\pi}\,\sin(\theta)\,(5\cos^2(\theta) - 1)\,\sin(\varphi)$
	2	$\sqrt{105/16\pi}\,\sin^2(\theta)\,\cos(\theta)\,\cos(2\varphi)$ $\sqrt{105/16\pi}\,\sin^2(\theta)\,\cos(\theta)\,\sin(2\varphi)$
	3	$\sqrt{35/32\pi}\,\sin^3(\theta)\,\cos(3\varphi)$ $\sqrt{35/32\pi}\,\sin^3(\theta)\,\sin(3\varphi)$

Table 1: *Spherical harmonics of order* $0, 1, 2$ *and* 3.

set of order two can for example be defined as:

$$B'_{2i}(\bar{u}) = B_{2i}(\bar{u}) - k_0 B_0(\bar{u}) \qquad i = (0,1,2,\ldots 5) \quad (5)$$

where $k_0 = 0.5442$ or $k_0 = 0.1225$.

3 Control of Orientation

To synthesize general filter responses in arbitrary orientations from the basis filters a necessary requirement is that that the basis filters of order $l = (0, 1, \ldots, N)$ support a synthetization of a corresponding basis filter in an arbitrary orientation. In this section, the interpolation functions for basis filters up to the third order are defined in terms of the filter orienting vectors, $\hat{n}_{li}$. The extension to an inclusion of filters of arbitrary order is also discussed.

3.1 Basis Filters of Order Zero

The basis filter of order zero is a single isotropic Laplace filter which is defined by the radial frequency response:

$$B_0 = G(\rho) \quad (6)$$

where $G(\rho)$ is assumed to be of bandpass type, i.e. $G(0) = 0$.

3.2 Basis filters of First Order

For symmetry reasons and to reduce the effects of noise, the basis filters are uniformly distributed on the unit sphere. For first order basis filters, which require three filters (eq. (3)), the natural choice is to direct these filters along the coordinate axis in the Fourier domain, i.e.

$$\hat{n}_{10} = (1,0,0) \;\; \hat{n}_{11} = (0,1,0) \;\; \hat{n}_{12} = (0,0,1) \quad (7)$$

A first order basis filter in an arbitrary direction $\hat{v} = (v_1, v_2, v_3)^T$ is expressed as (eq. (4)):

$$\begin{aligned} B_1(\bar{u}) &= G(\rho)\,(\hat{v}\cdot\hat{u}) = \rho^{-1}\,G(\rho)\,(\hat{v}\cdot\bar{u}) = \\ &= \rho^{-1}\,G(\rho)\,(v_1u_1 + v_2u_2 + v_3u_3) \end{aligned} \quad (8)$$

where $\bar{u} = (u_1, u_2, u_3)^T$ defines the signal vector and $\rho^2 = u_1^2 + u_2^2 + u_3^2$. The basis filters in the directions defined in eq. (7) are calculated in the same manner:

$$\begin{aligned} B_{10}(\bar{u}) &= G(\rho)\,(\hat{n}_{10}\cdot\hat{u}) = \rho^{-1}\,G(\rho)\,u_1 \\ B_{11}(\bar{u}) &= G(\rho)\,(\hat{n}_{11}\cdot\hat{u}) = \rho^{-1}\,G(\rho)\,u_2 \\ B_{12}(\bar{u}) &= G(\rho)\,(\hat{n}_{12}\cdot\hat{u}) = \rho^{-1}\,G(\rho)\,u_3 \end{aligned} \quad (9)$$

Let the vector $\bar{t}_1 = (t_{10}, t_{11}, t_{12})^T$ define the interpolation coefficients from the fixed basis filters to a corresponding filter in arbitrary orientation, i.e.:

$$B_1(\bar{u}) = t_{10}\,B_{10}(\bar{u}) + t_{11}\,B_{11}(\bar{u}) + t_{12}\,B_{12}(\bar{u}) \quad (10)$$

By substitution of eq. (8) and eq. (9) into eq. (10) the relation between the interpolation coefficients and the orientation of the synthesized filter is in matrix notation obtained as:

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_{10} \\ t_{11} \\ t_{12} \end{pmatrix} \quad (11)$$

To synthesize a first order filter in an arbitrary orientation $\hat{v}$, chose the interpolation vector $(t_{10}, t_{11}, t_{12})^T$ in eq. (10) to be equal to $\hat{v}$. The interpolation of a first order basis filter is consequently very simple, and the purpose to perform this extensive deduction is mainly to simplify a generalization to higher order basis filters.

3.3 Second Order Basis Filters

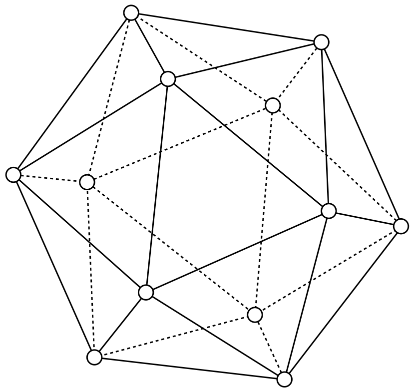

Figure 1: *The icosahedron.*

Let $B_2(\bar{u})$ denote the second order filter in an arbitrary orientation $\hat{v} = (v_1, v_2, v_3)^T$ such that

$$B_2(\bar{u}) = G(\rho)\,(\hat{v}\cdot\hat{u})^2 \quad (12)$$

This expression is decomposed as

$$\begin{aligned} B_2(\bar{u}) &= \rho^{-2}\,G(\rho)\,(v_1u_1 + v_2u_2 + v_3u_3)^2 = \\ &= \rho^{-2}\,G(\rho)\,(v_1^2u_1^2 + v_2^2u_2^2 + v_3^2u_3^2 + \\ &\quad 2v_1v_2u_1u_2 + 2v_1v_3u_1u_3 + 2v_2v_3u_2u_3) \end{aligned}$$

To simplify a later introduction of matrix notation, the vector $\bar{w}_2$ is defined as the v_iv_j terms of $B_2(\bar{u})$.

$$\bar{w}_2 = (v_1^2, v_2^2, v_3^2, 2v_1v_2, 2v_1v_3, 2v_2v_3)^T \quad (13)$$

From eq. (3) it follows that six linearly independent basis filters are required to interpolate a second order filter in an arbitrary orientation. In order to distribute six filters uniformly on the unit sphere, it is relevant to study the geometry of regular (platonic) polyhedrals. The icosahedron (fig. 1) has 12 vertices. Since these vertices are pairwise diametrically opposite, the coordinates of six vertices localized within the same half-sphere define the filter orienting vectors for the second order filters, i.e.

$$\begin{aligned} \hat{n}_{20} &= c\,(\;\; b, & -a, & \;\; 0\;)^T \\ \hat{n}_{21} &= c\,(\;\; b, & a, & \;\; 0\;)^T \\ \hat{n}_{22} &= c\,(\;\; 0, & b, & -a\;)^T \\ \hat{n}_{23} &= c\,(\;\; 0, & b, & \;\; a\;)^T \\ \hat{n}_{24} &= c\,(-a, & 0, & \;\; b\;)^T \\ \hat{n}_{25} &= c\,(\;\; a, & 0, & \;\; b\;)^T \end{aligned} \quad (14)$$

where

$$a = 2 \quad b = 1 + \sqrt{5} \quad c = (10 + 2\sqrt{5})^{-\frac{1}{2}}$$

The six fix basis filters:

$$B_{2i} = G(\rho)\,(\hat{n}_{2i}\cdot\hat{u})^2 \qquad i = (0,1,2,\ldots,5) \qquad (15)$$

are consequently expressed as:

$$\begin{aligned}
B_{20} &= \rho^{-2}\,G(\rho)\,c^2(b^2u_1^2 + a^2u_2^2 - 2abu_1u_2)\\
B_{21} &= \rho^{-2}\,G(\rho)\,c^2(b^2u_1^2 + a^2u_2^2 + 2abu_1u_2)\\
B_{22} &= \rho^{-2}\,G(\rho)\,c^2(b^2u_2^2 + a^2u_3^2 - 2abu_2u_3)\\
B_{23} &= \rho^{-2}\,G(\rho)\,c^2(b^2u_2^2 + a^2u_3^2 + 2abu_2u_3)\\
B_{24} &= \rho^{-2}\,G(\rho)\,c^2(a^2u_1^2 + b^2u_3^2 - 2abu_1u_3)\\
B_{25} &= \rho^{-2}\,G(\rho)\,c^2(a^2u_1^2 + b^2u_3^2 + 2abu_1u_3)
\end{aligned}$$

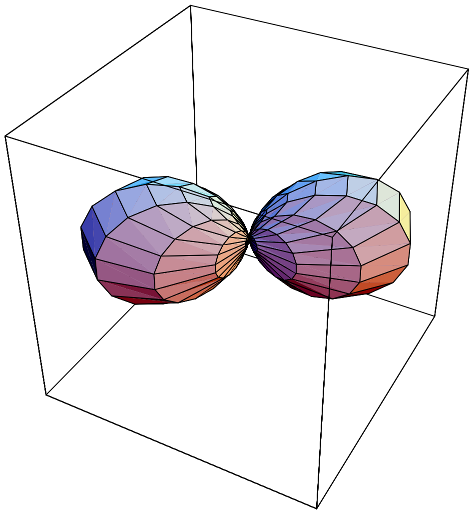

Figure 2: *Angular function of a second order basis filter in direction $\hat{n}_{21}$ in the Fourier domain.*

Finally let the the columns of the matrix A_2 as the coefficients for each of the six second order basis filters.

$$A_2 = c^2\begin{pmatrix} b^2 & b^2 & 0 & 0 & a^2 & a^2\\ a^2 & a^2 & b^2 & b^2 & 0 & 0\\ 0 & 0 & a^2 & a^2 & b^2 & b^2\\ -2ab & 2ab & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -2ab & 2ab\\ 0 & 0 & -2ab & 2ab & 0 & 0 \end{pmatrix}$$

A synthetization of a second order basis filter $B_2(\bar{u})$ in the orientation $\hat{v}$ from the six basis filters is equivalent to compute the interpolation vector $\bar{t}_2 = (t_{20}, t_{21}, \ldots, t_{25})^T$ that satisfies

$$B_2(\bar{u}) = \sum_{i=0}^{5} t_{2i}\,B_{2i}(\bar{u}) \qquad (16)$$

In matrix notation eq. (16) is expressed as

$$\bar{w}_2 = A_2\bar{t}_2 \qquad (17)$$

where the vector $\bar{w}_2$ and the matrix A_2 are defined by the orientation $\hat{v}$ of the synthesized filter and the orientation of the basis filters respectively.

Since the columns of A_2 are linearly independent, it is clear that A_2 is non-singular. The second order interpolation vector is finally obtained as:

$$\bar{t}_2 = A_2^{-1}\,\bar{w}_2 \qquad (18)$$

From the definition of A_2 it is furthermore obvious that the sum of all basis filters results in an isotropic filter, i.e.

$$\frac{1}{2}\sum_{i=0}^{5} B_{2i} = G(\rho) = B_0 \qquad (19)$$

The basis filter of order zero is consequently obtained from second order basis filters by application of the interpolation vector:

$$\bar{t}_1 = (1/2, 1/2, 1/2, 1/2, 1/2, 1/2)^T \qquad (20)$$

in eq. (16). For a basis filer set of order $N = 2$ it is consequently sufficient to compute 3 first order and 6 second order filter responses which results in 9 real filters.

3.4 Third Order Basis Filters

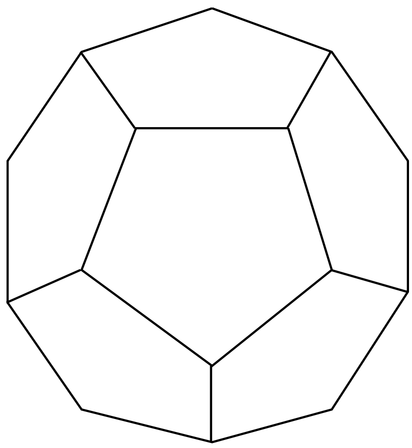

Figure 3: *The dodecahedron.*

An extension of the basis filter set to the third order requires an additional $(l+1)(l+2)/2 = 10$ filters. These filters also support the three first order filters, resulting in a total of 16 filters. Ten filters can be equally spread in a unit sphere, if the filter directions correspond to the main diagonals of a dodecahedron, see fig. 3. The icosahedron and the dodecahedron are, according to [6, 18] reciprocal polyhedrals. This means that the centre of a face of a icosahedron corresponds to a vertex of the dodecahedron and vice versa. An icosahedron has 12 vertices and 20 faces, while the relation for the dodecahedron is the opposite. The orientation of each face in the icosahedron is defined by the sum of the three vectors that define the surrounding three vertices. Ten filter orienting vectors $(\hat{n}_{30}, \hat{n}_{31}\ldots\hat{n}_{39})$ that are located in the same half-sphere can then be obtained by a careful combination of the filter orienting vectors in

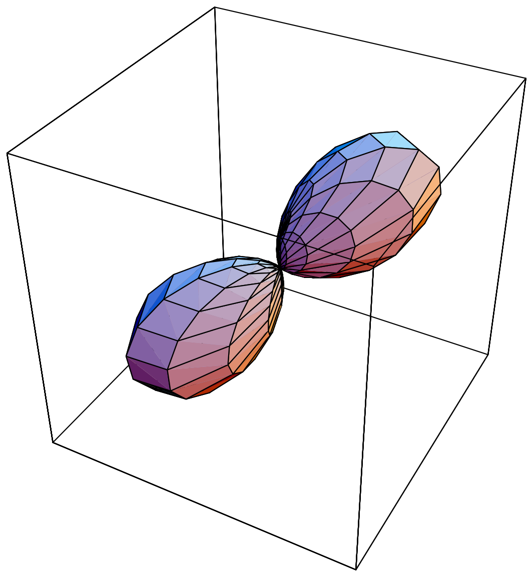

Figure 4: *Angular function of third order basis filter in direction $\hat{n}_{37}$ in the Fourier domain. Note that this filter function is odd as opposed to the second order filter in fig. 2.*

eq. (14).

$$\begin{array}{lclcl}
\hat{n}_{30} & = & k\,(\hat{n}_{21}+\hat{n}_{22}-\hat{n}_{25}) & = & k\,(\;d,\;\;0,\;-b\;)^T \\
\hat{n}_{31} & = & k\,(\hat{n}_{21}+\hat{n}_{22}+\hat{n}_{26}) & = & k\,(\;d,\;\;0,\;\;b\;)^T \\
\hat{n}_{32} & = & k\,(\hat{n}_{23}+\hat{n}_{24}+\hat{n}_{22}) & = & k\,(\;b,\;\;d,\;\;0\;)^T \\
\hat{n}_{33} & = & k\,(\hat{n}_{23}+\hat{n}_{24}-\hat{n}_{21}) & = & k\,(-b,\;\;d,\;\;0\;)^T \\
\hat{n}_{34} & = & k\,(\hat{n}_{25}+\hat{n}_{26}+\hat{n}_{24}) & = & k\,(\;0,\;\;b,\;\;d\;)^T \\
\hat{n}_{35} & = & k\,(\hat{n}_{25}+\hat{n}_{26}-\hat{n}_{23}) & = & k\,(\;0,\;-b,\;\;d\;)^T \\
\hat{n}_{36} & = & k\,(\hat{n}_{21}+\hat{n}_{26}-\hat{n}_{23}) & = & k\,(\;f,\;-f,\;\;f\;)^T \\
\hat{n}_{37} & = & k\,(\hat{n}_{22}+\hat{n}_{24}+\hat{n}_{26}) & = & k\,(\;f,\;\;f,\;\;f\;)^T \\
\hat{n}_{38} & = & k\,(\hat{n}_{24}+\hat{n}_{25}-\hat{n}_{21}) & = & k\,(-f,\;\;f,\;\;f\;)^T \\
\hat{n}_{39} & = & k\,(\hat{n}_{25}-\hat{n}_{22}-\hat{n}_{23}) & = & k\,(-f,\;-f,\;\;f\;)^T
\end{array}$$

where

$$d = a + 2b \quad f = a + b \quad k = \frac{1}{\sqrt{3}(a+b)}$$

In agreement with the second order basis filters the $\bar{w}_3$-vector and the A_3 matrix are computed from $\bar{v}$ and the 10 filter orienting vectors as

$$\begin{array}{lcl}
\bar{w}_3 & = & (v_1^3,\, v_2^3,\, v_3^3,\, 3v_1^2 v_2,\, 3v_1^2 v_3, \\
 & & 3v_1 v_2^2,\, 3v_1 v_3^2,\, 3v_2^2 v_3,\, 3v_2 v_3^3,\, 6v_1 v_2 v_3)^T
\end{array}$$

$$k^3 \begin{pmatrix}
d^3 & d^3 & b^3 & -b^3 & 0 & 0 & f^3 & f^3 & -f^3 & -f^3 \\
0 & 0 & d^3 & d^3 & b^3 & -b^3 & -f^3 & f^3 & f^3 & -f^3 \\
-b^3 & b^3 & 0 & 0 & d^3 & d^3 & f^3 & f^3 & f^3 & f^3 \\
0 & 0 & 3b^2d & 3b^2d & 0 & 0 & -3f^3 & 3f^3 & 3f^3 & -3f^3 \\
-3bd^2 & 3bd^2 & 0 & 0 & 0 & 0 & 3f^3 & 3f^3 & 3f^3 & 3f^3 \\
0 & 0 & 3bd^2 & -3bd^2 & 0 & 0 & 3f^3 & 3f^3 & -3f^3 & -3f^3 \\
3b^2d & 3b^2d & 0 & 0 & 0 & 0 & 3f^3 & 3f^3 & -3f^3 & -3f^3 \\
0 & 0 & 0 & 0 & 3b^2d & 3b^2d & 3f^3 & 3f^3 & 3f^3 & 3f^3 \\
0 & 0 & 0 & 0 & 3bd^2 & -3bd^2 & -3f^3 & 3f^3 & 3f^3 & -3f^3 \\
0 & 0 & 0 & 0 & 0 & 0 & -6f^3 & 6f^3 & -6f^3 & 6f^3
\end{pmatrix}$$

The interpolation vector $\bar{t}_3$ that provides a synthetization of a third order basis filter in an arbitrary orientation $\bar{v}$ such that:

$$B_3(\bar{u}) = G(\rho)\,(\hat{v}\cdot\hat{u})^3 = \sum_{i=0}^{9} t_{3i}\,B_{3i}(\bar{u}) \tag{21}$$

is according to the previous results given by

$$\bar{t}_3 = A_3^{-1}\,\bar{w}_3 \tag{22}$$

It is straightforward to show that the third order basis filters support a synthetization of a first order basis filter in arbitrary orientation $\hat{v}$ by insertion of

$$\bar{w}_1' = (v_1, v_2, v_3, v_2, v_3, v_1, v_1, v_3, v_2, 0)^T \tag{23}$$

instead of $\bar{w}_3$ in eq. (22) [1].

3.5 Higher Order basis Filters

It is clear that the basis filters and interpolation schemes developed in this chapter can be generalized to arbitrary order as well as to higher dimensions (e.g. time sequences of 3-D volumes). For $l \geq 4$ there are no matching regular polyhedra in 3-D. It is consequently impossible to distribute more than 10 filters equally on a unit sphere. This requirement is, however, optional as it is sufficient that the resulting A_l-matrix is non-singular. For a robust computation, it is preferable that the basis filters are approximately uniformly distributed.

4 Filter Synthesis

In this section a method to synthesize quadrature filters and to control the position of the resulting filters are briefly presented. A more detailed description is found in [1]. Quadrature filters provide a phase independent magnitude and are frequently used in computer vision [16, 15, 8, 5, 9].

Since the interpolation functions support a synthesis of the basis filters in arbitrary orientations, it is sufficient to consider a synthetization of the target filter in a single orientation.

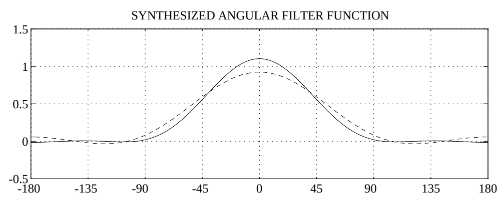

Figure 5: *Angular filter function in the Fourier domain for a synthesized quadrature filter in the u_3-direction as a function of θ for $N = 2$ (dashed) and $N = 3$ (solid).*

To simplify the analysis, consider a general axially symmetric 3-D filter in the direction of the u_3-axis which can be expressed solely as a function of θ:

$$F(\theta) = G(\rho)\sum_{n=0}^{N} a_n \cos^n(\theta) \qquad \bar{a} = (a_0, a_1 \ldots a_N)^T$$

where the coefficient vector $\bar{a}$ defines the angular envelope of the synthesized filter. To obtain the phase invariant property which is fundamental for quadrature filters, it is essential to compute the $\bar{a}$-vector that minimize energy contribution E_1 of $F(\theta)$ in the 'rear' half-sphere of the Fourier domain

$$E_1 = \int_0^{2\pi}\int_{\pi/2}^{\pi} [\sum_{n=0}^{N} a_n \cos^n(\theta)\,]^2 \sin(\theta)\, d\theta\, d\varphi$$

under the condition that the total energy contribution is constant. This approach leads to an eigenvalue problem which can be solved by conventional methods and is related to prolate spheroidals. The $\bar{a}$-vector that in the above sense provides the best approximation of a quadrature filter is consequently given by the eigenvector that corresponds to the least eigenvalue. For a basis filter set of $N = 2$ and $N = 3$ this results in

$$N = 2: \quad \bar{a} \;=\; (\,0.082,\, 0.433,\, 0.409\,)^T$$
$$N = 3: \quad \bar{a} \;=\; (\,0.020,\, 0.221,\, 0.524,\, 0.338\,)^T$$

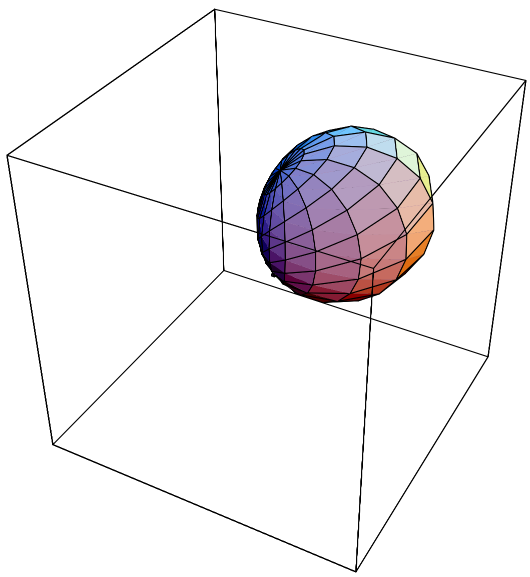

Figure 6: *Angular function in the Fourier domain for quadrature filter synthesized form a basis filter set of order $N = 2$ (9 basis filters).*

In fig. 5 the corresponding angular filter functions for a synthesized quadrature filter in the u_3 direction are illustrated as a function of θ for a basis filter set of order $N = 2$ and $N = 3$. Note that a second order basis filter set which only requires 9 filters provides a fair approximation of a quadrature filter. For a third order basis filter set the energy contribution from the rear half-sphere of the Fourier domain is negligible and the filter envelope becomes sharper. It is consequently possible to control both the orientation and the angular lobewidth of the synthesized filter within this approach. In fig. 6 the angular functions of a synthesized quadrature filter of order $N = 2$ are illustrated as a 3-D plot.

Control of Position

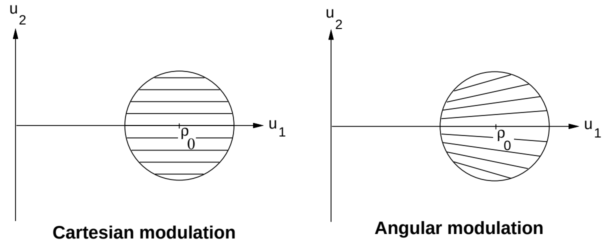

Figure 7: *Cartesian and angular modulation of a quadrature filter in the Fourier domain.*

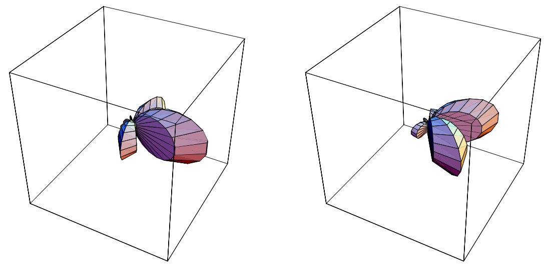

Figure 8: *Angular functions corresponding to real (left) and imaginary (right) part of a synthesized dual quadrature filter in the Fourier domain. The filter is oriented along the u_1-axis and modulated in the φ-direction. These filters are synthesized from a basis filter set of order $N = 3$ which requires 16 real convolution kernels.*

So far the proposed basis filters have provided a control of the orientation and to some extent the lobewidth of the synthesized filters. In the initial discussion control of the position of the filter in the spatial domain were discussed. Can this feature be accomplished within the basis filter set? Control of the position (translation) of the synthesized filters in the spatial domain corresponds in the Fourier domain to a complex modulation in the same direction. The polar separable basis filters do unfortunately not support a cartesian modulation.

Under certain restrictions on the bandwidth of the radial frequency response $G(\rho)$ and the angular lobewidth of the synthesized filter, a shift in the spatial domain can be approximated by a modulation in the angular direction [1], see fig. 7.

Such an angular modulation of the filter envelope can be accomplished by the proposed basis filters. The filter in fig. 6 can for example be modulated along any great circle on the unit sphere that intersects the filter orienting vector $\hat{v}$.

Since these modulated quadrature filters consist of an even real and an odd imaginary part in both the Fourier domain and in the spatial domain, these filters are denoted dual quadrature filters. In fig. 8 the even (real) and odd (imaginary) part of a dual quadrature filter in the Fourier domain is illustrated. These filters are synthesized from a basis filter set of order $N = 3$ (16 basis filters) and are directed along the u_1-axis and modulated in the φ-direction. This type of filters provide a novel approach for curvature/acceleration estimation and for detection of line and plane ends.

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