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Conformal Einstein spaces and Bach tensor generalizations in n dimensions

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Abstract

In this thesis we investigate necessary and sufficient conditions for an n-dimensional space, $n \geq 4$, to be locally conformal to an Einstein space. After reviewing the classical results derived in tensors we consider the four-dimensional spinor result of Kozameh, Newman and Tod. The involvement of the four-dimensional Bach tensor (which is divergence-free and conformally well-behaved) in their result motivates a search for an n-dimensional generalization of the Bach tensor B_{ab} with the same properties. We strengthen a theorem due to Belfagón and Jaén and give a basis (U_{ab}, V_{ab} and W_{ab}) for all n-dimensional symmetric, divergence-free 2-index tensors quadratic in the Riemann curvature tensor. We discover the simple relationship $B_{ab} = \frac{1}{2}U_{ab} + \frac{1}{6}V_{ab}$ and show that the Bach tensor is the unique tensor with these properties in four dimensions. Unfortunately we have to conclude, in general that there is no direct analogue in higher dimension with all these properties.

Nevertheless, we are able to generalize the four-dimensional results due to Kozameh, Newman and Tod to n dimensions. We show that a generic space is conformal to an Einstein space if and only if there exists a vector field satisfying two conditions. The explicit use of dimensionally dependent identities (some of which are newly derived in this thesis) is also exploited in order to make the two conditions as simple as possible; explicit examples are given in five and six dimensions using these tensor identities.

For n dimensions, we define the tensors $\mathfrak{b}_{abc}$ and $\mathfrak{B}_{ab}$, and we show that their vanishing is a conformal invariant property which guarantees that the space with non-degenerate Weyl tensor is a conformal Einstein space.

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Jonas Bergman, Linköping, 20 September 2004

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Chapter 1

Introduction and outline of the thesis

Within semi-Riemannian geometry there are classes of spaces which have special significance from geometrical and/or physical viewpoints; e.g., flat spaces with zero Riemann curvature tensor, conformally flat spaces (i.e., spaces conformal to flat spaces) with Weyl tensor equal to zero, Einstein spaces with trace-free Ricci tensor equal to zero. There are a number of both physical and geometrical reasons to study conformally Einstein spaces (i.e., spaces conformal to Einstein spaces), and it has been a long-standing classical problem to find simple characterizations of these spaces in terms of the Riemann curvature tensor.

Therefore, in this thesis, we will investigate necessary and sufficient conditions for an n-dimensional space, $n \geq 4$, to be locally conformal to an Einstein space, a subject studied since the 1920s. Global properties will not be considered here.

The first results in this field are due to Brinkmann [6], [7], but also Schouten [36] has contributed to the subject; they both considered the general n-dimensional case. Nevertheless, the set of conditions they found is large, and not useful in practice.

Later, in 1964, Szekeres [39] introduced spinor tools into the problem and proposed a partial solution in four dimensions using spinors, restricting the space to be Lorentzian, i.e., to have signature -2. Nevertheless, the spinor conditions he found are hard to analyse and complicated to translate into tensors. Wünsch [43] pointed out a mistake in Szekeres's paper which means that his conditions are only necessary.

However, in 1985, Kozameh, Newman and Tod [27] continued with the spinor approach and found a much simpler set consisting of only two independent necessary and sufficient conditions for four-dimensional spaces; however, the price they paid for this simplicity was that the result was restricted to a subspace of the most general class of spaces − those for which one of the scalar invariants of the Weyl tensor is non-zero, i.e.,

$$J = \frac{1}{2}\left(C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{ab} - i^{*}C^{ab}{}_{cd}{}^{*}C^{cd}{}_{ef}{}^{*}C^{ef}{}_{ab}\right) \neq 0 \quad . \tag{1.1}$$

One of their conditions is the vanishing of the Bach tensor B_{ab}; in four dimensions this tensor has a number of nice properties.
The condition $J \neq 0$ in the result of Kozameh *et al.* [27] has been relaxed, also using spinor methods, by Wünsch [43], [44], by adding a third condition to the set found by Kozameh *et al.* This still leaves some spaces excluded; in particular the case when the space is of Petrov type N, although Czapor, McLenaghan and Wünsch [12] have some results in the right direction.
The spinor formalism is the natural tool for general relativity in four dimensions in a Lorentzian space [32], [33] since it has built in both four dimensions and signature -2; on the other hand, it gives little guidance on how to generalize to n-dimensional semi-Riemannian spaces. However, using a more differential geometry point of view, Listing [28] recently generalized the result of Kozameh *et al.* [27] to n-dimensional semi-Riemannian spaces having non-degenerate Weyl tensor. Listing's results have been extended by Edgar [13] using the Cayley-Hamilton Theorem and dimensionally dependent identities [14], [29].
There have been other approaches to this problem. For example, Kozameh, Newman and Nurowski [26] have interpreted and studied the necessary and sufficient condition for a space to be conformal to an Einstein space in terms of curvature restrictions for the corresponding Cartan conformal connection. Also Baston and Mason [3], [4], working with a twistorial formulation of the Einstein equations, found a different set of necessary and sufficient conditions. However, we shall restrict ourselves to a classical semi-Riemannian geometry approach.

In this thesis we are going to try and find n-dimensional tensors, $n \geq 4$, generalizing the Bach tensor in such a way that as many good properties of the four-dimensional Bach tensor as possible are carried over to the n-dimensional generalization. We shall also investigate how these generalizations of the Bach tensor link up with conformal Einstein spaces.

The first part of this thesis will review the classical tensor results; Chapters 5 to 8 will review and extend a number of the results in both spinors and tensors during the last 20 years. In the remaining chapters we will present some new results and also discuss the directions where this work can develop in the future. We have also included four appendices in which we have collected some old and developed some new results needed in the thesis, but to keep the presentation as clear as possible we have chosen to summarize these at the end.

The outline of the thesis is as follows:

We begin in Chapter 2 by fixing the conventions and notation used in the thesis and giving some useful relations and identities. The chapter ends by reviewing and proving the classical result that a space is conformally flat if and only if the Weyl tensor is identically zero.

In Chapter 3 Einstein spaces and conformal Einstein spaces are introduced and the conformal Einstein equations are derived. Some of the earlier results in the field are also mentioned.

In Chapter 4 the Bach tensor B_{ab} in four dimensions is defined and various attempts to find an n-dimensional counterpart are investigated. We strengthen a theorem due to Belfagón and Jeán and give a basis (U_{ab}, V_{ab} and W_{ab}) for all n-dimensional symmetric, divergence-free 2-index tensors quadratic in the Riemann curvature tensor. We discover the simple relationship $B_{ab} = \frac{1}{2}U_{ab} + \frac{1}{6}V_{ab}$ between the four-dimensional Bach tensor and these tensors, and show that this is the only 2-index tensor (up to constant rescaling) which in four dimensions is symmetric, divergence-free and quadratic in the Riemann curvature tensor. We also demonstrate that there is no useful analogue in higher dimensions.

Chapter 5 deals with the four-dimensional result for spaces in which $J \neq 0$ due to Kozameh, Newman and Tod, and both explicit and implicit results in their paper are proven and discussed. We also explore a little further the relationship between spinor and tensor results.

In Chapter 6 and Chapter 7 the recent work of Listing in spaces with non-degenerate Weyl tensors is reviewed; Chapter 6 deals with the four-dimensional case and Chapter 7 with the n-dimensional case.

In Chapter 8 we look at the extension of Listing's result due to Edgar using the Cayley-Hamilton Theorem and dimensionally dependent identities.

In Chapter 9 the concept of a generic Weyl tensor and a generic space is defined. The results of Kozameh, Newman and Tod are generalized and generic results presented. We show that an n-dimensional generic space is conformal to an Einstein space if and only if there exists a vector field satisfying two conditions. The explicit use of dimensionally dependent identities is also exploited in order to make these two conditions as simple as possible; explicit examples are given in five and six dimensions.

In Chapter 10, for n dimensions, we define the tensors $\mathfrak{b}_{abc}$ and $\mathfrak{B}_{ab}$, whose vanishing guarantees a space with non-degenerate Weyl tensor being a conformal Einstein space. We show that $\mathfrak{b}_{abc}$ is conformally invariant in all spaces with non-degenerate Weyl tensor, and that $\mathfrak{B}_{ab}$ is conformally weighted with weight -2, but only in spaces with non-degenerate Weyl tensor where $\mathfrak{b}_{abc} = 0$. We also show that the Listing tensor $\mathfrak{L}_{ab}$ is conformally invariant in all n-dimensional spaces with non-degenerate Wely tensor.

In the final chapter we briefly summarize the thesis and discuss different ways of continuing this work and possible applications of the results given in the earlier chapters.

Appendix A deals with the representations of the Weyl spinor/tensor as matrices and discusses the Cayley-Hamilton Theorem for matrices and tensors.

In Appendix B dimensionally dependent identities are discussed. A number

of new tensor identities in five and six dimensions suitable for our purpose are derived; these identities are exploited in Chapter 9.
In Appendix C, in four dimensions, we look at the Weyl scalar invariants and derive relations between the two complex invariants naturally arising from spinors and the standard four real tensor invariants.
The last appendix briefly comments on the computer tools used for some of the calculations in this thesis.

At an early stage of this investigation we became aware of the work of Listing [28], who had also been motivated to generalize the work of Kozameh *et al.* [27]. So, although we had already anticipated some of the Listing's generalizations independently, we have reviewed these generalizations as part of his work in Chapters 6 and 7.

When we were writing up this thesis (May 2004) a preprint by Gover and Nurowski [16] appeared on the http://arxiv.org/ . The first part of this preprint obtains some of the results which we have obtained in Chapter 9 in essentially the same manner; however, they do not make the link with dimensionally dependent identities, which we believe makes these results more useful. The second part of this preprint deals with conformally Einstein spaces in a different manner based on the tractor calculus associated with the normal Cartan bundle. Out of this treatment emerges the results on the conformal behavior of $\mathfrak{b}_{abc}$ and $\mathfrak{B}_{ac}$, which we obtained in a more direct manner in Chapter 10. Due to the very recent appearance of [16] we have not referred to this preprint in our thesis, since all of our work was done completely independently of it.

Chapter 2

Preliminaries

In this chapter we will briefly describe the notation and conventions used in this thesis, but for a more detailed description we refer to [32] and [33]. We also review and prove the classical result that a space is conformally flat if and only if the Weyl tensor of the space is identically zero.

2.1 Conventions and notation

All manifolds we consider are assumed to be differentiable and equipped with a symmetric non-degenerate bilinear form $g_{ab} = g_{ba}$, i.e. a metric. No assumption is imposed on the signature of the metric unless explicitly stated, and we will be considering semi-Riemannian (or pseudo-Riemannian) spaces in general; we will on occasions specialize to (proper) Riemannian spaces (metrics with positive definite signature) and Lorentzian spaces (metrics with signature $(+-\ldots-)$).

All connections, ∇, are assumed to be Levi-Civita, i.e. metric compatible, and torsion-free, e.g. $\nabla_a g_{bc} = 0$, and $\big(\nabla_a\nabla_b - \nabla_b\nabla_a\big)f = 0$ for all scalar fields f, respectively.

Whenever tensors are used we will use the abstract index notation, see [32], and when spinors are used we again follow the conventions in [32].

The Riemann curvature tensor is constructed from second order derivatives of the metric but can equivalently be defined as the four-index tensor field R_{abcd} satisfying

$$2\nabla_{[a}\nabla_{b]}\omega_c = \nabla_a\nabla_b\omega_c - \nabla_b\nabla_a\omega_c = -R_{abc}{}^{d}\omega_d \qquad (2.1)$$

for all covector fields ω_a, and it has the following algebraic properties, $R_{abcd} = R_{[ab][cd]} = R_{cdab}$; it satisfies the first Bianchi identity, $R_{[abc]d} = 0$, and it also satisfies the second Bianchi identity, $\nabla_{[a}R_{bc]de} = 0$.

From the Riemann curvature tensor (2.1) we define the Ricci (curvature) tensor, R_{ab}, by the contraction

$$R_{ab} = R_{acb}{}^{c} \tag{2.2}$$

and the Ricci scalar, R, from the contracted Ricci tensor

$$R = R_{a}{}^{a} = R_{ab}{}^{ab} \quad . \tag{2.3}$$

For dimensions $n \geq 3$ the Weyl (curvature) tensor or the Weyl conformal tensor, C_{abcd}, is defined as the trace-free part of the Riemann curvature tensor,

$$C_{abcd} = R_{abcd} - \frac{2}{n-2}\Big(g_{a[c}R_{d]b} - g_{b[c}R_{d]a}\Big) + \frac{2}{(n-1)(n-2)} R g_{a[c}g_{d]b} \tag{2.4}$$

and as is obvious from above when $R_{ab} = 0$ the Riemann curvature tensor reduces to the Weyl tensor. The Weyl tensor has all the algebraic properties of the Riemann curvature tensor, i.e. $C_{abcd} = C_{[ab][cd]} = C_{cdab}, C_{a[bcd]} = 0$, and in addition is trace-free, i.e. $C_{abc}{}^{a} = 0$. It is also well known that the Weyl tensor is identically zero in three dimensions.

For clarity we note that due to our convention in defining the Riemann curvature tensor (2.1) we have for an arbitrary tensor field $H^{b\ldots d}{}_{f\ldots h}$ that

$$\begin{aligned}\big(\nabla_i\nabla_j - \nabla_j\nabla_i\big) H^{b\ldots d}{}_{f\ldots h} = {} & R_{ijb_0}{}^{b} H^{b_0\ldots d}{}_{f\ldots h} + \ldots + R_{ijd_0}{}^{d} H^{b\ldots d_0}{}_{f\ldots h} \\ & - R_{ijf}{}^{f_0} H^{b\ldots d}{}_{f_0\ldots h} - \ldots - R_{ijh}{}^{h_0} H^{b\ldots d}{}_{f\ldots h_0}\end{aligned} \tag{2.5}$$

and (2.5) is sometimes referred to as the Ricci identity.

We will use both the conventions in the literature for denoting covariant derivatives, e.g., both the “nabla” and the semicolon, $\nabla_a v \equiv v_{;a}$. Note however the difference in order of the indices in each case, $\nabla_a\nabla_b v \equiv v_{;ba}$.

For future reference we write out the twice contracted (second) Bianchi identity,

$$\nabla^a R_{ab} - \frac{1}{2}\nabla_b R = 0 \quad , \tag{2.6}$$

the second Bianchi identity in terms of the Weyl tensor

$$0 = R_{ab[cd;e]} = C_{ab[cd;e]} + \frac{1}{(n-3)} g_{a[c}C_{de]b}{}^{f}{}_{;f} + \frac{1}{(n-3)} g_{b[c}C_{ed]a}{}^{f}{}_{;f} \quad , \tag{2.7}$$

the second contracted Bianchi identity for the Weyl tensor,

$$\nabla_d C_{abc}{}^{d} = \frac{(n-3)}{(n-2)}\Big(-2\nabla_{[a}R_{b]c} + \frac{1}{(n-1)} g_{c[b}\nabla_{a]}R\Big) \quad . \tag{2.8}$$

Note that (2.8) also can be written

$$\nabla_d C_{abc}{}^d = -\frac{(n-3)}{(n-2)} C_{cba} \tag{2.9}$$

where $C_{abc} = \Big(-2\nabla_{[a} R_{b]c} + \frac{1}{(n-1)} g_{c[b}\nabla_{a]} R \Big)$ is the *Cotton tensor*. The Cotton tensor plays an important role in the study of thee-dimensional spaces [5] ,[15], [21].

We also have the divergence of (2.8)

$$\begin{aligned} C_{abcd;}{}^{db} =& \frac{(n-3)}{(n-2)} R_{ac;}{}^{b}{}_{b} - \frac{(n-3)}{2(n-1)} R_{;ac} + \frac{(n-3)n}{(n-2)^2} R_{ab} R^b{}_c \\ & - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - \frac{(n-3)}{(n-2)^2} g_{ac} R_{bd} R^{bd} \\ & - \frac{n(n-3)}{(n-1)(n-2)^2} R R_{ac} - \frac{(n-3)}{2(n-1)(n-2)} g_{ac} R_;{}^{b}{}_{b} \\ & + \frac{(n-3)}{(n-1)(n-2)^2} g_{ac} R^2 \quad . \end{aligned} \tag{2.10}$$

Using the second Bianchi identity for the Weyl tensor, the Ricci identity and finally decomposing the Riemann tensor into the Weyl tensor, the Ricci tensor and the Ricci scalar we get the following identity

$$\nabla_{[e} \nabla^d C_{ab]cd} = \frac{(n-3)}{(n-2)} R_{[ab|c|}{}^f R_{e]f} = \frac{(n-3)}{(n-2)} C_{[ab|c|}{}^f R_{e]f} \quad . \tag{2.11}$$

In an n-dimensional space, letting $H^{\{\Omega\}}{}_{a_1\ldots a_p} = H^{\{\Omega\}}{}_{[a_1\ldots a_p]}$ denote any tensor with an arbitrary number of indices schematically denoted by $\{\Omega\}$, plus a set of $p \leq n$ completely antisymmetric indices $a_1 \ldots a_p$, we define the *(Hodge) dual* $\overset{*}{H}{}^{\{\Omega\}}{}_{a_{p+1}\ldots a_n}$ with respect to $a_1 \ldots a_p$ by

$$\overset{*}{H}{}^{\{\Omega\}}{}_{a_{p+1}\ldots a_n} = \frac{1}{p!} \eta_{a_1\ldots a_n} H^{\{\Omega\} a_1 \ldots a_p} \tag{2.12}$$

where η is the totally antisymmetric normalized tensor. Sometimes the * is placed over the indices onto which the operation acts, e.g. $H^{\{\Omega\}}{}_{\overset{*}{a_{p+1}\ldots a_n}}$.

In the special case of taking the dual of a double two-form $H_{abcd} = H_{[ab][cd]}$ in four dimensions there are two ways to perform the dual operation; either acting on the first pair of indices, or the second pair. To separate the two we define *the left dual* and the *the right dual* as

$$^*H_{ij}{}^{cd} = \frac{1}{2} \eta_{abij} H^{abcd} \tag{2.13}$$

and

$$H^{*ab}{}_{ij} = \frac{1}{2}\eta_{cdij} H^{abcd} \tag{2.14}$$

respectively.

We are going to encounter highly structured products of Weyl tensors and to get a neater notation we follow [42] and make the following definition:

Definition 2.1.1. For a trace-free $(2,2)$-form T_{abcd}, i.e. for a tensor such that

$$T^{a}{}_{bad} = 0 \quad , \quad T_{abcd} = T_{[ab]cd} = T_{ab[cd]} \tag{2.15}$$

an expression of the form

$$\underbrace{T^{ab}{}_{c_1 d_1} T^{c_1 d_1}{}_{c_2 d_2} \ldots T^{c_{m-2} d_{m-2}}{}_{c_{m-1} d_{m-1}} T^{c_{m-1} d_{m-1}}{}_{ef}}_{m} \tag{2.16}$$

where the indices a, b, e and f are free, is called a *chain (of the zeroth kind) of length* m and is written[1] $T[m]^{ab}{}_{ef}$.

Hence, we have for instance that $C[3]^{ab}{}_{cd} = C^{ab}{}_{ij} C^{ij}{}_{kl} C^{kl}{}_{cd}$.

On occasions we will use matrices and these will always be written in bold capital letters, e.g. $\mathbf{A}$. In this context $\mathbf{O}$ and $\mathbf{I}$ will denote the zero- and the identity matrix respectively and we will use square brackets to represent the operation of taking the trace of a matrix, e.g. $[\mathbf{A}]$ means the trace of the matrix $\mathbf{A}$.

Throughout this thesis we will only be considering spaces of dimension $n \geq 4$. This is because in two dimensions all spaces are Einstein spaces and in three dimensions all spaces are conformally flat.

2.2 Conformal transformations

Definition 2.2.1. Two metrics g_{ab} and $\widehat{g}_{ab}$ are said to be *conformally related* if there exists a smooth scalar field $\Omega > 0$ such that

$$\widehat{g}_{ab} = \Omega^2 g_{ab} \tag{2.17}$$

holds.

The metric $\widehat{g}_{ab}$ is said to arise from a conformal transformation of g_{ab}. Clearly we have $\widehat{g}^{ab} = \Omega^{-2} g^{ab}$, since then $\widehat{g}^{ab}\widehat{g}_{bc} = g^{ab} g_{bc} = \delta^a_c$.

[1] In [42] this is denoted $\underset{0}{T}[m]^{ab}{}_{ef}$.

To the rescaled metric $\widehat{g}_{ab}$ there is a unique symmetric connection, $\widehat{\nabla}$, compatible with $\widehat{g}_{ab}$, i.e. $\widehat{\nabla}_c \widehat{g}_{ab} = 0$. The relation between the two connections can be found in [32] and acting on an arbitrary tensor field $H^{b\ldots d}{}_{f\ldots h}$ it is

$$\begin{aligned}\widehat{\nabla}_a H^{b\ldots d}{}_{f\ldots h} =& \nabla_a H^{b\ldots d}{}_{f\ldots h} + Q_{ab_0}{}^b H^{b_0\ldots d}{}_{f\ldots h} + \ldots + Q_{ad_0}{}^d H^{b\ldots d_0}{}_{f\ldots h} \\ & - Q_{af}{}^{f_0} H^{b\ldots d}{}_{f_0\ldots h} - \ldots - Q_{ah}{}^{h_0} H^{b\ldots d}{}_{f\ldots h_0} \end{aligned} \tag{2.18}$$

with

$$Q_{ab}{}^c = 2\Upsilon_{(a}\delta^c_{b)} - g_{ab}\Upsilon^c \tag{2.19}$$

where $\Upsilon_a = \Omega^{-1}\nabla_a\Omega = \nabla_a(\ln\Omega)$ and $\Upsilon^a = g^{ab}\Upsilon_b$.

The relations between the tensors defined in (2.1) - (2.4) and their hatted counterparts, i.e. the ones constructed from $\widehat{g}_{ab}$, are[2]

$$\begin{aligned}\widehat{R}_{abc}{}^d =& R_{abc}{}^d - 2\delta^d_{[a}\nabla_{b]}\Upsilon_c + 2g_{c[a}\nabla_{b]}\Upsilon^d - 2\delta^d_{[b}\Upsilon_{a]}\Upsilon_c \\ & + 2\Upsilon_{[a}g_{b]c}\Upsilon^d + 2g_{c[a}\delta^d_{b]}\Upsilon_e\Upsilon^e \quad , \end{aligned} \tag{2.20}$$

$$\begin{aligned}\widehat{R}_{ab} =& R_{ab} + (n-2)\nabla_a\Upsilon_b + g_{ab}\nabla_c\Upsilon^c \\ & -(n-2)\Upsilon_a\Upsilon_b + (n-2)g_{ab}\Upsilon_c\Upsilon^c \quad , \end{aligned} \tag{2.21}$$

$$\widehat{R} = \Omega^{-2}\Big(R + 2(n-1)\nabla_c\Upsilon^c + (n-1)(n-2)\Upsilon_c\Upsilon^c\Big) \quad , \tag{2.22}$$

and

$$\widehat{C}_{abc}{}^d = C_{abc}{}^d \quad . \tag{2.23}$$

Note that the positions of the indices in all equations (2.20) - (2.23) are crucial since we raise and lower indices with different metrics, e.g. $\widehat{C}_{abcd} = \widehat{g}_{de}\widehat{C}_{abc}{}^e = \Omega^2 g_{de} C_{abc}{}^e = \Omega^2 C_{abcd}$.

Definition 2.2.2. A tensor field $H^{b\ldots d}{}_{f\ldots h}$ is said to be *conformally well-behaved* or *conformally weighted (with weight ω)* if under the conformal transformation (2.17), $\widehat{g}_{ab} = \Omega^2 g_{ab}$, there is a real number w such that

$$H^{b\ldots d}{}_{f\ldots h} \to \widehat{H}^{b\ldots d}{}_{f\ldots h} = \Omega^\omega H^{b\ldots d}{}_{f\ldots h} \quad . \tag{2.24}$$

If $\omega = 0$ then $H^{b\ldots d}{}_{f\ldots h}$ is said to be *conformally invariant.*

Following [32] we introduce the tensor

$$P_{ab} = -\frac{1}{(n-2)}R_{ab} + \frac{1}{2(n-1)(n-2)}Rg_{ab} \tag{2.25}$$

[2] These can be found, for instance, in [40], but note that Wald is using a different definition than (2.1) to define the Riemann curvature tensor.

and with the notation (2.25) we can express (2.2) - (2.4) as

$$C_{ab}{}^{cd} = R_{ab}{}^{cd} + 4P_{[a}{}^{[c}g_{b]}{}^{d]} \quad , \tag{2.26}$$

$$R_{ab} = -(n-2)P_{ab} - g_{ab}P_c{}^c \quad , \tag{2.27}$$

and

$$R = -2(n-1)P_c{}^c = -2(n-1)P \quad . \tag{2.28}$$

The contracted second Bianchi identity for the Weyl tensor (2.8) can now be written

$$\nabla_d C_{abc}{}^d = 2(n-3)\nabla_{[a}P_{b]c} \quad . \tag{2.29}$$

Note that P_{ab} is essentially R_{ab} with a different trace term added and that P_{ab} simply replaces R_{ab} to make equations such as (2.26) and (2.29) simpler than (2.4) and (2.8), and hence to make calculations simpler. Furthermore, under a conformal transformation,

$$\widehat{P}_{ab} = P_{ab} + \Upsilon_a\Upsilon_b - \nabla_a\Upsilon_b - \frac{1}{2}g_{ab}\Upsilon_c\Upsilon^c \quad , \tag{2.30}$$

and

$$\widehat{P} = \Omega^{-2}\big(P - \nabla_c\Upsilon^c - \frac{(n-2)}{2}\Upsilon_c\Upsilon^c\big) \quad , \tag{2.31}$$

which are simpler than the corresponding equations (2.21) and (2.22).

2.3 Conformally flat spaces

Definition 2.3.1. A space is called *flat* if its Riemann curvature tensor vanishes, $R_{abcd} = 0$.

Since flat spaces are well understood we would like a simple condition on the geometry telling us when there exists a conformal transformation making the space flat. Hence we define,

Definition 2.3.2. A space is called *conformally flat* if there exists a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ such that the Riemann curvature tensor in the space with metric $\widehat{g}_{ab}$ vanishes, i.e., $\widehat{R}_{abcd} = 0$.

From (2.20) we know that a space is conformally flat if and only if

$$\begin{aligned} 0 =& R_{abc}{}^d - 2\delta^d_{[a}\nabla_{b]}\Upsilon_c + 2g_{c[a}\nabla_{b]}\Upsilon^d - 2\delta^d{}_{[b}\Upsilon_{a]}\Upsilon_c \\ &+ 2\Upsilon_{[a}g_{b]c}\Upsilon^d + 2g_{c[a}\delta^d_{b]}\Upsilon_e\Upsilon^e \end{aligned} \tag{2.32}$$

for some gradient vector field Υ^a. Raising one index in equation (2.32) we can rewrite this as

$$\begin{aligned} 0 &= R_{ab}{}^{cd} + 4\delta^{[c}_{[a}\nabla_{b]}\Upsilon^{d]} + 4\Upsilon_{[a}\delta^{[c}_{b]}\Upsilon^{d]} + 2\delta^{[c}_{[a}\delta^{d]}_{b]}\Upsilon_e\Upsilon^e \\ &= R_{ab}{}^{cd} + 4P_{[a}{}^{[c}\delta^{d]}_{b]} \end{aligned} \tag{2.33}$$

where from (2.30) with $\widehat{P}_{ab} = 0$ we have

$$P_{ab} = \nabla_a \Upsilon_b - \Upsilon_a \Upsilon_b + \frac{1}{2} g_{ab} \Upsilon_c \Upsilon^c \tag{2.34}$$

and

$$P_{[ab]} = 0 \quad . \tag{2.35}$$

We will now prove the important classical theorem[3]

Theorem 2.3.1. *A space is conformally flat if and only if its Weyl tensor is zero.*

Proof. The necessary part follows immediately from the conformal properties of the Weyl tensor (2.23).

To prove the sufficient part we break up the proof into two steps. First we shall show that if there exists some symmetric 2-tensor P_{ab} in a space such that

$$0 = R_{ab}{}^{cd} + 4P_{[a}{}^{[c}\delta^{d]}_{b]} \tag{2.36}$$

$$P_{ab} = \nabla_a K_b - K_a K_b + \frac{1}{2} g_{ab} K_c K^c; \quad P_{[ab]} = 0 \tag{2.37}$$

for some vector field K^a, then the space is conformally flat.

From (2.37) it follows that $\nabla_{[a} K_{b]} = 0$ which means that locally K_a is a gradient vector field,

$$K_a = \nabla_a \Phi \tag{2.38}$$

for some scalar field Φ, and substitution of P_{ab} in (2.37) with this gradient expression for K_a into (2.36) gives

$$\begin{aligned} 0 =& R_{abc}{}^d - 2\delta^d_{[a}\nabla_{b]}\nabla_c\Phi + 2g_{c[a}\nabla_{b]}\nabla^d\Phi - 2\delta^d_{[b}\big(\nabla_{a]}\Phi\big)\nabla_c\Phi \\ &+ 2\big(\nabla_{[a}\Phi\big)g_{b]c}\nabla^d\Phi + 2g_{c[a}\delta^d_{b]}\big(\nabla_e\Phi\big)\nabla^e\Phi \quad , \end{aligned}$$

i.e., (2.20) with $\Upsilon_a = \nabla_a \Phi$, which implies that $\widehat{R}_{abcd} = 0$ and so the space is conformally flat, $\widehat{g} = e^{2\Phi} g_{ab}$.

Secondly we shall show that if $C_{abcd} = 0$ then (2.36) and (2.37) are satisfied. From (2.26) it follows immediately that if $C_{abcd} = 0$,

$$0 = R_{ab}{}^{cd} + 4P_{[a}{}^{[c}\delta^{d]}_{b]} \tag{2.39}$$

---

[3] The necessary part is originally due to Weyl [41], and the sufficient part to Schouten [37].

i.e. (2.36) is satisfied. Also, if $C_{abcd} = 0$, we know from the second Bianchi identity (2.29) that

$$\nabla_{[a} P_{b]c} = 0 \quad . \tag{2.40}$$

To check if (2.37) can be satisfied we calculate the integrability condition of (2.37) which is

$$\begin{aligned} 0 &= \nabla_{[c} P_{a]b} - \nabla_{[c} \nabla_{a]} \Upsilon_b + \Upsilon_b \nabla_{[c} \Upsilon_{a]} + \Upsilon_{[a} \nabla_{c]} \Upsilon_b - \Upsilon^e g_{b[a} \nabla_{c]} \Upsilon_e \\ &= \nabla_{[c} P_{a]b} + \left(\frac{1}{2} R_{ca}{}^{be} + 2 P_{[c}{}^{[b} \delta^{e]}_{a]} \right) \Upsilon_e \end{aligned} \tag{2.41}$$

But from (2.39) and (2.40) it follows that this condition is identically satisfied. Hence we conclude that (2.37) is a consequence of (2.36).

To summarize, we have shown that $C_{abcd} = 0$ implies (2.36), which in turn implies (2.36) and (2.37) (with the help of the Bianchi identities), meaning that the space is conformally flat. $\square$

Chapter 3

Conformal Einstein equations and classical results

In this chapter we will define conformal Einstein spaces and derive the conformal Einstein equations in n dimensions. We will also give a short summary of the classical results due to Brinkmann [6] and Schouten [36].

3.1 Einstein spaces

Definition 3.1.1. An n-dimensional space is said to be an *Einstein space* if the trace-free part of the Ricci tensor is identically zero, i.e.

$$R_{ab} - \frac{1}{n} g_{ab} R = 0 \quad . \tag{3.1}$$

Expressing this condition using the P_{ab} tensor we get an expression having the same algebraic structure

$$(n-2)\Big(P_{ab} - \frac{1}{n} g_{ab} P\Big) = 0 \quad , \tag{3.2}$$

and we also note that in an Einstein space (2.25) becomes

$$P_{ab} = -\frac{1}{2n(n-1)} g_{ab} R \quad . \tag{3.3}$$

From the contracted Bianchi identity (2.6) we find for an Einstein space that

$$0 = 2\nabla^a R_{ab} - \nabla_b R = -\frac{(n-2)}{n} \nabla_b R \quad , \tag{3.4}$$

i.e. the Ricci scalar must be constant.

3.2 Conformal Einstein spaces

Definition 3.2.1. An n-dimensional space with metric g_{ab} is a *conformal Einstein space* (or *conformally Einstein*) if there exists a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ such that in the conformal space with metric $\widehat{g}_{ab}$

$$\widehat{R}_{ab} - \frac{1}{n}\widehat{g}_{ab}\widehat{R} = 0 \quad , \tag{3.5}$$

or equivalently

$$\widehat{P}_{ab} - \frac{1}{n}\widehat{g}_{ab}\widehat{P} = 0 \quad . \tag{3.6}$$

Note that from (2.22) and (3.4) we have that

$$\widehat{R} = \Omega^{-2}\Big(R + 2(n-1)\nabla_c\Upsilon^c + (n-1)(n-2)\Upsilon_c\Upsilon^c\Big) = \text{ constant} \quad , \tag{3.7}$$

where we used the notation introduced in Chapter 2.2. Further, equation (3.5) is equivalent to

$$\begin{aligned} &R_{ab} - \frac{1}{n}g_{ab}R + (n-2)\nabla_a\Upsilon_b - \frac{(n-2)}{n}g_{ab}\nabla_c\Upsilon^c \\ &- (n-2)\Upsilon_a\Upsilon_b + \frac{(n-2)}{n}g_{ab}\Upsilon_c\Upsilon^c = 0 \quad , \end{aligned} \tag{3.8}$$

and (3.6) to

$$P_{ab} - \frac{1}{n}g_{ab}P - \nabla_a\Upsilon_b + \frac{1}{n}g_{ab}\nabla_c\Upsilon^c + \Upsilon_a\Upsilon_b - \frac{1}{n}g_{ab}\Upsilon_c\Upsilon^c = 0 \tag{3.9}$$

respectively. (3.8) or (3.9) is often referred to as the (n-dimensional) *conformal Einstein equations.*

Taking a derivative of (3.7) gives the relations

$$\begin{aligned} 0 &= \nabla_a R - 2R\Upsilon_a - 4(n-1)\Upsilon_a\nabla_c\Upsilon^c - 2(n-1)(n-2)\Upsilon_a\Upsilon_c\Upsilon^c \\ &\quad + 2(n-1)\nabla_a\nabla_c\Upsilon^c + 2(n-1)(n-2)\Upsilon^c\nabla_a\Upsilon_c \\ &= \nabla_a P - 2P\Upsilon_a + 2\Upsilon_a\nabla_c\Upsilon^c + (n-2)\Upsilon_a\Upsilon_c\Upsilon^c - \nabla_a\nabla_c\Upsilon^c \\ &\quad - (n-2)\Upsilon^c\nabla_a\Upsilon_c \end{aligned} \tag{3.10}$$

and using this, the first integrability condition of (3.9) is calculated to be

$$\nabla_{[a}P_{b]c} + \frac{1}{2}C_{abcd}\Upsilon^d = 0 \quad , \tag{3.11}$$

or, using (2.29),

$$\nabla^d C_{abcd} + (n-3)\Upsilon^d C_{abcd} = 0 \quad . \tag{3.12}$$

Taking another derivative and using (2.29) again we have

$$\nabla^b\nabla_{[a}P_{b]c} + \frac{1}{2}P^{bd}C_{abcd} + (n-4)\Upsilon^b\nabla_{[a}P_{b]c} = 0 \tag{3.13}$$

and clearly both (3.11) and (3.13) are necessary conditions for a space to be conformally Einstein. Obviously we could get additional necessary conditions by taking higher derivatives.

The last equation (3.13) can also be written

$$\nabla^b \nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-4) \Upsilon^b \Upsilon^d C_{abcd} = 0 \quad (3.14)$$

and in dimension $n = 4$ this condition (3.14) reduces to a condition only on the geometry, and is independent of Υ^a. This condition,

$$B_{ac} \equiv \nabla^b \nabla^d C_{abcd} - \frac{1}{2} R^{bd} C_{abcd} = 0 \quad , \quad (3.15)$$

is a necessary, but not a sufficient, condition for a four-dimensional space to be conformally Einstein.

Note that if (3.8) holds for any vector field K^a,

$$\begin{aligned} R_{ab} - \frac{1}{n} g_{ab} R + (n-2) K_a K_b - \frac{(n-2)}{n} g_{ab} \nabla_c K^c \\ - (n-2) K_a K_b + \frac{(n-2)}{n} g_{ab} K_c K^c = 0 \quad , \end{aligned} \quad (3.16)$$

and remembering that R_{ab} is symmetric, then by antisymmetrising we get $\nabla_{[a} K_{b]} = 0$, i.e. that K^a is locally a gradient. Hence we have that a space is locally a conformal Einstein space *if and only if* (3.16) holds for some vector field K^a. Given that P_{ab} is defined by (2.25) this same statement also holds for

$$P_{ab} - \frac{1}{n} g_{ab} P - \nabla_a K_b + \frac{1}{n} g_{ab} \nabla_c K^c + K_a K_b - \frac{1}{n} g_{ab} K_c K^c = 0 \quad . \quad (3.17)$$

3.3 The classical results

In 1924 Brinkmann [6] found necessary and sufficient conditions for a space to be conformally Einstein. In his approach he derived a large set of differential equations involving Υ^a and by exploiting both existence and compatibility of this derived set he was able to formulate necessary and sufficient conditions. However, from his results it is hard to get a constructive set of necessary and sufficient conditions, and his results are not very useful in practice.

Brinkmann later also studied in detail some special cases of conformal Einstein spaces [7].

Schouten [36] used a slightly different approach and looked directly at the explicit form of integrability condition for the conformal Einstein equations, but he did not go beyond Brinkmann's results as regards sufficient conditions. Schouten found the necessary condition (3.13) which we will return to in the following chapters.

Chapter 4

The Bach tensor in four dimensions and possible generalizations

In this chapter we will take a closer look at the four-dimensional version of the conformal Einstein equations introduced in the previous chapter. We will consider the Bach tensor B_{ab} and derive and discuss its properties.

A theorem stating that in n dimensions there only exists three independent symmetric, divergence-free 2-index tensors (U_{ab}, V_{ab} and W_{ab}) quadratic in the Riemann curvature tensor is proven, extending a result due to Balfagón and Jaén [2]. The properties of these tensors are investigated, and we obtain the new result that $B_{ab} = \frac{1}{2}U_{ab} + \frac{1}{6}V_{ab}$.

We also seek possible generalizations of the Bach tensor in n dimensions.

4.1 The Bach tensor in four dimensions

From (3.8), (3.9) in the previous chapter we know that in four dimensions the conformal Einstein equations are

$$\begin{aligned} R_{ab} - \frac{1}{4}g_{ab}R + 2\nabla_a\Upsilon_b - g_{ab}\nabla_c\Upsilon^c \\ - 2\Upsilon_a\Upsilon_b + g_{ab}\Upsilon_c\Upsilon^c = 0 \end{aligned} \quad (4.1)$$

or

$$P_{ab} - \frac{1}{4}g_{ab}P - \nabla_a\Upsilon_b + \frac{1}{4}g_{ab}\nabla_c\Upsilon^c + \Upsilon_a\Upsilon_b - \frac{1}{4}g_{ab}\Upsilon_c\Upsilon^c = 0 \quad . \quad (4.2)$$

The necessary conditions (3.11) and (3.13) in four dimensions become

$$\nabla_{[a}P_{b]c} + \frac{1}{2}C_{abcd}\Upsilon^d = 0 \tag{4.3}$$

and

$$\nabla^b\nabla_{[a}P_{b]c} + \frac{1}{2}P^{bd}C_{abcd} = 0 \tag{4.4}$$

respectively. The left hand side of this last equation (4.4) defines, as in (3.15), the tensor B_{ac},

$$B_{ac} = \nabla^b\nabla_{[a}P_{b]c} + \frac{1}{2}P^{bd}C_{abcd} \quad , \tag{4.5}$$

which can also be written as

$$B_{ac} = \nabla^b\nabla^d C_{abcd} - \frac{1}{2}R^{bd}C_{abcd} \quad , \tag{4.6}$$

and we see that the necessary condition (4.4) then can be formulated as $B_{ac} = 0$. The tensor B_{ab} is called the Bach tensor and was first discussed by Bach [1].

As seen above, the origin of the Bach tensor is in an integrability condition for a four-dimensional space to be conformal to an Einstein space. The Bach tensor is a tensor built up from pure geometry, and thereby captures necessary features of a space being conformally Einstein in an intrinsic way.

It is obvious from the definition of B_{ab} (4.6) that the Bach tensor is symmetric, trace-free and quadratic in the Riemann curvature tensor.

Definition 4.1.1. A tensor is said to be *quadratic in the Riemann curvature tensor* if it is a linear combination of products of two Riemann curvature tensors and/or a linear combination of second derivatives of the Riemann curvature tensor [11].

Calculating the divergence of B_{ab} from (4.6) we get, after twice using (2.5) to switch the order of the derivatives,

$$\begin{aligned}\nabla^c B_{ac} =& \frac{1}{3}R_a{}^c\nabla_c R + R^{bc}\nabla_c R_{ba} - \frac{1}{2}R^{bc}\nabla_a R_{bc} + \frac{1}{2}\nabla^b\nabla_c\nabla^c R_{ba} \\ & - R_{aebd}\nabla^b R^{de} - \frac{1}{12}\nabla_a\nabla_c\nabla^c R - \frac{1}{6}\nabla_c\nabla_a\nabla^c R \\ =& 0 \end{aligned} \tag{4.7}$$

i.e. B_{ab} is divergence-free[1].

[1] This was first noted by Hesselbach [22].

Under a conformal transformation, $\widehat{g}_{ab} = \Omega^2 g_{ab}$, we see after some calculation that

$$
\begin{aligned}
\widehat{B}_{ac} &= \widehat{\nabla}^b \widehat{\nabla}_d \widehat{C}_{abc}{}^d - \frac{1}{2}\widehat{R}^b{}_d \widehat{C}_{abc}{}^d \\
&= \widehat{\nabla}^b \big(\nabla_d C_{abc}{}^d + \Upsilon_d C_{abc}{}^d\big) - \frac{1}{2}\Omega^{-2}\big(R^b{}_d + 2\nabla^b \Upsilon_d \\
&\qquad + \delta^b_d \nabla_c \Upsilon^c - 2\Upsilon^b \Upsilon_d + 2\delta^b_d \Upsilon_c \Upsilon^c\big) C_{abc}{}^d \\
&= \Omega^{-2}\big(\nabla^b \nabla^d C_{abcd} - \frac{1}{2} R^{bd} C_{abcd}\big) \\
&= \Omega^{-2} B_{ac}
\end{aligned} \tag{4.8}
$$

so that B_{ac} is conformally weighted[2] with weight -2.

We can also express the Bach tensor (4.6) in an alternative form in terms of the Weyl tensor, the Ricci tensor and the Ricci scalar, using the four-dimensional version of (2.10),

$$
\begin{aligned}
\nabla^b \nabla^d C_{abcd} =& \frac{1}{2} R_{ac;}{}^b{}_b - \frac{1}{6} R_{;ac} - \frac{1}{12} g_{ac} R_{;}{}^b{}_b + R_{ab} R^b{}_c - \frac{1}{2} R^{bd} C_{abcd} \\
& - \frac{1}{3} R R_{ac} - \frac{1}{4} g_{ac} R_{bd} R^{bd} + \frac{1}{12} g_{ac} R^2 \quad ,
\end{aligned} \tag{4.9}
$$

and we find

$$
\begin{aligned}
B_{ac} =& \frac{1}{2} R_{ac;b}{}^b - \frac{1}{12} g_{ac} R_{;b}{}^b - \frac{1}{6} R_{;ac} - R^{bd} C_{abcd} \\
& + R_{ab} R^b{}_c - \frac{1}{4} g_{ac} R_{db} R^{bd} - \frac{1}{3} R R_{ac} + \frac{1}{12} g_{ac} R^2 \quad .
\end{aligned} \tag{4.10}
$$

For completeness we also give the Bach tensor expressed in spinor language

$$
B_{ab} = B_{AA'BB'} = 2\big(\nabla^C{}_{A'} \nabla^D{}_{B'} + \Phi^{CD}{}_{A'B'}\big)\Psi_{ABCD} \quad . \tag{4.11}
$$

To summarize, the Bach tensor B_{ab} in four dimensions (given by (4.5), (4.6), (4.10) or (4.11)) is symmetric, trace-free, quadratic in the Riemann curvature tensor, divergence-free, and is conformally weighted with weight -2.

Although it is only in four dimensions that the Bach tensor has been defined and has these (nice) properties, it is natural to ask if there is an n-dimensional counterpart to the Bach tensor. Unfortunately as we shall see in the next section, it is easy to show that if we simply carry over the form of the Bach tensor given in (4.6) or (4.10) into $n > 4$ dimensions, it does not retain all these useful properties. So, in the subsequent sections we look to see if there is a generalization which retains as many as possible of the useful properties that the Bach tensor has in four dimensions.

[2] This result is originally due to Haantjes and Schouten [20].

4.2 Attempts to find an n-dimensional Bach tensor

Before we begin looking for an n-dimensional Bach tensor we note that, using the notation from Chapter 2.2, we can derive the two useful relations

$$\widehat{\nabla}^b\widehat{\nabla}^d\widehat{C}_{abcd} = \Omega^{-2}\Big(\nabla^b\nabla^d C_{abcd} + (n-3)C_{abcd}\nabla^b\Upsilon^d + (n-4)\Upsilon^d\nabla^b C_{abcd} + (n-4)\Upsilon^b\nabla^d C_{abcd} + (n-3)(n-5)\Upsilon^b\Upsilon^d C_{abcd}\Big) \tag{4.12}$$

and

$$\widehat{R}^{bd}\widehat{C}_{abcd} = R^{bd}C_{abcd} + (n-2)C_{abcd}\nabla^b\Upsilon^d - (n-2)C_{abcd}\Upsilon^b\Upsilon^d \quad , \tag{4.13}$$

which are used extensively in this chapter.

If we simply carry over the tensor in (4.6) to arbitrary n dimensions, and label this tensor $\underset{1}{B}_{ac}$,

$$\underset{1}{B}_{ac} = \nabla^b\nabla^d C_{abcd} - \frac{1}{2}R^{bd}C_{abcd} \quad , \tag{4.14}$$

we find that

$$\underset{1}{B}_{ac;}{}^{a} = \frac{(n-4)}{2(n-2)}\Big(C_{abcd}R^{bd}{}_{;}{}^{a} + \frac{(n-3)}{(n-2)}\Big(R^{ab}R_{bc;a} - R^{bd}R_{bd;c} - \frac{1}{2(n-1)}R^a{}_c R_{;a} + \frac{1}{2(n-1)}RR_{;c}\Big)\Big) \tag{4.15}$$

and

$$\widehat{\underset{1}{B}}_{ac} = \Omega^{-2}\Big(\underset{1}{B}_{ac} + (n-4)\Big(\Upsilon^b\nabla^d C_{abcd} + \Upsilon^d\nabla^b C_{abcd} + \frac{1}{2}C_{abcd}\nabla^b\Upsilon^d + (2n-7)C_{abcd}\Upsilon^b\Upsilon^d\Big)\Big) \quad , \tag{4.16}$$

i.e., that its divergence-free and conformally well behaved properties do in general not carry over into $n > 4$ dimensions. Similarly the alternative form from (4.10),

$$\underset{2}{B}_{ac} = \frac{1}{2}R_{ac;b}{}^{b} - \frac{1}{12}g_{ac}R_{;b}{}^{b} - \frac{1}{6}R_{;ac} - R^{bd}C_{abcd} + R_{ab}R^b{}_c - \frac{1}{4}g_{ac}R_{db}R^{bd} - \frac{1}{3}RR_{ac} + \frac{1}{12}g_{ac}R^2 \quad , \tag{4.17}$$

also fails to have these properties in general in $n > 4$ dimensions since

$$\underset{2}{B}_{ac;}{}^{a} = \frac{(n-4)}{(n-2)}\Big(R^{ab}R_{bc;a} - R^{bd}R_{bd;c} - \frac{(n+1)}{6(n-1)}R^{a}{}_{c}R_{;a} + \frac{1}{2(n-1)}RR_{;c}\Big) \tag{4.18}$$

and

$$\begin{aligned}\widehat{\underset{2}{B}}_{bc} =&\Omega^{-2}\underset{2}{B}_{bc} + (n-4)\Omega^{-2}\Bigg(- \frac{(n-2)}{2}\Upsilon^a\nabla^d C_{abcd} - \frac{(n-2)}{2}\Upsilon^d\nabla^a C_{abcd} \\ &- \frac{(n-3)(n-2)}{2}C_{abcd}\Upsilon^a\Upsilon^d + \frac{(3n-10)}{6}\Theta R_{bc} \\ &+ \frac{(n-3)}{2}\Big(g_{bc}R_{ad}\Theta^{ad} - \Theta_{ab}R^{a}{}_{c} - R_{ab}\Theta^{a}{}_{c}\Big) + \frac{(8n-15)}{6(n-1)}R\Upsilon_b\Upsilon_c \\ &- \frac{(3n-5)}{4(n-1)}\Big(\Upsilon_b\nabla_c R + \Upsilon_c\nabla_b R\Big) - \frac{(n-7)(3n-5)}{12(n-1)}g_{bc}\Upsilon^a\nabla_a R \\ &+ \frac{(3n^2-10n+5)}{6(n-1)}R\Theta_{bc} + \frac{(3n-5)(n-7)}{12(n-1)}g_{bc}R\Upsilon_a\Upsilon^a \\ &- \frac{(n-3)(n-2)}{2}\Big(\Theta_{ab}\Theta^{a}{}_{c} - \frac{1}{2}g_{bc}\Theta_{ad}\Theta^{ad}\Big) - \frac{(3n-10n)}{6}g_{bc}R\Theta \\ &+ \frac{(3n^2-16n+19)}{3}\Theta\Theta_{bc} - \frac{(3n-5)}{2}\Big(\Upsilon_c\nabla_b\Theta + \Upsilon_b\nabla_c\Theta - \frac{1}{3}\nabla_b\nabla_c\Theta\Big) \\ &- \frac{(3n-5)}{6}g_{bc}\nabla^a\nabla_a\Theta + \frac{(n-7)(3n-5)}{6}g_{bc}\Big(\Theta\Upsilon_a\Upsilon^a - \Upsilon^a\nabla_a\Theta\Big) \\ &+ (3n-5)\Theta\Upsilon_b\Upsilon_c - \frac{(6n^2-41n+53)}{12}g_{bc}\Theta^2\Bigg)\end{aligned} \tag{4.19}$$

where $\Theta_{ab} = \nabla_a\Upsilon_b - \Upsilon_a\Upsilon_b + \frac{1}{2}g_{ab}\Upsilon_c\Upsilon^c$ and $\Theta = \Theta^{a}{}_{a} = \nabla^a\Upsilon_a + \frac{(n-2)}{2}\Upsilon_a\Upsilon^a$.

Going back to the origins of the Bach tensor as an integrability condition for conformal Einstein spaces (3.14),

$$\nabla^b\nabla^d C_{abcd} - \frac{(n-3)}{(n-2)}R^{bd}C_{abcd} - (n-3)(n-4)\Upsilon^b\Upsilon^d C_{abcd} = 0 \tag{4.20}$$

suggests considering the tensor

$$\begin{aligned}\underset{3}{B}_{ac} &= \nabla^b\nabla^d C_{abcd} - \frac{(n-3)}{(n-2)}R^{bd}C_{abcd} \\ &= \underset{1}{B}_{ac} - \frac{(n-4)}{2(n-2)}R^{bd}C_{abcd} \quad .\end{aligned} \tag{4.21}$$

But once again, for dimensions $n > 4$, we see that

$$\begin{aligned}\underset{3}{B}^{a}{}_{c;a} = \frac{(n-4)(n-3)}{(n-2)^2}\Big(&R^{ab}R_{bc;a} - R^{bd}R_{bd;c} \\ &- \frac{1}{2(n-1)}R^{a}{}_{c}R_{;a} + \frac{1}{2(n-1)}RR_{;c}\Big)\end{aligned} \tag{4.22}$$

and

$$\underset{3}{\widehat{B}}_{ac} = \Omega^{-2}\Bigg(\underset{3}{B}_{ac} + (n-4)\Big[\Upsilon^b\nabla^d C_{abcd} + \Upsilon^d\nabla^b C_{abcd} + (n-3)C_{abcd}\Upsilon^b\Upsilon^d\Big]\Bigg) \tag{4.23}$$

so this tensor is neither divergence-free nor conformally well-behaved in general.

So we need to look elsewhere for possible generalization of the Bach tensor to $n > 4$ dimensions.

4.3 The tensors U_{ab}, V_{ab} and W_{ab}

Many years ago Gregory [17] discovered two symmetric divergence-free tensors in four dimensions, and later Collinson [11] added a third. Recently Robinson [34] and Balfagón and Jaén [2] have shown that these three tensors have direct counterparts in $n > 4$ dimension with the same properties.

We shall first of all show that these three tensors, U_{ab}, V_{ab} and W_{ab}, are the only three tensors with these properties and then also examine in more detail their structure and properties.

Balfagón and Jaén [2] have proven the following theorem[3]:

Theorem 4.3.1. *In an arbitrary n-dimensional semi-Riemannian manifold:*

(a) There exist 14 independent and quadratic in Riemann, four-index divergence-free tensors.

(b) There are no totally symmetric, quadratic in Riemann, and divergence-free four-index tensors

(c) The complete family of quadratic in Riemann, and divergence-free

[3]Note that this is a quotation from their paper [2], but here expressed using our conventions.

four-index tensors T^{abcd} totally symmetric in (bcd) is

$$T^{abcd} = a_S T_S^{abcd} + a_R T_R^{abcd}; \tag{4.24}$$

$$T_S^{abcd} = Q^{a(bcd)}; \tag{4.25}$$

$$\begin{aligned} Q^{abcd} = & -\frac{1}{3}g^{ac}R^d{}_i R^{ib} - 2R^{ab;dc} + \frac{4}{3}R^{bd;ac} - \frac{4}{3}g^{ac}R^{bd;i}{}_i \\ & + 2g^{ac}R^b{}_i{}^{;di} + \frac{4}{3}R^b{}_i R^{acid} + 2R^{aibj}R^c{}_i{}^d{}_j - \frac{1}{2}g^{ac}R_{ij}{}^d{}_k R^{ijbk} \end{aligned} \tag{4.26}$$

$$T_R^{abcd} = X^{a(b}g^{cd)}; \tag{4.27}$$

$$X^{ab} = KU^{ab} + LV^{ab} - \frac{1}{4}W^{ab} \tag{4.28}$$

$$U^{ab} = -G^{ab;s}{}_s - 2G^{sb;a}{}_s + 2G^a{}_p R^{pb} - \frac{1}{2}g^{ab}G_{pq}R^{pg} \tag{4.29}$$

$$V^{ab} = -R^{;ab} + g^{ab}R^{;s}{}_s - RS^{ab} \tag{4.30}$$

$$W^{ab} = G^{apqr}R^b{}_{pqr} - \frac{1}{4}g^{ab}G^{mpqr}R_{mpqr} \tag{4.31}$$

$$G_a{}^c = G_{ab}{}^{cb} \tag{4.32}$$

$$G^{ab}{}_{cd} = R^{ab}{}_{cd} - 4\delta^{[a}_{[c}S^{b]}{}_{d]} \tag{4.33}$$

$$S^{ab} = \frac{1}{4}g^{ab}R \tag{4.34}$$

where a_S, a_R, K and L are four independent constants[4].

First note that there are some differences in signs in (4.26), (4.29) and (4.30) compared to the original definitions in [2]. This is due to our definition of the Riemann curvature tensor (2.1) which differs from the one in [2]. A change of convention makes the change $R_{abcd} \to -R_{abcd}$, meaning that there is only going to be a difference in sign for the terms created from a odd number of Riemann tensors (e.g. here exactly only one). However, our definition agrees with the one in [34] up to an overall sign.

Secondly, by a “divergence-free” four index tensor Balfagón and Jaén mean a tensor T^{abcd} such that $\nabla_a T^{abcd} = 0$, i.e. a tensor divergence-free on the first index. Hence, in Theorem 4.3.1 (a) states that there exist only 14 independent such tensors, (b) states that none of these are totally symmetric, $T^{abcd} = T^{(abcd)}$ and (c) gives all tensors T^{abcd} such that $T^{abcd} = T^{a(bcd)}$ and $\nabla_a T^{abcd} = 0$.

[4] T_R is the tensor found by Robinson [34] and T_S the tensor found by Sachs [35].

Note that Theorem 4.3.1 implies that the tensors U_{ab}, V_{ab} and W_{ab} are all divergence-free. This follows from the divergence-free property and the construction of the tensor T_R^{abcd} via (4.27) and (4.28).

We are specially interested in the tensors U_{ab}, V_{ab} and W_{ab}, and to see their inner structure and their properties we write them out in terms of the Riemann curvature tensor, the Ricci tensor and the Ricci scalar,

$$\begin{aligned} U_{bc} =& 2(n-3)R_{abcd}R^{ad} - (n-3)R_{bc;a}{}^{a} + \frac{(n-3)}{2} g_{bc}R_{ad}R^{ad} \\ &+ (n-3)RR_{cb} + \frac{(n-3)}{2} g_{bc}R_{;a}{}^{a} - \frac{(n-3)}{4} g_{bc}R^2 \quad , \end{aligned} \tag{4.35}$$

$$V_{bc} = -R_{;bc} + g_{bc}R_{;a}{}^{a} - RR_{bc} + \frac{1}{4} g_{bc}R^2 \quad , \tag{4.36}$$

$$\begin{aligned} W_{bc} =& R_b{}^{ade}R_{cade} - \frac{1}{4} g_{bc}R^{fade}R_{fade} + 2R^{ad}R_{abcd} + RR_{bc} \\ &- 2R_{ab}R^{a}{}_{c} + g_{bc}R_{ad}R^{ad} - \frac{1}{4} g_{bc}RR \quad . \end{aligned} \tag{4.37}$$

When we later study the conformal behavior of U_{ab}, V_{ab} and W_{ab} it is useful to have them expressed in terms of the Weyl tensor, the Ricci tensor and the Ricci scalar,

$$\begin{aligned} U_{bc} =& 2(n-3)R^{ad}C_{abcd} + (n-3)R_{bc;a}{}^{a} - \frac{(n-3)}{2} g_{bc}R_{;a}{}^{a} \\ &+ \frac{(n-6)(n-3)}{2(n-2)} g_{bc}R_{de}R^{de} + \frac{4(n-3)}{(n-2)} R_{ab}R^{a}{}_{c} \\ &+ \frac{(n-3)(n^2-5n+2)}{(n-1)(n-2)} RR_{bc} - \frac{(n-3)(n^2-3n-6)}{4(n-1)(n-2)} g_{bc}R^2 \quad , \end{aligned} \tag{4.38}$$

$$V_{bc} = -R_{;bc} + g_{bc}R_{;d}{}^{d} - RR_{bc} + \frac{1}{4} g_{bc}R^2 \quad , \tag{4.39}$$

$$\begin{aligned} W_{bc} =& C_b{}^{ade}C_{cade} - \frac{1}{4} g_{bc}C^{fade}C_{fade} + \frac{2(n-4)}{(n-2)} C_{abcd}R^{ad} \\ &- \frac{2(n-3)(n-4)}{(n-2)^2} R_{ab}R^{a}{}_{c} + \frac{(n-3)(n-4)}{(n-2)^2} g_{bc}R_{ad}R^{ad} \\ &+ \frac{n(n-3)(n-4)}{(n-1)(n-2)^2} RR_{bc} - \frac{(n+2)(n-3)(n-4)}{4(n-1)(n-2)^2} g_{bc}R^2 \quad . \end{aligned} \tag{4.40}$$

From (4.35) - (4.37) (or (4.38) - (4.40)) it is obvious that U_{ab}, V_{ab} and W_{ab} are symmetric and quadratic in the Riemann curvature tensor in all dimensions. We note from (4.40) that $W_{bc} = 0$ in four dimensions because in four dimensions $C_b{}^{ade}C_{cade} - \frac{1}{4}g_{bc}C^{fade}C_{fade} = 0$, see Appendix B. This fact was not noticed by Collinson [11] but was subsequently pointed out in [34] and [2].

By taking the trace of (4.38) - (4.40) we have

$$U_a{}^a = -\frac{(n-3)(n-2)}{2}R_{;a}{}^a + \frac{(n-4)(n-3)}{2}R_{ad}R^{ad} - \frac{(n-4)(n-3)}{4}R^2 \quad , \tag{4.41}$$

$$V_a{}^a = (n-1)R_{;a}{}^a + \frac{(n-4)}{4}R^2 \quad , \tag{4.42}$$

$$W_a{}^a = -\frac{(n-4)}{4}C^{fade}C_{fade} + \frac{(n-3)(n-4)}{(n-2)}R_{ad}R^{ad} - \frac{n(n-3)(n-4)}{4(n-1)(n-2)}R^2 \quad . \tag{4.43}$$

A simple direct calculation would confirm that U_{ab}, V_{ab} and W_{ab} are all divergence-free, but we have already noted that this can be deduced from Theorem 4.3.1.

It is easily checked that the three tensors U_{ab}, V_{ab} and W_{ab} are independent and an obvious question is whether there are any more such tensors; we shall now show that there are not.

Given any symmetric and divergence-free tensor, Y_{ab}, quadratic in the Riemann curvature tensor we see that the tensor $Y^{a(b}g^{cd)}$ is a four-index tensor which is totally symmetric over (bcd), quadratic in the Riemann curvature tensor and divergence-free (on the first index). Hence we know from Theorem 4.3.1 (c), that there exist constants a_S, and a_R such that

$$Y^{a(b}g^{cd)} = a_S T_S^{abcd} + a_R T_R^{abcd} \tag{4.44}$$

holds. Taking the trace over c and d of (4.44) using the facts that

$$\begin{aligned} g_{cd}T_R^{abcd} &= g_{cd}X^{a(bcd)} = g_{cd}\Big(KU^{a(b} + LV^{a(b} + \frac{1}{4}W^{a(b}\Big)g^{cd)} \\ &= (n+2)\Big(KU^{ab} + LV^{ab} + \frac{1}{4}W^{ab}\Big) \quad , \end{aligned} \tag{4.45}$$

where K and L are constants fixed by (4.44), and

$$\begin{aligned} g_{cd}T_S^{abcd} =& g_{cd}Q^{a(bcd)} = -\frac{4}{3}R^a{}_cR^{bc} + \frac{8}{9}R_;{}^{ab} - \frac{14}{9}R^{ab}{}_;{}^c{}_c \\ &- \frac{1}{9}g^{ab}R_{;d}{}^d + \frac{16}{9}R_{cd}R^{adbc} + \frac{5}{3}R^{adef}R^b{}_{def} \\ &- \frac{1}{9}g^{ab}R_{cd}R^{cd} - \frac{1}{6}g^{ab}R_{cdef}R^{cdef} \quad , \end{aligned} \tag{4.46}$$

noting that (4.46) actually can be written as a linear combination of U_{ab}, V_{ab} and W_{ab},

$$g_{cd}T_S^{abcd} = -\frac{14}{9(n-3)}U^{ab} - \frac{8}{9}V^{ab} + \frac{2}{3}W^{ab} \quad , \tag{4.47}$$

we find that

$$\begin{aligned} Y^{ab} =& \Big(a_R K - \frac{14a_S}{9(n-2)(n-3)}\Big)U^{ab} + \Big(a_R L - \frac{8a_S}{9(n-2)}\Big)V^{ab} \\ &+ \Big(\frac{a_R}{4} + \frac{2a_S}{3(n-2)}\Big)W^{ab} \quad , \end{aligned} \tag{4.48}$$

i.e., that Y^{ab} is a linear combination of U_{ab}, V_{ab} and W_{ab}. We summarize this result in the following theorem

Theorem 4.3.2. *In an n-dimensional space there are only three independent symmetric and divergence-free 2-index tensors quadratic in the Riemann curvature tensor, e.g., U_{ab}, V_{ab} and W_{ab}.*

Before we investigate the relations between the four-dimensional Bach tensor and U_{ab}, V_{ab} and W_{ab} we note that under the conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ the tensors U_{ab}, V_{ab} and W_{ab} transform according to

$$\begin{aligned} \widehat{U}_{bc} =& \Omega^{-2}U_{bc} + \Omega^{-2}(n-3)\Big[2(n-2)C_{abcd}\Theta^{ad} + (n-6)\Upsilon^a\nabla_a R_{bc} \\ &+ 2\Upsilon^a\nabla_c R_{ab} + 2\Upsilon^a\nabla_b R_{ac} + R_{bc}\Big(2(n-4)\Theta - (n-4)\Upsilon^a\Upsilon_a\Big) \\ &+ R_{ab}\Big(4\Theta^a{}_c - n\Upsilon^a\Upsilon_c\Big) + R_{ac}\Big(4\Theta^a{}_b - n\Upsilon^a\Upsilon_b\Big) \\ &+ g_{bc}R_{ef}\Big((n-6)\Theta^{ef} + 2\Upsilon^e\Upsilon^f\Big) - \Upsilon_b\nabla_c R - \Upsilon_c\nabla_b R \\ &- \frac{(n-6)}{2}g_{bc}\Upsilon^a\nabla_a R + R\Big(2\Upsilon_b\Upsilon_c + \frac{(n^2-5n+2)}{(n-1)}\Theta_{bc}\Big) \\ &+ g_{bc}R\Big(-\frac{(n^2-5n+2)}{(n-1)}\Theta + \frac{(n-6)}{2}\Upsilon_a\Upsilon^a\Big) \\ &+ g_{bc}\Big(2(n-2)\Upsilon^e\Upsilon^f\Theta_{ef} - (n-2)\nabla^a\nabla_a\Theta \\ &+ (n-2)(n-6)\Upsilon_a\Upsilon^a\Theta - \frac{(n-2)(2n-11)}{2}\Theta^2 \\ &\quad + \frac{(n-2)(n-6)}{2}\Theta_{ad}\Theta^{ad} - (n-2)(n-6)\Upsilon^a\nabla_a\Theta\Big) \\ &+ (n-2)\nabla^a\nabla_a\Theta_{bc} + (n-6)(n-2)\Upsilon^a\nabla_a\Theta_{bc} - 2(n-2)\Upsilon_b\nabla^a\Theta_{ac} \\ &- 2(n-2)\Upsilon_c\nabla^a\Theta_{ab} + 2(n-2)\Upsilon^a\nabla_b\Theta_{ac} + 2(n-2)\Upsilon^a\nabla_c\Theta_{ab} \\ &+ \Theta_{ab}\Big(4(n-2)\Theta^a{}_c - n(n-2)\Upsilon_c\Upsilon^a\Big) - n(n-2)\Theta_{ac}\Upsilon^a\Upsilon_b \end{aligned}$$

$$+ \Theta_{bc}\Big(2(n-2)(n-4)\Theta - (n-2)(n-4)\Upsilon_a\Upsilon^a\Big)$$
$$+ 2(n-2)\Theta\Upsilon_b\Upsilon_c\Bigg] \quad , \tag{4.49}$$

$$\begin{aligned}\widehat{V}_{bc} =& \Omega^{-2}V_{bc} + \Omega^{-2}\Big[- 2(n-1)\Theta R_{bc} - 6R\Upsilon_c\Upsilon_b - (n-4)R\Theta_{bc} \\ & - (n-7)g_{bc}R\Upsilon_a\Upsilon^a + (n-4)g_{bc}R\Theta + 3\Upsilon_c\nabla_b R + 3\Upsilon_b\nabla_c R \\ & + (n-7)g_{bc}\Upsilon^a\nabla_a R - 2(n-1)\nabla_b\nabla_c\Theta + 2(n-1)g_{bc}\nabla^a\nabla_a\Theta \\ & + 6(n-1)\Upsilon_c\nabla_b\Theta + 6(n-1)\Upsilon_b\nabla_c\Theta + 2(n-1)(n-7)g_{bc}\Upsilon^a\nabla_a\Theta \\ & + (n-1)(n-7)g_{bc}\Theta^2 - 2(n-1)(n-7)\Theta g_{bc}\Upsilon_a\Upsilon^a \\ & - 12(n-1)\Theta\Upsilon_b\Upsilon_c - 2(n-1)(n-4)\Theta\Theta_{bc}\Big] \quad ,\end{aligned} \tag{4.50}$$

$$\begin{aligned}\widehat{W}_{bc} =& \Omega^{-2}W_{bc} + \Omega^{-2}(n-4)\Big[2C_{abcd}\Theta^{ad} - \frac{2(n-3)}{(n-2)}R_{ab}\Theta^a{}_c \\ & - \frac{2(n-3)}{(n-2)}R_{ac}\Theta^a{}_b - 2(n-3)\Theta_{ab}\Theta^a{}_c \\ & + \frac{2(n-3)}{(n-2)}g_{bc}\Theta_{ad}R^{ad} + (n-3)g_{bc}\Theta_{ad}\Theta^{ad} \\ & + \frac{n(n-3)}{(n-1)(n-2)}R\Theta_{bc} + \frac{2(n-3)}{(n-2)}R_{bc}\Theta \\ & + 2(n-3)\Theta_{bc}\Theta - (n-3)g_{bc}\Theta^2 \\ & - \frac{n(n-3)}{(n-1)(n-2)}g_{bc}R\Theta\Big] \quad ,\end{aligned} \tag{4.51}$$

where $\Theta_{ab} = \nabla_a\Upsilon_b - \Upsilon_a\Upsilon_b + \frac{1}{2}g_{ab}\Upsilon_c\Upsilon^c$ and $\Theta = \Theta^a{}_a = \nabla^a\Upsilon_a + \frac{(n-2)}{2}\Upsilon_a\Upsilon^a$.

It is easily seen from (4.49) - (4.51) that U_{bc} and V_{bc} are not conformally well-behaved in general in four dimensions (where $W_{bc} = 0$).

4.4 Four-dimensional Bach tensor expressed in U_{ab}, V_{ab} and W_{ab}

Since U_{ab}, V_{ab} and W_{ab} constitute a basis for all 2-index symmetric divergence-free tensors quadratic in the Riemann curvature tensor, and the Bach tensor B_{ab} has these properties, we must be able to express the Bach tensor (4.6) in four dimensions in terms of U_{ab} and V_{ab}, remembering $W_{ab} = 0$ in four dimensions.

In four dimensions we know from (4.38) - (4.39)

$$U_{bc} = R_{bc;a}{}^{a} - \frac{1}{2} g_{bc} R_{;a}{}^{a} + 2C_{abcd} R^{ad} + 2R_{ab} R_c{}^{a} - \frac{1}{2} g_{bc} R_{ad} R^{ad} - \frac{1}{3} R R_{bc} + \frac{1}{12} g_{bc} R^2 \tag{4.52}$$

$$V_{bc} = -R_{;bc} + g_{bc} R_{;a}{}^{a} - R R_{bc} + \frac{1}{4} g_{bc} R R \tag{4.53}$$

Comparing these equations with (4.10) we conclude that we have the relation

$$B_{bc} = \frac{1}{2} U_{bc} + \frac{1}{6} V_{bc} \quad . \tag{4.54}$$

The numerical relationship between the tensors V_{bc} and U_{bc} could also be found using the trace-free property of the Bach tensor. Making the ansatz

$$B_{bc} = \alpha U_{bc} + \beta V_{bc} \tag{4.55}$$

we see from (4.41) and (4.42) in four dimensions that

$$B_b{}^{b} = -\alpha R_{;b}{}^{b} + 3\beta R_{;b}{}^{b} = -(\alpha - 3\beta) R_{;b}{}^{b} = 0 \tag{4.56}$$

and hence in general we must have $3\alpha = \beta$.

This link between the Bach tensor B_{ab} and U_{ab} and V_{ab} in four dimensions does not seem to have been noted before.

4.5 An n-dimensional tensor expressed in U_{ab}, V_{ab} and W_{ab}

If we consider the tensor

$$B_{bc} = \frac{1}{2} U_{bc} + \frac{1}{6} V_{bc} \tag{4.57}$$

in $n > 4$ dimensions, it is clearly divergence-free due to the properties of U_{ab} and V_{ab}, but when we examine its conformal properties we find, after a lot of work and rearranging,

$$\begin{aligned} \frac{1}{2}\widehat{U}_{bc} + \frac{1}{6}\widehat{V}_{bc} = \Omega^{-2}\Big(\frac{1}{2}U_{bc} + \frac{1}{6}V_{bc}\Big) \\ + (n-4)\Omega^{-2}\Big[-\frac{1}{2}(n-2)(n-3) C_{abcd} \Upsilon^a \Upsilon^d \\ + 2(n-3)\Upsilon^a \nabla_a R_{bc} - \frac{(n-3)}{2}\Upsilon^a \nabla_c R_{ab} - \frac{(n-3)}{2}\Upsilon^a \nabla_b R_{ac} \\ - \frac{(n-3)}{2} R_{ab} \Theta^a{}_c - \frac{(n-3)}{2} R_{ac} \Theta^a{}_b + \frac{(n-3)}{2} g_{bc} R_{ad} \Theta^{ad} \end{aligned}$$

$$
\begin{aligned}
&+\frac{(3n-7)}{3}R_{bc}\Theta-\frac{1}{2}\Upsilon_b\nabla_c R-\frac{1}{2}\Upsilon_c\nabla_b R-\frac{(3n-17)}{12}g_{bc}\Upsilon^a\nabla_a R\\
&-\frac{(3n-10)}{6}g_{bc}R\Theta+\frac{(n-7)3(n-5)}{2(n-1)}R\Upsilon_b\Upsilon_c\\
&+\frac{(n-7)(3n-5)}{12(n-1)}g_{bc}R\Upsilon_a\Upsilon^a+\frac{(n-2)(n-3)}{4}g_{bc}\Theta_{ad}\Theta^{ad}\\
&-\frac{(n-2)(n-3)}{2}(n-2)\Theta_{ab}\Theta^a{}_c+\frac{(3n^2-16+19)}{3}\Theta\Theta_{bc}\\
&+\frac{(3n-5)}{2}\nabla_b\nabla_c\Theta-\frac{(3n-5)}{2}g_{bc}\nabla^a\nabla_a\Theta-3\frac{(3n-5)}{2}\Upsilon_b\nabla_c\Theta\\
&-3\frac{(3n-5)}{2}\Upsilon_c\nabla_b\Theta-(n-7)\frac{(3n-5)}{6}g_{bc}\Upsilon^a\nabla_a\Theta\\
&+2\frac{(3n-5)}{2}\Theta\Upsilon_b\Upsilon_c+(n-7)\frac{(3n-5)}{6}g_{bc}\Theta\Upsilon_a\Upsilon^a\\
&-\frac{(6n^2-41n+53)}{12}g_{bc}\Theta^2\Big] \quad , \qquad (4.58)
\end{aligned}
$$

again getting a tensor that is not conformally well-behaved except in four dimensions.

The n-dimensional analogous integrability condition that gave rise to the four-dimensional Bach tensor is (3.14),

$$
\nabla^b\nabla^d C_{abcd}-\frac{(n-3)}{(n-2)}R^{bd}C_{abcd}-(n-3)(n-4)\Upsilon^b\Upsilon^d C_{abcd}=0 \qquad (4.59)
$$

and taking only the terms built up from pure geometry and quadratic in the Riemann curvature tensor, e.g., the first and second terms, suggests that we study

$$
\underset{3}{B}_{ac}=\nabla^b\nabla^d C_{abcd}-\frac{(n-3)}{(n-2)}R^{bd}C_{abcd} \quad . \qquad (4.60)
$$

Although we have already shown that this tensor is neither conformally well-behaved nor divergence-free in $n>4$ dimensions, it will be instructive to investigate its relationship to the tensors U_{ab}, V_{ab} and W_{ab}.

Using (2.10) we can equivalently express $\underset{3}{B}_{ac}$ in the decomposed form

$$
\begin{aligned}
\underset{3}{B}_{ac}=&\frac{(n-3)}{(n-2)}R_{ac;}{}^b{}_b-\frac{(n-3)}{2(n-1)}R_{;ac}-\frac{(n-3)}{2(n-1)(n-2)}g_{ac}R_{;}{}^b{}_b\\
&+\frac{(n-3)n}{(n-2)^2}R_{ab}R^b{}_c-\frac{2(n-3)}{(n-2)}R^{bd}C_{abcd}\\
&-\frac{(n-3)}{(n-2)^2}g_{ac}R_{bd}R^{bd}-\frac{n(n-3)}{(n-1)(n-2)^2}RR_{ac}\\
&+\frac{(n-3)}{(n-1)(n-2)^2}g_{ac}R^2 \quad . \qquad (4.61)
\end{aligned}
$$

This clearly reduces to the ordinary Bach tensor in four dimensions, but to investigate its other properties we first try to express $\underset{3}{B}_{ac}$ in terms of U_{ac}, V_{ac} and W_{ac}. Doing this we find

$$\begin{aligned}\underset{3}{B}_{ac} =& \frac{1}{(n-2)}U_{ac} + \frac{(n-3)}{2(n-1)}V_{ac} - \frac{1}{2}W_{ac} \\ &+ \frac{1}{2}\big(C_a{}^{bde}C_{cbde} - \frac{1}{4}g_{ac}C^{fbde}C_{fbde}\big) \\ &- \frac{(n-4)}{(n-2)}R^{bd}C_{abcd} \quad ,\end{aligned} \tag{4.62}$$

and as we have already noted (and is easily confirmed directly) this tensor is only divergence-free in four dimensions, and not for $n > 4$ dimensions.

However, defining the tensor $\underset{4}{B}_{ac}$,

$$\begin{aligned}\underset{4}{B}_{ac} &= \underset{3}{B}_{ac} - \frac{1}{2}\big(C_a{}^{bde}C_{cbde} - \frac{1}{4}g_{ac}C^{fbde}C_{fbde}\big) + \frac{(n-4)}{(n-2)}R^{bd}C_{abcd} \\ &= \underset{1}{B}_{ac} - \frac{1}{2}\big(C_a{}^{bde}C_{cbde} - \frac{1}{4}g_{ac}C^{fbde}C_{fbde}\big) + \frac{(n-4)}{2(n-2)}R^{bd}C_{abcd} \\ &= \frac{1}{(n-2)}U_{ac} + \frac{(n-3)}{2(n-1)}V_{ac} - \frac{1}{2}W_{ac} \quad ,\end{aligned} \tag{4.63}$$

we indeed get a tensor quadratic in the Riemann curvature tensor which is symmetric and divergence-free in all dimensions, and it collapses to the original Bach tensor in four dimensions.

To investigate the conformal properties of $\underset{4}{B}_{ac}$ (and thereby also the four-dimensional B_{ac}) we use (4.49) - (4.51) and find

$$\begin{aligned}\underset{4}{\widehat{B}}_{ac} =& \Omega^{-2}\Bigg(\underset{4}{B}_{ac} + (n-4)\Big[\Upsilon^b\nabla^d C_{abcd} + \Upsilon^d\nabla^b C_{abcd} \\ &+ C_{abcd}\nabla^b\Upsilon^d + (n-4)C_{abcd}\Upsilon^b\Upsilon^d\Big]\Bigg) \quad .\end{aligned} \tag{4.64}$$

From (4.64) we see that, in general, it is *only* in four dimensions that $\underset{4}{B}_{ac}$ is conformally well-behaved. Hence, in general, there is no obvious n-dimensional symmetric and divergence-free tensor which is quadratic in the Riemann curvature tensor and generalizes the Bach tensor in four dimensions, which also is of good conformal weight.

We can now ask more generally if it is possible to construct *any* n-dimensional 2-index tensor of good conformal weight from U_{ab}, V_{ab} and W_{ab}, i.e. a tensor which is symmetric, divergence-free, quadratic in the Riemann curvature tensor and of good conformal weight. To investigate this we look at

$\alpha U_{ab} + \beta V_{ab} + \gamma W_{ab}$, where α, β and γ are arbitrary constants,

$$
\begin{aligned}
\alpha\widehat{U}_{bc} + \beta\widehat{V}_{bc} + \gamma\widehat{W}_{bc} = {} & \Omega^{-2}\Big(\alpha U_{bc} + \beta V_{bc} + \gamma W_{bc}\Big) \\
& + \Omega^{-2}(n-4)\Big[2\gamma C_{abcd}\Theta^{ad} - \alpha(n-2)(n-3)C_{abcd}\Upsilon^a\Upsilon^d \\
& \quad + \alpha(n-3)\Big(2\Upsilon^a\nabla_a R_{bc} - \Upsilon^a\nabla_c R_{ab} - \Upsilon^a\nabla_b R_{ac}\Big) \\
& \quad + (n-3)\Big(\alpha - \beta\frac{1}{(n-3)} + \gamma\frac{n}{(n-1)(n-2)}\Big)R\Theta_{bc} \\
& \quad + 2(n-3)\Big(\alpha(n-2) - \beta\frac{(n-1)}{(n-3)} + \gamma\Big)\Theta\Theta_{bc} \\
& \quad - (n-3)\Big(\alpha(n-2) + 2\gamma\Big)\Theta_{ab}\Theta^a{}_c \\
& \quad + (n-3)\Big(\alpha\frac{(n-2)}{2} + \gamma\Big)g_{bc}\Theta_{ad}\Theta^{ad} + \gamma\frac{1}{(n-2)}R_{bc}\Theta \\
& \quad - (n-3)\Big(\alpha + \gamma\frac{2}{(n-2)}\Big)\Big(R_{ab}\Theta^a{}_c + R_{ac}\Theta^a{}_b - g_{bc}R_{ad}\Theta^{ad}\Big) \\
& \quad - \Big(\alpha(n-3) - \beta + \gamma\frac{n(n-3)}{(n-1)(n-2)}\Big)g_{bc}R\Theta + \gamma(n-3)g_{bc}\Theta^2\Big] \\
& + \Omega^{-2}\Big(\alpha(n-2)(n-3) - 2\beta(n-1)\Big)\Big[\frac{3}{(n-1)}R\Upsilon_b\Upsilon_c \\
& \quad + \frac{(n-7)}{(n-1)}g_{bc}R\Upsilon_a\Upsilon^a + \nabla_b\nabla_c\Theta + 6\Theta\Upsilon_b\Upsilon_c - (n-7)g_{bc}\Upsilon^a\nabla_a\Theta \\
& \quad - 3\Big(\Upsilon_b\nabla_c\Theta + \Upsilon_c\nabla_b\Theta\Big) - g_{bc}\nabla^a\nabla_a\Theta + (n-7)g_{bc}\Theta\Upsilon_a\Upsilon^a\Big] \\
& + \Omega^{-2}\Big[2(n-3)\Big(\alpha(n-3) - \beta\frac{(n-1)}{(n-3)}\Big)R_{bc}\Theta \\
& \quad - \Big(\alpha\frac{(n-2)(n-3)(2n-11)}{2} - \beta(n-1)(n-7)\Big)g_{bc}\Theta^2 \\
& \quad - \Big(\alpha(n-3) - 3\beta\Big)\Upsilon_b\nabla_c R - \Big(\alpha(n-3) - 3\beta\Big)\Upsilon_c\nabla_b R \\
& \quad - \Big(\alpha\frac{(n-3)(n-6)}{2} - \beta(n-7)\Big)g_{bc}\Upsilon^a\nabla_a R\Big] \quad . \qquad (4.65)
\end{aligned}
$$

Note that for $n = 4$, remembering that $W_{bc} = 0$, and so we can put $\gamma = 0$, (4.65) reduces to

$$
\begin{aligned}
\alpha\widehat{U}_{bc}+\beta\widehat{V}_{bc} = {} & \Omega^{-2}\Big(\alpha U_{bc} + \beta V_{bc}\Big) \\
& + \Omega^{-2}\Big(\alpha - 3\beta\Big)\Big\{2R\Upsilon_b\Upsilon_c - 2g_{bc}R\Upsilon_a\Upsilon^a + 2\nabla_b\nabla_c\Theta \\
& - 2g_{bc}\nabla^a\nabla_a\Theta - 6\Big(\Upsilon_b\nabla_c\Theta + \Upsilon_c\nabla_b\Theta\Big) + 6g_{bc}\Upsilon^a\nabla_a\Theta \\
& + 12\Theta\Upsilon_b\Upsilon_c - 6g_{bc}\Theta\Upsilon_a\Upsilon^a + 2R_{bc}\Theta + 3g_{bc}\Theta^2 - \Upsilon_b\nabla_c R \\
& - \Upsilon_c\nabla_b R + \frac{3}{2}g_{bc}\Upsilon^a\nabla_a R\Big\} \quad , \qquad (4.66)
\end{aligned}
$$

and if this expression is to be conformally well-behaved we clearly must have $\alpha - 3\beta = 0$. Hence we see that the only linear combination of U_{ab}, V_{ab} and W_{ab} which is conformally well-behaved in four dimension is, up to a constant factor, the Bach tensor (4.54).

For $n > 4$ we note that there will be terms with the factor $(n-4)$ involving Weyl tensor component in (4.65); but no other terms in (4.65) have similar Weyl tensor components. Hence, in general it will be impossible for these terms in Weyl to cancel out and so it will be impossible for this expression with any non-trivial values of α, β and γ to be conformally well-behaved[5] . Hence, if the expression is to be conformally well-behaved, the only solution of (4.65) is the trivial solution, i.e. $\alpha = \beta = \gamma = 0$ and we have

Theorem 4.5.1. *For $n = 4$ there is only one (up to constant rescaling) 2-index tensor which is symmetric, divergence-free, conformally well-behaved and quadratic in the Riemann curvature tensor, i.e.,*

$$B_{ab} = \frac{1}{2}U_{ab} + \frac{1}{6}V_{ab} \quad . \qquad (4.67)$$

For $n > 4$, in general, there is no symmetric and divergence-free 2-index tensor quadratic in the Riemann curvature tensor which is of good conformal weight.

[5]With a strategic choice of a few metrics a simple calculation using GRTII [18] easily confirms this.

Chapter 5

The Kozameh-Newman-Tod four-dimensional result and the Bach tensor

Szekeres [39] used spinor methods to attack the problem of finding necessary and sufficient conditions for a space being conformally Einstein. Indeed he found a set of conditions written in spinor language, but the equations are difficult to analyze and rather complicated when translated into tensors[1].

However, in 1985 Kozameh, Newman and Tod [27] found a much simpler and more useful set of conditions for spaces for which the complex scalar invariant

$$\begin{aligned} J &= \Psi_{ABCD}\Psi^{CDEF}\Psi_{EF}{}^{AB} \\ &= \frac{1}{8}\Big(C_{ab}{}^{cd}C_{cd}{}^{ef}C_{ef}{}^{ab} - i^{*}C_{ab}{}^{cd*}C_{cd}{}^{ef*}C_{ef}{}^{ab}\Big) \end{aligned} \tag{5.1}$$

of the Weyl spinor/tensor is nonzero. Their set involves the Bach tensor introduced in the previous chapters.

In this chapter we will use a tensor/spinor approach to review both the explicit and the implicit four-dimensional results of KNT[2] [27]. Since spinors are used in this chapter we are restricted to four-dimensional spaces with signature $(+---)$. To some extent the presentation here follows the original paper by KNT [27], although some of our proofs are more direct.

[1] In fact there is a small mistake in [39] pointed out by Wünsch [43] making some of the results in [39] incorrect and the conditions only necessary.

[2] Kozameh, Newman and Tod.

5.1 Two useful lemmas

In the proof of the KNT result the following two lemmas play a key role:

Lemma 5.1.1. *Given a skew-symmetric real tensor $F^{ab} = F^{[ab]}$, then, provided $J \neq 0$, the only solution of the equation*

$$C_{abcd}F^{cd} = 0 \tag{5.2}$$

is $F^{cd} = 0$.

Lemma 5.1.2. *Given a symmetric and trace-free real tensor $H^{ab} = H^{(ab)}$, then, provided $J \neq 0$, the only solution of*

$$C_{abcd}H^{ad} = 0 \tag{5.3}$$

and

$$^{*}C_{abcd}H^{ad} = 0 \tag{5.4}$$

is $H^{ad} = 0$.

The proof of these two lemmas can be given simultaneously using spinors:

Proof. A skew-symmetric real tensor $F^{ab} = F^{[ab]}$ can be written in spinor language as

$$F_{ab} = F_{AA'BB'} = \phi_{AB}\varepsilon_{A'B'} + \bar{\phi}_{A'B'}\varepsilon_{AB} \tag{5.5}$$

where $\phi_{AB} = \phi_{(AB)}$ is a symmetric spinor. A symmetric real trace-free tensor $H^{ab} = H^{(ab)}$ can be written as

$$H_{ab} = H_{AA'BB'} = \phi_{ABA'B'} \tag{5.6}$$

where $\phi_{ABA'B'} = \bar{\phi}_{(AB)(A'B')}$.

In this notation equation (5.2) and (5.3), (5.4) become

$$\Psi_{ABCD}\phi^{CD} = 0 \tag{5.7}$$

and

$$\Psi_{ABCD}\phi^{CD}{}_{A'B'} = 0 \tag{5.8}$$

respectively, Ψ_{ABCD} being the Weyl spinor. Hence we see that the primed indices play no role in (5.8) and we only need to consider the equation

$$\Psi^{AB}{}_{CD}\phi^{CD} = 0 \tag{5.9}$$

and show that under the assumption that $J \neq 0$ this implies $\phi_{CD} = 0$. Note that we can consider

$$\phi^{AB} \longrightarrow \Psi^{AB}{}_{CD}\phi^{CD} \tag{5.10}$$

as a linear mapping from symmetric 2-index spinors to symmetric 2-index spinors and hence we can study the properties of the mapping via its matrix representation $\boldsymbol{\Psi}$ (see Appendix A.3). So, we can represent (5.9) by the matrix equation

$$\boldsymbol{\Psi x} = \mathbf{0} \quad , \tag{5.11}$$

where $\boldsymbol{x}$ is the vector representation of ϕ^{AB}. If $\det \boldsymbol{\Psi} \neq \mathbf{0}$, then the only solution of equation (5.11) is the trivial one, $\boldsymbol{x} = \mathbf{0}$.

From (A.25) we have that

$$\det(\boldsymbol{\Psi}) = \frac{1}{3} J \tag{5.12}$$

and hence under the assumption $J \neq 0$ the only solution to (5.9) is the trivial solution, i.e. $\phi_{CD} = 0$. This completes the proof of both lemmas. □

Using tensors the first of these lemmas, Lemma 5.1.1, can easily be proved in an analogous manner as when using spinors. This is because the Weyl tensor C_{abcd} in this case can be looked upon as a linear mapping taking 2-forms to 2-forms. Hence, as shown in appendix A, the properties of the mapping can be studied using its matrix representation, $\boldsymbol{C}$, and from Appendix C we know that

$$\det(\boldsymbol{C}) = \frac{1}{9} J \bar{J} \tag{5.13}$$

and hence if $J \neq 0$, then $\det(\boldsymbol{C}) \neq 0$ and the only solution of (5.2) is the trivial one.

The second lemma, Lemma 5.1.2, is harder to prove just using tensor methods, and is a good example of the power of using spinors when working in four dimensions; the result is almost trivial to prove in spinors, but is much harder using tensors.

5.2 C-spaces and conformal C-spaces

Szekeres [39] introduced a class of spaces called *C-spaces* and this class is defined by

Definition 5.2.1. A space is a *C-space* if the space has a divergence-free Weyl tensor[3], i.e.,

$$\nabla^d C_{abcd} = 0 \quad . \tag{5.14}$$

Since we are interested in conformal spaces we also have the definition

[3]Sometimes a Weyl tensor satisfying (5.14) is called a *harmonic Weyl tensor*; see for instance [28].

Definition 5.2.2. A space is a *conformal C-space* if there exists a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ such that $\widehat{\nabla}^d \widehat{C}_{abcd} = 0$.

The condition $\widehat{\nabla}^d \widehat{C}_{abcd} = 0$ written out in the space with metric g_{ab} is

$$\nabla^d C_{abcd} + \Upsilon^d C_{abcd} = 0 \tag{5.15}$$

where $\Upsilon^d = \nabla^d(\ln \Omega)$.

On the other hand if

$$\nabla^d C_{abcd} + K^d C_{abcd} = 0 \tag{5.16}$$

holds for *any* vector field K^a we see that by taking ∇^c of equation (5.16) and then using (5.16) we have

$$\nabla^c \nabla^d C_{abcd} + \nabla^c K^d C_{abcd} - K^c K^d C_{abcd} = 0 \quad . \tag{5.17}$$

Using the facts that the last term in this expression (5.17) vanishes and that (independently of dimension) $\nabla^c \nabla^d C_{abcd} = 0$ we find

$$\nabla^c K^d C_{abcd} = \nabla^{[c} K^{d]} C_{abcd} = 0 \quad . \tag{5.18}$$

Hence from (5.18) and Lemma 5.1.1 we conclude that in a space in which $J \neq 0$ it follows that $\nabla^{[c} K^{d]} = 0$, locally giving us a gradient vector from which the conformal factor can be found. So, we have the first part of the KNT result [27],

Theorem 5.2.1. *A space in which $J \neq 0$ is locally conformal to a C-space if and only if there exists a vector field K^a such that*

$$\nabla^d C_{abcd} + K^d C_{abcd} = 0 \quad . \tag{5.19}$$

Furthermore, K^a is unique.

To see that the vector K^d in Theorem 5.2.1 is unique suppose there exists another vector ξ^a satisfying (5.19), i.e.,

$$\nabla^d C_{abcd} + \xi^d C_{abcd} = 0 \quad . \tag{5.20}$$

Subtracting (5.20) from (5.19) and setting $\eta^a = K^a - \xi^a$ we then have

$$\eta^d C_{abcd} = 0 \quad . \tag{5.21}$$

Multiplying this with an arbitrary vector field ζ^c,

$$\zeta^c \eta^d C_{abcd} = \zeta^{[c} \eta^{d]} C_{abcd} = 0 \quad , \tag{5.22}$$

and since $J \neq 0$ this equation implies that $\zeta^{[c}\eta^{d]} = 0$ for all vector fields ζ^a. But this can only be true if $\eta^a = 0$, i.e., if $K^a = \xi^a$, and hence Theorem 5.2.1 is proved[4].

We can have alternative versions of Theorem 5.2.1. Consider the four-dimensional identities[5],

$$C^{ibcd}C_{jbcd} = \frac{1}{4}\delta^i_j C^{abcd}C_{abcd} \quad , \tag{5.23}$$

$$C^{ib}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{jb} = \frac{1}{4}\delta^i_j C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{ab} \quad , \tag{5.24}$$

$${}^*C^{ibcd}C_{jbcd} = \frac{1}{4}\delta^i_j {}^*C^{abcd}C_{abcd} \quad , \tag{5.25}$$

$${}^*C^{ib}{}_{cd}{}^*C^{cd}{}_{ef}{}^*C^{ef}{}_{jb} = \frac{1}{4}\delta^i_j {}^*C^{ab}{}_{cd}{}^*C^{cd}{}_{ef}{}^*C^{ef}{}_{ab} \quad . \tag{5.26}$$

If $C_{abcd}C^{abcd} \neq 0$, we can multiply (5.19) by C^{abce} to obtain

$$C^{abce}\nabla^d C_{abcd} + \frac{1}{4}K^e C_{abcd}C^{abcd} = 0 \quad . \tag{5.27}$$

By dividing (5.27) with $C^2 = C_{abcd}C^{abcd}$ we are actually able to express the vector in pure geometric terms:

$$K^e = -4C^{abce}\nabla^d C_{abcd}/C^2 \quad . \tag{5.28}$$

Analogously, we get from (5.24) - (5.26), provided the corresponding scalar invariant of the Weyl tensor is nonzero,

$$K^e = -\,4{}^*C^{abce}\nabla^d C_{abcd}/C^*C \quad , \tag{5.29}$$

$$K^e = -\,4C^{abkl}C_{kl}{}^{ce}\nabla^d C_{abcd}/C^3 \quad , \tag{5.30}$$

$$K^e = -\,4{}^*C^{abkl}{}^*C_{kl}{}^{ce}\nabla^d C_{abcd}/{}^*C^3 \quad . \tag{5.31}$$

We can now restate Theorem 5.2.1 as

Theorem 5.2.2. *A space is conformal to a C-space if*

$$\nabla^d C_{abcd} + K^d C_{abcd} = 0 \tag{5.32}$$

holds with K^a defined by (at least) one of the equations (5.28) - (5.31).

We comment on the case when $J = 0$ in the last section of this chapter.

[4]There is another way to prove the uniqueness of K^a using $J \neq 0$ to solve for K^a explicitly, and this is illustrated in Chapter 8.

[5]See appendix B for their derivation.

5.3 Conformal Einstein spaces

We know from Chapter 3 that a four-dimensional space is conformally Einstein if and only if there exists a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ such that

$$R_{ab} - \frac{1}{4}g_{ab}R - 2\nabla_a\Upsilon_b + \frac{1}{2}g_{ab}\nabla_c\Upsilon^c - 2\Upsilon_a\Upsilon_b + \frac{1}{2}g_{ab}\Upsilon_c\Upsilon^c = 0 \qquad (5.33)$$

where $\Upsilon_a = \nabla_a(\ln\Omega)$.

The first integrability condition of (5.33) is, as calculated in (3.12),

$$\nabla^d C_{abcd} + \Upsilon^d C_{abcd} = 0 \quad , \qquad (5.34)$$

meaning that conformal Einstein spaces constitute a subclass of conformal C-spaces, and that being a conformal C-space is a necessary condition for a space to be conformally Einstein;

Theorem 5.3.1. *A conformal Einstein space is also conformal to a C-space.*

On the other hand, in four dimensions we found another necessary condition for a space being conformally Einstein in (3.15), the vanishing of the Bach tensor,

$$B_{bc} = \nabla^a\nabla^d C_{abcd} - \frac{1}{2}R^{ad}C_{abcd} = 0 \quad , \qquad (5.35)$$

and in fact (5.34) together with (5.35) are also sufficient provided $J \neq 0$.

To prove this we will use Lemma 5.1.2. First observe that if $J \neq 0$ then by the same argument as was used to prove Theorem 5.2.1 it follows that equation (5.34) is satisfied with $\Upsilon_a = \nabla_a(\ln\Omega)$ for some scalar field Ω. Hence, differentiating (5.34),

$$\nabla^a\nabla^d C_{abcd} + \left(\nabla^a\Upsilon^d - \Upsilon^a\Upsilon^d\right)C_{abcd} = 0 \qquad (5.36)$$

and subtracting (5.35) we have

$$C_{abcd}\left(R^{ad} + 2\nabla^a\Upsilon^d - 2\Upsilon^a\Upsilon^d\right) = 0 \quad . \qquad (5.37)$$

This equation (5.37) will serve as the first equation in Lemma 5.1.2 with $H^{ad} = R^{ad} + 2\nabla^a\Upsilon^d - 2\Upsilon^a\Upsilon^d$.

To obtain the second equation in Lemma 5.1.2 we first take the dual of equation (5.34),

$$\nabla^{d*}C_{abcd} + \Upsilon^{d*}C_{abcd} = 0 \qquad (5.38)$$

and then differentiate this with ∇^a to find

$$\nabla^a\nabla^{d*}C_{abcd} + \left(\nabla^a\Upsilon^d - \Upsilon^a\Upsilon^d\right)^* C_{abcd} = 0 \quad . \qquad (5.39)$$

By taking the divergence of the dual of the contracted Bianchi identity (2.8),

$$0 = \nabla^b \nabla_d {}^*C_{abc}{}^d + \eta_{ab}{}^{pq} \Big(\nabla^b \nabla_{[p} R_{q]c} - \frac{1}{3} g_{c[q} \nabla^b \nabla_{p]} R \Big) \quad , \qquad (5.40)$$

and using the Ricci identity and (2.5) to decompose the Riemann curvature tensor into its irreducible parts we get

$$\nabla^a \nabla^{d*} C_{abcd} - \frac{1}{2} R^{ad*} C_{abcd} = 0 \quad . \qquad (5.41)$$

Hence subtracting (5.41) from (5.39) gives

$${}^*C_{abcd} \big(R^{ad} + 2\nabla^a \Upsilon^d - 2\Upsilon^a \Upsilon^d \big) = 0 \quad , \qquad (5.42)$$

i.e., the second equation in Lemma 5.1.2.

Equations (5.37) and (5.42) applied to Lemma 5.1.2 now give us that

$$R^{ad} + 2\nabla^a \Upsilon^d - 2\Upsilon^a \Upsilon^d = \frac{1}{4} g^{ad} T \quad , \qquad (5.43)$$

with T being the trace of the left hand side of (5.43). Hence (5.43) is equivalent to

$$R_{ab} - \frac{1}{4} g_{ab} R + 2\nabla_a \Upsilon_b + \frac{1}{2} g_{ab} \nabla_c \Upsilon^c - 2\Upsilon_a \Upsilon_b + \frac{1}{2} g_{ab} \Upsilon_c \Upsilon^c = 0 \quad . \qquad (5.44)$$

Since we know $\Upsilon_a = \nabla_a (\ln \Omega)$ it follows from Section 3.2 that the space is a conformal Einstein space, and so we have proven the main result in KNT [27],

Theorem 5.3.2. *A space in which $J \neq 0$ is a conformal Einstein space if and only if there exists a vector field K^a such that*

$$\nabla^d C_{abcd} + K^d C_{abcd} = 0 \quad , \qquad (5.45)$$

$$B_{bc} = \nabla^a \nabla^d C_{abcd} - \frac{1}{2} R^{ad} C_{abcd} = 0 \quad . \qquad (5.46)$$

Note that this set of conditions naturally divide into two sets; condition (5.45) selects the class of C-spaces, and condition (5.46) specifies a particular subclass of the C-spaces.

The natural question arises if both the conditions (5.45) and (5.46) in Theorem 5.3.2 really are needed, or if one of them is redundant. However, as pointed out in [27], the Newman-Kerr metric (subject to the parameter restrictions $a = 0$ and $e \neq 0$) provides an example of a space where $J \neq 0$ and although it has non-vanishing Bach tensor, $B_{ab} \neq 0$, it is a conformal C-space. Hence it is not enough that a space is conformal to a C-space to be conformally Einstein. Also Kozameh *et al.* [27] provide an argument, suggested by R. Geroch and G. T. Horowitz, based on counting initial data, showing that the conditions (5.45) and (5.46) are independent, and hence both needed. But this falls short of providing a specific counterexample to the possibility of one of (5.45) or (5.46) being redundant.

On the other hand, Nurowski and Plebanánski [31] recently found a type N metric of the Feferman class which has vanishing Bach tensor, $B_{ab} = 0$, but which is not conformal to an Einstein space[6]. Hence, in the case of a general space, it is not enough that the Bach tensor vanishes for a space to be conformal to an Einstein space. However, we emphasize that this metric has[7] $J = 0$ and so is not strictly relevant to our theorem. Therefore the possibility of a zero Bach tensor together with the condition $J \neq 0$ being sufficient for a conformally Einstein space is still open.

The condition (5.46), the vanishing of the Bach tenor, has been discussed in different contexts by a number of authors [25], [27], [30]. Also note that in an alternative theorem to Theorem 5.3.2 for the existence of conformal Einstein spaces by Baston and Mason [3], [4] for spaces in which $I \neq 0$ the Bach tensor is chosen to be zero, but the condition (5.45) is replaced with another condition formulated in spinors as a restriction on the Weyl spinor.

5.4 $J = 0$

When we put the condition $J \neq 0$ on a space-time we exclude space-times of Petrov type III, N, and some particular cases of type I. In these other cases the conditions in Theorem 5.3.2 do not provide necessary and sufficient conditions on a space being conformally Einstein, i.e., it is not enough to demand that the space is a conformal C-space and the vanishing of the Bach tensor.

Later Wünsch [44], also using a spinor approach, derived necessary and sufficient conditions for space-times of type III to be conformally Einstein. The conditions Wünsch found are the two in Theorem 5.3.2 plus an additional one involving the vanishing of a scalar defined by four contractions between a trace-free and symmetric tensor constructed from the geometry and a preferred vector constructed from a special choice of spin basis for space-times of Petrov type III. That these conditions are not sufficient for a space-time of Petrov type N to be conformally Einstein is provided by the example of generalized plane wave space-times [43].

The Petrov type N case is still unsolved, but in the pursuit of the solution Czapor, McLenaghan and Wünsch [12] derived necessary and sufficient conditions for a space-time of Petrov type N to be conformally related

[6] The spaces in the Feferman class have the property that they are not conformal to a Einstein space.

[7] This is a property of all type N metrics.

to an empty space ($R_{ab} = 0$), as well as some sufficient conditions for a space-time of Petrov type N to be conformal to a C-space.

Chapter 6

Listing's result in four dimensions

In this chapter we will describe the recent generalization of the KNT result in Chapter 5 due to Listing [28]. In his paper he looks at the problem with more emphasis on the differential geometry point of view, which makes some of his quantities and equations awkward to translate into tensors or spinors. We will restrict ourselves to four-dimensional spaces throughout this chapter.

6.1 Non-degenerate Weyl tensor

When we proved the results of KNT the two Lemmas 5.1.1 and 5.1.2 played a key role and to prove these we had to impose a restriction on the spaces under consideration. The restriction was found from a spinor point of view and was $J \neq 0$.

However, it is well known that locally we can look upon the Weyl tensor as a linear mapping between two-forms, i.e., given $F^{ab} = F^{[ab]}$ then

$$F^{ab} \longrightarrow C^{ab}{}_{cd}F^{cd} \quad . \tag{6.1}$$

Thereby it is possible to translate the Weyl tensor into a matrix, $\mathbf{C}$ (see appendix A), and to study properties of the mapping (6.1) using this representation of the Weyl tensor.

It can be shown, see Appendix C, that in four dimensions the condition $J \neq 0$ is equivalent to the condition $\det(\mathbf{C}) \neq 0$, meaning that imposing $\det(\mathbf{C}) \neq 0$ the mapping (6.1) is injective. Hence, given any two-form $F^{ab} = F^{[ab]}$ the only solution of the equation $C^{ab}{}_{cd}F^{cd} = 0$ in a space where $\det(\mathbf{C}) \neq 0$ is $F^{cd} = 0$, and in such a space, Lemma 5.1.1 of KNT holds[1].

Definition 6.1.1. If $\det(\mathbf{C}) \neq 0$ the Weyl tensor is said to be *non-degenerate.*

The fact that the Weyl tensor is non-degenerate means that there is an inverse, i.e., that there exists a tensor field $D^{ab}{}_{cd}$ such that

$$D^{ab}{}_{cd}C^{cd}{}_{ef} = \delta^a_{[e}\delta^b_{f]} \quad , \tag{6.2}$$

where $\delta^a_{[e}\delta^b_{f]}$ is the "identity" in Λ^2. When the Weyl tensor $C^{ab}{}_{cd}$ is considered as a matrix $\mathbf{C}$, then the inverse, $D^{ab}{}_{cd}$, is the matrix $\mathbf{C}^{-1}$. In practice the tensor $D^{ab}{}_{cd}$ is not explicitly known and to interpret $\mathbf{C}^{-1}$ in tensor notation cannot easily be done[2].

6.2 Conformal C-spaces

In the spaces where $\det(\mathbf{C}) \neq 0$ we can use $D^{ab}{}_{cd}$ to get an alternative characterization of conformal C-spaces to the one presented in the previous chapter.

From Theorem 5.2.1 we know that a space for which $\det(\mathbf{C}) \neq 0$ is conformally a C-space if and only if there exist a vector field K^a such that

$$\nabla^d C^{ab}{}_{cd} + K^d C^{ab}{}_{cd} = 0 \quad . \tag{6.3}$$

Multiplying (6.3) with $D^{ef}{}_{ab}$ and using (6.2) we have

$$\begin{aligned} 0 &= D^{ef}{}_{ab}\nabla^d C^{ab}{}_{cd} + K^d D^{ef}{}_{ab}C^{ab}{}_{cd} \\ &= D^{ef}{}_{ab}\nabla^d C^{ab}{}_{cd} + K^d \delta^e_{[c}\delta^f_{d]} \quad , \end{aligned} \tag{6.4}$$

and by taking the trace over c and f and solving for the vector field we find

$$K^e = \frac{2}{3} D^{ec}{}_{ab}\nabla^d C^{ab}{}_{cd} \quad . \tag{6.5}$$

Hence, using $D^{ab}{}_{cd}$ we are able to get an expression for the vector field K^a and we can now formulate the corresponding theorem to Theorem 5.2.2,

Theorem 6.2.1. *A space having a non-degenerate Weyl tensor is locally conformal to a C-space if and only if* (6.3) *holds with* K^a *defined by* (6.5).

Note that the central difference between Theorem 5.2.2 and Theorem 6.2.1 is the expressions defining the vector field K^a.

[1] Also, since $\det(\mathbf{C}) \neq 0 \Leftrightarrow J \neq 0$, in fact Lemma 5.1.2 of KNT holds, but how to translate the Weyl tensor to a matrix and how to interpret it in this case is not clear.

[2] We will see in Chapter 8 that we can avoid the use of the tensor $D^{ab}{}_{cd}$.

6.3 Conformal Einstein spaces

To prove Theorem 5.3.2 of KNT we needed both the Lemmas 5.1.1 and 5.1.2, and indeed, as noted in the previous section, if the Weyl tensor is non-degenerate both these lemmas hold.

However, in a space with non-degenerate Weyl tensor we can always define the vector field K^e as done in (6.5), i.e. such that

$$K^e = \frac{2}{3} D^{ec}{}_{ab} \nabla^d C^{ab}{}_{cd} \tag{6.6}$$

and this give us the following characterization of the conformal Einstein spaces:

Theorem 6.3.1. *A space with non-degenerate Weyl tensor is locally conformal to an Einstein space if and only if the vector field K^a defined in* (6.6) *satisfies*

$$R_{ab} - \frac{1}{4} g_{ab} R + 2\nabla_a K_b + \frac{1}{2} g_{ab} \nabla_c K^c - 2K_a K_b + \frac{1}{2} g_{ab} K_c K^c = 0 \quad . \tag{6.7}$$

Proof. If (6.7) holds we see by skewing (6.7) that $\nabla_{[a} K_{b]} = 0$, i.e. that K^a locally is a gradient from which the conformal factor can be found and hence the space is conformally Einstein.

Conversely, if the space is conformally Einstein equation (6.7) is satisfied for some (gradient) vector field K^a. Also, from Theorem 5.3.1 we have that the space is conformally a C-space, and Theorem 6.2.1 then tells us that this vector is unique and hence is the one given by (6.6). □

Note the difference between Theorem 5.3.2 and Theorem 6.3.1 though both have the same generic setting ($\det(\mathbf{C}) \neq 0$ or equivalently $J \neq 0$). The two ingredients in Theorem 5.3.2 are the conditions that the space is a conformal C-space and the vanishing of the Bach tensor: the first condition is the first integrability condition for a space being conformally Einstein, and the second one arise from taking a derivative of the first one.

In Theorem 6.3.1 on the other hand the condition of being a conformal Einstein space and an explicit expression for the vector field K^a are the conditions. However, in the construction of K^a we use the condition that the space is a conformal C-space, and hence K^a “contains” the first integrability condition for the space being conformally Einstein.

Chapter 7

Listing's result in n dimensions

In this chapter we briefly comment on the extension of the definition of the non-degenerate Weyl tensor to arbitrary n dimensions. We will also give the n-dimensional counterparts of Theorem 6.2.1 and Theorem 6.3.1 from the previous chapter as well as Theorem 5.2.1 and Theorem 5.3.1 in Chapter 5. The theorems are essentially the same as those given by Listing [28] although the formulation and notation differs somewhat.

7.1 Non-degenerate Weyl tensor

The concept of a non-degenerate Weyl tensor introduced in Chapter 6.1 transfers directly to arbitrary n dimensions and we define

Definition 7.1.1. An n-dimensional space is said to have a *non-degenerate Weyl tensor* if the determinant of the matrix $\mathbf{C}$ associated with the linear mapping

$$F^{ab} \longrightarrow C^{ab}{}_{cd}F^{cd} \quad , \quad F^{cd} = F^{[cd]} \tag{7.1}$$

is non-zero, i.e., if $\det \mathbf{C} \neq 0$.

It follows directly that from the definition 7.1.1 that in a space with non-degenerate Weyl tensor the equation

$$C^{ab}{}_{cd}F^{cd} = 0 \quad , \quad F^{cd} = F^{[cd]} \tag{7.2}$$

only has the trivial solution, $F^{cd} = 0$.

In a space with non-degenerate Weyl tensor there exist a tensor field $D^{ab}{}_{cd}$ such that

$$D^{ab}{}_{cd}C^{cd}{}_{ef} = \delta^a_{[e}\delta^b_{f]} \quad , \tag{7.3}$$

where $\delta^a_{[e}\delta^b_{f]}$ is the "identity" in the space of two-forms.

7.2 Conformal C-spaces

Before we formulate and prove the n-dimensional version of Theorem 6.2.1 we first note that the four-dimensional Theorem 5.2.1 has a natural extension to n-dimensional spaces with non-degenerate Weyl tensor:

Theorem 7.2.1. *An n-dimensional space with non-degenerate Weyl tensor is locally conformal to a C-space if and only if there exists a vector field K^a such that*

$$\nabla^d C^{ab}{}_{cd} + (n-3)K^d C^{ab}{}_{cd} = 0 \quad . \tag{7.4}$$

Furthermore, K^a is unique.

Proof. The proof is analogous to the four-dimensional theorem. If the space is a conformal C-space there exists a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ such that $\widehat{\nabla}^d \widehat{C}^{ab}{}_{cd} = 0$ holds, i.e., expressed in the space with metric g_{ab} such that (7.4) holds with $\Upsilon^a = \nabla^a(\ln\Omega)$.

If on the other hand (7.4) holds we see by differentiating (7.4), using $\nabla^c\nabla^d C^{ab}{}_{cd} = 0$ and (7.4) again, that

$$\nabla^c K^d C^{ab}{}_{cd} = \nabla^{[c} K^{d]} C^{ab}{}_{cd} = 0 \quad , \tag{7.5}$$

i.e., provided the that the Weyl tensor is non-degenerate that $\nabla^{[c}K^{d]} = 0$. Hence K^d is locally a gradient from which the conformal factor can be found.

Clearly a vector field satisfying (7.4) must be unique since in a given space with non-degenerate Weyl tensor both $C^{ab}{}_{cd}$ and $D^{ab}{}_{cd}$ are uniquely determined. Hence, using $D^{ef}{}_{ab}$ on (7.4), taking a trace over c and f and solving for K^e we find

$$K^e = \frac{2}{(n-1)(n-3)} D^{ec}{}_{ab} \nabla^d C^{ab}{}_{cd} \quad , \tag{7.6}$$

which clearly must be unique. □

As a matter of fact the previous proof also proves the n-dimensional version of Theorem 6.2.1:

Theorem 7.2.2. *An n-dimensional space having a non-degenerate Weyl tensor is locally conformal to a C-space if and only if*

$$\nabla^d C^{ab}{}_{cd} + (n-3)K^d C^{ab}{}_{cd} = 0 \tag{7.7}$$

holds with the vector field K^a defined by

$$K^e = \frac{2}{(n-1)(n-3)} D^{ec}{}_{ab} \nabla^d C^{ab}{}_{cd} \quad . \tag{7.8}$$

7.3 Conformal Einstein spaces

We close this chapter with the n-dimensional version of Theorem 6.2.1 and again the proof is very similar to the four-dimensional one.

First note that an n-dimensional space is conformally Einstein if and only if there is a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ such that

$$R_{ab} - \frac{1}{n} g_{ab} R + (n-2)\nabla_a \Upsilon_b - \frac{(n-2)}{n} g_{ab} \nabla_c \Upsilon^c \\ - (n-2)\Upsilon_a \Upsilon_b + \frac{(n-2)}{n} g_{ab} \Upsilon_c \Upsilon^c = 0 \quad , \tag{7.9}$$

where $\Upsilon^a = \nabla^a (\ln \Omega)$, and that the first integrability condition of (7.9), as found in (3.12), is

$$\nabla^d C^{ab}{}_{cd} + (n-3)\Upsilon^d C^{ab}{}_{cd} = 0 \quad ; \tag{7.10}$$

hence we have the n-dimensional version of Theorem 5.3.1

Theorem 7.3.1. *A conformal Einstein space is conformal to a C-space.*

Finally we have Listing's n-dimensional result,

Theorem 7.3.2. *An n-dimensional space with a non-degenerate Weyl tensor is locally conformal to an Einstein space if and only if the vector field K^a defined by*

$$K^e = \frac{2}{(n-1)(n-3)} D^{ec}{}_{ab} \nabla^d C^{ab}{}_{cd} \tag{7.11}$$

satisfies

$$R_{ab} - \frac{1}{n} g_{ab} R + (n-2)\nabla_a K_b - \frac{(n-2)}{n} g_{ab} \nabla_c K^c \\ - (n-2) K_a K_b + \frac{(n-2)}{n} g_{ab} K_c K^c = 0 \quad . \tag{7.12}$$

Proof. If (7.12) holds we see by skewing (7.12) that $\nabla_{[a} K_{b]} = 0$, i.e., that K^a is locally a gradient from which the conformal factor can be found and hence the space is conformally Einstein.

Conversely, if the space is conformally Einstein, equation (7.12) is satisfied for some (gradient) vector field Υ^a. Also, from Theorem 7.3.1 we have that the space is conformally a C-space, and Theorem 7.2.2 then tells us that this vector field is unique and hence is the one given by (7.8). □

Chapter 8

Edgar's result in n dimensions

In this chapter we shall show how Edgar [13] reformulated the results of KNT and Listing and showed how to get an explicit expression for the vector field K^a in Theorems 7.2.2, 7.3.2 without using the inverse map of Listing.

The procedure (essentially using the Cayley-Hamilton Theorem) gives us a generic expression in n dimensions, but due to dimensionally dependent identities we are able to get expressions of considerable lower order. Examples in 4, 5 and 6 dimensions are discussed in detail.

8.1 Using the Cayley-Hamilton Theorem

We show how we can get around the implicit construction of the vector field K^a in the work of Listing. The non-degeneracy condition we impose on the Weyl tensor, formulated in the matrix representation of the Weyl tensor, is $\det(\mathbf{C}) \neq 0$, e.i., the same as used by KNT and Listing.

First we prove the following useful lemma:

Lemma 8.1.1. *In an n-dimensional space in which* $\det(\mathbf{C}) \neq 0$ *the inhomogeneous algebraic equation for the vector field* ξ^d

$$C^{ab}{}_{cd}\xi^d = H^{ab}{}_c \tag{8.1}$$

has the unique solution,

$$\begin{aligned}\xi^a = &\frac{2}{(n-1)c_N} H^{ef}{}_b \Big(c_0 C[N-1]^{ab}{}_{ef} + c_2 C[N-3]^{ab}{}_{ef} \\ &+ c_3 C[N-4]^{ab}{}_{ef} + \ldots + c_{N-2} C^{ab}{}_{ef} \Big) \quad ,\end{aligned} \tag{8.2}$$

where $N = n(n-1)/2$. The coefficients c_0, c_1, ..., c_{N-2}, c_N are the usual characteristic coefficients of the Cayley-Hamilton Theorem given in (A.6).

Proof. If we consider the Cayley-Hamilton Theorem for the $N \times N$ matrix $\mathbf{C}$, see Appendix A (A.5), in tensor notation, with the characteristic coefficients given by (A.6), remembering that $c_1 = 0$ since $\mathbf{C}$ is trace-free,

$$\begin{aligned} & c_0 C[N]^{ab}{}_{cd} + c_2 C[N-2]^{ab}{}_{cd} + c_3 C[N-3]^{ab}{}_{cd} + \ldots \\ & \ldots + c_{N-2} C[2]^{ab}{}_{cd} + c_{N-1} C[1]^{ab}{}_{cd} + c_N \delta^a_{[c} \delta^b_{d]} = 0 \quad , \end{aligned} \tag{8.3}$$

we obtain by multiplying by ξ_a

$$\begin{aligned} & \xi_a \Big(c_0 C[N]^{ab}{}_{cd} + c_2 C[N-2]^{ab}{}_{cd} + c_3 C[N-3]^{ab}{}_{cd} + \ldots \\ & \ldots + c_{N-2} C[2]^{ab}{}_{cd} + c_{N-1} C[1]^{ab}{}_{cd} \Big) + c_N \xi_{[c} \delta^b_{d]} \\ =& \xi_a C^{ab}{}_{ef} \Big(c_0 C[N-1]^{ef}{}_{cd} + c_2 C[N-3]^{ef}{}_{cd} + c_3 C[N-4]^{ef}{}_{cd} + \ldots \\ & \ldots + c_{N-2} C[1]^{ef}{}_{cd} \Big) + c_{N-1} \xi_a C^{ab}{}_{cd} + c_N \xi_{[c} \delta^b_{d]} = 0 \quad . \end{aligned} \tag{8.4}$$

Taking the trace over b and d, remembering that the Weyl tensor is trace-free we find

$$\begin{aligned} & \xi_a C^{ab}{}_{ef} \Big(c_0 C[N-1]^{ef}{}_{cb} + c_2 C[N-3]^{ef}{}_{cb} + c_3 C[N-4]^{ef}{}_{cb} + \ldots \\ & \ldots + c_{N-2} C[1]^{ef}{}_{cb} \Big) + \frac{(n-1)}{2} c_N \xi_c = 0 \quad . \end{aligned} \tag{8.5}$$

Since $c_N = (-1)^N \det(\mathbf{C})$ and $\det(\mathbf{C}) \neq 0$ there is no problem dividing with c_N and a trivial rearranging of the terms in this last equation gives the required expression (8.2). □

Using Lemma 8.1.1 on

$$\nabla^d C^{ab}{}_{cd} + (n-3) K^d C^{ab}{}_{cd} = 0 \quad , \tag{8.6}$$

i.e., setting $H^{ab}{}_c = K^d C^{ab}{}_{cd} = -\frac{1}{(n-3)} \nabla^d C^{ab}{}_{cd}$, we can directly reformulate Theorem 6.3.1 as

Theorem 8.1.1. *A n-dimensional space having a non-degenerate Weyl tensor is locally conformal to an Einstein space if and only if the vector field K^a defined by*

$$\begin{aligned} K^a =& \frac{2}{(n-1)(n-3)c_N} \nabla^d C^{ef}{}_{db} \Big(c_0 C[N-1]^{ab}{}_{ef} + c_2 C[N-3]^{ab}{}_{ef} \\ & + c_3 C[N-4]^{ab}{}_{ef} + \ldots + c_{N-2} C^{ab}{}_{ef} \Big) \quad , \end{aligned} \tag{8.7}$$

where $N = n(n-1)/2$ satisfies

$$R_{ab} - \frac{1}{n}g_{ab}R + (n-2)\nabla_a K_b - \frac{(n-2)}{n}g_{ab}\nabla_c K^c$$
$$- (n-2)K_a K_b + \frac{(n-2)}{n}g_{ab}K_c K^c = 0 \quad . \tag{8.8}$$

Note that substituting (8.7) into (8.8) gives necessary and sufficient conditions defined in terms only depending on the geometry.

Four dimensions

In four dimensions $N = 6$, and from (8.7) in Theorem 8.1.1 we get the following expression for the vector field

$$K^a = \frac{2}{3c_6}\nabla^d C^{ef}{}_{db}\Big(c_0 C[5]^{ab}{}_{ef} + c_2 C[3]^{ab}{}_{ef}$$
$$+ c_3 C[2]^{ab}{}_{ef} + c_4 C^{ab}{}_{ef}\Big) \quad . \tag{8.9}$$

It is clear that this expression involves powers of the Weyl tensor up to the sixth order, but since the vector of the theorems is unique there must be a link between the expression (8.9) and the ones given by KNT, (5.28) - (5.31), and Listing, (6.5). This link has been exploited by Edgar [13].

Higher dimensions

Theorem 8.1.1 supplies an explicit expression for the vector field K^a in all dimensions $n \geq 4$. For example, in six dimensions where $N = 15$ we have

$$K^a = \frac{2}{15c_{15}}\nabla^d C^{ef}{}_{db}\Big(c_0 C[14]^{ab}{}_{ef} + c_2 C[12]^{ab}{}_{ef} + c_3 C[11]^{ab}{}_{ef}$$
$$+ \ldots + c_{11} C[3]^{ab}{}_{ef} + c_{12} C[2]^{ab}{}_{ef} + c_{13} C^{ab}{}_{ef}\Big) \quad , \tag{8.10}$$

and clearly this expression contains powers of the Weyl tensor up to fifteenth order.

In general, Theorem 8.1.1 gives an expression of the vector field K^a for an n-dimensional space containing powers of the Weyl tensor up to order $n(n-1)/2$. Hence, the generic expression for K^a quickly grows in size as the dimension becomes higher and higher.

However, in four dimensions we have seen that the generic expression (8.9), having powers of the Weyl tensor up to sixth order, can be replaced with an expression with much lower powers in the Weyl tensor obtained from dimensionally dependent identities. In the next section we will extend this approach to spaces of arbitrary dimension n.

8.2 Using dimensionally dependent identities

The method used by KNT to get their vector field[1] K^a was to assume that (at least) one of the scalar invariants of the Weyl tensor is non-zero and use one of the dimensionally dependent identities (5.23) - (5.26) on the equation

$$\nabla^d C_{abcd} + K^d C_{abcd} = 0 \quad . \tag{8.11}$$

All four identities (5.23) - (5.26) are, as discussed in appendix B, a consequence of the four-dimensional identity

$$C^{[ab}{}_{[cd}\delta^{i]}_{j]} = 0 \tag{8.12}$$

and they arise when we multiply (8.12) with more Weyl tensors.

The structure of these identities are of the type

$$L\{C[m]^i{}_j\} = \frac{1}{4}\delta^i_j L\{C[m]\} \tag{8.13}$$

where $L\{C[m]^i{}_j\}$ represents a 2-index tensor consisting of a linear combination of products of m Weyl tensors, and $L\{C[m]\} \equiv L\{C[m]^i{}_i\}$ represents a linear combination of scalar products of m Weyl tensors.

In general, an n-dimensionally dependent identity arising from the n-dimensional analogue to (8.12) (see appendix B) has the structure

$$L\{C[m]^i{}_j\} = \frac{1}{n}\delta^i_j L\{C[m]\} \quad . \tag{8.14}$$

Hence, for $L\{C[m]\} \neq 0$, we can use (8.14) on the n-dimensional analogue to (8.11),

$$\nabla^d C_{abcd} + (n-3)K^d C_{abcd} = 0 \tag{8.15}$$

to get the following expression for the vector field K^a

$$K^i = nL\{C[m]^i{}_j\}K^j / L\{C[m]\} \quad , \tag{8.16}$$

where all the terms involving the vector field K^a on the right hand side – which will each contain a factor of the form $C^{ab}{}_{cj}K^j$ – can be replaced using (8.15), i.e.,

$$C^{ab}{}_{cj}K^j = -\frac{1}{(n-3)}\nabla^j C^{ab}{}_{cj} \quad . \tag{8.17}$$

Using the notation introduced above we are now able to state the following theorem generalizing Theorem 5.3.2 of KNT,

[1] The same method was also used by Listing in [28] in his examples.

Theorem 8.2.1. *An n-dimensional space with a non-degenerate Weyl tensor restricted by $L\{C[m]\} \neq 0$, where $L\{C[m]\}$ is associated with an identity of the form*

$$L\{C[m]^i{}_j\} = \frac{1}{n}\delta^i_j L\{C[m]\} \quad , \tag{8.18}$$

is locally conformally related to an Einstein space if and only if the vector field K^a defined by

$$K^i = nL\{C[m]^i{}_j\}K^j / L\{C[m]\}, \tag{8.19}$$

with the appropriate substitutions

$$C^{ab}{}_{cj}K^j = -\frac{1}{(n-3)}\nabla^j C^{ab}{}_{cj} \tag{8.20}$$

satisfies

$$\begin{aligned} R_{ab} - \frac{1}{n}g_{ab}R + (n-2)\nabla_a K_b - \frac{(n-2)}{n}g_{ab}\nabla_c K^c \\ -(n-2)K_a K_b + \frac{(n-2)}{n}g_{ab}K_c K^c = 0 \quad . \end{aligned} \tag{8.21}$$

A five-dimensional example

In Appendix B.2 we derive the following five-dimensional identity having the structure (8.18),

$$\begin{aligned} &5C^{ijkf}C_{ijkh}C^{ab}{}_{cd}C^{cd}{}_{ab} \\ &-8C^{ijkg}C_{ijkh}C^{ab}{}_{cg}C^{cf}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{gh}C^{ef}{}_{ab} \\ &+8C^{ijkg}C_{ijke}C^{af}{}_{bh}C^{be}{}_{ag} - 8C^{ijkg}C^{ae}{}_{bh}C_{ijke}C^{bf}{}_{ag} \\ &= (C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab})\delta^f_h \end{aligned} \tag{8.22}$$

and multiplying this with K^h, using $C^{ab}{}_{cj}K^j = -\frac{1}{2}\nabla^j C^{ab}{}_{cj}$, we find

$$\begin{aligned} &5C^{ijkf}C^{ab}{}_{cd}C^{cd}{}_{ab}\nabla^h C_{ijkh} \\ &-8C^{ijkg}C^{ab}{}_{cg}C^{cf}{}_{ab}\nabla^h C_{ijkh} - 4C^{ijkg}C_{ijke}C^{ef}{}_{ab}\nabla^h C^{ab}{}_{gh} \\ &+8C^{ijkg}C_{ijke}C^{be}{}_{ag}\nabla^h C^{af}{}_{bh} - 8C^{ijkg}C_{ijke}C^{bf}{}_{ag}\nabla^h C^{ae}{}_{bh} \\ &= -2(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab})K^f \quad . \end{aligned} \tag{8.23}$$

We know from Appendix B.2 that the scalar on the right hand side is not identically zero and so for all spaces where

$$(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab}) \neq 0 \quad , \tag{8.24}$$

the necessary and sufficient condition in five dimensions in Theorem 8.2.1 is obtained by substituting

$$\begin{aligned}K^f =&\Big(-\frac{5}{2}C^{ijkf}C^{ab}{}_{cd}C^{cd}{}_{ab}\nabla^h C_{ijkh} + 4C^{ijkg}C^{ab}{}_{cg}C^{cf}{}_{ab}\nabla^h C_{ijkh}\\ &+ 2C^{ijkg}C_{ijke}C^{ef}{}_{ab}\nabla^h C^{ab}{}_{gh} - 4C^{ijkg}C_{ijke}C^{be}{}_{ag}\nabla^h C^{af}{}_{bh}\\ &+ 4C^{ijkg}C_{ijke}C^{bf}{}_{ag}\nabla^h C^{ae}{}_{bh}\Big)\\ &/(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab})\end{aligned} \tag{8.25}$$

into

$$R_{ab} - \frac{1}{5}g_{ab}R + 3\nabla_a K_b - \frac{3}{5}g_{ab}\nabla_c K^c - 3K_aK_b + \frac{3}{5}g_{ab}K_cK^c = 0 \quad . \tag{8.26}$$

Six-dimensional examples

First we give an analogous example to the five-dimensional case. In Appendix B.3 we calculate that

$$\begin{aligned}&C_{aj}{}^{bc}C^{ag}{}_{de}C_{bc}{}^{de} - 2C_{aj}{}^{bg}C^{ac}{}_{de}C_{bc}{}^{de} - 4C_{aj}{}^{bc}C^{dg}{}_{be}C_{cd}{}^{ae}\\ &= \frac{1}{6}\Big(C_{ab}{}^{cd}C_{ef}{}^{ab}C_{cd}{}^{ef} - 4C_{ab}{}^{cd}C^{ae}{}_{cf}C_{de}{}^{bf}\Big)\delta^g_j \quad ,\end{aligned} \tag{8.27}$$

and multiplying this with K^j, using $C^{ab}{}_{cj}K^j = -\frac{1}{3}\nabla^j C^{ab}{}_{cj}$, and solving for K^g we find

$$\begin{aligned}K^g =&\Big(- 2C^{ag}{}_{de}C_{bc}{}^{de}\nabla^j C_{aj}{}^{bc} + 4C^{ac}{}_{de}C_{bc}{}^{de}\nabla^j C_{aj}{}^{bg}\\ &+ 8C^{dg}{}_{be}C_{cd}{}^{ae}\nabla^j C_{aj}{}^{bc}\Big)\\ &/\Big(C_{ab}{}^{cd}C_{ef}{}^{ab}C_{cd}{}^{ef} - 4C_{ab}{}^{cd}C^{ae}{}_{cf}C_{de}{}^{bf}\Big) \quad .\end{aligned} \tag{8.28}$$

We know from Appendix B.3 that the scalar on the right hand side is not identically zero and so for all spaces where

$$(C_{ab}{}^{cd}C_{ef}{}^{ab}C_{cd}{}^{ef} - 4C_{ab}{}^{cd}C^{ae}{}_{cf}C_{de}{}^{bf}) \neq 0 \quad , \tag{8.29}$$

the necessary and sufficient condition in six dimensions in Theorem 8.2.1 is obtained by substituting (8.28) into

$$R_{ab} - \frac{1}{6}g_{ab}R + 4\nabla_a K_b - \frac{2}{3}g_{ab}\nabla_c K^c - 4K_aK_b + \frac{2}{3}g_{ab}K_cK^c = 0 \quad . \tag{8.30}$$

Secondly, we close this section with another six-dimensional example using a six-dimensional identity, discussed in Appendix B.4, due to Lovelock [29].

This identity involves the tensor $A_{ijk}{}^{abc}$ which is quadratic in the Weyl tensor and defined by

$$A_{ijk}{}^{abc} = 4C_{[ij}{}^{h[a}C_{k]h}{}^{bc]} \quad . \tag{8.31}$$

The analogous identity to (8.18) is (B.20),

$$\begin{aligned} &A_{abi}{}^{cde}A_{cde}{}^{abj} + 3A_{abi}{}^{abc}A_{cde}{}^{dej} + 6A_{abi}{}^{acd}A_{cde}{}^{bej} \\ &- 3A_{abc}{}^{ade}A_{dei}{}^{bcj} - A_{abc}{}^{abc}A_{dei}{}^{dej} + 6A_{abc}{}^{abd}A_{dei}{}^{cej} \\ &+ \frac{1}{6}A_{abc}{}^{abc}A_{def}{}^{def}\delta^j{}_i + \frac{3}{2}A_{abc}{}^{ade}A_{def}{}^{bcf}\delta^j{}_i \\ &- \frac{3}{2}A_{abc}{}^{abd}A_{def}{}^{cef}\delta^j{}_i - \frac{1}{6}A_{abc}{}^{def}A_{def}{}^{abc}\delta^j{}_i = 0, \end{aligned} \tag{8.32}$$

and multiplying by K^i and solving for K^j we have (provided we can perform the division)

$$\begin{aligned} K^j =& 6\Big(K^iA_{abi}{}^{cde}A_{cde}{}^{abj} + 3K^iA_{abi}{}^{abc}A_{cde}{}^{dej} + 6K^iA_{abi}{}^{acd}A_{cde}{}^{bej} \\ &- 3A_{abc}{}^{ade}K^iA_{dei}{}^{bcj} - A_{abc}{}^{abc}K^iA_{dei}{}^{dej} + 6A_{abc}{}^{abd}\Upsilon^iA_{dei}{}^{cej}\Big) \\ &/(A_{abc}{}^{def}A_{def}{}^{abc} - A_{abc}{}^{abc}A_{def}{}^{def} \\ &\qquad - 9A_{abc}{}^{ade}A_{def}{}^{bcf} + 9A_{abc}{}^{abd}A_{def}{}^{cef}) \quad . \end{aligned} \tag{8.33}$$

Hence in this case we get the necessary and sufficient condition in six dimension in Theorem 8.2.1 by first substituting all the terms involving K^i on the right hand side via

$$K^iA_{iab}{}^{cde} = \nabla^iC_{i[a}{}^{p[c}C_{b]p}{}^{de]} - \nabla^iC_{ip}{}^{[de}C_{ab}{}^{c]p} \tag{8.34}$$

and all the other terms on the right hand side via

$$A_{ijk}{}^{abc} = 4C_{[ij}{}^{h[a}C_{k]h}{}^{bc]} \quad . \tag{8.35}$$

into (8.33) and then this expression into

$$R_{ab} - \frac{1}{6}g_{ab}R + 4\nabla_aK_b - \frac{2}{3}g_{ab}\nabla_cK^c - 4K_aK_b + \frac{2}{3}g_{ab}K_cK^c = 0 \quad . \tag{8.36}$$

Chapter 9

Generalizing the Bach tensor in n dimensions

In this chapter we will introduce the concept of a generic space and seek to find n-dimensional generalizations of the Bach tensor. We will also generalize the KNT four-dimensional result to arbitrary n-dimensional generic spaces and give explicit examples in five and six dimensions.

9.1 A generic Weyl tensor

To generalize the KNT result we need to impose a non-degeneracy condition on the Weyl tensor generalizing the four-dimensional condition $J \neq 0$. Guided by the two lemmas (5.1.1) and (5.1.2) we now give the definition:

Definition 9.1.1. A Weyl tensor is said to be *generic* if it locally has the two properties

a) given a real two-form $F^{cd} = F^{[cd]}$, the only solution of $C_{abcd}F^{cd} = 0$ is $F^{cd} = 0$,

b) given a real symmetric and trace-free tensor $H^{ad} = H^{(ad)}$, $H^e{}_e = 0$ the only solution of

$$C_{abcd}H^{ad} = 0$$
$$^*C_{i_1 i_2 \ldots i_{n-2} acd}H^{ad} = 0$$

is $H^{ad} = 0$.

Note that any generic Weyl tensor is non-degenerate − the converse in not true.

Definition 9.1.2. A space having a generic Weyl tensor is called a *generic space*.

From the definitions above we note that in four dimensions a generic space is equivalent to a space having a non-degenerate Weyl tensor, e.g., $\det(\mathbf{C}) \neq 0$ or $J \neq 0$.

9.2 The generalization of the KNT result

In this section we will generalize the KNT four-dimensional Theorem 5.3.2 to arbitrary n-dimensional generic spaces.

Theorem 9.2.1. *An n-dimensional generic space is locally conformal to an Einstein space if and only if there exists a vector field K^a such that*

$$\nabla^d C_{abcd} + (n-3)K^d C_{abcd} = 0 \tag{9.1}$$

and

$$\nabla^b \nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-4)K^b K^d C_{abcd} = 0 \quad . \tag{9.2}$$

Proof. The proof has an analogous structure to the four-dimensional proof of Theorem 5.3.2. An n-dimensional space is conformal to an Einstein space if there exists a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$ such that

$$\begin{aligned} R_{ab} - \frac{1}{n} g_{ab} R + (n-2)\nabla_a \Upsilon_b - \frac{(n-2)}{n} g_{ab} \nabla_c \Upsilon^c \\ - (n-2)\Upsilon_a \Upsilon_b + \frac{(n-2)}{n} g_{ab} \Upsilon_c \Upsilon^c = 0 \end{aligned} \tag{9.3}$$

holds with $\Upsilon^a = \nabla^a(\ln \Omega)$, Ω some scalar field.

In Chapter 3.2 we found that the first integrability condition of (9.3) is (9.1) with $K_a = \nabla_a(\ln \Omega)$, and that taking another derivative of (9.1) gives us (9.2). Hence (9.1) and (9.2) are necessary conditions.

Conversely if (9.1) and (9.2) hold for a vector field K^a we see that differentiating (9.1) and using (9.1) gives

$$\nabla^c \nabla^d C_{abcd} + (n-3)\nabla^c K^d C_{abcd} - (n-3)^2 K^c K^d C_{abcd} = 0 \quad . \tag{9.4}$$

Since the last term is zero and $\nabla^c \nabla^d C_{abcd} = 0$ we have from (9.4) that $\nabla^c K^d C_{abcd} = \nabla^{[c} K^{d]} C_{abcd} = 0$, and from the Definition 9.1.1 of generic spaces it follows that $\nabla^{[c} K^{d]} = 0$ and hence that K^a is locally a gradient vector from which the conformal factor can be found.

Differentiating (9.1) and again using (9.1) we find

$$\nabla^b \nabla^d C_{abcd} + (n-3)\nabla^b K^d C_{abcd} - (n-3)^2 K^b K^d C_{abcd} = 0 \quad . \tag{9.5}$$

Subtracting this from equation (9.2) gives

$$(n-3)\Big(R^{bd}+(n-2)\nabla^b K^d-(n-2)K^b K^d\Big)C_{abcd}=0 \quad , \tag{9.6}$$

and this equation will serve as the first equation in Definition 9.1.1b with $H^{ad}=R^{ad}+(n-2)\nabla^a K^d-(n-2)K^a K^d$.

Next we take the left dual of equation (9.1),

$$\nabla^{d*}C_{i_1i_2\ldots i_{n-2}cd}+(n-3)K^{d*}C_{i_1i_2\ldots i_{n-2}cd}=0 \tag{9.7}$$

and after differentiating this with $\nabla^{i_{n-2}}$ we have

$$\begin{aligned}&\nabla^{i_{n-2}}\nabla^{d*}C_{i_1i_2\ldots i_{n-2}cd}+(n-3)\nabla^{i_{n-2}}K^{d*}C_{i_1i_2\ldots i_{n-2}cd}\\&+(n-3)K^d\nabla^{i_{n-2}*}C_{i_1i_2\ldots i_{n-2}cd}=0 \quad .\end{aligned} \tag{9.8}$$

Using (2.7) and (9.1) the third term in this expression can be rewritten as

$$\begin{aligned}K^d\nabla^{i_{n-2}*}C_{i_1i_2\ldots i_{n-2}cd}&=\frac{1}{2}\eta_{i_1i_2\ldots i_{n-2}}{}^{ab}K^d\nabla^{i_{n-2}}C_{abcd}\\&=\frac{1}{2}\eta_{i_1i_2\ldots i_{n-3}}{}^{i_{n-2}ab}K^d\nabla_{[i_{n-2}}C_{ab]cd}\\&=-\frac{1}{2}\eta_{i_1i_2\ldots i_{n-3}}{}^{i_{n-2}ab}\frac{1}{(n-3)}K^d\Big(g_{c[a}C_{bi_{n-2}]d}{}^f{}_{;f}\\&\qquad\qquad+g_{d[a}C_{i_{n-2}b]c}{}^f{}_{;f}\Big)\\&=\frac{1}{2}\eta_{i_1i_2\ldots i_{n-3}}{}^{i_{n-2}ab}K^dK^f\Big(g_{c[a}C_{bi_{n-2}]df}\\&\qquad\qquad+g_{d[a}C_{i_{n-2}b]cf}\Big)\\&=-\frac{1}{2}\eta_{i_1i_2\ldots i_{n-3}}{}^{i_{n-2}ab}K^fK_{[i_{n-2}}C_{ab]cf}\\&=-\frac{1}{2}\eta_{i_1i_2\ldots i_{n-2}}{}^{ab}K^dK^{i_{n-2}}C_{abcd}\\&=-K^dK^{i_{n-2}*}C_{i_1i_2\ldots i_{n-2}cd}\end{aligned} \tag{9.9}$$

and hence equation (9.8) is equivalent to

$$\begin{aligned}&\nabla^{i_{n-2}}\nabla^{d*}C_{i_1i_2\ldots i_{n-2}cd}+(n-3)\nabla^{i_{n-2}}K^{d*}C_{i_1i_2\ldots i_{n-2}cd}\\&-(n-3)K^dK^{i_{n-2}*}C_{i_1i_2\ldots i_{n-2}cd}=0 \quad .\end{aligned} \tag{9.10}$$

Taking the the dual of the contracted Bianchi identity (2.8),

$$\begin{aligned}&\nabla^{d*}C_{i_1i_2\ldots i_{n-2}cd}\\&=\frac{(n-3)}{(n-2)}\frac{1}{2}\eta_{i_1i_2\ldots i_{n-2}}{}^{ab}\Big(-2\nabla_{[a}R_{b]c}+\frac{1}{(n-1)}g_{c[b}\nabla_{a]}R\Big) \quad ,\end{aligned} \tag{9.11}$$

and then the divergence with respect to i_{n-2} of this, using the Ricci identity and decomposing the Riemann curvature tensor into its irreducible parts we find

$$\nabla^{i_{n-2}}\nabla^{d*}C_{i_1i_2\ldots i_{n-2}cd} = \frac{(n-3)}{(n-2)}{}^*C_{i_1i_2\ldots i_{n-3}i_{n-2}cd}R^{i_{n-2}d} \quad . \tag{9.12}$$

If we now use (9.12) to eliminate $\nabla^{i_{n-2}}\nabla^{d*}C_{i_1i_2\ldots i_{n-2}cd}$ in (9.10) we have

$$\begin{aligned}&\frac{(n-3)}{(n-2)}{}^*C_{i_1i_2\ldots i_{n-3}i_{n-2}cd}R^{i_{n-2}d} + (n-3)\nabla^{i_{n-2}}K^{d*}C_{i_1i_2\ldots i_{n-2}cd}\\ &-(n-3)K^dK^{i_{n-2}*}C_{i_1i_2\ldots i_{n-2}cd} = 0\end{aligned} \tag{9.13}$$

or equivalently

$$(n-3)\Big(R^{i_{n-2}d} + (n-2)\nabla^{i_{n-2}}K^d - (n-2)K^dK^{i_{n-2}}\Big){}^*C_{i_1i_2\ldots i_{n-2}cd} = 0 \quad , \tag{9.14}$$

giving us the second equation in Definition 9.1.1b.

We now assume that the space is generic. This means from Definition 9.1.1 and the equations (9.6) and (9.14) with

$$H^{ad} = R^{ad} + (n-2)\nabla^aK^d - (n-2)K^aK^d \tag{9.15}$$

that

$$R^{ad} + (n-2)\nabla^aK^d - (n-2)K^aK^d = \frac{1}{n}g^{ad}T \quad , \tag{9.16}$$

where T is the trace of the left hand side, e.g.,

$$\begin{aligned}&R_{ab} - \frac{1}{n}g_{ab}R + (n-2)\nabla_aK_b - \frac{(n-2)}{n}g_{ab}\nabla_cK^c\\ &-(n-2)K_aK_b + \frac{(n-2)}{n}g_{ab}K_cK^c = 0 \quad .\end{aligned} \tag{9.17}$$

Since K^a is locally a gradient vector the space is conformally Einstein. This completes the proof. □

As in Chapter 5.3, when we proved the four-dimensional version of Theorem 9.2.1, it is relevant to ask if both of the two conditions (9.1) and (9.2) really are needed in generic spaces, or if one of them is redundant. It is clear from the four-dimensional case that there exist spaces which are conformal C-spaces having a non-vanishing Bach tensor, and from these four-dimensional spaces we can easily construct corresponding spaces in higher dimension. However, it is still an open question whether condition (9.2) alone might be a necessary and sufficient condition in generic spaces.

9.3 n dimensions using generic results

In Theorem 9.2.1 we considered the symmetric 2-index tensor

$$\nabla^b\nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-4) K^b K^d C_{abcd} \tag{9.18}$$

and this tensor clearly reduces to the original Bach tensor in four dimensions so we know that this tensor is divergence-free and conformally well-behaved in four dimensions. We will examine its conformal and divergence-free properties in $n > 4$ dimensions in the next chapter.

As already mentioned a generic space has non-degenerate Weyl tensor and hence in generic spaces we can use the tensor D_{abcd} introduced in Chapter 7 to get an explicit expression for the vector K^a in Theorem 9.2.1. Using this notation we can restate Theorem 9.2.1 as

Theorem 9.3.1. *An n-dimensional generic space is locally conformal to an Einstein space if and only if*

$$\nabla^d C_{abcd} + (n-3) K^d C_{abcd} = 0 \tag{9.19}$$

and

$$\nabla^b\nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-4) K^b K^d C_{abcd} = 0 \quad , \tag{9.20}$$

where K^a is the vector field defined by

$$K^e = \frac{2}{(n-1)(n-3)} D^{ec}{}_{ab} \nabla^d C^{ab}{}_{cd} \quad . \tag{9.21}$$

Alternatively, using Edgar's approach in Chapter 8 we have that

Theorem 9.3.2. *An n-dimensional generic space is locally conformal to an Einstein space if and only if*

$$\nabla^d C_{abcd} + (n-3) K^d C_{abcd} = 0 \tag{9.22}$$

and

$$\nabla^b\nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-4) K^b K^d C_{abcd} = 0 \quad , \tag{9.23}$$

where K^a is the vector field defined by

$$\begin{aligned} K^a = & \frac{2}{(n-1)(n-3)c_N} \nabla^d C^{ef}{}_{db} \big(c_0 C[N-1]^{ab}{}_{ef} + c_2 C[N-3]^{ab}{}_{ef} \\ & + c_3 C[N-4]^{ab}{}_{ef} + \ldots + c_{N-2} C^{ab}{}_{ef} \big) \quad , \end{aligned} \tag{9.24}$$

c_i being the usual characteristic coefficients of the Cayley-Hamilton theorem (given in Appendix A*).*

Although Theorem 9.3.2 gives an explicit expression for the vector field K^a, we know from the discussion in Chapter 8 that in higher dimensions this expression becomes very large and difficult to use. However, we saw in Chapter 5 that we can get expressions of considerable lower order in the Weyl tensors using dimensional dependent identities, and using the notation in Chapter 8 we can reformulate Theorem 9.2.1 as

Theorem 9.3.3. *An n-dimensional generic space with a Weyl tensor restricted by* $L\{C[m]\} \neq 0$*, where* $L\{C[m]\}$ *is associated with an identity of the form*

$$L\{C[m]\}^i{}_j = \frac{1}{n}\delta^i_j L\{C[m]\} \quad , \tag{9.25}$$

is locally conformal to an Einstein space if and only if

$$\nabla^d C_{abcd} + (n-3)K^d C_{abcd} = 0 \tag{9.26}$$

and

$$\nabla^b\nabla^d C_{abcd} - \frac{(n-3)}{(n-2)}R^{bd}C_{abcd} - (n-3)(n-4)K^b K^d C_{abcd} = 0 \quad , \tag{9.27}$$

where the vector field K^a *is defined by*

$$K^i = nL\{C[m]\}^i{}_j K^j / L\{C[m]\}, \tag{9.28}$$

with the appropriate substitutions

$$C^{ab}{}_{cj}K^j = -\frac{1}{(n-3)}\nabla^j C^{ab}{}_{cj} \quad . \tag{9.29}$$

To illustrate Theorem 9.3.2 we close this chapter with explicit examples in five and six dimensions.

9.4 Five-dimensional spaces using dimensional dependent identities

We saw in Chapter 8.2 that using the five-dimensional identity (B.13) on the five-dimensional version of (9.26),

$$\nabla^d C_{abcd} + 2K^d C_{abcd} = 0 \quad , \tag{9.30}$$

assuming $(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab}) \neq 0$ and using $C^{ab}{}_{cj}K^j = -\frac{1}{2}\nabla^j C^{ab}{}_{cj}$, we find

$$\begin{aligned} K^f =& \Big(-\frac{5}{2}C^{ijkf}C^{ab}{}_{cd}C^{cd}{}_{ab}\nabla^h C_{ijkh} + 4C^{ijkg}C^{ab}{}_{cg}C^{cf}{}_{ab}\nabla^h C_{ijkh} \\ &+ 2C^{ijkg}C_{ijke}C^{ef}{}_{ab}\nabla^h C^{ab}{}_{gh} - 4C^{ijkg}C_{ijke}C^{be}{}_{ag}\nabla^h C^{af}{}_{bh} \\ &+ 4C^{ijkg}C_{ijke}C^{bf}{}_{ag}\nabla^h C^{ae}{}_{bh}\Big) \\ &/(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab}) \quad . \end{aligned} \tag{9.31}$$

Hence the necessary and sufficient condition for a five-dimensional generic space with $(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab}) \neq 0$ to be conformally related to an Einstein space is that

$$\nabla^d C_{abcd} + 2K^d C_{abcd} = 0 \tag{9.32}$$

and

$$\nabla^b \nabla^d C_{abcd} - \frac{2}{3} R^{bd} C_{abcd} - 2K^b K^d C_{abcd} = 0 \tag{9.33}$$

hold, where K^a is given by (9.31).

9.5 Six-dimensional spaces using dimensional dependent identities

In six dimensions, equation (9.26) become

$$\nabla^d C_{abcd} + 3K^d C_{abcd} = 0 \quad , \tag{9.34}$$

and, as in Chapter 8.2, after the use of the six-dimensional dependent identity (B.15) and $C^{ab}{}_{cj}K^j = -\frac{1}{3}\nabla^j C^{ab}{}_{cj}$, we find that

$$\begin{aligned} K^g = & -2C^{ag}{}_{de}C_{bc}{}^{de}\nabla^j C_{aj}{}^{bc} + 4C^{ac}{}_{fde}C_{bc}{}^{de}\nabla^j C_{aj}{}^{bg} \\ & + 8C^{dg}{}_{be}C_{cd}{}^{ae}\nabla^j C_{aj}{}^{bc} \\ & /(C_{ab}{}^{cd}C_{ef}{}^{ab}C_{cd}{}^{ef} - 4C_{ab}{}^{cd}C^{ae}{}_{cf}C_{de}{}^{bf}) \quad , \end{aligned} \tag{9.35}$$

providing $(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab}) \neq 0$.

So, a six-dimensional generic space, in which the scalar

$$(C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab}) \neq 0 \tag{9.36}$$

is conformally related to an Einstein space if and only if

$$\nabla^d C_{abcd} + 3K^d C_{abcd} = 0 \tag{9.37}$$

and

$$\nabla^b \nabla^d C_{abcd} - \frac{3}{4} R^{bd} C_{abcd} - 6K^b K^d C_{abcd} = 0 \tag{9.38}$$

hold, where K^a is given by (9.35).

Chapter 10

Conformal properties of different tensors

In Chapter 4 we showed that some obvious attempts for an n-dimensional generalization of the Bach tensor were not conformally well-behaved nor divergence-free in dimensions $n > 4$. Now we have found another generalization of this tensor, $\mathcal{B}_{ac}$,

$$\mathcal{B}_{ac} = \nabla^b \nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-4) K^b K^d C_{abcd} \quad (10.1)$$

with important properties, and in this chapter we will investigate conformal properties of this and related tensors.

10.1 The tensors $\mathfrak{b}_{abc}$ and $\mathfrak{B}_{ac}$ and their conformal properties in generic spaces

The tensor $\mathcal{B}_{ac}$ is, as the original four-dimensional Bach tensor B_{ac}, motivated by the second integrability condition for a space being conformal to an Einstein space; it is found by taking a derivative of

$$\nabla^d C_{abcd} + (n-3) K^d C_{abcd} = 0 \quad (10.2)$$

and is given by

$$\mathcal{B}_{ac} = \nabla^b \nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-4) K^b K^d C_{abcd} = 0 \quad , \quad (10.3)$$

where we replaced the vector field Υ^a connected with a conformal transformation with the vector field K^a.

In general we cannot discuss the conformal properties of the form (10.3) for $\mathcal{B}_{ac}$ because of the non-geometric vector K^a. However, we encountered $\mathcal{B}_{ac}$ as one of the two sufficient and necessary conditions for a space to be conformal to an Einstein space, so we look at these two conditions together and check their conformal behavior.

In a space with non-degenerate Weyl tensor let us define the vector field $\underset{G}{K}^a$ by

$$\underset{G}{K}^a = \frac{2}{(n-1)(n-3)} D^{ai}{}_{jk} \nabla^d C^{jk}{}_{id} \quad , \tag{10.4}$$

where $D^{ab}{}_{cd}$ is the tensor satisfying

$$D^{ab}{}_{cd} C^{cd}{}_{ef} = \delta^a_{[e} \delta^b_{f]} \quad . \tag{10.5}$$

Hence, in a space with non-degenerate Weyl tensor, guided by (10.2), we will define

$$\begin{aligned} \mathfrak{b}_{abc} &= \nabla^d C_{abcd} + \underset{G}{K}^d C_{abcd} \\ &= \nabla^d C_{abcd} + \frac{2}{(n-1)} C_{abcd} D^{di}{}_{jk} \nabla^l C^{jk}{}_{il} \end{aligned} \tag{10.6}$$

and, guided by $\mathcal{B}_{ac}$ in (10.3) we define,

$$\begin{aligned} \mathfrak{B}_{ac} =& \nabla^b \nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} - (n-3)(n-5) \underset{G}{K}^b \underset{G}{K}^d C_{abcd} \\ =& \nabla^b \nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} \\ & - \frac{4(n-4)}{(n-1)^2(n-3)} D^{bi}{}_{jk} \nabla^l C^{jk}{}_{il} D^{dr}{}_{pq} \nabla^m C^{pq}{}_{rm} C_{abcd} \quad . \end{aligned} \tag{10.7}$$

Since the right hand side of equation (10.5) is conformally invariant under a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$, we have

$$\delta^a_{[e} \delta^b_{f]} = \widehat{D}^{ab}{}_{cd} \widehat{C}^{cd}{}_{ef} = \Omega^{-2} \widehat{D}^{ab}{}_{cd} C^{cd}{}_{ef} \quad , \tag{10.8}$$

and multiplying this with $D^{ef}{}_{gh}$, solving for $\widehat{D}^{ab}{}_{cd}$, we find that the conformal behavior of $D^{ab}{}_{cd}$ is

$$\widehat{D}^{ab}{}_{cd} = \Omega^2 D^{ab}{}_{cd} \quad . \tag{10.9}$$

Using (10.5) and the conformal property of $D^{ab}{}_{cd}$ as well as the well-known ones for the Weyl tensor we see that the conformal behavior of $\mathfrak{b}_{abc}$ is

$$\begin{aligned}
\widehat{\mathfrak{b}}_{abc} &= \widehat{\nabla}^d \widehat{C}_{abcd} + \frac{2}{(n-1)} \widehat{C}_{abcd} \widehat{D}^{di}{}_{jk} \widehat{\nabla}^l \widehat{C}^{jk}{}_{il} \\
&= \nabla^d C_{abcd} + (n-3)\Upsilon^d C_{abcd} \\
&\quad + \frac{2}{(n-1)} C_{abcd} D^{di}{}_{jk} \Big(\nabla^l C^{jk}{}_{il} + (n-3)\Upsilon^l C^{jk}{}_{il} \Big) \\
&= \nabla^d C_{abcd} + \frac{2}{(n-1)} C_{abcd} D^{di}{}_{jk} \nabla^l C^{jk}{}_{il} \\
&\quad + (n-3)\Upsilon^d C_{abcd} + \frac{2(n-3)}{(n-1)} \Upsilon^l C_{abcd} D^{di}{}_{jk} C^{jk}{}_{il} \\
&= \mathfrak{b}_{abc} \quad ,
\end{aligned} \tag{10.10}$$

i.e., that $\mathfrak{b}_{abc}$ is conformally invariant.

Turning to the tensor $\mathfrak{B}_{bc}$ we find after some calculation that under a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$, we have

$$\begin{aligned}
\widehat{\mathfrak{B}}_{ac} = \Omega^{-2}\mathfrak{B}_{ac} + (n-4)\Omega^{-2} \Bigg(& \Upsilon^b \Big(C_{abcd;}{}^d + (n-3) K^d C_{abcd} \Big) \\
& + \Upsilon^d \Big(C_{abcd;}{}^b + (n-3) K^b C_{abcd} \Big) \Bigg) \quad ,
\end{aligned} \tag{10.11}$$

meaning that in general $\mathfrak{B}_{bc}$ is *not* conformally well-behaved.

However, in a space with non-degenerate Weyl tensor we can use (10.6) to rewrite (10.11) as

$$\widehat{\mathfrak{B}}_{ac} = \Omega^{-2}\mathfrak{B}_{ac} + (n-4)\Omega^{-2} \Bigg(\Upsilon^b \mathfrak{b}_{abc} + \Upsilon^d \mathfrak{b}_{acd} \Bigg) \quad , \tag{10.12}$$

and it is clear from (10.12) that under the condition that $\mathfrak{b}_{abc} = 0$ we see that $\mathfrak{B}_{bc}$ is in fact conformally weighted with weight -2. We collect these results in the following theorem:

Theorem 10.1.1. *In a space with non-degenerate Weyl tensor the tensor*

$$\mathfrak{b}_{abc} = \nabla^d C_{abcd} + \frac{2}{(n-1)} C_{abcd} D^{di}{}_{jk} \nabla^l C^{jk}{}_{il} \tag{10.13}$$

is conformally invariant but the tensor

$$\begin{aligned}
\mathfrak{B}_{ac} =& \nabla^b \nabla^d C_{abcd} - \frac{(n-3)}{(n-2)} R^{bd} C_{abcd} \\
& - \frac{4(n-4)}{(n-1)^2(n-3)} D^{bi}{}_{jk} \nabla^l C^{jk}{}_{il} D^{dr}{}_{pq} \nabla^m C^{pq}{}_{rm} C_{abcd}
\end{aligned} \tag{10.14}$$

is in general not. However, if $\mathfrak{b}_{abc} = 0$*, then* $\mathfrak{B}_{bc}$ *is conformally weighted with weight* -2.

Note that, remembering that generic spaces are subspaces of the class of spaces with non-degenerate Weyl tensor, and using the notation (10.6) and (10.7) we can reformulate the sufficient and necessary condition for a space being conformal to an Einstein space, i.e., Theorem (9.2.1), as

Theorem 10.1.2. *A generic space is conformal to an Einstein space if and only if the conformal invariant condition hold,*

$$\mathfrak{b}_{abc} = 0 \tag{10.15}$$

and

$$\mathfrak{B}_{bc} = 0 \quad . \tag{10.16}$$

10.2 The tensor $\mathfrak{L}_{ab}$ and its conformal properties

Motivated by Listing's n-dimensional result (Theorem (7.3.2)) we will define, in a space with non-degenerate Weyl tensor, the tensor

$$\begin{aligned}\mathfrak{L}_{ab} =& R_{ab} - \frac{1}{n} g_{ab} R + (n-2)\nabla_a \underset{G}{K}_b + \frac{(n-2)}{n} g_{ab} \nabla_c \underset{G}{K}^c \\ & - (n-2)\underset{G}{K}_a \underset{G}{K}_b + \frac{(n-2)}{n} g_{ab} \underset{G}{K}_c \underset{G}{K}^c \quad ,\end{aligned} \tag{10.17}$$

where $\underset{G}{K}^a$ is defined by (10.4) .

Hence, in a space with non-degenerate Weyl tensor, the tensor $\mathfrak{L}_{ab}$ is built up purely from geometry, and we find that under a conformal transformation $\widehat{g}_{ab} = \Omega^2 g_{ab}$, using

$$\begin{aligned}\underset{G}{\widehat{K}}^e &= \frac{2}{(n-1)(n-3)} \widehat{D}^{ec}{}_{ab} \widehat{\nabla}^d \widehat{C}^{ab}{}_{cd} \\ &= \frac{2}{(n-1)(n-3)} D^{ec}{}_{ab} \Big(\nabla^d C^{ab}{}_{cd} + (n-3)\Upsilon^d C^{ab}{}_{cd} \Big) \\ &= \frac{2}{(n-1)(n-3)} D^{ec}{}_{ab} \nabla^d C^{ab}{}_{cd} + \Upsilon^d \frac{2}{(n-1)} D^{ec}{}_{ab} C^{ab}{}_{cd} \\ &= \underset{G}{K}^e - \Upsilon^e \quad ,\end{aligned} \tag{10.18}$$

that

$$\begin{aligned}
\widehat{\mathfrak{L}}_{ab} &= \widehat{R}_{ab} - \frac{1}{n}\widehat{g}_{ab}\widehat{R} + (n-2)\widehat{\nabla}_a \widehat{\underset{G}{K}}_b - \frac{(n-2)}{n}\widehat{g}_{ab}\widehat{\nabla}_c \widehat{\underset{G}{K}}^c \\
&\quad - (n-2)\widehat{\underset{G}{K}}_a \widehat{\underset{G}{K}}_b + \frac{(n-2)}{n}\widehat{g}_{ab}\widehat{\underset{G}{K}}_c \widehat{\underset{G}{K}}^c \\
&= R_{ab} + (n-2)\nabla_a \Upsilon_b + g_{ab}\nabla_c \Upsilon^c - (n-2)\Upsilon_a\Upsilon_b + (n-2)g_{ab}\Upsilon_c\Upsilon^c \\
&\quad - \frac{1}{n}g_{ab}\Big(R + 2(n-1)\nabla_c\Upsilon^c + (n-1)(n-2)\Upsilon_c\Upsilon^c\Big) \\
&\quad + (n-2)\Big(\nabla_a \underset{G}{K}_b - \nabla_a\Upsilon_b - 2\underset{G}{K}_{(a}\Upsilon_{b)} + 2\Upsilon_a\Upsilon_b + g_{ab}\Upsilon^c\big(\underset{G}{K}_c - \Upsilon_c\big)\Big) \\
&\quad - \frac{(n-2)}{n}g_{ab}\Big(\nabla_c \underset{G}{K}^c - \nabla_c\Upsilon^c + (n-2)\Upsilon^c \underset{G}{K}_c - (n-2)\Upsilon_c\Upsilon^c\Big) \\
&\quad - (n-2)\Big(\underset{G}{K}_a - \Upsilon_a\Big)\Big(\underset{G}{K}_b - \Upsilon_b\Big) + \frac{(n-2)}{n}g_{ab}\Big(\underset{G}{K}_c - \Upsilon_c\Big)\Big(\underset{G}{K}^c - \Upsilon^c\Big) \\
&= \mathfrak{L}_{ab} \quad , \qquad (10.19)
\end{aligned}$$

i.e., that $\mathfrak{L}_{ab}$ is conformally invariant, and we have

Theorem 10.2.1. *In a space with non-degenerate Weyl tenor, the tensor* $\mathfrak{L}_{ab}$ *defined by equation* (10.17)*, with all the* $\underset{G}{K}^a$ *substituted using* (10.4)*, is conformally invariant.*

Chapter 11

Concluding remarks and future work

In this thesis we began by reviewing the developments of the problem of finding necessary and sufficient conditions for a space to be conformal to an Einstein space.

Since the Bach tensor featured in some four-dimensional results, we looked to see if there was an obvious n-dimensional analogue of the Bach tensor.

A theorem due to Belfagón and Jeán was strengthened and a basis, consisting of three tensors, for all n-dimensional symmetric, divergence-free 2-index tensors which are also quadratic in the Riemann curvature tensor, was presented. We were able to show that the Bach tensor is a linear combination of these tensors in four dimensions, and that the Bach tensor is the only (up to constant rescaling) 2-index tensor in four dimensions which is symmetric, divergence-free, conformally well-behaved, and quadratic in the Riemann curvature tensor; however, there was no such n-dimensional result. Specifically, we have shown for dimensions $n > 4$, in general, that there is no 2-index symmetric and divergence-free tensor quadratic in the Riemann curvature tensor which is also of good conformal weight.

So rather than focus on a generalization of the Bach tensor itself we looked for a generalization of the four-dimensional KNT result, which provided two necessary and sufficient conditions (one of which involves the Bach tensor) for the existence of conformal Einstein spaces.

We have introduced the concept of a generic Weyl tensor and a generic space, which is stronger than the KNT condition $J \neq 0$, and so we were able to generalize the KNT four-dimensional result to n dimensions, $n \geq 4$. We found that a generic space is locally conformal to an Einstein space if and only if two conformally invariant conditions hold:

$$\mathfrak{b}_{abc} = 0 \tag{11.1}$$

and

$$\mathfrak{B}_{ab} = 0 \quad . \tag{11.2}$$

The tensor $\mathfrak{B}_{ab}$ in a sense generalizes the Bach tensor B_{ab} to n dimensions. We have also shown that the tensors $\mathfrak{b}_{abc}$ and $\mathfrak{L}_{ab}$ are conformally invariant in spaces with non-degenerate Weyl tensor, and that for spaces in which $\mathfrak{b}_{abc} = 0$ the tensor $\mathfrak{B}_{ab}$ is conformally weighted with weight -2.
Explicit examples of these new results have been given in four, five and six dimensions using (in the five- and six-dimensional cases) newly derived tensor identities.
As well as obtaining a number of new identities in Appendix B, we also found in Appendix C the explicit relation between the two complex scalar invariants of the Weyl spinor and four real invariants of the Weyl tensor.

There are some questions that would be interesting to address in the future. We have already mentioned that a generic Weyl tensor also is a non-degenerate Weyl tensor, but not *vice-versa*[1]. The natural questions arise: what is the difference between these two, and how much bigger is the class of spaces having non-degenerate Weyl tensor compared to the class of generic spaces? Is there a way to extend the results for the generic class so that they apply to the larger class of spaces having non-degenerate Weyl tensor? Is there a way of extending such results even to a larger class than the class of spaces having non-degenerate Weyl tensor, i.e., to spaces where $J = 0$.
We have pointed out that the question of whether $B_{ab} = 0$ in a space having non-degenerate Weyl tensor in four dimension is alone a necessary condition for conformal Einstein spaces is still open. We suspect it is not so, and it would be good to find a precise counterexample.
We have not distinguished between positive definite and Lorentz metrics; it would appear that when we make this distinction we can get stronger results for the positive definite cases. The Lorentz case is well covered in four dimensions, but not in higher dimensions. The conditions put on the Weyl tensor in four dimensions are related to the Petrov classification, but in n dimensions, $n > 4$, no such classification is known. However, recently Coley, Milson, Pravda and Pravdová [8], [9], [10] developed a classification of the Weyl tensor in higher dimension using aligned null vectors of various orders of alignment. It would be interesting to see how the conditions for a generic space look in their formalism, and if their formalism gives rise to other natural necessary and sufficient conditions in higher dimensions.
Finally, we note that the very recent work of Gover and Nurowski [16] offers wider possibilities within differential geometry; an obvious next step is to try and exploit our new identities in higher dimensions within these developments.

[1] In four dimensions they are equivalent.

Appendix A

The Cayley-Hamilton Theorem and the translation of the Weyl tensor/spinor to a matrix

In this appendix we will review the Cayley-Hamilton Theorem and see how we can translate the Wely tensor and/or the Weyl spinor to a matrix. We we also formulate the Cayley-Hamilton Theorem in tensor notation.

A.1 The Cayley-Hamilton Theorem

The Cayley-Hamilton Theorem[1] states that a matrix satisfies its characteristic polynomial, i.e., for a $n \times n$ matrix $\mathbf{A}$ with characteristic polynomial

$$p(x) = a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n \quad , \tag{A.1}$$

$p(\mathbf{A}) = 0$.

It is a well-know fact [19], [38] that the relation between the determinant of a $n \times n$ matrix $\mathbf{A}$ and the coefficient a_n in its characteristic polynomial is

$$(-1)^n \det(\mathbf{A}) = a_n \quad . \tag{A.2}$$

A linear transformation on an n-dimensional vector space V given by an $n \times n$ matrix $\mathbf{A}$ can be interpreted as a tensor $A^a{}_b$ and *vice-versa*. In this context the Cayley-Hamilton Theorem can be written

$$A^{c_1}{}_{[c_1} A^{c_2}{}_{c_2} \cdots A^{c_{n-1}}{}_{c_{n-1}} A^{c_n}{}_{c_n} \delta^a{}_{b]} = 0 \quad , \tag{A.3}$$

[1]See [38] for a general proof.

i.e., as a consequence of antisymmetrising over $n+1$ indices, and one can calculate the coefficients $a_1, a_2, \ldots, a_n$ of the characteristic polynomial in terms of traces of the matrix from

$$a_k = A^{[i_1}{}_{i_1} A^{i_2}{}_{i_2} \ldots A^{i_k]}{}_{i_k}, \quad k = 1 \ldots n \tag{A.4}$$

(a_0=1).

So, for an $n \times n$ matrix $\mathbf{A}$ the Cayley-Hamilton Theorem is given by

$$a_0 \mathbf{A}^n + a_1 \mathbf{A}^{n-1} + a_2 \mathbf{A}^{n-2} + \ldots + a_{n-2} \mathbf{A}^2 + a_{n-1} \mathbf{A} + a_n \mathbf{I} = 0 \quad , \tag{A.5}$$

where $\mathbf{I}$ is the $n \times n$ identity matrix, and denoting the trace of a matrix with square brackets the characteristic coefficients are given by [19]

$$\begin{aligned}
&a_0 = 1, \quad a_1 = -[\mathbf{A}], \quad a_2 = -\frac{1}{2}\left([\mathbf{A}^2] - \left([\mathbf{A}]\right)^2\right), \\
&a_3 = -\frac{1}{3!}\left(2[\mathbf{A}^3] - 3[\mathbf{A}][\mathbf{A}^2] + \left([\mathbf{A}]\right)^3\right), \\
&a_4 = -\frac{1}{4!}\left(6[\mathbf{A}^4] - 3\left([\mathbf{A}^2]\right)^2 - 8[\mathbf{A}][\mathbf{A}^3] + [\mathbf{A}^2]\left([\mathbf{A}]\right)^2 - \left([\mathbf{A}]\right)^4\right), \\
&\ldots, \\
&a_n = -\frac{1}{n!}\left((n-1)![\mathbf{A}^n] + \ldots + \ldots\right) \quad .
\end{aligned} \tag{A.6}$$

Note that if the matrix is trace-free we will have $[\mathbf{A}] = 0$ in the expressions for the coefficients a_k above, and especially, $a_1 = 0$.

For future reference we write out two special cases of particular interest for us.

The case where $n = 3$ and the matrix is trace-free.

In the case of a 3×3 trace-free matrix $\mathbf{A}$ the Cayley-Hamilton Theorem becomes

$$a_0 \mathbf{A}^3 + a_2 \mathbf{A} + a_3 \mathbf{I} = \mathbf{0} \quad , \tag{A.7}$$

where

$$a_0 = 1, \quad a_2 = -\frac{1}{2}[\mathbf{A}^2], \quad a_3 = -\frac{1}{3}[\mathbf{A}^3] \quad , \tag{A.8}$$

i.e.,

$$\mathbf{A}^3 - \frac{1}{2}[\mathbf{A}^2]\mathbf{A} - \frac{1}{3}[\mathbf{A}^3]\mathbf{I} = \mathbf{0} \quad . \tag{A.9}$$

The determinant in terms of traces of $\mathbf{A}$ is

$$\det(\mathbf{A}) = \frac{1}{3}[\mathbf{A}^3] \quad . \tag{A.10}$$

The case where $n=6$ and the matrix is trace-free.

In the case of a 6×6 trace-free matrix $\mathbf{A}$ the Cayley-Hamilton Theorem states

$$a_0\mathbf{A}^6+a_2\mathbf{A}^4+a_3\mathbf{A}^3+a_4\mathbf{A}^2+a_5\mathbf{A}+a_6\mathbf{I}=\mathbf{0}\quad , \tag{A.11}$$

where

$$\begin{aligned}
&a_0=1,\quad a_2=-\frac{1}{2}[\mathbf{A}^2],\quad a_3=-\frac{1}{3}[\mathbf{A}^3],\\
&a_4=-\frac{1}{4}\left([\mathbf{A}^4]-\frac{1}{2}\Big([\mathbf{A}^2]\Big)^2\right),\quad a_5=-\frac{1}{5}\left([\mathbf{A}^5]-\frac{5}{6}[\mathbf{A}^2][\mathbf{A}^3]\right),\\
&a_6=-\frac{1}{6}\left([\mathbf{A}^6]-\frac{3}{4}[\mathbf{A}^2][\mathbf{A}^4]-\frac{1}{3}\Big([\mathbf{A}^3]\Big)^2+\frac{1}{8}\Big([\mathbf{A}^2]\Big)^3\right)\quad .
\end{aligned} \tag{A.12}$$

The formula expressing the determinant of $\mathbf{A}$ in terms of traces is then

$$\det(\mathbf{A})=-\frac{1}{6}\left([\mathbf{A}^6]-\frac{3}{4}[\mathbf{A}^2][\mathbf{A}^4]-\frac{1}{3}\Big([\mathbf{A}^3]\Big)^2+\frac{1}{8}\Big([\mathbf{A}^2]\Big)^3\right)\quad . \tag{A.13}$$

A.2 Translation of $C^{ab}{}_{cd}$ to a matrix $\mathbf{C}^A{}_B$

The n-dimensional Weyl tensor can be translated to a $N\times N$ trace-free matrix, where $N=n(n-1)/2$. We write this as

$$\mathbf{C}^A{}_C=C^{ab}{}_{cd}\quad , \tag{A.14}$$

where $A=[ab]$ and $C=[cd]$ so that $A,C=1,2,\ldots,N$. In doing this we *only* use the fact that the Weyl tensor is a double two-form, i.e., that $C^{ab}{}_{cd}=C^{[ab]}{}_{[cd]}$, *not* the other symmetries, $C^{ab}{}_{cd}=C_{cd}{}^{ab}$ and $C_{[abc]d}=0$. Also note that, in this construction, we are not defining a metric for the N-dimensional space.

In the case of $n=6$, the Cayley-Hamilton Theorem applied to $\mathbf{C}$ gives

$$\begin{aligned}
&\mathbf{C}^6-\frac{1}{2}[\mathbf{C}^2]\mathbf{C}^5-\frac{1}{3}[\mathbf{C}^3]\mathbf{C}^4\\
&-\frac{1}{4}\left([\mathbf{C}^4]-\frac{1}{2}\Big([\mathbf{C}^2]\Big)^2\right)\mathbf{C}^3-\frac{1}{5}\left([\mathbf{C}^5]-\frac{5}{6}[\mathbf{C}^2][\mathbf{C}^3]\right)\mathbf{C}\\
&-\frac{1}{6}\left([\mathbf{C}^6]-\frac{3}{4}[\mathbf{C}^2][\mathbf{C}^4]-\frac{1}{3}\Big([\mathbf{C}^3]\Big)^2+\frac{1}{8}\Big([\mathbf{C}^2]\Big)^3\right)\mathbf{I}=\mathbf{0}\quad ,
\end{aligned} \tag{A.15}$$

and from (A.13) we have

$$\det(\mathbf{C}) = -\frac{1}{6}\left([\mathbf{C}^6] - \frac{3}{4}[\mathbf{C}^2][\mathbf{C}^4] - \frac{1}{3}\Big([\mathbf{C}^3]\Big)^2 + \frac{1}{8}\Big([\mathbf{C}^2]\Big)^3\right) \quad . \qquad \text{(A.16)}$$

In four dimensions on the other hand, there is a more efficient representation of the Weyl tensor using duals. In this case the Weyl tensor can be represented via the trace-free complex 3×3 matrix $\boldsymbol{\mathcal{C}}$ representing $C^{ab}{}_{cd} + i^*C^{ab}{}_{cd}$, where ${}^*C^{ab}{}_{cd} = \frac{1}{2}\eta^{abij}C_{ijcd}$ is the left dual of the Weyl tensor.

So, in four dimensions, the Cayley-Hamilton Theorem (A.9) applied to the trace-free matrix $\boldsymbol{\mathcal{C}}$ gives

$$\boldsymbol{\mathcal{C}}^3 - \frac{1}{2}[\boldsymbol{\mathcal{C}}^2]\boldsymbol{\mathcal{C}} - \frac{1}{3}[\boldsymbol{\mathcal{C}}^3]\mathbf{I} = \mathbf{0} \qquad \text{(A.17)}$$

and according to (A.10), the determinant in terms of traces of $\boldsymbol{\mathcal{C}}$ is

$$\det(\boldsymbol{\mathcal{C}}) = \frac{1}{3}[\boldsymbol{\mathcal{C}}^3] \quad . \qquad \text{(A.18)}$$

A.3 Translation of $\Psi^{AB}{}_{CD}$ to a matrix $\mathbf{\Psi}$

In four dimensions we can represent the Weyl spinor as a complex matrix, and this is done in detail in for instance [33]. The information of the Weyl tensor C_{abcd} in spinor languages is described by the fully symmetric Weyl spinor $\Psi_{ABCD} = \Psi_{(ABCD)}$, and writing the Weyl spinor in the form $\Psi^{AB}{}_{CD}$ we may regard it as a linear transformation on the (three-dimensional) complex space of symmetric spinors $\phi^{AB} = \phi^{(AB)}$,

$$\phi^{AB} \to \Psi^{AB}{}_{CD}\phi^{CD} \quad , \qquad \text{(A.19)}$$

and hence, analogous to the previous sections, can be described by a 3×3 complex matrix $\mathbf{\Psi}$.

Looking at the eigenvalue problem for the mapping (A.19),

$$\Psi^{AB}{}_{CD}\phi^{CD} = \lambda\phi^{AB} \quad , \qquad \text{(A.20)}$$

we see by expressing (A.20) in components with respect to a specific basis[2] that an eigenvalue λ of equation (A.20) also is an eigenvalue in the ordinary

[2]See [33] for details

sense of the matrix $\mathbf{\Psi}$. Given that λ_1, λ_2 and λ_3 are the three eigenvalues of $\mathbf{\Psi}$ we have

$$\lambda_1 + \lambda_2 + \lambda_3 = \Psi_{AB}{}^{AB} = 0 \tag{A.21}$$

$$\lambda_1^2 + \lambda_2^2 + \lambda_3^2 = \Psi_{AB}{}^{CD}\Psi_{CD}{}^{AB} = I \tag{A.22}$$

$$\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = \Psi_{AB}{}^{CD}\Psi_{CD}{}^{EF}\Psi_{EF}{}^{AB} = J \tag{A.23}$$

where I and J are the two complex scalar invariants of the Weyl spinor. By cubing (A.21) and subtracting 3 times the product of (A.21) and (A.22) we get

$$J = 3\lambda_1\lambda_2\lambda_3 \quad , \tag{A.24}$$

i.e., that

$$\det(\mathbf{\Psi}) = \frac{1}{3}J \quad . \tag{A.25}$$

We note that the Weyl spinor has therefore two complex or four real independent scalar invariants. This result can be transfered to the Weyl tensor, where a direct proof is more complicated.

Appendix B

Dimensionally dependent tensor identities

The dimensionally dependent tensor identities are a class of identities based on the trivial idea that antisymmetrising a tensor expression over more indices than the number of dimensions makes the expression vanish.

In the late 1960's Lovelock [29] studied some apparently unrelated tensor identities and showed that they where all special cases of two dimensionally dependent identities. Lovelocks results have been generalized by Edgar and Höglund [14] and they proved the following theorem:

Theorem B.0.1. *Let $T_{a_1\dots a_k}{}^{b_1\dots b_m}$ be antisymmetric in the upper and lower indices respectively and $d = n - k - m + 1 \geq 0$. Then*

$$0 = T_{[a_1\dots k_k}{}^{[b_1\dots b_m}\delta^{b_{m+1}}_{a_{k+1}}\dots\delta^{b_{m+d}]}_{a_{k+d}]} \Longleftrightarrow T_{a_1\dots a_k}{}^{b_1\dots b_m} \text{ is trace-free.}$$

We will use this basic theorem to derive dimensionally dependent identities suitable for our purpose. Although the process of deriving these identities results in a vast number of identities, only those relevant for us will be accounted for here.

B.1 Four-dimensional identities

Theorem B.0.1 tells us that for the trace-free Weyl tensor $C^{ab}{}_{cd}$ in four dimensions, we have

$$C^{[ab}{}_{[cd}\delta^{i]}_{j]} = 0 \quad . \tag{B.1}$$

Expanding the left hand side and using the antisymmetry of C gives

$$\begin{aligned} 0 = &C^{ab}{}_{cd}\delta^i{}_j - C^{ab}{}_{jd}\delta^i{}_c - C^{ab}{}_{cj}\delta^i{}_d \\ &+C^{bi}{}_{cd}\delta^a{}_j + C^{ai}{}_{jd}\delta^b{}_c - C^{ib}{}_{jc}\delta^a{}_d \\ &+C^{ia}{}_{cd}\delta^b{}_j + C^{ib}{}_{jd}\delta^a{}_c + C^{ia}{}_{jc}\delta^b{}_d \quad . \end{aligned}$$

Multiplication by $C^{cd}{}_{ab}$ and using the trace-free properties of the Weyl tensor yields the well-know four-dimensional identity

$$C^{ibcd}C_{jbcd} = \frac{1}{4}\delta^i_j C^{abcd}C_{abcd} \quad . \tag{B.2}$$

Using this identity (B.2) we can multiply (B.1) by $C_{ab}{}^{ef}C_{ef}{}^{cd}$ and then simplify this to

$$C^{ib}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{jb} = \frac{1}{4}\delta^i_j C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{ab} \quad . \tag{B.3}$$

We can also in a similar manner derive the identities

$$^*C^{ibcd}C_{jbcd} = \frac{1}{4}\delta^i_j {}^*C^{abcd}C_{abcd} \quad , \tag{B.4}$$

$$^*C^{ib}{}_{cd}{}^*C^{cd}{}_{ef}{}^*C^{ef}{}_{jb} = \frac{1}{4}\delta^i_j {}^*C^{ab}{}_{cd}{}^*C^{cd}{}_{ef}{}^*C^{ef}{}_{ab} \quad , \tag{B.5}$$

starting with the dual version of (B.1).

Note that the scalars on the right of the equations (B.2) - (B.5) are four real invariants of the Weyl tensor [33]; since these are independent they can be chosen as the basis for the invariants.

Also note that (B.2) is the only identity with two free indices one can construct from (B.1) involving a product of two Weyl tensors.

B.2 Five-dimensional identities

In five dimensions, from Theorem B.0.1, the analogue to (B.1) is

$$C^{[ab}{}_{[cd}\delta^e_h\delta^{f]}_{i]} = 0 \quad , \tag{B.6}$$

which has eight free indices. Note that our "five-dimensional identity" (B.6) is also valid for four dimensions and that clearly we cannot get a quadratic expression for $C^{ab}{}_{cd}$ involving only two free indices.

The *only identity* with two free indices cubic in the Weyl tensor is given by

$$C^{ab}{}_{pe}C^{gp}{}_{cd}C_{[ab}{}^{[cd}\delta^e_g\delta^{f]}_{h]} = 0 \quad . \tag{B.7}$$

Expanding, collecting, and using the trace-free property gives

$$\begin{aligned} &C^{af}{}_{bc}C^{bc}{}_{de}C^{de}{}_{ah} - 2C^{af}{}_{bh}C^{bc}{}_{de}C^{de}{}_{ac} - 4C^{af}{}_{bc}C^{bd}{}_{eh}C^{ce}{}_{ad} \\ &= \frac{1}{5}\Big(C^{ab}{}_{cd}C^{cd}{}_{eg}C^{eg}{}_{ab} - 4C^{ab}{}_{cd}C^{ce}{}_{ag}C^{dg}{}_{be}\Big)\delta^f{}_h \end{aligned} \tag{B.8}$$

and the structures of (B.8) and (B.2) appear to be analogous. However, it turns out that the right hand side of (B.7) is identically zero for $n \leq 5$. This can be realized by looking at the five-dimensional scalar identity closely related to (B.7)

$$C^{ab}{}_{cd}C^{gh}{}_{ef}C_{[ab}{}^{[cd}\delta^{e}_{g}\delta^{f]}_{h]} = 0 \quad , \tag{B.9}$$

which gives[1]

$$4C^{ab}{}_{cd}C^{ce}{}_{af}C^{df}{}_{be} = C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{ab} \quad . \tag{B.10}$$

Since we are interested in an identity with two free indices this case is not useful for our purpose.

Nevertheless we note that (B.8) and (B.10) yields an interesting 2-index five-dimensional identity which will be useful in other contexts,

$$C^{af}{}_{bc}C^{bc}{}_{de}C^{de}{}_{ah} - 2C^{af}{}_{bh}C^{bc}{}_{de}C^{de}{}_{ac} - 4C^{af}{}_{bc}C^{bd}{}_{eh}C^{ce}{}_{ad} = 0 \quad . \tag{B.11}$$

On the other hand, if we consider a quartic identity in five dimensions,

$$C^{ijkg}C_{ijke}C^{ab}{}_{cd}C_{[ab}{}^{[cd}\delta^{e}_{g}\delta^{f]}_{h]} = 0 \quad , \tag{B.12}$$

we obtain

$$\begin{aligned} &5C^{ijkf}C_{ijkh}C^{ab}{}_{cd}C^{cd}{}_{ab} \\ &-8C^{ijkg}C_{ijkh}C^{ab}{}_{cg}C^{cf}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{gh}C^{ef}{}_{ab} \\ &+8C^{ijkg}C_{ijke}C^{af}{}_{bh}C^{be}{}_{ag} - 8C^{ijkg}C^{ae}{}_{bh}C_{ijke}C^{bf}{}_{ag} \\ &= (C^{ijkg}C_{ijkg}C^{ab}{}_{cd}C^{cd}{}_{ab} - 4C^{ijkg}C_{ijke}C^{ab}{}_{cg}C^{ce}{}_{ab})\delta^{f}{}_{h} \quad . \end{aligned} \tag{B.13}$$

which again appears to have analogous structure to (B.2). To determine if the right hand side of (B.13) is identically zero, one could either look on all of the quartic scalar identities or try to find a counter example. Unlike in the cubic case where there was only one possible scalar identity (B.10), there will be a number of quartic scalar identities in five dimensions, so rather than starting to find all quartic scalar identities, a simple counter example[2] shows that the right hand side cannot be identically zero.

B.3 Six-dimensional identities

In six dimensions, from Theorem B.0.1, the analogue to (B.1) and (B.6) is

$$C^{[ab}{}_{[cd}\delta^{e}_{h}\delta^{f}_{i}\delta^{g]}_{j]} = 0 \tag{B.14}$$

[1] This five-dimensional scalar identity was also noted in [24], where it was obtained from the five-dimensional identity $C^{ab}{}_{[cd}C^{cd}{}_{ef}C^{ef}{}_{ab]} = 0$.

[2] Using the Maple package GRTensorII [18] with the metric
$ds^2 = \frac{dr^2}{1-\frac{1}{2m}-\frac{\Lambda r^2}{3}+\frac{e^2}{r^2}} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 - \left(1-\frac{1}{2m}-\frac{\Lambda r^2}{3}+\frac{e^2}{r^2}\right)dt^2 + dx^2$
provides such a counterexample.

and this expression has 10 free indices. Hence there is only one possibility to create an identity with two free indices involving three Weyl tensors. Expanding (B.14) and multiplying with $C_{ab}{}^{cd}C_{ef}{}^{hi}$ we get

$$\begin{aligned} &C_{aj}{}^{bc}C^{ag}{}_{de}C_{bc}{}^{de} - 2C_{aj}{}^{bg}C^{ac}{}_{de}C_{bc}{}^{de} - 4C_{aj}{}^{bc}C^{dg}{}_{be}C_{cd}{}^{ae} \\ &\quad = \frac{1}{6}\Big(C_{ab}{}^{cd}C_{ef}{}^{ab}C_{cd}{}^{ef} - 4C_{ab}{}^{cd}C^{ae}{}_{cf}C_{de}{}^{bf}\Big)\delta^{g}{}_{j} \quad . \end{aligned} \tag{B.15}$$

Unlike for the case of the five-dimensional 2-index identity (B.8), there is no related scalar identity cubic in the Weyl tensor, analogous to (B.9). So therefore we do not need to worry about the possibility of the right hand side of (B.15) being identically zero, but we confirm this fact by a counterexample[3], and so we have an identity which has precisely the structure (B.2).

B.4 Lovelock's quartic six-dimensional identity

There are few explicit examples in the literature of identities for the Weyl tensor in higher dimensions, but in the original paper [29] introducing dimensionally dependent identities Lovelock gives an explicit example of a six-dimensional 2-index tensor identity quartic in the Weyl tensor. It involves the double three-form, $H_{abc}{}^{efg}$,

$$\begin{aligned} H_{abc}{}^{def} &= H_{[abc]}{}^{[def]} \quad , \\ H_{abc}{}^{dec} &= 0 \quad , \end{aligned} \tag{B.16}$$

which in six dimensions (and lower) satisfies the identity

$$H_{abi}{}^{def}H_{def}{}^{abj} = \frac{1}{6}\delta_i^j H_{abc}{}^{def}H_{def}{}^{abc} \quad . \tag{B.17}$$

Making the choice

$$\begin{aligned} H_{ijk}{}^{abc} =& A_{ijk}{}^{abc} - \frac{9}{4(n-4)}A_{r[jk}{}^{r[bc}\delta^{a]}{}_{i]} \\ &+ \frac{18}{(n-4)(n-3)}A_{rs[i}{}^{rs[c}\delta^{a}{}_{j}\delta^{b]}{}_{k]} \\ &+ \frac{6}{(2-n)(3-n)(4-n)}A_{rst}{}^{rst}\delta^{[a}{}_{[i}\delta^{b}{}_{j}\delta^{c]}{}_{k]} \end{aligned} \tag{B.18}$$

[3] Again, using Maple and GRTensorII [18] with the metric
$ds^2 = \frac{dr^2}{1-\frac{1}{2m}-\frac{\Lambda r^2}{3}+\frac{e^2}{r^2}} + r^2 d\theta^2 + r^2\sin^2(\theta)d\phi^2 - \left(1 - \frac{1}{2m} - \frac{\Lambda r^2}{3} + \frac{e^2}{r^2}\right)dt^2 + dx^2 + dy^2$
provides such a counterexample.

(B.17) gives a quartic identity for the Weyl tensor with

$$A_{ijk}{}^{abc} = 4C_{[ij}{}^{h[a}C_{k]h}{}^{bc]} \quad . \tag{B.19}$$

Substituting (B.18) into (B.17) reveals

$$\begin{aligned}
&A_{abi}{}^{cde}A_{cde}{}^{abj} + 3A_{abi}{}^{abc}A_{cde}{}^{dej} + 6A_{abi}{}^{acd}A_{cde}{}^{bej} \\
&- 3A_{abc}{}^{ade}A_{dei}{}^{bcj} - A_{abc}{}^{abc}A_{dei}{}^{dej} + 6A_{abc}{}^{abd}A_{dei}{}^{cej} \\
&+ \frac{1}{6}A_{abc}{}^{abc}A_{def}{}^{def}\delta^j{}_i + \frac{3}{2}A_{abc}{}^{ade}A_{def}{}^{bcf}\delta^j{}_i \\
&- \frac{3}{2}A_{abc}{}^{abd}A_{def}{}^{cef}\delta^j{}_i - \frac{1}{6}A_{abc}{}^{def}A_{def}{}^{abc}\delta^j{}_i = 0 \quad ,
\end{aligned} \tag{B.20}$$

i.e., an identity of the structure analogous to (B.2).

Appendix C

Weyl scalar invariants

In this Appendix in four dimensions we derive relations between the two complex scalar invariants of the Weyl spinor and the four real standard invariants of the Weyl tensor. It will be useful first to recollect some notation:

$\mathbf{C}, \mathbf{C}^n$	(bold letter) a matrix and a power of a matrix
$[\mathbf{C}^n]$	the trace of the matrix $\mathbf{C}^n$
$\mathbf{0}$	the zero matrix
$\mathbf{I}$	the identity matrix
$C[2]^{ab}{}_{cd} = C^{ab}{}_{ij}C^{ij}{}_{cd}$, $C[n]^{ab}{}_{cd} = \ldots$	a chain of n Weyl tensors
$C[2] = C^{ab}{}_{cd}C^{cd}{}_{ab}$, $C[n] = \ldots$	the double trace taken over a chain of n Weyl tensors

C.1 Weyl scalar invariants in 4 dimensions.

In four dimensions, we know from Appendix A that we can consider the Weyl tensor $C^{ab}{}_{cd}$ as a trace-free 6×6 matrix, $\mathbf{C}$, noting that this representation will yield the five independent trace invariants $[\mathbf{C}^2]$, $[\mathbf{C}^3]$, $[\mathbf{C}^4]$, $[\mathbf{C}^5]$ and $[\mathbf{C}^6]$; but because of the Cayley-Hamilton theorem

$$\mathbf{C}^6 - \frac{1}{2}[\mathbf{C}^2]\mathbf{C}^5 - \frac{1}{3}[\mathbf{C}^3]\mathbf{C}^4 - \frac{1}{4}\left([\mathbf{C}^4] - \frac{1}{2}\left([\mathbf{C}^2]\right)^2\right)\mathbf{C}^3$$
$$- \frac{1}{5}\left([\mathbf{C}^5] - \frac{5}{6}[\mathbf{C}^2][\mathbf{C}^3]\right)\mathbf{C}$$
$$- \frac{1}{6}\left([\mathbf{C}^6] - \frac{3}{4}[\mathbf{C}^2][\mathbf{C}^4] - \frac{1}{3}\left([\mathbf{C}^3]\right)^2 + \frac{1}{8}\left([\mathbf{C}^2]\right)^3\right)\mathbf{I} = \mathbf{0} \quad , \qquad \text{(C.1)}$$

any higher trace invariant will be dependent on these five. These five trace invariants translate directly into the five Weyl scalar invariants $C[2], C[3]$, $C[4], C[5]$ and $C[6]$ as follows

$$
\begin{aligned}
&[\mathbf{C}^2] = \mathbf{C}^A{}_C \mathbf{C}^C{}_A = C^{ab}{}_{cd} C^{cd}{}_{ab} = C[2], \\
&[\mathbf{C}^3] = \mathbf{C}^A{}_C \mathbf{C}^C{}_E \mathbf{C}^E{}_A = C^{ab}{}_{cd} C^{cd}{}_{ef} C^{ef}{}_{ab} = C[3], \\
&\ldots, \\
&[\mathbf{C}^6] = \mathbf{C}^A{}_C \mathbf{C}^C{}_E \ldots \mathbf{C}^K{}_A = C^{ab}{}_{cd} C^{cd}{}_{ef} \ldots C^{kl}{}_{ab} = C[6] \quad . \qquad \text{(C.2)}
\end{aligned}
$$

However it is well known that there are only four independent scalar invariants for the Weyl tensor in four dimensions. This reduction from five to four is due to the first Bianchi identity $C_{a[bcd]} = 0$, which is not incorporated in the 6×6 matrix representation; so clearly there must be one relationship between the five invariants $C[2], C[3], C[4], C[5]$ and $C[6]$.

An alternative, more efficient, representation for the Weyl tensor is via the trace-free complex 3×3 matrix, $\boldsymbol{\mathcal{C}}$ which represents $\frac{1}{2}\left(C^{ab}{}_{cd} + i^* C^{ab}{}_{cd}\right)$, where ${}^*C^{ab}{}_{cd} = \frac{1}{2}\eta^{abij} C_{ijcd}$ is the dual tensor to $C^{ab}{}_{cd}$.

Note that for the Weyl tensor

$$
{}^*C_{abcd} = C^*{}_{abcd}, \quad {}^{**}C_{abcd} = {}^*C^*{}_{abcd} = -C_{abcd} \qquad \text{(C.3)}
$$

and that

$$
\begin{aligned}
{}^*C^{abcd\,*}C_{efgh} =& \frac{1}{2}\eta^{abij} C_{ij}{}^{cd} \frac{1}{2}\eta_{efpq} C^{pq}{}_{gh} \\
=& -6 g_e{}^{[a} g_f{}^b g_p{}^i g_q{}^{j]} C_{ij}{}^{cd} C^{pq}{}_{gh} \\
=& -C_{ef}{}^{cd} C^{ab}{}_{gh} + C_{pf}{}^{cd} C^{pb}{}_{gh} g_e{}^a \\
& - C_{ep}{}^{cd} C^{bp}{}_{gh} g_f{}^a + C_{ep}{}^{cd} C^{ap}{}_{gh} g_f{}^b + C_{pf}{}^{cd} C^{ap}{}_{gh} g_e{}^b \\
& + \frac{1}{2} C_{pq}{}^{cd} C^{pq}{}_{gh} g_e{}^b g_f{}^a - \frac{1}{2} C_{pq}{}^{cd} C^{pq}{}_{gh} g_e{}^a g_f{}^b \quad . \qquad \text{(C.4)}
\end{aligned}
$$

We are using the property that an expression with two duals (or more generally any even number of duals) can be written equivalently without *any* duals. For instance it follows from (C.4) that

$$
{}^*C^{ab}{}_{cd}{}^*C^{cd}{}_{ab} = -C^{ab}{}_{cd} C^{cd}{}_{ab} \quad . \qquad \text{(C.5)}
$$

The trace-free property of the complex matrix $\boldsymbol{\mathcal{C}}$ incorporates the trace-free property of the Weyl tensor as well as the first Bianchi identity since

$$
{}^*C^{ab}{}_{bd} = \frac{1}{2}\eta^{abij} C_{ijbd} = \frac{1}{2}\eta^{abij} C_{[ijb]d} \quad .
$$

Due to the Cayley-Hamilton Theorem for the matrix trace-free $\boldsymbol{\mathcal{C}}$ (A.7),

$$c_o\boldsymbol{\mathcal{C}}^3 + c_2\boldsymbol{\mathcal{C}} + c_3\mathbf{I} = 0 \quad ,$$

there are only two complex trace invariants of this complex three-dimensional trace-free matrix, $I = [\boldsymbol{\mathcal{C}}^2]$ and $J = [\boldsymbol{\mathcal{C}}^3]$, and so four real invariants are obtained directly. (These two complex invariants also follow easily from spinor considerations, as shown in Appendix A)

Clearly there must be simple direct relationships between I, J and $C[2]$, $C[3]$, $C[4]$,$C[5]$ and $C[6]$, and just writing out I, J in terms of the Weyl tensor and its dual we get

$$\begin{aligned} I =&[\boldsymbol{\mathcal{C}}^2] = \frac{1}{4}\big(C^{ab}{}_{cd} + i^*C^{ab}{}_{cd}\big)\big(C^{cd}{}_{ab} + i^*C^{cd}{}_{ab}\big) \\ =&\frac{1}{2}\Big(C^{ab}{}_{cd}C^{cd}{}_{ab} + iC^{ab}{}_{cd}{}^*C^{cd}{}_{ab}\Big) \end{aligned} \tag{C.6}$$

and

$$\begin{aligned} J =&[\boldsymbol{\mathcal{C}}^3] = \frac{1}{8}\big(C^{ab}{}_{cd} + i^*C^{ab}{}_{cd}\big)\big(C^{cd}{}_{ef} + i^*C^{cd}{}_{ef}\big)\big(C^{ef}{}_{ab} + i^*C^{ef}{}_{ab}\big) \\ =&\frac{1}{2}\Big(C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{ab} - i^*C^{ab}{}_{cd}{}^*C^{cd}{}_{ef}{}^*C^{ef}{}_{ab}\Big) \quad . \end{aligned} \tag{C.7}$$

We would now like to eliminate the duals in these expressions and we do this by using (C.4). So, for the complex term in (C.6) we find,

$$\begin{aligned} \big({}^*C_{ab}{}^{cd}C_{cd}{}^{ab}\big)^2 =&{}^*C_{ab}{}^{cd}C_{cd}{}^{ab*}C_{ef}{}^{gh}C_{gh}{}^{ef} \\ =&-2C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{gh}C^{gh}{}_{ab} - 4C^{ab}{}_{cd}C^{efcd}C_{afgh}C_{be}{}^{gh} \\ =&-2C[4] - 4C^{ab}{}_{cd}C^{efcd}C_{afgh}C_{be}{}^{gh} \quad . \end{aligned} \tag{C.8}$$

Since it is known that there are *only* four scalar invariants of the Weyl tensor in four dimensions, the second term must be expressible in terms of $C[2], C[3]$, $C[4], C[5], C[6]$. Note that since this term is quartic in the Weyl tensor it must be a linear combination of $(C[2])^2$ and $C[4]$. To find this linear combination

$$C^{ab}{}_{cd}C^{efcd}C_{afgh}C_{be}{}^{gh} = \alpha\ C[4] + \beta\Big(C[2]\Big)^2 \tag{C.9}$$

(α and β constants) we can use dimensionally dependent identities or we can start by making the ansatz (C.9), calculate[1] the expression for a few metrics, and thereby determine α and β.

Doing this we find

$$C^{ab}{}_{cd}C^{efcd}C_{afgh}C_{be}{}^{gh} = \frac{1}{2}\ C[4] - \frac{1}{4}\Big(C[2]\Big)^2 \tag{C.10}$$

[1]Here the computer tools are very useful, for example GRTensor II [18].

and hence we get from (C.8) that

$$\left({}^*C_{ab}{}^{cd}C_{cd}{}^{ab}\right)^2 = -4C[4] + \Big(C[2]\Big)^2 \quad . \tag{C.11}$$

Using similar methods as described above we can find an expression for ${}^*C^{ab}{}_{cd}{}^*C^{cd}{}_{ef}{}^*C^{ef}{}_{ab}$ in (C.7) in terms of $C[2], C[3]$, $C[4], C[5], C[6]$

$$\left({}^*C_{ab}{}^{cd*}C_{cd}{}^{ef*}C_{ef}{}^{ab}\right)^2 = -6C[6] + \Big(C[3]\Big)^2 + \frac{9}{2}C[2]C[4] - \frac{3}{4}\Big(C[2]\Big)^3 \tag{C.12}$$

and also

$$\left({}^*C_{ab}{}^{cd*}C_{cd}{}^{ef*}C_{ef}{}^{ab}\right)\left({}^*C_{gh}{}^{ij}C_{ij}{}^{gh}\right) = \frac{12}{5}C[5] - C[2]C[3] \quad . \tag{C.13}$$

From (C.11), (C.12) and (C.13) we now find

$$\begin{aligned} I =&[\boldsymbol{\mathcal{C}}^2] = \frac{1}{4}\left(C^{ab}{}_{cd} + i^*C^{ab}{}_{cd}\right)\left(C^{cd}{}_{ab} + i^*C^{cd}{}_{ab}\right) \\ =&\frac{1}{2}\Big(C^{ab}{}_{cd}C^{cd}{}_{ab} + iC^{ab}{}_{cd}{}^*C^{cd}{}_{ab}\Big) = \frac{1}{2}\left(C[2] + i\sqrt{-4C[4] + \Big(C[2]\Big)^2}\right) \end{aligned} \tag{C.14}$$

and

$$\begin{aligned} J =&[\boldsymbol{\mathcal{C}}^3] = \frac{1}{8}\left(C^{ab}{}_{cd} + i^*C^{ab}{}_{cd}\right)\left(C^{cd}{}_{ef} + i^*C^{cd}{}_{ef}\right)\left(C^{ef}{}_{ab} + i^*C^{ef}{}_{ab}\right) \\ =&\frac{1}{2}\Big(C^{ab}{}_{cd}C^{cd}{}_{ef}C^{ef}{}_{ab} - i^*C^{ab}{}_{cd}{}^*C^{cd}{}_{ef}{}^*C^{ef}{}_{ab}\Big) \\ =&\frac{1}{2}\left(C[3] - i\sqrt{-6C[6] + \Big(C[3]\Big)^2 + \frac{9}{2}C[2]C[4] - \frac{3}{4}\Big(C[2]\Big)^3}\right) \quad . \end{aligned} \tag{C.15}$$

It follows directly that

$$I\bar{I} = -C[4] + \frac{1}{2}\Big(C[2]\Big)^2 \tag{C.16}$$

$$J\bar{J} = -\frac{3}{2}C[6] + \frac{1}{2}\Big(C[3]\Big)^2 + \frac{9}{8}C[2]C[4] - \frac{3}{16}\Big(C[2]\Big)^3 \quad . \tag{C.17}$$

Using

$$\begin{aligned} &\Big(\left({}^*C_{ab}{}^{cd*}C_{cd}{}^{ef*}C_{ef}{}^{ab}\right)\left({}^*C_{gh}{}^{ij}C_{ij}{}^{gh}\right)\Big)^2 \\ &= \left({}^*C_{ab}{}^{cd}C_{cd}{}^{ab}\right)^2\left({}^*C_{ab}{}^{cd*}C_{cd}{}^{ef*}C_{ef}{}^{ab}\right)^2 \end{aligned} \tag{C.18}$$

we obtain the relationship between $C[2], C[3], C[4], C[5]$ and $C[6]$,

$$\left(\frac{12}{5}C[5] - C[2]C[3]\right)^2$$
$$= \left(- 6C[6] + \left(C[3]\right)^2 + \frac{9}{2}C[2]C[4] - \frac{3}{4}\left(C[2]\right)^2\right)\left(- 4C[4] + \left(C[2]\right)^2\right). \tag{C.19}$$

This confirms that $C[2], C[3], C[4], C[5]$ and $C[6]$ are not independent.

From the Cayley-Hamilton Theorem for a 6×6 trace-free matrix we have an expression for $\det(\mathbf{C})$ in terms of $[\mathbf{C}^2], [\mathbf{C}^3], [\mathbf{C}^4], [\mathbf{C}^5]$ and $[\mathbf{C}^6]$

$$\det(\mathbf{C}) = -\frac{1}{6}\left([\mathbf{C}^6] - \frac{3}{4}[\mathbf{C}^2][\mathbf{C}^4] - \frac{1}{3}\left([\mathbf{C}^3]\right)^2 + \frac{1}{8}\left([\mathbf{C}^2]\right)^3\right) \quad ,$$

which enables us to obtain an expression in terms of $C[2], C[3], C[4]$, $C[5]$ and $C[6]$

$$\det(\mathbf{C}) = -\frac{1}{6}\left(C[6] - \frac{3}{4}C[2]C[4] - \frac{1}{3}\left(C[3]\right)^2 + \frac{1}{8}\left(C[2]\right)^3\right) \quad .$$

Hence we can confirm the relationship between $\det(\mathbf{C})$ and J,

$$\det(\mathbf{C}) = \frac{1}{9}J\bar{J} \quad . \tag{C.20}$$

Appendix D

Computer tools

In this thesis we have carried out some long calculations, often with expressions involving several hundreds of terms. To perform these calculations two software packages have proven particularly useful, GRTensor II and Tensign. In this appendix we give a short introduction to each of them and how they have been used.

D.1 GRTensor II

On http://grtensor.phy.queensu.ca/ or http://grtensor.org/ , where also GRTensor II can be downloaded free of charge, one can read:

"GRTensor II is a computer algebra package for performing calculations in the general area of differential geometry. Its purpose is the calculation of tensor components on curved spacetimes specified in terms of a metric or set of basis vectors. The package contains a library of standard definitions of a large number of commonly used curvature tensors, as well as the Newman-Penrose formalism. The standard object libraries are easily expandable by a facility for defining new tensors. Calculations can be carried out in spaces of arbitrary dimension, and in multiple spacetimes simultaneously. Though originally designed for use in the field of general relativity, GRTensorII is useful in many other fields. GRTensor II is not a stand alone package, but requires an algebraic engine. The program was originally developed for MapleV.GRTensorII runs with all versions of Maple, Maple V Release 3 to Maple 9.5. A limited version (GRTensorM) has been ported to Mathematica."

In this thesis we have mainly used GRTensor II to prove negative results via explicit counterexamples.

D.2 Tensign

Tensign is a program developed by Anders Höglund for handling and manipulating indices associated with tensor expressions. The program aids the user in keeping track of all details in calculations, such as coefficients, symmetries, signs, trace properties etc., and is an invaluable tool for simplifying, expanding and rewriting tensor expressions.

In Tensign one can define tensors with any kind of symmetries and of arbitrary order. In addition, all other properties, e.g., relations to other tensors or trace properties, are defined by rules which can be applied to an expression or part of an expression. A defined rule can only be applied to a specific type of tensor, making it impossible to make mistakes in the calculations.

One of the strengths with Tensign is that the user has full control over the calculation at all times, and all operations are, and *have* to be, made "by hand" – the program is not "smart" in the sense that it does any calculations, simplifications or other operations by itself.

In this thesis Tensign was used to derive the identities in Appendix B and the conformal calculations in Chapter 4

To learn more about Tensign, its interface, etc, see either [23] or visit the homepage http://www.lysator.liu.se/~andersh/tensign/ .

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