On the Equivariance of the Orientation and the Tensor Field Representation

Klas Nordberg Hans Knutsson Gösta Granlund
Computer Vision Laboratory, Department of Electrical Engineering
Linköping University, S-581 83 LINKÖPING, SWEDEN
Phone: +46-13-28 16 34, Telefax: +46-13-13 85 26

Abstract

The tensor representation has proven a successful tool as a mean to describe local multi-dimensional orientation. In this respect, the tensor representation is a map from the local orientation to a second order tensor. This paper investigates how variations of the orientation are mapped to variation of the tensor, thereby giving an explicit equivariance relation. The results may be used in order to design tensor based algorithms for extraction of image features defined in terms of local variations of the orientation, e.g. multi-dimensional curvature or circular symmetries. It is assumed that the variation of the local orientation can be described in terms of an orthogonal transformation group. Under this assumption a corresponding orthogonal transformation group, acting on the tensor, is constructed. Several correspondences between the two groups are demonstrated.

1 Introduction

The tensor representation for orientation was introduced by [Knutsson, 1989] as a tool for managing orientation representation of images with dimensionality greater than two. The representation may be employed for arbitrary dimensionality, even though theoretical investigations and practical implementations have been carried out only for images of dimensionality two, three and four, see [Knutsson et al., 1992a] and [Knutsson et al., 1992b]. The main idea is to let the eigensystem of a symmetric and positive semidefinite tensor, in practice corresponding to an $n \times n$ matrix, represent the orientation structure of a neighbourhood. As a simple example, consider the case of two-dimensional images. The local orientation of an image neighbourhood is then represented by a 2×2 tensor $\mathbf{T}$. Due to its symmetry, the tensor can be decomposed as

$$\mathbf{T} = \lambda_1 \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^\star + \lambda_2 \, \hat{\mathbf{e}}_2 \, \hat{\mathbf{e}}_2^\star, \tag{1}$$

where $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$ are orthonormal eigenvectors of $\mathbf{T}$ with corresponding eigenvalues $\{\lambda_1, \lambda_2\}$. The positive semidefiniteness of $\mathbf{T}$ implies that the eigenvalues can be ordered such that $\lambda_1 \geq \lambda_2 \geq 0$. The tensor representation suggested by [Knutsson, 1989] uses the eigenvalues of $\mathbf{T}$ to describe both the energy content and the orientation structure of the neighbourhood and it uses the eigenvectors to describe the direction of the orientation. This is exemplified with the following three ideal cases.

- $\lambda_1 = \lambda_2 > 0$. The neighbourhood is isotropic, i.e. does not contain any oriented structure.
- $\lambda_1 > 0, \ \lambda_2 = 0$. The neighbourhood contains a dominant orientation which is perpendicular to $\hat{\mathbf{e}}_1$.
- $\lambda_1 = \lambda_2 = 0$. The neighbourhood contains no energy.

The three-dimensional case is almost as simple. Here, the local orientation is represented by a 3×3 tensor $\mathbf{T}$ which is decomposed as

$$\mathbf{T} = \lambda_1 \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^\star + \lambda_2 \, \hat{\mathbf{e}}_2 \, \hat{\mathbf{e}}_2^\star + \lambda_3 \, \hat{\mathbf{e}}_3 \, \hat{\mathbf{e}}_3^\star. \tag{2}$$

Again, $\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ are orthonormal eigenvectors of $\mathbf{T}$ with corresponding eigenvalues $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0$. The orientation representation is exemplified with the following four ideal cases.

- $\lambda_1 = \lambda_2 = \lambda_3 > 0$. The neighbourhood is isotropic.
- $\lambda_1 = \lambda_2 > 0, \ \lambda_3 = 0$. The neighbourhood contains iso-curves with one dominant orientation which is perpendicular to the plane spanned by $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$.

- $\lambda_1 > 0,\ \lambda_2 = \lambda_3$. The neighbourhood contains iso-surfaces with one dominant orientation. The iso-surfaces are perpendicular to $\hat{\mathbf{e}}_1$.
- $\lambda_1 = \lambda_2 = \lambda_3 = 0$. The neighbourhood contains no energy.

[Knutsson et al., 1992a] describes how tensors with the above characteristics may be constructed using the responses from quadrature filters.

In general, an arbitrary variation of the orientation structure will be reflected both in the eigenvalues and the eigenvectors of $\mathbf{T}$. As an example, the variation may indicate a transition from a line to a plane structure in three dimensions. In the following, however, it is assumed that there is no variation in the eigenvalues of $\mathbf{T}$, implying that the character of the orientation structure is constant and only the orientation changes. Furthermore it is assumed that this variation can be described as some type of transformation, $\mathbf{A}$, acting on the eigenvectors of the tensor, i.e. acting on the orientation. As an example, consider a two-dimensional image containing a circle. An image neighbourhood on the circle will then contain a dominant orientation which is described by a vector $\hat{\mathbf{e}}$ perpendicular to the circle segment and, hence, represented by a tensor $\mathbf{T} = \hat{\mathbf{e}}\hat{\mathbf{e}}^\star$. When moving along the circle, the dominant orientation will change with a speed determined by the radius of the circle. In fact, this variation can be described as a rotation of the vector $\hat{\mathbf{e}}$. Consequently, also the representation tensors along the circle will vary somehow. The relations between the variation of the vectors and that of the tensors is, however, not apparent.

Generally, $\mathbf{A}$ may be of arbitrary type, but in the following section it is assumed to be a linear operator, corresponding to an $n \times n$ matrix. For instance, in the above example $\mathbf{A}$ would be represented by a rotation matrix. Let $\{\hat{\mathbf{e}}_k\}$ denote the eigenvectors of $\mathbf{T}$. According to the above, $\{\hat{\mathbf{e}}_k\}$ will change between two image points, x_0 to x_1, as

$$\{\hat{\mathbf{e}}_k\}_{x_1} = \mathbf{A}\{\hat{\mathbf{e}}_k\}_{x_0}. \tag{3}$$

The representation tensor, $\mathbf{T}$, is a function of the eigensystem, i.e.

$$\mathbf{T} = \mathbf{T}(\{\hat{\mathbf{e}}_k\}), \tag{4}$$

and will therefore transform according to

$$\mathbf{T}_{x_1} = \mathbf{T}(\mathbf{A}\{\hat{\mathbf{e}}_k\}_{x_0}). \tag{5}$$

Since $\mathbf{T}$ is a linear combination of outer products of the eigenvectors, Equation (5) may be rewritten as

$$\mathbf{T}_{x_1} = \mathbf{A}\,\mathbf{T}_{x_0}\,\mathbf{A}^\star, \tag{6}$$

Though correct, this description of how $\mathbf{T}$ transforms is of little practical use and it would be much more convenient to find an operator $\mathcal{A}$, corresponding to $\mathbf{A}$, such that

$$\mathbf{T}_{x_1} = \mathcal{A}[\,\mathbf{T}_{x_0}\,]. \tag{7}$$

The concept of *equivariance* was introduced by [Wilson and Knutsson, 1988] and [Wilson and Spann, 1988] in order to make a formal theory for feature representation. It implies that transformations of a feature are reflected in transformations of the representation. In view of the previous discussion, $\mathbf{A}$ and $\mathcal{A}$ form such a pair of transformations called *equivariance operators*. The purpose of this paper is to establish a pair of equivariance operators for the tensor representation of orientation. The results may be used in order to design tensor based algorithms for extraction of features defined in terms of local variation of the orientation, e.g. curvature or circular symmetries.

2 Derivation of results

Let V and $V^\star$ denote a vector space and its corresponding dual space, both of the type $\mathbb{R}^n$ for some integer n. The vector space $V \overset{*}{\otimes} V$ is then the set of all linear maps from from V to itself. For convenience, elements of V are referred to as vectors whereas elements of $V \overset{*}{\otimes} V$ are referred to as tensors. Let $\mathbf{v}$ be an arbitrary vector and define

$$\mathbf{T} = \mathbf{v}\,\mathbf{v}^\star, \tag{8}$$

where the $\star$-sign indicates transpose. This implies that $\mathbf{T}$ is a tensor. We will now consider the case where $\mathbf{v}$ is a function of a real variable x, defined as

$$\mathbf{v} = \mathbf{v}(x) = e^{x\mathbf{H}}\,\mathbf{v}_0, \tag{9}$$

where $\mathbf{H}$ is an anti-Hermitian tensor. The exponential function is here defined in terms of the familiar Taylor series,

$$e^{x\mathbf{H}} = \mathbf{I} + x\mathbf{H} + \tfrac{x^2}{2}\,\mathbf{H}^2 + \ldots \tag{10}$$

valid for any tensor $\mathbf{H}$. This functions has the interesting property of mapping anti-Hermitian tensors to orthogonal tensors, see e.g. [Nordberg, 1992]. Hence, for $x \in \mathbb{R}$, Equation (10) defines a continuous set of orthogonal tensors which in fact forms a group.

The motivation for introducing the exponential function is that it provides a mean to realize both rotations and other quite general orthogonal transformations. It can be proved, see e.g. [Nordberg, 1992], that any $n \times n$ anti-Hermitian tensor, $\mathbf{H}$, can be decomposed as

$$\mathbf{H} = \sum_{k=1}^{n} i\,\lambda_k\,\mathbf{f}_k\,\mathbf{f}_k^\star, \tag{11}$$

where $\{\mathbf{f}_k,\ k = 1, \ldots, n\}$ are orthonormal eigenvectors of $\mathbf{T}$ with corresponding eigenvalues $i\lambda_k$ where $\lambda_k \in \mathbb{R}$. It should be noted that in general the eigenvectors $\mathbf{f}_k$ are complex implying that the $\star$-operation also must include complex conjugation. Furthermore, these vectors are not proper elements of V but rather a complexified version thereof. For all cases of interest for this paper, $\mathbf{H}$ is real which implies that its eigenvectors as well as their corresponding eigenvalues come in complex conjugate pairs. Furthermore, the real and imaginary part of the eigenvectors are orthogonal, at least for eigenvectors with non-zero eigenvalues. Hence, each pair of complex conjugate eigenvectors, $\mathbf{f}_k$ and $\bar{\mathbf{f}}_k$, will define a two-dimensional subspace of V. The corresponding eigenvalues, $\pm i\lambda_k$, will then determine the relative speed by which $e^{x\mathbf{H}}$ rotates the projection of $\mathbf{v}$ on the subspace. In practice, we are often interested in operators $e^{x\mathbf{H}}$ which are periodic in the parameter x. If normalizing the period to 2π, this implies that all λ_k are integers. For more details on this subject see e.g. [Nordberg, 1992]. Hence, it is possible to construct an orthogonal operator which rotates the projection of $\mathbf{v}$ on arbitrary orthogonal two-dimensional subspaces of V, the angular velocity relative to x being arbitrary integers, by choosing $\mathbf{H}$ appropriately.

Insertion of Equation (9) into the right hand side of Equation (8) gives

$$\mathbf{T} = \mathbf{T}(x) = e^{x\mathbf{H}}\,\mathbf{v}_0\,\mathbf{v}_0^\star\,e^{-x\mathbf{H}}. \tag{12}$$

If $\mathbf{v}$ describes the orientation of an image neighbourhood, and Equation (9) describes how the vector changes when moving along some path in the image, then Equation (12) will describe how the orientation tensor changes along the same path. In the form presented by Equation (12), however, the variation of $\mathbf{T}$ with respect to x is quite obscure. The right hand side consists of the matrix product between a variable orthogonal tensor, the tensor $\mathbf{v}_0\mathbf{v}_0^\star$ and the transpose of the first orthogonal tensor. There is, however, another way of expressing this product. Let the parameter x have some fixed value x_0. The mapping $\mathcal{Q} : V \overset{*}{\otimes} V \to V \overset{*}{\otimes} V$, defined as

$$\mathcal{Q}[\,\mathbf{X}\,] = e^{x_0\mathbf{H}}\,\mathbf{X}\,e^{-x_0\mathbf{H}}, \tag{13}$$

is then a linear map. Allowing x to vary implies that $\mathcal{Q}$ is a function of x and suggest the notation $\mathcal{Q}(x)$ instead of $\mathcal{Q}$. The introduction of the map $\mathcal{Q}(x)$ means that the right hand side of Equation (12) can be rewritten as

$$\mathbf{T}(x) = \mathcal{Q}(x)[\,\mathbf{v}_0\,\mathbf{v}_0^\star\,]. \tag{14}$$

Hence, the tensor $\mathbf{T}$ is the image of $\mathbf{v}_0\mathbf{v}_0^\star$ under the map $\mathcal{Q}(x)$.

We will now investigate the structure of $\mathcal{Q}$. A scalar product on $V \overset{*}{\otimes} V$ is defined by the function $s : V \overset{*}{\otimes} V \times V \overset{*}{\otimes} V \to \mathbb{R}$, where

$$s(\mathbf{X}, \mathbf{Y}) = \mathrm{trace}[\,\mathbf{X}^\star\mathbf{Y}\,]. \tag{15}$$

This gives

$$\begin{aligned}
&s(\mathcal{Q}(x)\,\mathbf{X}, \mathcal{Q}(x)\,\mathbf{Y}) = \\
&\mathrm{trace}[\,(\mathcal{Q}(x)\,\mathbf{X})^\star\,\mathcal{Q}(x)\mathbf{Y}\,] = \\
&\mathrm{trace}[\,(e^{x\mathbf{H}}\,\mathbf{X}\,e^{-x\mathbf{H}})^\star\,e^{x\mathbf{H}}\,\mathbf{Y}\,e^{-x\mathbf{H}}\,] = \\
&\mathrm{trace}[\,e^{x\mathbf{H}}\,\mathbf{X}^\star e^{-x\mathbf{H}}\,e^{x\mathbf{H}}\,\mathbf{Y}\,e^{-x\mathbf{H}}\,] = \\
&\mathrm{trace}[\,e^{-x\mathbf{H}}\,e^{x\mathbf{H}}\,\mathbf{X}^\star\,e^{-x\mathbf{H}}\,e^{x\mathbf{H}}\,\mathbf{Y}\,] = \\
&\mathrm{trace}[\,\mathbf{X}^\star\mathbf{Y}\,] = s(\mathbf{X}, \mathbf{Y}),
\end{aligned} \tag{16}$$

which implies that $\mathcal{Q}(x)$ is an orthogonal transformation on $V \overset{*}{\otimes} V$ for all $x \in \mathbb{R}$. Evidently, $\mathcal{Q}(x)\mathcal{Q}(y) = \mathcal{Q}(x+y)$ which implies that the set of all $\mathcal{Q}$ forms a group under composition of transformations. Hence, if $\mathbf{v}$ is subject to an orthogonal transformation group, then $\mathbf{T}$ is subject to an orthogonal transformation group as well. Let $\mathbf{H}$ be a fixed and anti-Hermitian tensor. A linear map $\mathcal{H} : V \overset{*}{\otimes} V \to V \overset{*}{\otimes} V$, is then defined by

$$\mathcal{H}[\,\mathbf{X}\,] = \mathbf{H}\,\mathbf{X} - \mathbf{X}\,\mathbf{H}. \tag{17}$$

Using the notation $\mathbf{H}^0 = \mathbf{I}$, where $\mathbf{I}$ is the identity tensor, the following equation, which is easily proved by induction, gives an explicit form for repeated applications of $\mathcal{H}$ on $\mathbf{X}$

$$\mathcal{H}^k[\,\mathbf{X}\,] = \sum_{l=0}^{k} \binom{k}{l} \mathbf{H}^{k-l}\,\mathbf{X}\,(-\mathbf{H})^l. \tag{18}$$

This equation is valid for all integers $k \geq 0$, using the convention that $\mathcal{H}^0 = \mathcal{I}$ where $\mathcal{I}$ is the identity map on $V \overset{*}{\otimes} V$. With this results at hand, insertion of Equation (10) into Equation (13) gives

$$\begin{aligned}
&\mathcal{Q}(x)[\,\mathbf{X}\,] = \\
&\left[\sum_{k=0}^{\infty} \frac{x^k}{k!}\,\mathbf{H}^k\right] \mathbf{X} \left[\sum_{l=0}^{\infty} \frac{(-x)^l}{l!}\,\mathbf{H}^l\right] = \\
&\sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{x^{k+l}}{k!\,l!}\,\mathbf{H}^k\,\mathbf{X}\,(-\mathbf{H})^l = \\
&\sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{x^k}{(k-l)!\,l!}\,\mathbf{H}^{k-l}\,\mathbf{X}\,(-\mathbf{H})^l = \\
&\sum_{k=0}^{\infty} \frac{x^k}{k!} \sum_{l=0}^{k} \binom{k}{l} \mathbf{H}^{k-l}\,\mathbf{X}\,(-\mathbf{H})^l = \\
&\left[\sum_{k=0}^{\infty} \frac{x^k}{k!}\,\mathcal{H}^k\right] [\,\mathbf{X}\,] = e^{x\mathcal{H}}\,[\,\mathbf{X}\,].
\end{aligned} \tag{19}$$

Inserted into Equation (14) this gives

$$\mathbf{T}(x) = e^{x\mathcal{H}}[\,\mathbf{v}_0\mathbf{v}_0^\star\,]. \tag{20}$$

Hence, if the vector $\mathbf{v}$ is transformed by the operator $e^{x\mathbf{H}}$, then the tensor $\mathbf{T}$ is transformed by the operator $e^{x\mathcal{H}}$, where $\mathcal{H}$ is defined by Equation (17).

We have now established a correspondence between the transformations of the vector $\mathbf{v}$ and of the tensor $\mathbf{T}$, given by Equations (9), (17) and (20). In this form, however, the correspondence is quite implicit and does not reveal any interesting properties. By examining the eigensystem of $\mathbf{H}$ and $\mathcal{H}$ and how they are related, much more information can be obtained. The eigensystem of $\mathbf{H}$ has already been treated in the text accompanying Equation (11). When considering the eigensystem of $\mathcal{H}$ it is natural to use the term *eigentensor* for any tensor which after the mapping of $\mathcal{H}$ equals itself times a scalar constant. Assume that $\{\mathbf{f}_k,\ k = 1, \ldots, n\}$ and $\{i\lambda_k\}$ is the eigensystem of $\mathbf{H}$. It is then straightforward to prove that

$$\mathcal{H}[\,\mathbf{f}_k\mathbf{f}_l^\star\,] = i\,(\lambda_k - \lambda_l)\,\mathbf{f}_k\mathbf{f}_l^\star. \tag{21}$$

Hence, any tensor of the type $\mathbf{f}_k\mathbf{f}_l^\star$ is an eigentensor of $\mathcal{H}$ with corresponding eigenvalue $i(\lambda_k - \lambda_l)$. In fact, these tensors are the only eigentensors of $\mathcal{H}$. More details on the eigensystem of $\mathcal{H}$ is found in [Nordberg, 1992], Section 5.6. Since the eigenvectors and eigenvalues of $\mathbf{H}$ come in complex conjugate pairs, so must the eigentensors and the corresponding eigenvalues of $\mathcal{H}$ as well. Consequently, $\mathcal{H}$ constitutes an anti-Hermitian map from $V \overset{*}{\otimes} V$ to itself. This result could also have been derived using the scalar product defined by Equation (15). In the same way as each pair of eigenvectors of $\mathbf{H}$ defines a two-dimensional subspace of V, each pair of eigentensors, $\mathbf{f}_k\,\mathbf{f}_l^\star$ and $\bar{\mathbf{f}}_k\,\bar{\mathbf{f}}_l^\star$, will define a two-dimensional subspace of $V \overset{*}{\otimes} V$. Consequently, the operator $e^{x\mathcal{H}}$ rotates the projection of $\mathbf{T}$ on each such subspace with a relative speed determined by $\lambda_k - \lambda_l$.

To summarize, if the transformation properties of $\mathbf{v}$ can be accounted to an orthogonal operator $e^{x\mathbf{H}}$, then $\mathbf{T}$ is transformed by the operator $e^{x\mathcal{H}}$. The eigensystem of the anti-Hermitian tensor $\mathbf{H}$ describes how $\mathbf{v}$ is transformed in terms of how its projections on two-dimensional subspaces of V, defined by the eigenvectors, rotate with a speed relative to x determined by the corresponding eigenvalues. Given the eigensystem of $\mathbf{H}$ it is possible to construct the eigensystem of $\mathcal{H}$. The eigentensors of $\mathcal{H}$ are simply the outer product between any possible choice of two eigenvectors of $\mathbf{H}$, i.e. $\mathbf{f}_k\mathbf{f}_l^\star$, and the corresponding eigenvalues are the differences $i(\lambda_k - \lambda_l)$. The eigentensors, $\mathbf{f}_k\mathbf{f}_l^\star$, will define a number of two-dimensional subspaces of $V \overset{*}{\otimes} V$ and $\mathbf{T}$ will transform according to rotations in each such subspace with relative speed $(\lambda_k - \lambda_l)$.

3 Examples

The previous section showed a correspondence between the transformation of $\mathbf{v}$ and that of $\mathbf{T}$ in terms of the eigensystems of $\mathbf{H}$ and $\mathcal{H}$, two anti-Hermitian mapping in V and $V \overset{*}{\otimes} V$ respectively. In this section the correspondence is exemplified for the cases $n = 2$ and $n = 3$.

The two-dimensional case

Assume $n = 2$. Orthogonal transformations of $\mathbf{v}$ are then simple two-dimensional rotations. With

$$\hat{\mathbf{f}}_1 = [\tfrac{\mathbf{f}_1 + i\mathbf{f}_2}{\sqrt{2}}], \qquad \hat{\mathbf{f}}_2 = [\tfrac{\mathbf{f}_1 - i\mathbf{f}_2}{\sqrt{2}}], \tag{22}$$

where $\mathbf{f}_1$ and $\mathbf{f}_2$ are two arbitrary orthonormal vectors in $\mathbb{R}^2$, the anti-Hermitian tensor $\mathbf{H}$ which defines the transformation of $\mathbf{v}$ can be expressed as

$$\mathbf{H} = i\,\hat{\mathbf{f}}_1\,\hat{\mathbf{f}}_1^\star - i\,\hat{\mathbf{f}}_2\,\hat{\mathbf{f}}_2^\star. \tag{23}$$

The operator $e^{x\mathbf{H}}$ will then rotate any vector in $\mathbb{R}^2$ around the origin by the angle x. Furthermore, the eigenvalues of $\mathbf{H}$ are $\pm i$ and, according to the results from the previous section, this implies that $\mathcal{H}$, the anti-Hermitian map which governs the transformation of $\mathbf{T}$, has the following eigensystem.

Eigentensor	Eigenvalue
$\hat{\mathbf{f}}_1\,\hat{\mathbf{f}}_1^\star$	0
$\hat{\mathbf{f}}_2\,\hat{\mathbf{f}}_2^\star$	0
$\hat{\mathbf{f}}_1\,\hat{\mathbf{f}}_2^\star$	$2i$
$\hat{\mathbf{f}}_2\,\hat{\mathbf{f}}_1^\star$	$-2i$

(24)

It is easy to prove that independently of the choice of $\mathbf{f}_1$ and $\mathbf{f}_2$, this amounts to

Eigentensor	Eigenvalue
$\frac{1}{2}\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$	0
$\frac{1}{2}\begin{pmatrix} 1 & -i \\ i & 1 \end{pmatrix}$	0
$\frac{e^{2i\alpha}}{2}\begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}$	$2i$
$\frac{e^{-2i\alpha}}{2}\begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}$	$-2i$

(25)

where α is a constant determined by the choice of $\mathbf{f}_1$ and $\mathbf{f}_2$. The exponential factors in front of the last two eigentensors can, however, be omitted since it is the eigenspaces of $\mathcal{H}$ which are of interest rather than specific eigentensors. The first eigentensor pair of $\mathcal{H}$, with eigenvalue 0, defines a two-dimensional subspace of $V\overset{*}{\otimes}V$ which is spanned by the tensors

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{26}$$

Since the eigenvalues are 0, this implies that the projection of $\mathbf{T}$ on this subspace is invariant with respect to the parameter x. The second eigentensor pair, with eigenvalues $\pm 2i$, defines a two-dimensional subspace of $V\overset{*}{\otimes}V$ which is spanned by the tensors

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{27}$$

Since the eigenvalues are $\pm 2i$, this implies that the projection of $\mathbf{T}$ on this subspace rotates with twice the speed of $\mathbf{v}$. Hence, the tensor representation of two-dimensional orientation is in fact a type of double angle representation. This representation was introduced by [Granlund, 1978] who suggested that a two-dimensional vector should be used to represent the orientation by constructing the representation vector such that it rotates with twice the speed of the orientation. In the tensor case a 2×2 tensor, corresponding to a four dimensional vector, is used instead and it is the projection of the tensor on a specific two-dimensional subspace which rotates with twice the speed of the orientation.

The three-dimensional case

Assume $n = 3$. Any orthogonal transformation of $\mathbf{v}$ is then described by a two-dimensional plane in which the projection of $\mathbf{v}$ is rotated by the angle x. Let $\{\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3\}$ be an orthonormal of set of vectors such that $\mathbf{f}_1$ and $\mathbf{f}_2$ span the plane of rotation. With $\hat{\mathbf{f}}_1$ and $\hat{\mathbf{f}}_2$ as defined by Equation (22), the anti-Hermitian tensor can be written

$$\mathbf{H} = i\,\hat{\mathbf{f}}_1\,\hat{\mathbf{f}}_1^\star - i\,\hat{\mathbf{f}}_2\,\hat{\mathbf{f}}_2^\star + 0\cdot\mathbf{f}_3\mathbf{f}_3^\star \tag{28}$$

The eigenvalues of $\mathbf{H}$ are thus $\pm i$ and 0. Hence, the eigensystem of $\mathcal{H}$ is

Eigentensor	Eigenvalue
$\hat{\mathbf{f}}_1\hat{\mathbf{f}}_1^\star,\ \hat{\mathbf{f}}_2\hat{\mathbf{f}}_2^\star,\ \mathbf{f}_3\mathbf{f}_3^\star$	0
$\hat{\mathbf{f}}_1\mathbf{f}_3^\star,\ \mathbf{f}_3\hat{\mathbf{f}}_2^\star$	i
$\mathbf{f}_3\hat{\mathbf{f}}_1^\star,\ \hat{\mathbf{f}}_2\mathbf{f}_3^\star$	$-i$
$\hat{\mathbf{f}}_1\hat{\mathbf{f}}_2^\star$	$2i$
$\hat{\mathbf{f}}_2\hat{\mathbf{f}}_1^\star$	$-2i$

(29)

According to the above, the projection of $\mathbf{T}$ on the two-dimensional subspace defined by $\hat{\mathbf{f}}_1\hat{\mathbf{f}}_2^\star$ and $\hat{\mathbf{f}}_2\hat{\mathbf{f}}_1^\star$ will rotate will twice the speed of the parameter x. Hence, in the three-dimensional case, the orthogonal transformation of **T** will depend on the plane of rotation of **v**. As an example consider a cylinder in three dimensions. If cutting the cylinder with a plane perpendicular to its axes of symmetry, the result will be a circle. The normal vectors of the cylinder along the circle will lie in the plane. If moving along the circle, the normal vectors will transform according to a rotation in the plane. The three-dimensional rotation is determined by the three orthonormal vectors $\mathbf{f}_1, \mathbf{f}_2$ and $\mathbf{f}_3$, where the first two span the plane of rotation and the third is perpendicular to it. The orientation tensor, **T**, along the circular path will, according to the above, transform according to rotations in different two-dimensional subspaces of $V \overset{*}{\otimes} V$ with relative speed $2, 1$ and 0. In the case of a cylinder, however, the projection of the tensor on the planes with relative speed 1 and 0 vanishes. The tensor will only have non-vanishing projection on the two-dimensional subspace defined by $\hat{\mathbf{f}}_1 \hat{\mathbf{f}}_2^\star$ and $\hat{\mathbf{f}}_2 \hat{\mathbf{f}}_1^\star$ and this projection rotates with twice the speed of x.

4 Discussion

This paper have demonstrated an equivariance of the orientation and the tensor field representation. The equivariance is based on the assumption that there is one and the same transformation **A** which acts on all eigenvectors of the representation tensor. This transformation is furthermore assumed to form an orthogonal operator group, $e^{x\mathbf{H}}$. These assumptions may of course not valid for any possible type of variation between two adjacent neighbourhoods of an image. If valid, however, then Section 2 have proved the possibility to construct an orthogonal operator group, $e^{x\mathcal{H}}$, which acts on the representation tensor **T**. The latter group then describes the equivalent transformation of **T** relative to the former of the orientation. Both **H** and $\mathcal{H}$ are anti-Hermitian maps, on V and $V \overset{*}{\otimes} V$ respectively, and their eigensystems are closely related. As a general results from Section 2, we see that if $i\lambda$ is the eigenvalue of **H** with largest absolute value then the corresponding value for $\mathcal{H}$ is $2i\lambda$. Hence, in terms of rotations, there is always a projection of **T** on some two-dimensional subspace which rotates with twice the speed compared to the eigenvectors of **T**. In general, there are also projections of **T** which rotates in other subspaces with relative speed less than 2λ.

The tensor representation of orientation has inspired a number of algorithms for detection of local gradients in the tensor field, e.g. [Bårman, 1991], [Westin, 1991] or [Westin and Knutsson, 1992]. These algorithms are based on correlating the tensors in each neighbourhood with a fixed set of tensor filters. By choosing the filters appropriately and combining the filter outputs carefully, estimates of e.g. local curvature can be obtained. The results of this paper suggest that the variation of the local tensor field in an image may be seen as a consequence of the variation of the orientation field. Assuming that the latter is subject to an orthogonal transformation group, it has been proved that the representation tensor is subject to an orthogonal transformation group as well. This implies the possibility of designing algorithms for detection of local variation of orientation, e.g. three-dimensional curvature, based on the transformation characteristics of the representation tensor **T**. For a given feature, defined in terms of local variation of the orientation, it must first be established what transformation group acts on the orientation. If the transformation can be assumed to be orthogonal, the result will then be the eigensystem of **H**. Given this eigensystem, this paper explains how to form $\mathcal{H}$, describing the transformation group of the tensor **T**. Hence, one way of defining new algorithms is to search of image neighbourhoods in which the tensor transforms according to the transformations group $e^{x\mathcal{H}}$. Another way is to estimate the transformation group in each neighbourhood of the image. If the assumption of orthogonal groups is valid, this implies that $\mathcal{H}$ is estimated for each neighbourhood and also that this anti-Hermitian map may be used to represent the variation of the neighbourhood.

5 Acknowledege

This work was financially supported by the Swedish Board of Technical Deveopment.

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