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Optimization in Continuum Flow Problems

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Cover:
An optimized crack pattern in a drying porous material. The method used to calculate the pattern maximizes the moisture transport, from a high concentration reservoir underneath the cracked surface, towards the surroundings with a low moisture concentration.

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Abstract

This thesis concerns model development and optimization of continuum flow problems using techniques from the field of topology optimization. There are essentially two factors that motivate this work. The first is to study — on a qualitative level — naturally occurring flow structures, like river deltas and crack formations, from the point of view that they are (or might be) consequences of optimization principles. The second is to contribute to the development of the field of continuum flow optimization by investigating new areas where the topology optimization methodology can be applied.

The thesis is divided into two parts, where the first part serves as an introduction and a review of the topology optimization method and its application to different types of flows. A sample problem is derived in order to fix ideas and introduce terms and concepts. This is followed by a survey of the literature, summarizing the major developments in the area, while at the same time putting the contributions in this thesis into context.

The second part consists of a collection of five papers, treating the formulation and solution of optimization models for naturally occurring flow problems. These problem formulations include: finding the optimal topology of fluid channels in a permeable material; creating optimal bottom profiles for shallow water flow systems, and; establishing the optimal layout of cracks in a shrinking porous material. Several numerical examples are provided to demonstrate the models. Also, a framework for modelling planar multiphysic flow problems is presented at the very end.

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Last but not least, I am very grateful for having a family that has always been there to support me, and for all my friends who enrich my life outside the office walls in more ways then I can say.

Linköping, August 2008

Niclas Wiker

List of papers

This dissertation is based on the research papers listed below. They have been appended at the end of the thesis, and will be referred to by their Roman numerals in the text that follows. Each paper is printed in its originally published state, except for changes in formating to fit the layout of the thesis.

I. Wiker N, Klarbring A and Borrvall T. Topology optimization of regions of Darcy and Stokes flow. *International Journal for Numerical Methods in Engineering*, 69(7):1374–1404, 2007.

II. Wiker N, Klarbring A and Borrvall T. Simplified shallow water flow equations and their potential use in bottom profile optimization. *ZAMM Journal of Applied Mathematics and Mechanics*, 87(11–12):792–808, 2007.

III. Wiker N, Klarbring A and Borrvall T. Applications of bottom profile optimization in linear viscous shallow water flow. *Proceedings of the 7th World Congress on Structural and Multidisciplinary Optimization*, Seoul, Korea, May 2007.

IV. Wiker N, Klarbring A and Borrvall T. Topology optimization applied to crack patterns in drying porous material. 2008. (Submitted for publication.)

V. Wiker N and Klarbring A. A general multiphysic model for planar (2D) problems. LIU–IEI–R– –08/0026– –SE, Department of Management and Engineering, Linköping University, 2008.

Wiker is the main author of all papers and has performed all numerical experimentation, as well as contributed to the development of the mathematical models. Klarbring is a main contributor to the development of the models, and has provided the mathematical proofs in Paper I. Borrvall has been responsible for the implementation of the models in the in–house program *TO++*.

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1

Introduction

Different types of flow systems are found just about everywhere: in the water streaming in the ground, in lakes and rivers, and in the air in the atmosphere (fluid flow), in the dissolved sugar spreading in a cup of tea (diffusion), or in the transport of heat from stove plate to frying pan (heat flow), to mention just a few. On a macroscopic level, systems like these can, with a high degree of accuracy, be treated as continuous (as oppose to discrete) throughout their occupied region in space. In this thesis, mathematical models of such continuum flow systems are developed. Based on these models, optimization problems are then formulated using well established techniques from the field of *topology optimization*.

Very briefly, to fix ideas, in topology optimization one seeks the design or layout of material in a given domain, that is optimal with respect to some performance measure. Significant to the method, that makes it very powerful, is that the boundaries (e.g. holes) or the connectivity of material regions are not set *a priori*, but instead emerge as a result of the optimization process. A classical example, borrowed from the field of solid mechanics, is shown in Figure 1. Here, the domain (top picture) should be filled with 30 percent material and the question is where to put this material, i.e., to find the layout that maximizes the stiffness of the structure under the applied load (indicated by arrows). The result is seen in the bottom picture, and is often referred to in the literature as an MBB beam. Apart from flow systems and solid structures, the topology optimization method can be applied to more or less every field in engineering, including (but not limited to) wave propagation, bio–mechanics, and design of mechanisms. For a good overview of the field, the book by Bendsøe and Sigmund [2] is highly recommended.

The two main motives for performing the work presented in this thesis are, on the one hand, to contribute to the field of optimization of continuous

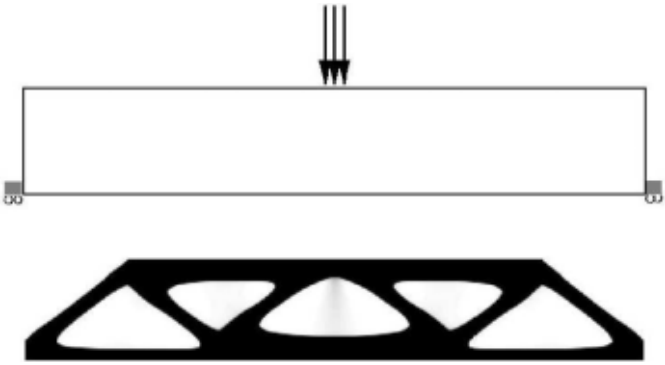

Figure 1: The continuous MBB beam problem. The design domain with boundary conditions (above) and the optimized structure (below). The figure is taken from [1].

flow systems, by applying the topology optimization methodology to new design problems and areas; on the other hand, to study naturally occurring flow systems, from the point of view that they are *results of optimization principles*. This view is also expressed by others, e.g. Bejan, which in [3] considers a number of natural systems, both animate (lungs, blood vessels) and inanimate (cross section of river channels, crystal dendrites), that he concludes to be "the result of a global process of optimization subject to global and local constraints" ([3], p. 6). The flow systems treated in this thesis, i.e., the tree–shaped structure of drainage basins and river deltas, and crack pattern in shrinking mud, have also been addressed to some extent in [3], but there using rather simple, one dimensional models. By considering more general continuum flow models, together with the techniques from topology optimization, it is believed that, with the increased detail and complexity of domain geometries that can be handled as a result, more insight into the nature of the problems is gained.

In the next chapter, a sample problem is derived in some detail in order to introduce the methodology used for formulating flow optimization problems. The focus is on the development of the flow model and the optimization problem, while other important issues, like the numerical implementation, are left out. Also, this chapter will serve as the basis for the survey that follows, where the major developments in the area are reviewed. The research contributions are appended at the end of the thesis as five self-contained papers. However, in order to put the papers into context, the main results have been summarized in the above mentioned survey.

2

The Darcy flow sample problem

The intention in this chapter is to describe the general steps involved when formulating an optimization problem for a fluid flow system. The problem under consideration has also been treated by Borrvall et al. [4], and Donoso and Sigmund [5] (these papers will be discussed in more detail in the next chapter). However, the flow model used in these works differs slightly from the one that will be presented below. The main reason for this is to derive a formulation that is representative for the majority of problems in the field, and therefore can act as a foundation for the review given in the next chapter.

The sample problem is stated as follows: a fluid (e.g. water) flows through a domain consisting of two porous materials, where one is more permeable (provides less resistance for the fluid) than the other. This type of flow is governed by Darcy's law (see [6]), and the aim here is to derive an optimization problem that can determine the optimal layout of these two materials for a certain problem formulation. This is a typical setup for a *topology optimization problem*.

The derivation of the optimization problem can be separated into a number of steps: first, the *state equations* describing the mechanical condition, or the state, of the system, is derived. Second, the *state problem* is defined by adding suitable *boundary conditions* to the state equations above. Third, the state problem is reformulated into *variational form*, which is needed for certain choices of *objective functions*, as well as to solve the problem numerically using the finite element method (not covered here). The fourth and last step is to define the optimization problem by choosing a specific objective function and deciding on appropriate *constraints* for the problem at hand.

2.1 State problem

This section covers the first three steps identified above, i.e., deriving the state equations, defining the state problem, and creating the variational formulation.

2.1.1 Derivation of the state equations

Let the *design domain* Ω be a sufficiently regular region in $\mathbb{R}^m$ with boundary $\partial\Omega$. The parameter m denotes the dimension of the domain and can be either 2 or 3. For the flow system considered here, for now, imagine that Ω contains a porous material with *spatially varying porosity*, i.e., the porosity varies inside Ω. A fluid flowing inside this domain obeys the equation of motion, which in local form reads (see [7], p. 101)

$$\operatorname{div} \boldsymbol{T} + \boldsymbol{b} = \rho \dot{\boldsymbol{u}}. \tag{1}$$

Here,

- $\operatorname{div} \boldsymbol{T}$ is the divergence of the Cauchy stress tensor $\boldsymbol{T}$ ($\operatorname{div} \boldsymbol{T} = \operatorname{tr}(\nabla \boldsymbol{T})$),
- $\boldsymbol{b}$ is the total volume force,
- ρ is the density,
- $\dot{\boldsymbol{u}}$ is the material time derivative of the velocity $\boldsymbol{u}$ ($\dot{\boldsymbol{u}} = \partial \boldsymbol{u}/\partial t + (\nabla \boldsymbol{u})\boldsymbol{u}$).

If one assumes that the porous material is fixed in Ω, the flow resistance can be modeled as a volume friction force acting on the fluid by the material. In this case, $\boldsymbol{b}$ can be split into two parts:

$$\boldsymbol{b} = \boldsymbol{f} + \boldsymbol{g}, \tag{2}$$

where $\boldsymbol{f}$ denotes friction force, and $\boldsymbol{g}$ represents all other forces on the system (including gravity). The law according to Darcy then states that $\boldsymbol{f}$ is proportional to the velocity of the fluid, i.e.,

$$\boldsymbol{f} = -\alpha \boldsymbol{u}, \qquad \alpha > 0, \tag{3}$$

where α is a parameter that depends on both the *dynamic viscosity* μ of the fluid and the *specific permeability* κ of the porous material, according to (see [6], p. 4)

$$\alpha = \frac{\mu}{\kappa}. \tag{4}$$

As seen, α is inversely proportional to κ, giving that α is large for a material with low permeability, i.e., small κ, and vice versa. This is the reason why α is often referred to as the *inverse permeability* of the material in the literature. From (3), it then follows that $\boldsymbol{f}$ increases as the permeability of the material decreases (increasing α), which corresponds well with intuition. Note also that, in agreement with the friction interpretation, $\boldsymbol{f}$ is directed opposite to the velocity. As a last remark, since the porosity of the material is assumed to vary in Ω, α must be a function of the coordinates. In the next section, when defining the optimization problem, this property will be used to introduce the shifting between regions of different permeability.

Next, assume that the density ρ is constant, implying that the motion of the fluid is *isochoric* ([7], p. 89), i.e., the volume is preserved. In order to take into account fluid sources within the domain, a trivial generalization of the usual definition of such a motion is needed. Let $\mathcal{P}$ be any subset of Ω with boundary $\partial\mathcal{P}$ and outward unit normal vector $\boldsymbol{n}$. Also, let s be a smooth function describing fluid *sources* within Ω. Then a motion is isochoric if

$$\int_{\partial\mathcal{P}} \boldsymbol{u} \cdot \boldsymbol{n} \, d\Gamma = \int_{\mathcal{P}} s \, d\Omega, \tag{5}$$

for all subsets $\mathcal{P}$ of Ω. Using the divergence theorem and the localization theorem ([7], p. 38) on (5), one has,

$$\operatorname{div} \boldsymbol{u} = s, \tag{6}$$

which should be enforced for the system — in addition to (1).

Further, assume that the fluid is *ideal*, that is, except for constant density, also assume that it is inviscid[1]. The stress tensor is then given by the simple relation

$$\boldsymbol{T} = -p\boldsymbol{I}, \tag{7}$$

where p is the pressure and $\boldsymbol{I}$ is the identity tensor. Considering the nature of the problem being modeled, i.e., a fluid flowing through a porous material, it is reasonable to assume that the flow is *slow and stationary*. This implies that

$$\dot{\boldsymbol{u}} = \boldsymbol{0}, \tag{8}$$

since $(\nabla\boldsymbol{u})\boldsymbol{u}$ will be small and can be ignored, and $\partial\boldsymbol{u}/\partial t$ is zero due to the stationary assumption. Using (2), (3), (7) and (8) in (1), the resulting

[1]The viscous properties (energy dissipation) of the system have already been included by assuming that the body force could be split into a friction force and other forces according to (2).

equation, together with (6), constitutes the *state equations* for the system, given by

$$\begin{aligned} \alpha \boldsymbol{u} + \nabla p &= \boldsymbol{g}, \\ \operatorname{div} \boldsymbol{u} &= s. \end{aligned} \tag{9}$$

As mentioned in the introduction to this chapter, the form (9) is not the conventional way to represent a Darcy flow system. Often, such as in [4] and [5], a single equation stated in the pressure is used. This formulation can easily be derived from the mixed formulation above, by simply eliminating $\boldsymbol{u}$ from the second equation in (9) by using the first equation. Assuming that $\boldsymbol{g}$ is constant gives

$$-\nabla \cdot \left(\frac{1}{\alpha} \nabla p \right) = s, \tag{10}$$

which is recognized as Poisson's equation.

2.1.2 Defining the state problem and variational form

The state equations (9) need to be completed with suitable boundary conditions in order to fully define the *state problem* of the system. Here, both essential (Dirichlet) and natural (Neumann) conditions will be used in order to keep the model as general as possible. To that end, let the boundary $\partial\Omega$ be split up into two parts, Γ_D and Γ_N, such that $\Gamma_D \cup \Gamma_N = \partial\Omega$. The subscript indicates which type of condition, Dirichlet or Neumann, should be specified on each part. For the Darcy flow system considered here, imposing a Dirichlet condition means specifying the normal velocity u_Γ on Γ_D, while a Neumann condition refers to a given traction vector $\boldsymbol{t}_\Gamma$ on Γ_N, respectively. The state problem is then defined as

$$\begin{aligned} \alpha \boldsymbol{u} + \nabla p &= \boldsymbol{g} && \text{in} \quad \Omega, \\ \operatorname{div} \boldsymbol{u} &= s && \text{in} \quad \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} &= u_\Gamma && \text{on} \quad \Gamma_D, \\ \boldsymbol{T}\boldsymbol{n} &= \boldsymbol{t}_\Gamma && \text{on} \quad \Gamma_N, \end{aligned} \tag{11}$$

where $\boldsymbol{n}$ is the outward unit normal to the boundary. Note that Cauchy's equation (see [7], p. 101) is used in the fourth equation in (11), which by (7) equals

$$-p\boldsymbol{n} = \boldsymbol{t}_\Gamma \quad \text{on} \quad \Gamma_N. \tag{12}$$

That is, specifying a traction condition means specifying the pressure on that boundary, and as indicated in (12), any imposed traction vector has to be

normal to the boundary, i.e., shear forces cannot be present in a Darcy flow problem.

The next step is to derive the *variational form* of (11) (with (12) replacing the fourth equation). In most cases when dealing with problems of this type, a mixed formulation is used (see e.g. [8]), where both the pressure and velocity are treated as unknowns. Thus, function spaces for both the velocity and the pressure are needed: define

$$\begin{aligned} \boldsymbol{\mathcal{U}} &= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = u_\Gamma \text{ on } \Gamma_D\}\,, \\ \boldsymbol{\mathcal{V}} &= \{\boldsymbol{v} \in \boldsymbol{H}(\mathrm{div};\Omega) : \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma_D\}\,, \\ \mathcal{Q} &= L^2(\Omega), \end{aligned}$$

where $\boldsymbol{\mathcal{U}}$ is the space of admissible velocities, $\boldsymbol{\mathcal{V}}$ the space of velocity test functions, and $\mathcal{Q}$ denotes the space of admissible pressures, as well as the corresponding test functions. The space $\boldsymbol{H}(\mathrm{div};\Omega)$ is defined as

$$\boldsymbol{H}(\mathrm{div};\Omega) = \left\{\boldsymbol{v} \in \left(L^2(\Omega)\right)^m : \mathrm{div}\,\boldsymbol{v} \in L^2(\Omega)\right\},$$

i.e., the space of vector functions in $(L^2(\Omega))^m$ whose divergence belongs to $L^2(\Omega)$ [2]. With this established, multiply the first equation of (11) by $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}$ and the second equation by $q \in \mathcal{Q}$, and integrate over Ω. Applying Green's theorem (where appropriate) then gives

$$\int_\Omega \alpha\boldsymbol{u}\cdot\boldsymbol{v}\,d\Omega - \int_\Omega p\,\mathrm{div}\,\boldsymbol{v}\,d\Omega = \int_\Omega \boldsymbol{g}\cdot\boldsymbol{v}\,d\Omega + \int_{\Gamma_N} (-p\boldsymbol{n})\cdot\boldsymbol{v}\,d\Gamma,$$

and

$$\int_\Omega (\mathrm{div}\,\boldsymbol{u})q\,d\Omega = \int_\Omega sq\,d\Omega.$$

Replacing $-p\boldsymbol{n}$ by using the boundary condition (12) and collecting the terms, the variational formulation becomes: Find $(\boldsymbol{u},p) \in \boldsymbol{\mathcal{U}} \times \mathcal{Q}$ such that

$$\begin{aligned} a_\alpha(\boldsymbol{u},\boldsymbol{v}) - b_{\mathrm{div}}(p,\boldsymbol{v}) &= \langle \boldsymbol{f},\boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v} \in \boldsymbol{\mathcal{V}}, \\ b_{\mathrm{div}}(q,\boldsymbol{u}) &= \langle s,q\rangle \quad \forall\,q \in \mathcal{Q}, \end{aligned} \tag{13}$$

[2] $L^2(\Omega)$ is the standard space of square–integrable functions on Ω, see [9].

where

$$a_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \alpha \boldsymbol{u} \cdot \boldsymbol{v} \, d\Omega, \tag{14a}$$

$$b_{\text{div}}(p, \boldsymbol{v}) = \int_\Omega p \operatorname{div} \boldsymbol{v} \, d\Omega, \tag{14b}$$

$$\langle \boldsymbol{f}, \boldsymbol{v} \rangle = \int_\Omega \boldsymbol{g} \cdot \boldsymbol{v} \, d\Omega + \int_{\Gamma_N} \boldsymbol{t}_\Gamma \cdot \boldsymbol{v} \, d\Gamma, \tag{14c}$$

$$\langle s, q \rangle = \int_\Omega sq \, d\Omega. \tag{14d}$$

Here, note that a subscript has been added to the bilinear form a_α in (13), in order to emphasize the dependence on the inverse permeability α.

As a_α is symmetric, there exists a corresponding convex minimization problem to (13), defined by the objective functional $\mathcal{J}_\alpha : \boldsymbol{H}(\text{div}; \Omega) \to \mathbb{R}$,

$$\mathcal{J}_\alpha(\boldsymbol{v}) = \frac{1}{2} a_\alpha(\boldsymbol{v}, \boldsymbol{v}) - \langle \boldsymbol{f}, \boldsymbol{v} \rangle, \tag{15}$$

and the set

$$\boldsymbol{\mathcal{U}}_{\text{div}} = \{ \boldsymbol{v} \in \boldsymbol{\mathcal{U}} : \text{div}(\boldsymbol{v}) = s \} .$$

The *state optimization problem* then reads:

$$\min_{\boldsymbol{v} \in \boldsymbol{\mathcal{U}}_{\text{div}}} \quad \mathcal{J}_\alpha(\boldsymbol{v}). \tag{16}$$

Worth noting is that if the forces $\boldsymbol{g}$ and $\boldsymbol{t}_\Gamma$ are zero (which is often assumed in practice), then (15) is a measure of the *dissipated power* in the system. Also, it is quite common in the literature that (15) is denoted the *total potential power functional* because of its resemblance to the expression for total potential energy in linear elasticity problems.

2.2 Optimization problem formulation

This section covers the last of the four steps identified in the introduction to this chapter—the definition of the optimization problem. This includes choosing an objective function and suitable constraints. The aim is a problem formulation that, in an optimal way, distributes two material regions with high and low permeability, respectively.

2.2.1 Introducing a design variable

The first step in formulating the optimization problem is to introduce an optimization variable, or *design variable*, that describes the distribution of high and low permeability regions inside Ω. Preferably, this variable should take discrete values, say 0 and 1, to indicate if a point in the domain contains high or low permeability material. However, requiring such a property introduces severe practical difficulties when it comes to solving the problem. To circumvent this, the design variable is allowed to vary continuously between the two extreme values. Then, in order to obtain a (near) discrete solution, intermediate values are penalized to make them undesirable in the optimization process.

To that end, let η denote the design variable with the following properties imposed:

$$\eta \in L^\infty(\Omega), \qquad \eta : \Omega \to [0, 1].$$

The desired shifting effect of material properties inside Ω is established by letting the inverse permeability variable α be influenced by η in such a way, that it becomes a function that varies between an upper and a lower value, representing the material parameters of the two regions. Using $\underline{\alpha}$ and $\overline{\alpha}$ to denote these two limits, the *interpolation function* $\alpha(\eta)$ is defined as

$$\alpha(\eta) : [0, 1] \to [\underline{\alpha}, \overline{\alpha}], \quad \text{such that} \quad \alpha(0) = \overline{\alpha},\ \alpha(1) = \underline{\alpha},$$

i.e., $\eta = 0$ corresponds to a material region with inverse permeability $\overline{\alpha}$ (high friction material), and $\eta = 1$ to a material region with $\underline{\alpha}$ (low friction material). With this definition of η, the integral

$$\int_\Omega \eta \, d\Omega,$$

can be interpreted as the amount of high permeability (low friction) material in the domain — a property that will be used in the following. Please note that, although it will not be indicated explicitly in the notation, due to the fact that α has been defined as a function of η, the bilinear form $a_{\alpha(\eta)}$ (and hence, $\mathcal{J}_{\alpha(\eta)}$) should from here on be considered as design dependent.

The explicit expression of $\alpha(\eta)$, defining the interpolation between the two parameter limits $\underline{\alpha}$ and $\overline{\alpha}$, should be chosen such that it forces the optimization process to deliver (near) discrete solutions, i.e., the domain should contain regions with $\underline{\alpha}$ and $\overline{\alpha}$ materials only. As mentioned above, one technique to do that is to penalize intermediate values of $\alpha(\eta)$, making them an uneconomical choice in the optimization process. This is known as the *SIMP*

method (solid isotropic material with penalization, see [2]), and is (probably) the most popular method in use at present. However, it is important to keep in mind that penalization increases the non–convexity of the optimization problem, which in turn significantly augments the risk of local optimal solutions. To be able to decide on a suitable interpolation function, knowledge about the character of the specific optimization problem formulation is needed. Specifying $\alpha(\eta)$ is therefore postponed until later.

2.2.2 Choosing an objective function

An *objective function* measures the property of the system that should be extremized (minimized or maximized). As will be evident in the next chapter, a very common choice is to use the value of the functional (15), evaluated at the solution to the state problem, $\boldsymbol{u}$. This is written as

$$\phi(\eta) = \min_{\boldsymbol{v}\in\boldsymbol{\mathcal{U}}_{\text{div}}} \mathcal{J}_{\alpha(\eta)}(\boldsymbol{v}) = \mathcal{J}_{\alpha(\eta)}(\boldsymbol{u}). \tag{17}$$

That is, in order to evaluate ϕ for a certain design η, one first has to solve the state problem (13) (or equivalently (16)) for that particular design.

Regarding the interpretation of (17), one has already been given at the end of Section 2.1: if the forces (body forces and tractions) are zero, (17) is a measure of the dissipated power in the system. Defining the optimization problem as to *minimize* ϕ, means finding the layout that minimizes this power. In the case of nonzero forces, the objective is to find the layout which minimizes the dissipated power *and* maximizes the flow at the applied forces. Another interpretation of (17) can be given by using the state problem (13) and assuming zero conditions on Γ_D, i.e., $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ on Γ_D. First, the zero condition implies that $\boldsymbol{\mathcal{U}} = \boldsymbol{\mathcal{V}}$, such that one can choose $\boldsymbol{v} = \boldsymbol{u}$ and $q = p$ in (13). Then, the b_{div} term in the first equation of (13) can be eliminated by using the second equation, giving $a_\alpha(\boldsymbol{u},\boldsymbol{u}) = \langle s, p\rangle + \langle \boldsymbol{f}, \boldsymbol{u}\rangle$. This inserted into (17) then gives

$$\phi(\eta) = \frac{1}{2}\left(\langle s, p\rangle - \langle \boldsymbol{f}, \boldsymbol{u}\rangle\right).$$

Again, assuming zero forces, together with restricting s to be constant, ϕ is then a measure of the total pressure driving the fluid. So minimizing ϕ means, in this context, minimizing the overall pressure drop in the system.

2.2.3 Considering constraints and interpolation function

From the interpretation of ϕ above, it follows that low friction regions ($\underline{\alpha}$, i.e., high permeability) are more favourable then high friction regions. This will

result in solutions where the entire domain is filled with low friction material only. To avoid such trivial solutions, a constraint that limits the amount of $\underline{\alpha}$–material is needed. With the definition of η made earlier, formulating such a constraint is straightforward:

$$\int_{\Omega} \eta \, d\Omega \leq \gamma |\Omega|, \tag{18}$$

where γ is the maximum allowed fraction of high permeability material, and $|\Omega|$ is the measure of the domain (area in 2D, volume in 3D). No further constraints need to be specified for the present formulation.

Regarding the explicit form of the interpolation function $\alpha(\eta)$, a first attempt to formulate the optimization problem is needed in order to continue the discussion. Gathering the objective function (17), the constraint (18), and the definition of the design variable above, gives

$$\min_{\eta \in \mathcal{H}_\eta} \quad \phi(\eta), \tag{19}$$

where $\mathcal{H}_\eta$ is the set of *feasible designs*, defined by

$$\mathcal{H}_\eta = \left\{ \eta \in L^\infty(\Omega) : 0 \leq \eta \leq 1 \text{ in } \Omega, \int_{\Omega} \eta \, d\Omega \leq \gamma |\Omega| \right\}.$$

From the definition of ϕ, it is clear that (19) has the character of a min–min problem. Hence, the discussion regarding the choice of interpolation function in connection to discrete solutions, outlined in Paper I, Section 5, is applicable here. It follows that *any concave* function, thus also a linear one, would produce completely discrete solutions. However, choosing such a function means introducing the maximum amount of penalty into the problem, and thus a high risk of local optimal solutions. In order to reduce this risk, a *convex* function should be chosen instead. Borrvall and Petersson [10] (their work will be treated in more detail in the next chapter) suggested the following strictly convex function:

$$\alpha(\eta) = \overline{\alpha} + (\underline{\alpha} - \overline{\alpha})\eta \frac{1+q}{\eta+q}, \tag{20}$$

where q is a parameter to control the degree of penalty. In Figure 2, equation (20) is plotted for different values of q ($q \in \{0.01, 0.1, 1\}$), and as seen, the penalty increases, i.e., $\alpha(\eta)$ becomes less convex, for larger q.

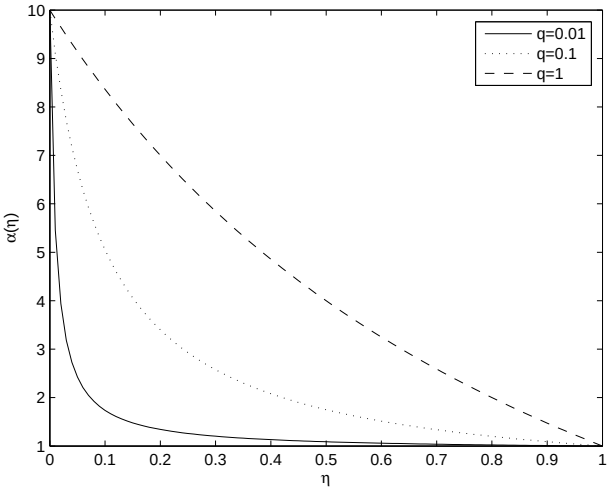

Figure 2: The interpolation function (20) for different values of the penalty parameter q. Here, $\overline{\alpha} = 10$ and $\underline{\alpha} = 1$.

2.2.4 Formulating the optimization problem

Before a definitive formulation of the optimization problem can be established, it is very important to consider the question of the existence of solutions. For structural optimization problems, this issue has been studied in a number of publications (see for example, the survey by Sigmund and Petersson [11], or the paper by Borrvall [12], including respective references) and the result is striking: a straightforward (naive) formulation of a topology optimization problem, like the one suggested in (19), is generally *not well–posed*, meaning that the feasible design set lacks closure. Unfortunately, the problem treated here is probably no exception — at least when compared to problems based on the form (10), treated in [4] and [5]. There are two ways to reformulate, i.e., introduce *regularization* into a problem formulation, such that well–posedness can be achieved. The first is to *relax* the problem and *extend* the feasible set with limits of solutions which are not included in the original set. The other one, that will be used here, is to *restrict* the problem by *excluding* solutions, to form a compact subset of the original set of feasible designs. A number of such restriction methods exist that can be applied to general problems (see e.g., [12] for an overview): here, the filter method proposed by Bruns and Tortorelli [13], and later proved by Bourdin [14], will

be used.

In this method, a new design variable ξ is introduced, defined by

$$\xi \in L^\infty(\Omega_\xi), \qquad \xi : \Omega_\xi \to [0,1],$$

in order to match the properties of η. The domain $\Omega_\xi \subset \mathbb{R}^m$ is an enlargement of Ω, on which the following *normalization requirement* holds:

$$\int_{\Omega_\xi} \psi(\boldsymbol{x}-\boldsymbol{y})\, d\boldsymbol{y} = 1, \quad \forall\, \boldsymbol{x} \in \Omega. \tag{21}$$

Here, ψ is a *filter function* given by,

$$\psi(\boldsymbol{x}-\boldsymbol{y}) = \frac{3}{\pi R^2} \max\left\{0, 1 - \frac{\|\boldsymbol{x}-\boldsymbol{y}\|}{R}\right\}, \quad \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m, \tag{22}$$

where the parameter $R > 0$ is the *radius* that characterizes the filter. The enlarged domain Ω_ξ is in practice created by adding a rim of width R around Ω. The two design variables η and ξ are then related through the *convolution product*,

$$\eta(\boldsymbol{x}) = \int_{\Omega_\xi} \psi(\boldsymbol{x}-\boldsymbol{y})\xi(\boldsymbol{y})\, d\boldsymbol{y} \equiv S(\xi)(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{23}$$

where S denotes the convolution operator. In essence, the method restricts too rapid variations in the design variable η by averaging the value at each point $\boldsymbol{x} \in \Omega$ with its neighboring points inside the filter radius R. Note that the optimization is actually performed on the variable ξ, but η will still be the one interpreted as the solution to the design problem.

By introducing this filtering technique on the design variable, a well–posed version of (19) is stated as

$$\min_{\xi \in \mathcal{H}_\xi} \quad \phi\left(S(\xi)\right), \tag{24}$$

where

$$\mathcal{H}_\xi = \left\{ \xi \in L^\infty(\Omega_\xi) : 0 \le \xi \le 1 \text{ in } \Omega_\xi, \int_\Omega S(\xi)\, dA \le \gamma|\Omega| \right\}.$$

To solve (24) means deciding on a finite element formulation for the state problem[3], discretizing the design variable[4], and choosing an optimization

[3]A number of different methods exist for solving this problem, see e.g., Roberts and Thomas [15], Masud and Huges [16], or Brezzi et al. [17].

[4]When using finite elements to solve the state problem, the simplest and most frequently used method is to approximate the design variable as constant on each element.

algorithm[5]. All of these are very important issues to consider in order to obtain useful and reliable results. However, they are outside the scope of this presentation and will not be addressed.

This said, one result will still be presented here in order to conclude this chapter. It is the rather well–known area–to–point flow problem studied by Bejan [3] and Borrvall et al. [4], which will be revisited once again. Consider the square shaped domain shown to the left in Figure 3. On Γ_D and Γ_N, zero normal velocity ($u_\Gamma = 0$) and zero pressure are prescribed ($\boldsymbol{t}_\Gamma = \mathbf{0}$), respectively. Also, inside the domain fluid is added everywhere, represented by setting $s = 1$ in (11). The goal is to find the distribution of high and low permeability material (the amount of high permeability material is limited by $\gamma = 0.3$ in (18)), which transports the added fluid from Ω to the outflow boundary Γ_N as effectively as possible (in the sense of (17)). The result is seen to the right in Figure 3, where channels of high permeability material have been created to facilitate the transport of the fluid.

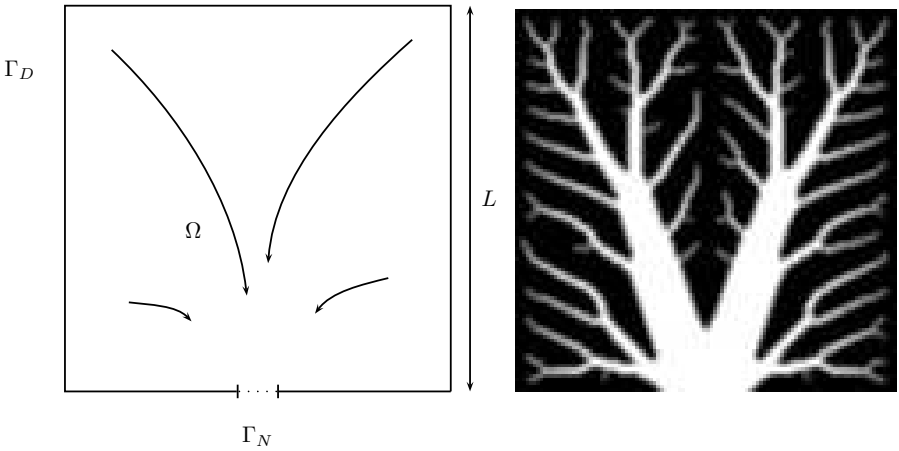

Figure 3: To the left: the design domain for the area–to–point flow sample problem. To the right: the optimal layout of high (white) and low (black) permeability material.

[5]A very powerful and popular method is *MMA* (Method of Moving Asymptotes), devised by Svanberg [18].

3

Developments

In this chapter, the developments in the area of continuous topology optimization flow problems will be reviewed. The innovative work by Borrvall and Petersson [10] is widely recognized as the initial step taken in this area. However, since the Darcy problem has already been treated in Chapter 2, it will serve as the starting point for the presentation.

3.1 Darcy flow

A rather concise treatment of the Darcy problem is given by Borrvall et al. in [4], and as mentioned in Section 2.1, the state equation used in this work is expressed by (10). The optimization problem is formulated analogous to the sample problem, i.e., minimizeing an objective function corresponding to (17) under a constraint on available high permeability material (cf. (18)). The numerical example provided is the optimal drainage problem seen in Figure 3.

In [5], Donoso and Sigmund studies a variety of physical problems, which can all be described by Poisson's equation (the meaning of variables, parameters, and boundary conditions, need of course to be adapted to the physical process under consideration). Among these, as seen in Section 2.1, is the Darcy flow problem in the form described by (10). Considering only this part of the paper, the two examples treated both rely on the fact that the velocity field of the fluid can be determined from the pressure according to (cf. (9) with $\boldsymbol{g} = \mathbf{0}$)

$$\boldsymbol{u} = -\frac{1}{\alpha}\nabla p.$$

In the first example, a body is submerged in a flowing fluid, and the design goal is to find the optimal shape which minimizes the dissipated power in the

fluid. In the second example, a simple kind of flow mechanism is designed by formulating a problem where the objective is to maximize the flow velocity in a certain direction in a small region in the middle of the domain.

3.2 Stokes flow

While the Darcy problem can be seen as a standard Poisson's problem where variables have been interpreted suitably to model porous media flow, the problem described by Borrvall and Petersson [10] introduces new ground in the field of topology optimization for a continuum, and constitutes the base from which research has been performed in a number of different directions. Here, the optimal distribution of a creeping fluid, governed by Stokes equations, and solid material is sought. The state equations (cf. (9)) used to model this system are given by

$$\begin{aligned} \alpha \boldsymbol{u} - \mu \Delta \boldsymbol{u} + \nabla p &= \boldsymbol{g}, \\ \operatorname{div} \boldsymbol{u} &= 0, \end{aligned} \tag{25}$$

where, compared to the classical form of the Stokes system, the first equation has been augmented with the term $\alpha \boldsymbol{u}$. To motivate this extra term, a planar (2D) flow model is derived , where the fluid is assumed to flow between two parallel plates. The derivation also works as a motivation to let the design variable η affect the parameter α in order to control the fluid–solid shifting in the domain, while the viscosity μ is kept constant. Hence, the different regions can be represented by setting α to zero for a fluid region (to retain Stokes equations), and infinity for a non–fluid, i.e., solid region. However, in [10], for practical reasons, these limits are replaced by a small ($\underline{\alpha} << \mu$) and a large ($\overline{\alpha} >> \mu$) value. As a consequence, the fluid velocity will not be zero in the solid parts, meaning that they should be interpreted as permeable regions. Also, since $\underline{\alpha} > 0$, the fluid parts will not be regions in pure Stokes flow.

For the optimization problem, the objective function is chosen in the same manner as for the sample problem in the previous chapter. The only difference is that the bilinear form a_α in $\mathcal{J}_\alpha$ is replaced by

$$a_\alpha(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \alpha \boldsymbol{u} \cdot \boldsymbol{v} \, d\Omega + \int_\Omega \mu \nabla \boldsymbol{u} \cdot \nabla \boldsymbol{v} \, d\Omega. \tag{26}$$

The goal is then to minimize the function corresponding to (17), under a constraint on available fluid. Similarly to the discussion in Chapter 2 regarding the explicit form of the interpolation function, it is shown in [10] that the

problem will be completely discrete valued for a linear function. As a result, in order to reduce the risk of local optimal solutions, Borrvall and Petersson proposed that the convex function given in (20) be used as interpolation function for $\alpha(\eta)$. An unexpected result from the analysis of the problem is that no regularisation is needed to prove the existence of solutions, something that is very unusual for topology optimization problems in general.

Several numerical examples are provided to demonstrate the proposed design method. One of the examples, the 90 degree pipe bend, is shown in Figure 4, and, as seen, the inlet and outlet are connected with a straight pipe — not, as most pipe bends in the fluid mechanics literature, torus shaped. In [10] this fact is explained as being a consequence of considering only Stokes flow in the problem formulation. Another example, the rugby ball, is classical

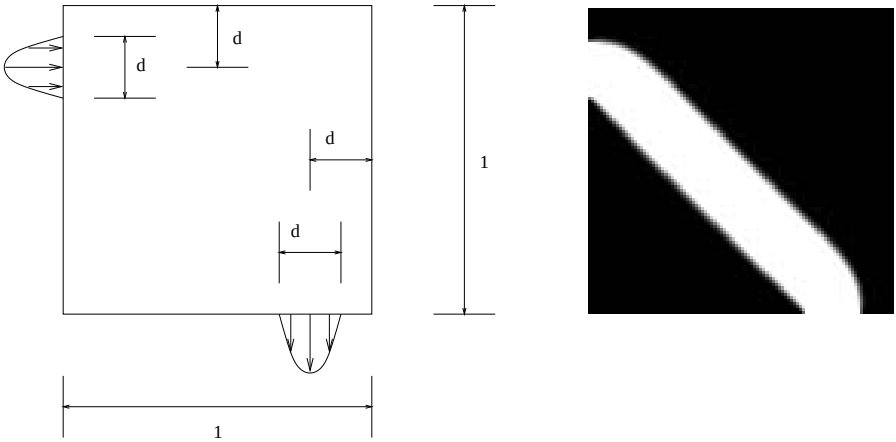

Figure 4: The design domain ($d = 0.2$) with boundary conditions (left picture) for the 90 degree pipe bend problem, and the optimal shape for a fluid in Stokes flow (right picture). The figure is taken from [10].

in the field of fluid shape optimization (see Pironneau [19] who treated this problem as early as 1974). Similar to the first of the two flow examples given in [5] in Section 3.1, the objective is to minimize the dissipated power for a body submerged in a fluid flow with prescribed velocity. The solution to this problem is a body shaped like a rugby ball.

As discussed above, only an approximation to the true solid–fluid topology optimization problem is addressed in [10]. This issue is given a rather rigorous mathematical treatment by Evgrafov in [20]. Here, the restrictions on the parameter α are relaxed, allowing it to attain the values 0 and ∞, and

the existence of solutions to this relaxed problem is investigated. The true discrete problem, where the design variable is only allowed to assume the values 0 or 1, is also analyzed. It is then proven that for a case when the objective is to minimize the function corresponding to (17) (suitably adapted to the state problem), both these problems are well–posed. However, this result does not necessarily carry over to other problems, where different objective functions are considered. In fact, these might very well be ill–posed and lack solutions.

3.2.1 Applications of Stokes flow optimization

There are a number of publications where attempts have been made to find new applications for the optimization method presented in [10]. In the master thesis by Höök [21], the method is applied to problems in microfluidics, i.e., fluid flow in very small channels. Two examples are studied: a four–way fluid intersection and a flow–cell in an apparatus of bio–chemical analysis. The aim is to improve the designs of the original microfluidic devices with respect to minimum pressure drop, something that was indeed successful.

Also related to this category is the work by Gersborg–Hansen et al. [22]. Here, the goal is to construct fluid distribution devices that fulfill prescribed outflow rates. This is accomplished by adding constraints on the outflow in the optimization problem. In order to validate the optimized design, a post–processing step is suggested where the design is analyzed using a standard fluid dynamic software.

In both of the works mentioned above, the discretization of the problems is rather coarse and (most of them) can be solved on a standard desktop computer. In contrast, for the 2D and 3D problems treated by Aage et al. [23] a much finer discretization is used ($\sim 10^6$ elements), resulting in large scale problems that require parallelized software running on a multi–processor computer to solve in a reasonable time. The examples presented are refinements of old problems solved by others, e.g. the classical rugby ball example covered in [10] or the fluid distribution device with prescribed outflow rates described in [22] (above). The solutions are well in agreement with earlier established results, but with much smoother fluid–solid interfaces due to the increased resolution.

3.2.2 Design dependent viscosity parameter μ

So far, only the α parameter has been assumed to depend on the design variable, while the viscosity parameter μ has been considered constant. In

the approach taken by Guest and Prévost [24], the focus is on formulating a convenient approximation to the original problem in [10]. In order to describe the process inside the domain, the solid and fluid phases are considered separately. The solid phase is treated as a porous medium with inverse permeability α, which is governed by Darcy's law (the medium is regarded as solid, if α is chosen large enough). For the fluid phase, only creeping flow is considered, hence Stokes equations are adequate. The flow through the domain is then described by combining these separate state equations as follows:

$$\begin{aligned} \alpha(\eta)\boldsymbol{u} - \mu(\eta)\Delta\boldsymbol{u} + \nabla p &= \boldsymbol{g}, \\ \operatorname{div}\boldsymbol{u} &= 0. \end{aligned} \tag{27}$$

Here,

$$\alpha(\eta) = \eta\overline{\alpha} \quad \text{and} \quad \mu(\eta) = (1-\eta)\overline{\mu},$$

where $\overline{\alpha}$ and $\overline{\mu}$ represent the material parameters for the Darcy and the Stokes regions, respectively. As seen, in contrast to the formulation in [10], the Darcy term is here allowed to disappear in the fluid region (represented by $\eta = 0$). More importantly, in the solid region, where $\eta = 1$, the Stokes part in (27) is also allowed to disappear since $\mu(1) = 0$. This normally requires special treatment for the interface between the two phases, something that is not considered in [24].

The objective function to be minimized is, in a sense, the same as the one used in [10], but with the exception that only the fluid region is considered (i.e., the term representing the dissipated power in the Darcy phase is left out), giving

$$\phi(\eta) = \frac{1}{2}\int_{\Omega} \mu\nabla\boldsymbol{u}\cdot\nabla\boldsymbol{u}\,d\Omega - \langle\boldsymbol{f},\boldsymbol{u}\rangle. \tag{28}$$

Together with the usual constraint on the available amount of fluid, several of the examples found in [10] are resolved, with comparable results. Some numerical difficulties, regarding local optimal solutions, are reported. These problems are claimed to be circumvented by introducing a filter strategy, comparable with the one described in Section 2.2. During the optimization process, the filter radius is gradually reduced until it no longer has any effect, thus creating seemingly 0–1 solutions. At the same time, starting from a low value, $\overline{\alpha}$ is successively increased during the process until the final value is reached.

It is claimed that the presented optimization problem is well–posed, *without* introducing regularization in the formulation *or* requiring that $\mu(\eta) > 0$, but a proof to support this statement is not provided. This issue will be addressed further below.

A similar approach is used in Paper I, presented in the second part of this thesis, but here the aim is to model a problem with true regions of Darcy and Stokes flow, and not, as in [24], as a numerical method to solve the original Stokes flow problem by Borrvall [10]. The derivation of the state equations follows the procedure outlined in Chapter 2, with the exception that the constitutive relation (7) is replaced by

$$\boldsymbol{T} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}(\boldsymbol{u}), \tag{29}$$

where $\boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}\left(\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T\right)$, to model an incompressible Newtonian fluid. The result is (almost) the same system as in [24], but with (6) replacing the second equation in (27). In addition, different interpolation functions are used, where $\alpha(\eta)$ is given by (20) and μ is defined by

$$\mu(\eta) = \underline{\mu} + (\overline{\mu} - \underline{\mu})\eta.$$

In the same way as in [24], α is allowed to be zero in Stokes regions by setting $\underline{\alpha} = 0$, while the corresponding property for Darcy regions is *not* admitted, by requiring $\underline{\mu} > 0$. Similarly to [10], the objective is to minimize the dissipated power (17) (with a_α replaced by (26)) under a constraint on the available amount of fluid. The method is evaluated on the area–to–point drainage flow problem described at the end of Section 2.2, with respect to various parameters and numerical strategies. A characteristic solution is shown in Figure 5.

Regrading the existence of solutions, a proof is provided in Paper I, where well–posedness for the optimization problem formulation is established. It is concluded that although the corresponding proof in Borrvall and Petersson [10] does not need regularization, the present one does. Another requirement is that $\mu(\eta) > 0$ for the proof to hold. This contradicts the statement in [24], claiming that neither of these properties are needed. Further analysis of the problem in Paper I shows that linear interpolations of state parameters will result in black and white (unfiltered) designs.

3.2.3 Constructing bottom profiles in Darcy–Stokes related flow

While not actually being a topology optimization problem, the same methodology is used in Paper II to propose a design method for creating optimal bottom profiles. The state equations used are recognized as the linearized 2D shallow water equations, where the variable describing the bottom profile of the domain emerges naturally from the derivation. The optimization problem is formulated so as to minimize an objective function corresponding to the total potential power functional, under a constraint on the amount of

Figure 5: The optimal layout of Darcy (black) and Stokes (white) regions for the area–to–point problem described at the end of Section 2.2. The solution is generated using the method described in Paper I.

total depth that is allowed to be distributed. The design method is closely related to the Darcy–Stokes topology optimization problem described in Paper I, and many results, e.g. the proof of the existence of solutions, carry over without need of modification. The real difference is that here a smooth, as opposed to a discrete 0–1, design is sought to represent the bottom profile of the flow problem.

The design method is demonstrated by a number of examples, where the influence on the design of different parameter values is studied. In Paper II, the area–to–point drainage problem described in Chapter 2 is revisited once again with a typical result shown in Figure 6. Also, a problem similar to the rugby ball example in [10] is solved, describing the sedimentation of bottom material around a fixed pole in a flowing fluid. Two more examples are given in Paper III, representing point–to–line and line–to–line class problems, respectively. The point–to–line problem is exemplified by mimicking the formation of a river delta, and for the line–to–line problem, channel formations of a spillway are imitated. From these examples it is concluded that for certain choices of parameter values, designs can be created that has similarities to flow structures found in nature.

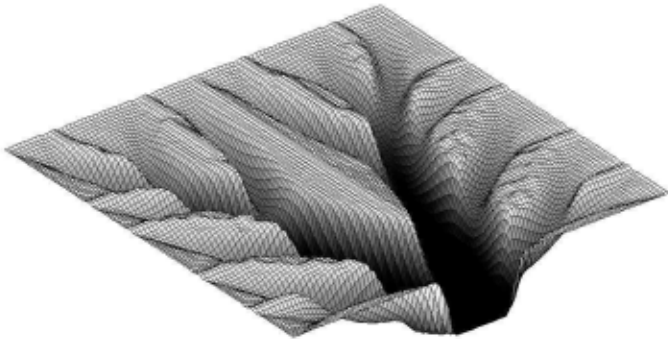

Figure 6: An optimal bottom profile for the area–to–point drainage problem described at the end of Section 2.2. The solution is generated using the method described in Paper II.

3.2.4 Designing porous materials with optimized properties

Combining homogenization theory with the topology optimization methodology to design materials with specific properties is an established method in solid structure optimization (see Bendsøe and Sigmund [2] for an introduction). The basic idea is to use topology optimization to determine the layout of the micro–structure of a material, such that the desired homogenized, i.e., effective, macroscopic properties of the material are met. This is known as an *inverse homogenization problem* in the literature and is the topic for Guest and Prévost [25]. In this work, the concept is extended to fluid flow systems in order to design the micro–structure of a periodic porous material with a certain given permeability. On the macroscopic scale, the flow through the material is governed by a slightly more generalized version of the Darcy law given in (9), where the specific permeability κ in (4) is replaced by a second order permeability tensor to allow for anisotropic material structures. On the microscopic scale, the periodic base cell consists of regions of solid material and fluid in Stokes flow, from where the effective permeability of the macroscopic material is calculated using homogenization.

The design problem is defined so as to find the layout of the fluid and solid regions in the base cell that maximizes the permeability of the macroscopic material. Except for the common upper limit on the available amount of fluid, a special constraint is used in the formulation requiring the resulting permeable material to be isotropic. Both two and three dimensional examples are solved. The designs for the 3D case show a base cell with symmetric flow channels running through the solid material, and it is noted in [25] that

the fluid–solid interface is similar to a so called minimal surface[1]. As for the optimal 2D designs, they are found to be cross–sections of the optimal designs of the 3D base cell.

3.3 Navier–Stokes flow

In order to handle more realistic flow cases, Borrvall and Petersson suggested in [10] that the state equation (25) should be generalized to Navier–Stokes equations. In the paper by Gersborg–Hansen et al. [26], this invitation is acknowledged. The corresponding state equations are derived in the same manner as in [10], but here the flow is assumed to obey the stationary Navier–Stokes equations instead. This gives

$$\begin{aligned} \alpha(\eta)\boldsymbol{u} - 2\mathrm{div}\,\boldsymbol{D}(\boldsymbol{u}) + Re(\nabla\boldsymbol{u})\boldsymbol{u} + \nabla p &= \boldsymbol{0}, \\ \mathrm{div}\,\boldsymbol{u} &= 0, \end{aligned} \tag{30}$$

where Re is the Reynolds number[2] (restricted to $0 \leq Re < 1000$ in [26]), and just as in [10], α is the only parameter that depend on the design.

Further, a generic optimization problem is defined, where different combinations of objective functions and constraints are tested. In the first example, the 90 degree pipe bend problem is revisited in order to study the influence of the Re number. The objective is to minimize the dissipated energy, defined by

$$\phi(\eta) = \frac{1}{2}\int_{\Omega} \left(\alpha(\eta)\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{D}(\boldsymbol{u})\cdot\boldsymbol{D}(\boldsymbol{u})\right)\, d\Omega, \tag{31}$$

under a constraint on available fluid. It is shown that for $Re \leq 50$ the effect on the design compared to the pure Stokes flow case is very small, but for larger values the torus shape starts to appear. In another example, a flow device is sought which reverts the fluid velocity at the center of a straight pipe. This is formulated so as to minimize the local velocity under constraints on both the dissipated energy and the fluid volume. Some numerical problems regarding instability are reported but are circumvented by avoiding too fine meshes.

Others that have also investigated the extension to stationary Navier–Stokes flow are Olesen et al. [27, 28]. Here, the focus is on implementation aspects, where the Navier-Stokes topology optimization problem is used to exemplify their method. Although the resulting equations are the same,

[1] A minimal surface is a surface connecting points or edges with the smallest possible area. Classical examples of minimal surfaces are soap films and bubbles.

[2] Here, note that both the velocity $\boldsymbol{u}$ and the pressure p should be regarded as dimensionless variables.

compared to [26], a different approach is used in [27] to derive the state problem (30), which more or less follows the procedure used in Paper I (cf. Chapter 2 with (7) replaced by (29)). However, the slow flow assumption is dropped (though still assuming steady flow), resulting in the additional $\nabla(\boldsymbol{u})\boldsymbol{u}$ term in the flow equation. With the introduction of the Reynolds number the state equations (30) are regained. As earlier, α is assumed to depend on the design variable and is used to interpolate between the solid and fluid regions. However, the solid phase is not interpreted as porous (as in Paper I) and thus $\overline{\alpha} \to \infty$ is assumed, but for numerical reasons set to a high but finite value. In the fluid phase, $\underline{\alpha} = 0$ is used, i.e. no fictitious porous material is introduced in the fluid region (cf. [10] where $\underline{\alpha} > 0$).

One of the two numerical examples given is the pipe problem with reversed flow, studied in [26]. Here, a prescribed pressure drop between the inlet and the outlet is used to drive the flow (instead of a prescribed velocity), and the influence of the Re number and inverse permeability $\overline{\alpha}$ is studied. It is shown that the value of $\overline{\alpha}$ strongly influences the design. In the second example, a four–terminal device is sought which minimizes the dissipated power under a constraint on the available amount of fluid. It is found that the solution depends strongly on the initial design guess, resulting from the fact that the problem is highly non–convex.

Continuing from the analysis in [20] of the Stoke flow problem, in [29] Evgrafov investigates theoretical aspects of the generalization to Navier–Stokes flow. Here, it is shown that a straightforward generalization of the Stokes problem treated above (again allowing the limiting values zero and infinity on α), will make the corresponding Navier–Stokes problem ill–posed. However, by introducing the filtering technique described in Chapter 2, as well as relaxing the incompressibility constraint on the fluid, the existence of solutions can be established. In addition, Evgrafov claims that this proof will hold for a variety of objective functions.

3.3.1 Alternative methods for flow simulation

A different approach to attack topology optimization in Navier–Stokes flow is taken by Pingen et al. [30]. Instead of solving the Navier–Stokes equations by some finite element or finite volume method, the flow system is simulated using the lattice Boltzmann method (LBM). Without going into detail, this method models the fluid as consisting of fictive particles that collide and propagate in a fixed lattice mesh, and is used to simulate low Mach num-

ber[3], incompressible flow problems. The main benefits of the LBM include the ability to handle complex geometries, and simple parallelization of the numerical scheme for solving large scale problems.

In [30], a minor modification is made to the standard LBM in order to model flow in a porous material, enabling it to be used in topology optimization formulations. The approach is compared to the standard Navier–Stokes formulation, and several examples are provided, including the pipe–bend and the rugby ball problems. However, although the results compare well with others, some numerical issues, such as the efficiency of the fluid flow solver, and problems with convergence of the optimization algorithm, need to be solved before the LBM can be considered competitive.

Another approach is explored by Evgrafov et al. [31]. Again, the goal is to approximate the fluid flow system in a topology optimization problem for Navier-Stokes related flows, but here using kinetic theory[4] instead. In this strictly mathematical paper, this theory is used to derive a model of the flow system that, at least formally, converges to the corresponding Navier-Stokes related system. From this, a topology optimization problem is formulated, for which proof of the existence of solutions is provided. The aim of the paper is to give more insight into the design problems solved previously in the literature, and it is found that a possible mathematical explanation can be given to the seemingly black–and–white solutions encountered in minimum dissipation power problems.

3.3.2 Changing fluid distribution description

The method presented in the sample problem (see Section 2.2), where the function η is used to describe the distribution of fluid and solid regions inside a domain, is presently the most frequently used method in topology optimization. An alternative method is to use *level sets* of a scalar function to describe the different material regions. Very briefly, a level set of the function Υ is the set of points $\boldsymbol{x}$ defined by

$$\{\boldsymbol{x} : \Upsilon(\boldsymbol{x}) = C\},$$

[3]The Mach number is a dimensionless number, defined as the ratio of the current flow speed and the speed of sound of the fluid.

[4]Kinetic theory is a statistical method that relates macroscopic properties (pressure, volume) of, for example, a gas, to the microscopic properties of its molecules.

where C is some given constant. The distribution of two material regions, A and B, in a domain can then be represented by Υ according to,

$$\begin{aligned} \Upsilon(\boldsymbol{x}) > 0 \quad & \text{if } \boldsymbol{x} \text{ belongs to material A,} \\ \Upsilon(\boldsymbol{x}) = 0 \quad & \text{if } \boldsymbol{x} \text{ belongs to the A–B interface,} \\ \Upsilon(\boldsymbol{x}) < 0 \quad & \text{if } \boldsymbol{x} \text{ belongs to material B.} \end{aligned}$$

Thus, $\boldsymbol{x}$ is either an A material point, a B material point, or a point on the interface of the different regions. The design problem is then to find the function Υ such that the level set $\{\boldsymbol{x} : \Upsilon(\boldsymbol{x}) = 0\}$ describes the optimal interface (curve in 2D, surface in 3D) that separates A and B. An advantage of the level set method compared to the traditional method is that the layout of material is decoupled from the discretization of the state problem, allowing smooth representation of the design boundaries.

A slight variation of this method (the basic principle described above still applies) for fluid flow problems is proposed by Pingen et al. [32]. The main reason is to investigate if a level set description of the material regions can increase the convergence rate of the optimization process. To evaluate the method, a simple 2D four-way flow intersection problem[5] is solved and compared to the traditional method, using both the Navier–Stokes and the lattice Boltzmann equations to model the underlying flow system. The design solutions reveal expected results, with only small differences in the layout between the two methods. Sadly, the expected increase in the convergence rate of the optimization problem for the level set material description could not be established.

3.3.3 Tackling real engineering problems

A major concern regarding the fluid flow optimization methods treated so far, is that they are restricted to flow systems with low to moderate Reynolds numbers ($Re < 1000$). This limits their use to rather specialized areas when considering solving real engineering problems. A heuristic method to approach this issue is suggested by Moos et al. [33] (see also Klimetzek et al. [34]) . The main idea behind this method is to identify regions with recirculation in the flow field, as these significantly increase the pressure loss in the system. Thus, by blocking the flow in such regions, minimum pressure loss fluid devices can be constructed. The procedure is built into a commercial

[5]The problem is defined as to find the layout connecting two inflow boundaries with two outflow boundaries (the inflows and the outflows are located opposite each other, respectively) on a square domain that minimizes the total pressure loss, under a constraint on the available amount of fluid.

iterative CFD[6] solver, where the recirculation regions are identified and filled with "numerical sand", i.e., decreasing the permeability of the region, inside the normal iteration loop. The method is terminated when the amount of "numerical sand" used in the domain has converged. In one of the examples provided, the now classical 2D 90 degree pipe bend, a significant (almost 60 percent) decrease in pressure loss compared to a manually designed torus shaped layout, was achieved. In another example, a 3D problem is solved representing the air duct for a climate control system in a car. Again, a large decrease (50 percent) in pressure drop was achieved compared to the original design. However, despite the quite promising results, the practical use of the method is limited: only minimum pressure loss problems can be solved, since other types of objective functions can not be handled.

Another approach, not limited to a specific objective function as in [34], is suggested by Othmer et al. [35][7]. Here, it is realized that in order to use general purpose gradient based optimization methods to solve the design problems, sensitivity information is needed — something that is rarely available in professional CFD solvers (yet). For a proof–of–concept, Othmer et al. implement an adjoint method (see [2] for an overview) into a research level CFD solver to calculate the sensitivities, where an automatic differentiation tool is used to preform the adjoint analysis. To show the flexibility of the implementation, a number of problems are solved using different objective functions in each case. In one example, two objectives are used for optimizing the same air duct: first for minimum dissipated power and then for uniformity of the flow through the outlet boundary. As expected, the resulting solutions differ considerably. Another objective function measures the angular momentum of the flow in a certain plane. This is used in an example where the intake port to a combustion chamber should be designed such that swirl motion is maximized (thus enhancing the mixing of air and fuel in the engine).

In [36], Othmer presents a refinement of the method. Instead of using automatic differentiation, expressions for the adjoint equations are derived theoretically and used for calculating the sensitivities. Also, the implementation platform is moved from the old research level code, to a professional CFD solver. In the two examples provided to verify the method, the 2D 90 degree pipe bend problem is optimized with respect to the first two objective functions given above, i.e., dissipated power and uniformity of the outflow. However, a more challenging problem is solved by Othmer et al. in [37]. Here,

[6] Computational Fluid Dynamics

[7] Actually, in the first part of the paper a method quite similar to the one used in [34] is treated, suffering from the same limitations.

a 3D fluid distribution device, with one inflow and four outflow boundaries, is optimized for minimal dissipated power. The domain was discretized using an impressive 7.5 million cells and solved for a Reynolds number of about 250000.

3.4 Multiphysic flow

The first steps to advance topology optimization of fluid flows to the multiphysic domain, were taken by Thellner in [38]. Based on the Stokes flow system from [10], he extended the problem to include a steady–state temperature field θ on top of the velocity and pressure fields. To account for this, corresponding state equations for the fluid and solid phase are introduced (in addition to (25) describing the flow) according to

$$\nabla \cdot (-k(\eta)\nabla\theta + c(\eta)\theta\boldsymbol{u}) = r. \tag{32}$$

Here,

- $k(\eta) = \overline{k} + (\underline{k} - \overline{k})\eta$ is a linear interpolation function for the heat conduction coefficients of the solid and fluid regions (if $k_{solid} > k_{fluid}$ then $\overline{k} = k_{solid}$ and $\underline{k} = k_{fluid}$),
- $c(\eta) = c\eta$ denotes the linear interpolation function for the specific heat of the fluid, and
- r represents heat sources in the domain.

Then, for $\eta = 1$, (32) represents the energy equation for the fluid region, and for $\eta = 0$ the corresponding equation for the solid region.

The optimization problem is defined so as to minimize a weighted objective function, consisting of the integral average of the temperature field and a function corresponding to (17), under the usual constraint on available fluid. By numerical experimentation, instabilities are observed in the solution, so the filter technique described in the sample problem is adopted. In the numerical examples provided, fluid devices that transport heat generated inside the domain are designed. One of these is shown in Figure 7, illustrating the influence of the inflow velocity.

Removing the design dependence in the temperature equation (32), the method is adopted by Gersborg–Hansen [39] to investigate the problem of finding the layout of a pipe that mixes two fluids with different inlet temperatures as effectively as possible. In this formulation, the measured property to be minimized is the difference between the outlet and the average of

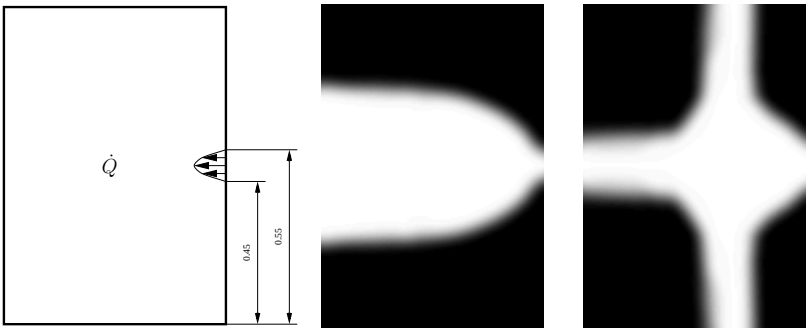

Figure 7: A fluid device transporting heat generated inside the domain. The left picture shows the design domain with boundary conditions. The inflow velocity for the design solution in the middle is ten times lower compared to the solution on the right. The figure is taken from [38].

the inlet temperatures, respectively, and constraints include upper limits on pressure drop and the available amount of fluid in the domain (although it is debatable whether or not the constraint on the fluid is actually needed in the formulation). The resulting design guides the fluid in a sinusoidal manner through the pipe, with a 15 percent decrease in objective value compared to a straight pipe.

A related extension, but instead based on the Navier-Stokes flow problem in [26], is briefly treated by Okkels et al. [40][8]. In a similar way as in [38], a state equation for the temperature field is added, but here only the heat capacity $c(\eta)$ is used as an interpolation function in (32) (i.e. k is assumed to be equal for both regions). Given a constraint on a maximum pressure drop between inlet and outlet, the goal is to design a fluidic device which minimizes the average temperature of a certain part of the domain where heat is added. In the single example provided, a simple cooling device is constructed.

3.4.1 From temperature to concentration

By merely changing the interpretation of the variables and choosing suitable values for the parameters, the state equation (32) can be used to describe the variation of the concentration of a substance in a domain. Letting θ represent the concentration, r express substance sources (or sinks), and k denote the

[8]This article also contains a part which has already been covered in [28].

diffusivity, as well as taking $c = 1$, one retrieves the equation used in Okkels and Bruus [41] to describe the nutrition variation in a reaction chamber for biological cell cultures. In this paper, the aim is to find the distribution of cell cultures such that the metabolic rate is maximized. The nutrition is carried by a bulk fluid governed by the Navier–Stokes equations, which is driven by a constant pressure drop between inlet and outlet in the reaction chamber. In the chamber, the cell cultures are located in regions of porous material where the nutrition is consumed. In the formulation of the optimization problem, no constraints need to be specified, since both extreme cases — the domain completely filled with either fluid or porous material — are non–optimal (in a fluid–only domain the metabolic rate is zero, and in a porous material–only domain the nutrition supply is cut off due to very low flow velocity). Design solutions for a simple square domain revealed a non–trivial distribution of the cell cultures that increased the metabolic rate in the bio–reactor by a factor of about 20, compared to a reference layout with a uniform cell distribution.

Integrating fluid flow with substance concentration is also the theme for Paper IV. In this case, however, the fluid is interpreted as air and the substance as moisture, and the goal is to develop a topology optimization problem in order to investigate crack formations in a drying periodic porous material. Starting from general 3D relations for the flow and the concentration variation, a planar model is developed for the air and moisture mixture, describing the out–of–plane (i.e., through the plane) flow of the mixture and the in–plane variation of the moisture concentration. The optimization problem is defined so as to maximize the average concentration in the domain, or equivalently to maximize the moisture flux to the surroundings, where the layout of the crack pattern, as well as a parameter describing the length scale of the domain, are taken as optimization variables. Two constraints are used, where the first one limits the total region that can be occupied by cracks, and the second one forces the crack layout to be connected. To solve this highly nonlinear problem, a rather intricate solution scheme is adopted. However, due to the simplifications made in the model, most crack patterns generated are more or less distorted versions of the hexagon pattern shown in Figure 8. Despite this, it is concluded from the two examples provided in the paper that certain characteristics in crack formations reported in the literature are captured by the method.

Apparent from the works reviewed so far in this section (and from earlier sections as well), is that many applications are successfully modelled as planar problems. However, in most cases this formulation is derived by simply omitting one of the dimensions in the more general three dimensional model. In the last paper presented in this dissertation, Paper V, an attempt is made to derive a general planar equation framework for fluid and

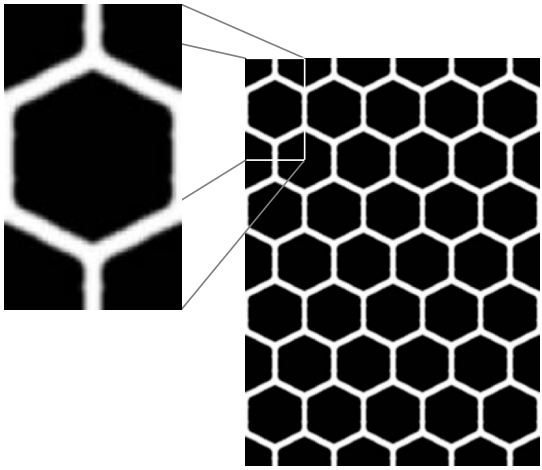

Figure 8: A typical crack pattern generated using the method described in Paper IV.

temperature/concentration flow problems. The advantage is that when using this framework to develop explicit flow models, certain effects from the out–of–plane boundary can be included in the model — something that normally would require solving the full 3D system. The use of the framework is demonstrated by deriving a slight generalization of the state equations used by Thellner [38].

3.4.2 Considering fluid flow and elasticity

With the aim of designing materials optimized for both maximum permeability *and* stiffness, Guest and Prévost extend their previous work ([25], see the end of Section 3.2) in [42] to also consider linear elasticity in the periodic base cell. Just as for the permeability, homogenization is used to compute the effective elasticity tensor of the macroscopic material. Since both permeability and stiffness should be considered in the optimization process, a weighted objective function is used, constructed from the homogenized permeability and elasticity tensors. The weight parameter then enables the designer to control the relative importance of each property in the resulting material. Following their previous work [25], the optimization problem formulation includes constraints on both isotropy of the macroscopic material properties, and an upper bound on the amount of material allowed to be distributed. The design method is tested on a 3D base cell, where the resulting layout is

shown to depend strongly on the weight parameter in the objective function.

Another example where a combination of Stokes flow and linear elasticity is used, is provided by Amstutz and Andrä [43]. This paper actually focuses on a new algorithm for the level set material description method, where a number of examples from different disciplines are solved to verify the approach. Briefly treating the flow–stiffness example, without considering the advances of the level set method (as that is outside the scope of this review), the objective is to determine the layout of channels inside a permeable ceramic filter. Because of a high pressure load on the outside boundary of the domain, the stiffness of the design needs to be considered in the optimization formulation, in addition to the flow. Using a domain shaped as an ellipse, the filter is optimized with respect to maximum flow rate, and in the resulting designs, material supports develops perpendicular to the major axis of the elliptic domain.

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