Quantum Paradoxes, Probability Theory, and Change of Ensemble

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Abstract

In this thesis, the question “What kind of models can be used to describe microcosmos?” will be discussed. Being difficult and very large in scope, the question has here been restricted to whether or not Local Realistic models can be used to describe Quantum-Mechanical processes, one of a collection of questions often referred to as Quantum Paradoxes. Two such paradoxes will be investigated using techniques from probability theory: the Bell inequality and the Greenberger-Horne-Zeilinger (GHZ) paradox.

A problem with the two mentioned paradoxes is that they are only valid when the detectors are 100% efficient, whereas present experimental efficiency is much lower than that. Here, an approach is presented which enables a generalization of both the Bell inequality and the GHZ paradox to the inefficient case. This is done by introducing the concept of change of ensemble, which provides both qualitative and quantitative information on the nature of the “loophole” in the 100% efficiency prerequisite, and is more fundamental in this regard than the efficiency concept. Efficiency estimates are presented which are easy to obtain from experimental coincidence data, and a connection is established between these estimates and the concept of change of ensemble.

The concept is also studied in the context of Franson interferometry, where the Bell inequality cannot immediately be used. Unexpected subtleties occur when trying to establish whether or not a Local Realistic model of the data is possible even in the ideal case. A Local Realistic model of the experiment is presented, but nevertheless, by introducing an additional requirement on the experimental setup it is possible to refute the mentioned model and show that no other Local Realistic model exists.
# Contents

1 Introduction and outline

2 Background
- The Einstein-Podolsky-Rosen (EPR) paradox
- Bohr’s response
- Hidden variables or not?
- Bohm’s version of the EPR paradox

3 Quantum-mechanical (QM) nonlocality
- Polarized light
- A very special light-source
- Two interpretations of QM
- Is QM really necessary?

4 Probability theory
- What is probability?
- Basic building-blocks
- Expectation values and correlations
- Random variables with subsets as domains

5 The Bell inequality
- The Bell inequality and its proof
- “Violation” of the Bell inequality
- The need for a generalized Bell inequality
- Generalization of the Bell inequality
- Detector-efficiency
- The 82.83% efficiency bound

6 A Local Hidden-Variable (LHV) model
- QM statistics
- Sinusoidal probabilities
- Independent errors
- Visibility problems
- The search for necessary and sufficient conditions
# 7 The Greenberger-Horne-Zeilinger (GHZ) paradox

7.1 The GHZ paradox and its proof ........................................ 53
7.2 Generalization of the GHZ paradox ................................... 54
7.3 Necessary and sufficient detector-efficiency conditions .......... 57
7.4 Independent errors .......................................................... 62

# 8 The Franson experiment

8.1 One-particle interferometry ............................................. 68
8.2 Two-particle interferometry ............................................. 71
8.3 A LHV model for the Franson experiment ................................ 74
8.4 Fast switching ............................................................... 77
8.5 A “chained” Bell inequality ............................................. 79

# 9 Interpretation

9.1 QM chance ................................................................. 83
9.2 Two probabilistic descriptions of QM ................................ 85
9.3 Change of ensemble ....................................................... 88
9.4 Interpretation of LHV models .......................................... 89

## Papers


## Bibliography

133
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Chapter 1

Introduction and outline

I think I can safely say that nobody understands quantum mechanics.

(Richard P. Feynman, 1967)

The following question has been widely discussed by philosophers since ancient times:

What kind of models can be used to describe micro-cosmos?

Interest in it was renewed with the birth of Quantum Mechanics (QM) at the start of the twentieth century. Given that the above question is difficult and very large in scope, the aim of this thesis is the somewhat less grand, but still interesting;

Is it possible to use Local Realistic models for Quantum-Mechanical processes?

Although the QM description agrees with experiment to a high degree of accuracy, some fundamental questions about interpretation of the description are motivated. In Chapter 2, a brief introduction will be given of the early discussion on the subject, and two different interpretations of QM.

For those not so interested in the historical background, there is a more practical introduction in Chapter 3 of the two mentioned interpretations and the concept of Local Realism. Having heard of polarized light should be enough to get you through the chapter, since QM formalism has been avoided there (and throughout the rest of the thesis, except in Chapter 9, see below). This choice was made to shorten the presentation, and to enable novices to read it. Thus, QM statistics will be motivated rather than derived, but the derivations can be found in standard textbooks of QM (see e.g., Peres, 1993), or in the case of Franson interferometry in Chapter 8, textbooks of Quantum Optics (e.g., Scully and Zubairy, 1997).

Present theoretical research in this field is largely devoted to ruling out models which cannot be used, and many physicists conjecture that in the end, only the QM description will remain. The statements used to distinguish bad models from good ones are usually on the form
Model properties

Statistical properties of measurement results.

The proof of such a statement most often entails probability theory, so in Chapter 4, a short introduction to probability theory will be given. A reader unfamiliar with the mathematical description of probability should pay special attention to the notion of random variable being a mapping between spaces.

One such result is recapitulated in Chapter 5, the Bell inequality (Bell, 1964). This result is the basis of much of the work done in this field, which is presently very active, both theoretically and experimentally. In the original Bell inequality, a prerequisite is that the detectors have to be 100% efficient. Here, following Larsson (1998a)*, the inequality is generalized by introducing the concept of change of ensemble, giving a generalization that is valid at lower levels of efficiency (down to 82.83%). Unfortunately, present experiments have efficiencies far below this bound (e.g., 5% in Weihs et al., 1998). The importance of efficient experiments is underlined in Chapter 6, where a Local Realistic model that mimics the QM statistics below 78.80% efficiency is presented, similar to that of Larsson (1999b)*.

There are quite a number of results in the spirit of Bell, and in Chapter 7 another such result is examined (Greenberger, Horne, and Zeilinger, 1989), using the approach of Larsson (1998b, 1999a)*. This result is especially interesting, because the model properties assumed yield a full contradiction with respect to the QM description. As expected, a generalization of this stronger contradiction yields a lower bound on the efficiency than in the Bell case.

In Chapter 8, we will look at the Franson experiment, a measurement setup that yields a formal violation of the Bell inequality. The setup is interesting, because the formal violation of the inequality does not rule out the kind of models that the Bell inequality is based upon; a model (100% efficient) is presented in the chapter. However, an additional prerequisite is enough to yield a violation of an extended version of the inequality. This chapter is based on Aerts, Kwiat, Larsson, and Zukowski (1999)*.

Finally, in Chapter 9, an attempt is made to interpret Local Realistic models including change of ensemble in the same manner as QM is interpreted in Chapter 2, following Larsson (2000)*. A short presentation is made of the differences between QM chance and (Kolmogorovian) probability theory as presented in Chapter 4. By restating the two interpretations from Chapter 2 and 3 in probabilistic language, a comparison is possible with the concept of change of ensemble in a Local Realistic model. This can then be used as an interpretation of QM.

*Reprints of these six papers can be found at the end of the thesis.
Chapter 2

Background

If, without in any way disturbing a system, we can predict with certainty (i.e., with a probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

(Einstein, Podolsky, and Rosen, 1935)

This chapter contains a brief review of the background to the concept of Quantum Paradoxes, i.e., results concerned with the possibility to use certain models for QM processes. The starting point is of course the classic Einstein, Podolsky, and Rosen (1935) paper and Bohr’s reply (1935), introducing the Copenhagen interpretation. The issue was thought to be closed by von Neumann’s “impossibility” proof (von Neumann, 1932, 1955) which will be mentioned, but fortunately this is not the end of the story. It is continued by the Bohm interpretation (Bohm, 1952a,b) showing that the assumptions used by von Neumann are too restrictive. At the end of the chapter, Bohm’s version of the Einstein-Podolsky-Rosen paradox (Bohm, 1951) will be presented. Of course, if one wants to go back to the sources, there is always the possibility of reading the papers in the journals they first appeared, but a collection of early papers may be found in Wheeler and Zurek (1983).

2.1 The Einstein-Podolsky-Rosen (EPR) paradox

In the EPR paper, the authors pose the question “Can the quantum-mechanical description of physical reality be considered complete?” This question is motivated by trying to judge the success of the theory, in the authors’ own words:

In attempting to judge the success of a physical theory, we may ask ourselves two questions: (1) “Is the theory correct?” and (2) “Is the description given by the theory complete?”
The first question is answered by making experiments and comparing the results to the predictions of the theory, and QM has indeed been very successful in this respect; the results coincide with QM predictions to a very high degree of accuracy. The second question is on the other hand a bit more difficult; what does it mean that the description is complete? The authors continue:

Whatever the meaning assigned to the term complete, the following requirement for a complete theory seems to be a necessary one: every element in the physical reality must have a counterpart in the physical theory.

So what, then, is an element of the physical reality? The answer has already been given at the start of this chapter, but it is very important so no harm will be done by repeating it here.

If, without in any way disturbing a system, we can predict with certainty (i.e., with a probability equal to unity) the value of a physical quantity, then there exists an element of physical reality corresponding to this physical quantity.

They proceed to discuss a gedankenexperiment (thought experiment) on a QM system consisting of two subsystems which are only allowed to interact between the time 0 and \( T \). After the time \( T \) no interaction is allowed, but if a measurement of a physical quantity is performed on one of the subsystems after the time \( T \), the QM description allows a certain prediction of the value of the corresponding physical quantity of the other subsystem. For example, measuring the position \( Q \) of one subsystem would enable us to predict the position of the other subsystem (likewise for the momentum \( P \)). Because of this certain prediction, the argument goes, the property is an element of reality. Depending on the measurement performed on the first subsystem, we can predict with certainty either property of the second subsystem without disturbing it, and therefore both the physical quantities \( P \) and \( Q \) are elements of reality. But, as the authors put it,

The usual conclusion from this [QM calculation] is that when the momentum \( P \) of a particle is known, its coordinate \( Q \) has no physical reality.

And therefore,

We are thus forced to conclude that the quantum-mechanical description of physical reality given by wave functions is not complete.

There is a final comment in their paper on the possible criticism of their reasoning, that it is not possible to predict the two properties \( P \) and \( Q \) simultaneously, since it is not possible to measure both at the first subsystem at the same time. So one could in principle argue that only one of the properties \( P \) and \( Q \) of the second subsystem belongs to the physical reality, depending on the measurement performed on the first subsystem. But, EPR’s response is that
This makes the reality of \( P \) and \( Q \) [at the second system] depend upon the process of measurement carried out on the first system, which does not disturb the second system in any way. No reasonable definition of reality would be expected to permit this.

And this is the concluding remark.

While we have thus shown that the wave function does not provide a complete description of the physical reality, we left open the question of whether or not such a description exists. We believe, however, that such a theory is possible.

### 2.2 Bohr’s response

A reaction to the EPR paper was quickly at hand, since Niels Bohr and his associates were at the time working (among other things) on what was later to be known as the Copenhagen interpretation. Léon Rosenfeld, a colleague of Bohr, later said (Rosenfeld, 1967):

> This onslaught came down upon us like a bolt from the blue. Its effect on Bohr was remarkable. . . . as soon as Bohr had heard my report of Einstein’s argument, everything else was abandoned: we had to clear up such a misunderstanding at once.

Bohr’s reply to the EPR paper was published in the same journal, in the subsequent volume, although in the same year (1935). The paper has the same title as the EPR paper, and the abstract reads:

> It is shown that a certain “criterion of reality” formulated in a recent article with the above title by A. Einstein, B. Podolsky, and N. Rosen contains an essential ambiguity when it is applied to quantum phenomena. In this connection a viewpoint termed “complementarity” is explained from which quantum-mechanical description of physical phenomena would seem to fulfill, within its scope, all rational demands of completeness.

The notion of complementarity is one of the corner-stones of the Copenhagen interpretation, stating that if we choose to measure \( P \) of a system, then the complementary property \( Q \) cannot be ascribed a meaningful value. The Bohr (1935) paper is a thorough description of the notion of complementarity and the reasons for using it. The analysis of this notion in Bohr’s argument leads to the conclusion that the definition of reality in the EPR paper is ambiguous.

From our point of view we now see that the wording of the above-mentioned criterion of physical reality proposed by Einstein, Podolsky and Rosen contains an ambiguity as regards the meaning of the expression “without in any way disturbing a system.” . . . even at this stage there is
essentially the question of an influence on the very conditions which define the possible types of predictions regarding the future behavior of the system. . . . It is just this entirely new situation as regards the description of physical phenomena, that the notion of complementarity aims at characterizing.

Bohr did not like the last comment in the EPR paper, because it seems to imply that there is something “unreasonable” in the notion of complementarity.

My main purpose in repeating these simple, and in substance well-known considerations, is to emphasize that in the phenomena concerned we are not dealing with an incomplete description characterized by the arbitrary picking out of different elements of physical reality at the cost of sacrificing other such elements, but with a rational discrimination between essentially different experimental arrangements and procedures which are suited either for an unambiguous use of the idea of a space location \( Q \), or for a legitimate application of the conservation theorem of momentum \( P \).

The central point of the argument is what it means to disturb something. Bohr argues that in principle, when measuring (e.g.) \( P \) on one subsystem, the other one is disturbed in an uncontrollable way so that whatever meaning \( Q \) had before the measurement, it is destroyed in the measurement process. At the time (in the late 1930s), this was widely accepted in the physics community, and Bohr was thought to have “set things straight” by this paper.

Recently, some authors have voiced the opinion that Bohr’s response is not as clear to the point as one might want (e.g., Mermin, 1985a; Baggott, 1992), and e.g., in Mermin (1985a) the following is said.

I learned my quantum metaphysics primarily through the writings of Heisenberg. As I understood it, the unavoidable, uncontrollable disturbances accompanying a measurement were local, “mechanical,” and not especially foreign to naive classical intuition (photons bumping into electrons in the course of a position measurement – that sort of thing). When I read the EPR paper (in the late 1950s), it gave me a shock. Bohr’s casual extension of Heisenberg’s straightforward view of a “disturbance” seemed to me radical and bold. That most physicists were not, apparently shocked at the time – that Bohr was generally and immediately viewed as having set things straight – surprised and perplexed me.

* Apparently, two pages have been interchanged in the reprint of Bohr’s paper in Wheeler and Zurek (1983). Finding out which is left as an exercise.
2.3 Hidden variables or not?

The wide acceptance of Bohr’s reply may in part depend upon the fact that at the time, there was what was thought to be a rigorous proof of the incompatibility of QM and the kind of complete description that EPR wanted. In Bohm (1952b) the following description of this result is given:

Von Neumann has studied the following question: “If the present mathematical formulation of the quantum theory and its usual probability interpretation are assumed to lead to absolutely correct results for every experiment that can ever be done, can quantum mechanical probabilities be understood in terms of any conceivable distribution over hidden parameters?” Von Neumann answers this question in the negative.

The wording hidden parameters (or hidden variables used in modern presentations) refers to some way of specifying the precise state, in this case both $P$ and $Q$, instead of using the “incomplete” QM description. The complete description would then be that of the hidden variables. Neither the precise result nor the precise prerequisites of von Neumann will be given here (see von Neumann, 1932, 1955), but one may appreciate the effect of such a statement: the physics community was certainly not going to take the idea of hidden variables seriously anymore.

There were proposals of using hidden variables, some before and some after 1932, but all were abandoned for one reason or another, more or less in connection with the von Neumann proof. That is, all were abandoned until 1952 when two papers appeared which proposed a hidden-variable interpretation of QM (Bohm, 1952a,b). The von Neumann result was simply not sufficiently general.

In these two papers, Bohm reinterprets the “wave function” as coexistent to the particles, and assumes that a particle moves in and is affected by a QM potential derived from the wave function. Then it is possible to assign definite values to the physical properties in question, so that a particle always has a definite position $P$ and momentum $Q$. These values are of course unknown, but this is only because their initial values are unknown, not for any deeper reason. In contrast, in the Copenhagen interpretation the particle does not have a definite position $Q$ nor a definite momentum $P$, the values do not exist until after measurement. There, the wave function only represents the particle’s probability of being in a certain position, and does not govern the particle’s movement in the manner described above (this is further developed in an example in Chapter 3 and also in the interpretational discussion in Chapter 9).

Then why, in Bohm’s interpretation, can we not measure both $P$ and $Q$? The answer is that the QM potential is directly affected by the measurement setup, so that setting up a measurement to measure $Q$ affects the particles so that the value of $P$ (being unknown to us) can become just about anything. The measurement of $Q$ renders us unable to determine what value $P$ had before the measurement, which is a difference compared to the Copenhagen interpretation where the value of $P$ cannot be said to exist at all. A somewhat peculiar property of the QM potential is that changes in the measurement setup propagates instantaneously (i.e., faster than light), and this is often referred to as
nonlocality. Another is that it does not fall off with distance, but can be quite large even
where the probability is small of finding the particle. But, in D. Bohm’s own words,

In any case, the mere possibility of such an interpretation proves that it is
not necessary for us to give up a precise, rational, and objective description
of individual systems at a quantum level of accuracy.

Twenty years had passed, however, and by this time the Copenhagen interpretation
was firmly settled in the physics community. This is still the case to the degree that it
is sometimes referred to as the orthodox interpretation.

2.4 Bohm’s version of the EPR paradox

Bohm’s interpretation above was in part motivated by the EPR paper, and its criticism
of the Copenhagen interpretation. Bohm made a thorough analysis of the EPR paradox,
thinking that the matter would surely be settled once and for all, but ended up in his
new interpretation of QM mentioned above. There is an aspect of the analysis itself
that is important in the remainder of this thesis. In his efforts to understand the EPR
paradox, Bohm made extensive use of a variation of it (Bohm, 1951).

The EPR-Bohm experiment is based upon spin. This is a property of a particle that
in QM takes half-integer values \(0, \frac{1}{2}, 1, \ldots\) and for instance, an electron has spin \(\frac{1}{2}\).
One may now measure how the spin is oriented along an axis, and in the electron case
the result would either be “up” or “down”. The interesting point is that spin orientations
along orthogonal axes (x, y, and z) are complementary, and thus cannot have values
simultaneously in the Copenhagen interpretation. The spin orientations along the three
axes are now our new candidates for \(P\) and \(Q\). The big improvement in the EPR-Bohm
experiment is that these spin orientations are easy to measure, in contrast to \(P\) and \(Q\).
There is for example only two possible answers, instead of the real number obtained
from a measurement of either \(P\) or \(Q\).

Imagine now that we have a pair of particles with total spin zero, which would
imply that their individual spins are opposite to each other, when measured along the
same axis. We are now in a position to make EPR-type “certain predictions” of the
spin orientation of one particle along an axis by measuring on the other particle along
the same axis. This variation of the EPR experiment made people believe, for the first
time, that it was not restricted to being a gedankenexperiment but actually could be
performed. It is now convenient to extend this to polarized light, but more on this in
the next chapter (the precise analogy will be described in Section 3.3).
Chapter 3

Quantum-mechanical (QM) nonlocality

And be warned: students are rarely told the whole truth about this theory. Instead they are fed the orthodox [Copenhagen] interpretation either by design or default... This is a great pity. Students have a right to know the truth, even if it is bizarre.

(Jim Baggott, 1992)

Only a short introduction will be given of the concepts of Local Realism to the novice in the field, in the context of the two-photon EPR-Bohm experiment. For a more complete discussion of the subject, there is a vast literature to choose from. Both novice and initiate would for instance benefit from reading Baggott (1992) which is a nice and quite comprehensive book on the subject. An author which is well-known for his clear and pedagogical presentation of the two-particle experiment is N. David Mermin, and some of the papers he has written can be found in Mermin (1990a). There are also often chapters in QM textbooks that introduce the subject (see e.g., Peres, 1993, Chapter 6).

The authority in the field is of course John S. Bell and his original paper (Bell, 1964) may be found in Wheeler and Zurek (1983), and that paper and some of his other work is collected in Bell (1987). The relativist would enjoy reading Maudlin (1994) which talks about the issue from the viewpoint of special relativity, while the philosopher may read Shimony (1993) where more philosophical aspects are discussed. This latter work consists of two volumes which are divided so that in the first volume general philosophic questions related to this issue are discussed, while in the second the specifics of the QM issues are examined in more detail. A more philosophically technical treatment may be found in Redhead (1987), but from the same author, a more easily penetrated book is Redhead (1995). Lastly, the probabilist will (probably) find a treatment to his liking in Suppes and Zanotti (1996).
3.1 Polarized light

First, some simple properties of polarized light will be pointed out. QM formalism has been avoided in this section to enable readers unfamiliar with that formalism to follow the reasoning. The phenomenon of polarized light is something people usually come in contact with in connection with polaroid sunglasses. The “measuring device” used in this section is a piece of polaroid which lets light polarized in one direction pass, and stops orthogonally polarized light from passing.

![Polarization filter](image)

(a) passes a vertical filter
(b) is absorbed in a horizontal filter

Figure 3.1: The behavior of vertically polarized light.

If the light is polarized at a $\pi/4$ angle to the filter, only half the light will pass. That is to say, after the filter the intensity of the light is halved, or in other words, only half of the energy present in the light-beam before the filter is allowed to pass through the filter.

![Figure 3.2](image)

Figure 3.2: Half of the vertically polarized light passes a $\pi/4$ filter.

The above is obtained from ordinary electro-magnetic field theory, and it is possible to derive a formula for the amount of light that passes through a filter as a function of the relative angle $\phi$ of the polarization of the light to the orientation of the filter, and the resulting amount is $\cos^2(\phi)$. That is, if $\phi = 0$, all the light passes ($\cos^2(0) = 1$; the polarization filter is directed along the polarization of the light), if $\phi = \pi/2$ none of the light passes ($\cos^2(\pi/2) = 0$; the filter is perpendicular to the polarization of the light), and if $\phi = \pi/4$ half of the light passes ($\cos^2(\pi/4) = 1/2$).

If instead we were to use the notion of photons or light packets*, one simplistic picture of this would be the following. If the photon is polarized “vertically”, it passes. If it instead is polarized “horizontally” it is absorbed in the filter.

*These packets of light were introduced by A. Einstein
If the polarization is at a \(\pi/4\) angle to the filter, only half of the energy of a light-beam passes it. The corresponding result in the photon picture is that the probability of a photon passing is \(\frac{1}{2}\).

More generally, for a relative angle \(\phi\) between the filter and the polarization of the photon, the probability of the photon passing is \(\cos^2(\phi)\). The picture described here where photons have a definite polarization is not quite the whole story, as we will see below. Nevertheless this picture is useful to explain some of the concepts we will meet further on. Now we have almost all we need to see what nonlocality is.

### 3.2 A very special light-source

The gedankenexperiment (thought experiment) used to look at nonlocality is as follows: in the experiment there is one very special light-source and two detectors. The light-source is special because it sends out pairs of photons where the pairs have the following property: if both the detectors are oriented in the same way, the result is the same, i.e., either both photons pass or both are absorbed in the filters.

An important point is that there are such light sources, or this reasoning would be without value. There are many ways to build such a light-source and there is quite a number of papers concerning this, e.g., the Aspect experiments (Aspect et al., 1981, 1982a,b) but more recently a better such source was proposed in Kwiat et al. (1995).

Now it is possible to see why the description in Section 3.1 cannot be the whole picture. Using that description of the polarization of the photon pair from the source
means that the arrows must point in the same direction, and if this is the same direction as the polarizing filters, the result is as in Fig. 3.6.

If, on the other hand, the polarization of the photons is orthogonal to the filter orientations, the result is as in Fig. 3.7.

There is no evident problem so far, but what if the polarization is neither along nor orthogonal to the orientations of the filters, e.g., at an angle of $\pi/4$ (see Fig. 3.8)? Looking at only one of the photons, the result would be that the probability of the photon passing is $1/2$, and the other photon would of course have the same probability of passing. But there is nothing to say that these events are correlated. In the simple picture used above, the probability of both photons passing would be $1/4$, and derived in the same manner, the probability of both photons being absorbed in the filters is also $1/4$. But then the pairs would not have the property that either both passes or both are absorbed.

The standard solution to this is to use QM, but as previously stated the formalism will be avoided in this thesis. Instead the two interpretations of QM mentioned in Chapter 2 will be briefly described.
Figure 3.8: The photons are polarized at a $\frac{\pi}{4}$ angle to the orientations of the filters. Here, there are four different equally likely combinations.

### 3.3 Two interpretations of QM

Let us first look at the Copenhagen interpretation, where there is no polarization state prior to the measurement. The picture before the measurement would therefore look something like Fig. 3.9.

The QM formalism then describes only what happens at the measurement. The probabilities of each measurement result are obtained from QM, but the QM description has no other meaning than this. The outcome of an experiment is decided at the instant of measurement (see Fig. 3.10), and it is meaningless to ask what happens before the measurement.†

The Copenhagen interpretation does not look at all like the simplistic picture presented in Section 3.1. But in the other interpretation, the Bohm interpretation from Section 2.4, the situation is different. There, the photons have a definite polarization also before the measurement, but there is a “QM potential” that affects the polarization as the photons approach the filters.

†The analogy between the above setting and the EPR-Bohm spin $\frac{1}{2}$ (electron) experiment in Section 2.4 is that the polarization of the photon when the filter is oriented in the $0$ angle is complementary to the polarization when the filter is oriented in the $\frac{\pi}{4}$ angle. These two cannot then have values simultaneously in the Copenhagen interpretation.
This potential is obtained from QM description of the photons and the resulting statistics coincides with the QM statistics. The potential guides the photon polarizations so that the situation in Fig. 3.8 is avoided, and again the pair has the property that either both pass the filters or both are absorbed. The problem here appears when changing the measurement setup. In the Bohm interpretation, the outcome at one filter does not only depend upon the initial polarization of the photon and the orientation of the filter in question, but also the orientation of the other filter. The QM potential reacts to any changes of the measurement setup and guides the polarizations differently.

This reaction is instantaneous, so that the propagation speed of changes in the QM potential is infinite. But then it is faster than the speed of light, and this is impossible according to relativity theory. This is often referred to as the “nonlocal” property of QM.

### 3.4 Is QM really necessary?

One would perhaps like to retain the simplistic picture used in Section 3.1, but the QM interpretations above indicates that this is not possible. It is possible to modify the simplistic model so that it yields the QM results (and the experimental data) in the case of equally oriented detectors. By making a slight change to the properties of the filter, we obtain the desired result.

The $\lambda$ is the “polarization” of the photon, and in this model it is uniformly distributed over all directions ($\lambda$ can have its tip anywhere on the circle). The source is modeled so that the $\lambda$'s of the two photons in a pair are opposite to each other. When the polarization is measured, the outcome is determined by the direction of $\lambda$: If it is directed up or down, the result is “vertical polarization” ("pass"). If it is directed right
or left, the result is “horizontal polarization” (= “absorption”). This model yields equal results for equally oriented filters.

However, when the filters are not parallel, the model does not follow the QM predictions; in QM, the probability of both photons passing is \( \frac{1}{2} \cos^2(\phi) \) (note the similar form of this probability to that of a photon passing the filter in Section 3.1). When using the model described above, the result is quite different: the probability would be \( 1 - \frac{2|\phi|}{\pi} \) for \( |\phi| \leq \pi/2 \), which is a straight line instead of the sinusoidal curve from QM. The calculation is easily done in Fig. 3.14, where one can immediately see that the probability decreases from 1 to 0 linearly with \( \phi \) when \( \phi \) increases from 0 to \( \pi/2 \).

The difference between the QM prediction and the one obtained from this model can be seen in Fig. 3.15.

We obtained equal results at equal filter orientation by a change to the simple model of photon polarization used previously in this chapter. However, it is not possible to change the model so that it yields the sinusoidal probability obtained from QM. This is proved by using “the Bell inequality” (Bell, 1964), a statistical inequality that is valid for a model of the above type, but is violated by the QM predictions for the above
measurement setup. That is, the Bell inequality is valid if we insist that the model should follow the two intuitive concepts used in construction of the above model:

(i) **Realism.** The polarization should be decided at the source (as opposed to the Copenhagen interpretation).

(ii) **Locality.** The result at one filter should not depend on the orientation of the other (as opposed to the Bohm interpretation).

We will look more closely at the Bell inequality in Chapter 5. First, a short presentation of probability theory is in order, as the inequality can most clearly be stated in that formalism.
Chapter 4

Probability theory

. . . . a basic course in probability should offer a broad perspective of the open field and prepare the student for various possibilities of study and research. To this aim he must acquire knowledge of ideas and practice in methods, and dwell with them long enough to reap the benefits.

(K. L. Chung, 1974)

This presentation is not meant to be a complete course in probability theory, but should be thought of as a short introduction.

To the layman, probability theory seems to be a convenient way of calculating probabilities. Many remember the probability theory they were taught in school as a way of counting outcomes, e.g., “there is an urn with seven black and three white marbles in it . . . “. In the presentation given here, the more mathematically formal way of probability theory will be stressed, but first some interpretations of the concept of probability will be presented.

4.1 What is probability?

As always, when encountering a concept, it is good to have an intuitive picture of what the concept “means”. Historically, the probability concept was most often used in connection with games of chance, and this gave rise to the “classical probability concept”.

If there are $n$ equally likely possibilities, of which one must occur,
and $w$ of them are regarded as favorable (a “win”),
then the probability of a “win” is $w/n$. 
This is where the counting of outcomes becomes important. One needs to find out how many possible outcomes there are and how many of them are favorable. A six-sided die as an example has equal probability that any one side faces up after a die roll, and the die “always” ends up with one of the sides facing up. Thus the probability of, say, rolling a 3 is $\frac{1}{6}$.

But what does the “classical probability concept” imply e.g. in a game of craps? What does probability mean in experiments? And if the various possibilities cannot be regarded as equally likely? These questions call for another concept, the “frequency interpretation”, and this is the most common intuitive picture of probability.

The probability of an event is the proportion of tries for which the event will occur in the long run.

This is in essence the statement: “If I roll the die again and again, I will roll a 3 on the average $\frac{1}{6}$ of the total number of rolls.” An interesting comment is that this does allow events of probability 0 (zero) to occur, as long as the proportion of that particular event approaches zero in the long run. So it is a misconception to think that events of probability zero never happens; they are just very very rare.

There are other interpretations of the probability concept, such as “subjective evaluations”, a concept which is concerned with one’s belief with regard to the uncertainties that are involved. This last concept is more applicable in situations where a decision is going to be made on the basis of a probability calculation. But since the below is concerned with the outcomes of physical experiments, the “frequency interpretation” is perhaps the best concept to use of the above.

### 4.2 Basic building-blocks

The mathematical foundations of probability theory is a subject which takes quite a bit of formalism to introduce. As has been said before, only the core concepts of the theory will be presented here. For a full explanation of these things, see one of the many textbooks on this subject (e.g., Chung, 1974). We will begin by defining the basic object of probability theory.

This basic object is a **probability space**, which consists of three different parts. One is the **sample space** $\Lambda$ which is a space of points describing an experiment, and this description may be rather complicated. For example, a point in this space may correspond to a full description of the tumbling of a die when rolled, complete with the position and momentum of the die from the instant one lets it go, to when it lies on the table in a certain orientation.

Another part of a probability space is a collection (a “family”) of sets denoted $\mathcal{F}$ of points in $\Lambda$, which are normally referred to as **events**. For instance, the set of die rolls resulting in a 3 is such an event. This family should have some properties which will not be described in detail, but the mathematical name for these properties is that $\mathcal{F}$ should be a $\sigma$-algebra.
The final part of this basic object is a probability measure $P$ on $\mathcal{F}$, i.e., a measure that tells us the probability (the “size”) of a set in $\mathcal{F}$. The probability of the set of die rolls resulting in a 3 would then be $\frac{1}{6}$. More formally this may be stated as below.

**Definition 4.1 (Probability space)**

A probability space is the triple $(\Lambda, \mathcal{F}, P)$ consisting of

\[
\begin{align*}
\Lambda, & \quad \text{the sample space,} \\
\mathcal{F}, & \quad \text{a } \sigma\text{-algebra of sets in } \Lambda, \\
P, & \quad \text{a probability measure on } \mathcal{F}.
\end{align*}
\]

The probability measure should be such that it follows

**Definition 4.2 (Kolmogorov axioms)**

(i) For any event $A \in \mathcal{F}$, we have $0 \leq P(A) \leq 1$

(ii) For the sample space $\Lambda$, we have $P(\Lambda) = 1$

(iii) For any sequence of events $A_1, A_2, \ldots \subset \mathcal{F}$ which are mutually exclusive, i.e., events for which $A_i \cap A_j = \emptyset$ when $i \neq j$, then

\[
P\left( \bigcup_{i=1}^{\infty} A_i \right) = \sum_{i=1}^{\infty} P(A_i).
\]

Since the sample space $\Lambda$ may contain a thorough description of a die roll, and all we really are interested in is the outcome, we introduce the concept of a random variable. This is a mapping from the sample space to (preferably) some number field, e.g., the real numbers. This means that in the sample space of the die roll above, the set of points where the result is a 3 is mapped onto the number 3. Which of course removes quite a lot of information, but on the other hand is much more simple to use.

**Definition 4.3 (Real-valued random variable)**

A real-valued random variable is a mapping from the sample space to the real numbers.

\[
X : \Lambda \to \mathbb{R}
\]

\[
\lambda \mapsto X(\lambda)
\]

The probability of rolling a 3 is then denoted as follows (on the right-hand side is a shortened notation which simplifies the expressions significantly).

\[
P\left( \{ \lambda \in \Lambda : X(\lambda) = 3 \} \right) = P(X = 3).
\] (4.1)
4.3 Expectation values and correlations

Another valid question is what the average value would be on a die roll. A simple calculation would give the answer: take the possible values and multiply them with the probability for each value. Add the resulting numbers, and we have the average die roll. For a six-sided die this is

\[ P(X = 1) + 2P(X = 2) + 3P(X = 3) + 4P(X = 4) + 5P(X = 5) + 6P(X = 6) \]
\[ = \frac{1 + 2 + 3 + 4 + 5 + 6}{6} = 3\frac{1}{2}. \]  

This is referred to as expectation value, i.e., the value one would expect to get. Note however that this is just saying that the result of a six-side die would on the average be \( 3\frac{1}{2} \), not that the result would be \( 3\frac{1}{2} \) most of the time (one can only roll integer numbers on a six-sided die). A mathematical definition of this concept would be as follows.

**Definition 4.4 (Expectation value)**

The expectation value of the real-valued random variable \( X \) is defined as

\[ E(X) = \int_{\Lambda} X(\lambda) dP(\lambda) = \int_{\Lambda} X dP. \]

The definition above contains rather a lot of theory, most notably measure and integration theory, and again this is out of the scope of this thesis. Expectation values have some nice properties, e.g. \( (X \) and \( Y \) are random variables, and \( a \) is a real constant)

\[ E(aX) = aE(X), \quad (4.3a) \]
\[ E(X) + E(Y) = E(X + Y), \quad (4.3b) \]
\[ |E(X)| \leq E(|X|). \quad (4.3c) \]

But what about products? Products are somewhat more difficult to use, because in general,

\[ E(XY) \neq E(X)E(Y). \quad (4.4) \]

This motivates the introduction of another concept, the covariance which is really the difference between the left-hand and right-hand side in (4.4).

**Definition 4.5 (Covariance)**

The covariance of two real-valued random variables \( X \) and \( Y \) is defined as

\[ \text{Cov}(X, Y) = E(XY) - E(X)E(Y). \]

It may seem difficult to obtain an intuitive picture of the above, but to simplify, assume for a moment that the expectation of \( X \) and \( Y \) is zero. It is now easier to see that the covariance is some sort of measure of dependence, because the expectation of the
product is positive if \( X \) on the average has the same sign as \( Y \), and negative if the sign of \( X \) tends to be opposite to the sign of \( Y \). However, since
\[
\text{Cov}(aX, Y) = E(aXY) - E(aX)E(Y) = a(E(XY) - E(X)E(Y)) = a \cdot \text{Cov}(X, Y),
\]
there is a problem; it seems that \( aX \) is more dependent of \( Y \) than \( X \) is. We would like the measure to show only the dependence, not being affected of the general size of the random variables. And this is why we need the last definition, that of correlation.

**Definition 4.6 (Correlation)**
The correlation of two real-valued random variables \( X \) and \( Y \) is defined as
\[
\rho(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Cov}(X, X) \text{Cov}(Y, Y)}}.
\]

Now the \( a \) from (4.5) cancels out if \( a > 0 \), so that \( \rho(aX, Y) = \rho(X, Y) \). Furthermore, \(-1 \leq \rho(X, Y) \leq 1\) can be derived from this, and (assuming for simplicity that \( E(X) = E(Y) = 0 \) and that \( \text{Cov}(X, X) = \text{Cov}(Y, Y) = 1 \))
\[
\rho(X, Y) = 1 \Rightarrow P(X = Y) = 1, \quad (4.6a)
\]
\[
\rho(X, Y) = -1 \Rightarrow P(X = -Y) = 1. \quad (4.6b)
\]

This measure tells us the level of linear dependence between the random variables, and the last observation shows the extremes, total correlation or total anti-correlation.

In many papers in physics journals the expectation of the product \( E(XY) \) is referred to as “correlation” and the reason for this is that in the simple cases used as a starting point in these papers, the random variables have expectation zero and can only take the values \( \pm 1 \) so that \( E(X^2) = E(Y^2) = 1 \). A short calculation then shows that the correlation is precisely the expectation of the product. In these papers, this terminology is retained even for cases when this ceases to be valid, and it will be used in this thesis as well as it has become the standard terminology in this context.

### 4.4 Random variables with subsets as domains

In the latter parts of the thesis, a generalization of the previous definition of random variable will be needed. This is for example to make it possible to generalize the Bell inequality in Chapter 5. We will need to have random variables with a domain that may be a proper subset of the whole sample space.

**Definition 4.7 (Real-valued random variable)**
A real-valued random variable is a mapping from a (possibly proper) subset \( \Lambda_X \in \mathcal{F} \) of the sample space to the real numbers.
\[
X : \Lambda_X \rightarrow \mathbb{R}, \quad \lambda \mapsto X(\lambda)
\]
This definition implies that our previous definition of expectation also must be modified to take into account that we cannot integrate on the whole of the sample space any longer. We must restrict the integration to the subset on which the random variable is defined. Then the measure used in the integration must also be changed to reflect that the set we are looking at is “the” probability space of the random variable in question.

**Definition 4.8 (Expectation value)**

The expectation value of the real-valued random variable $X$ is defined as

$$E_X(X) = \int_{\Lambda_X} X(\lambda) dP_X(\lambda) = \int_{\Lambda_X} X dP_X,$$

where the probability measure $P_X$ is defined on the trace of $\mathcal{F}$ on $\Lambda_X$ as

$$P_X(E) = P(E|\Lambda_X) = \frac{P(E \cap \Lambda_X)}{P(\Lambda_X)}.$$

This definition of an expectation also has some nice properties, e.g.,

$$E_X(aX) = aE_X(X), \quad (4.7a)$$

$$|E_X(X)| \leq E_X(|X|). \quad (4.7b)$$

But as can be seen in (4.8) below, there is a new problem when trying to add expectation values. To be able to calculate the expectation of $X + Y$, it is necessary to restrict the integration to the subset where the sum is defined, i.e., the set $\Lambda_{XY} = \Lambda_X \cap \Lambda_Y$ where both $X$ and $Y$ are defined.

$$E_{XY}(X) + E_{XY}(Y) = E_{XY}(X + Y), \quad (4.8)$$

It is generally the case that

$$E_X(X) + E_Y(Y) \neq E_{XY}(X + Y), \quad (4.9)$$

because of the fact that the domains of $X$ and $Y$ may be different. This will be investigated to some extent in the following chapters.
Chapter 5

The Bell inequality

Anyone who’s not bothered by Bell’s theorem has to have rocks in his head.

(N. David Mermin, 1985b, quoting a friend)

This chapter is based on Larsson (1998a)*, and will be concerned with the Bell inequality (Bell, 1964), the first result that stated the impossibility of constructing a Local Realistic (local hidden-variable, LHV) model that yields QM results. This inequality gave rise to a new research field in physics, which could be called “Experimental Metaphysics”. It was now possible to test Local Realism against QM, and more importantly, check if Nature allows a Local Realistic description. The experiments made to verify this have not been altogether conclusive, but they point quite decisively in a certain direction: Nature cannot be described by a Local Realistic model (see e.g., Aspect et al., 1981, 1982a,b; Weihs et al., 1998). There has been claims that the issue is settled by experimental evidence, but these claims are based on an additional assumption that will be avoided here (see further discussion in Section 9.3).

In Larsson (1998a)* the spin $\frac{1}{2}$ experiment described in Bell’s original paper has been chosen to adhere to his treatment of the problem, but in this thesis, it is rewritten in the polarized light setting (the violation is the same in both cases).

![Diagram of the two-photon EPR experiment]

Figure 5.1: The two-photon EPR experiment.

*A reprint of this paper can be found at the end of the thesis.
In Fig. 5.1, the source is the same source as discussed in Chapter 3, i.e., if both detectors have the same orientation, both photons in a pair yield the same result. The detectors are, however, not the simple filters described in Chapter 3, but slightly more sophisticated two-channel detectors which signal both “vertical” and “horizontal” polarization instead of the filters’ “vertical” or “nothing”.

The notation is as follows: $\lambda$ (the “hidden variable”) is a point in a probabilistic sample space $\Lambda$. The detector orientations which specify what directions are “horizontal” and “vertical” (i.e., orthogonal to the former) are labeled $\phi_A$ and $\phi_B$. Finally the measurement results are given by random variables labeled $A$ and $B'$ where unprimed letters correspond to detector 1 and primed ones to detector 2. These random variables are assigned the values $+1$ and $-1$ corresponding to “horizontal” and “vertical” polarization. I have, as in the papers, chosen to only discuss the “deterministic” case here, since the generalization to the “stochastic” case is straightforward. For a more in-depth discussion of this, see Stapp (1980) and Fine (1982).

It is also important to know how the locality prerequisite is enforced in experiments, and this is done by rapid random changes of the orientations $\phi_A$ and $\phi_B$, at a rate high enough to prohibit communication of the setting from one detector to the other (at or below the light-speed $c$). Usually, the rate is chosen to prohibit communication from a detector to the source; this is to prevent the possibility that the probability space $(\Lambda, \mathcal{F}, P)$ depends on the detector settings. An enforcement of this kind is difficult to achieve in practice; it is attempted in Aspect et al. (1982a) but the variation there is pseudo-random, which can be criticized. A more recent experiment is Weihs et al. (1998), where the random number generators used at the detector are “more truly” random to their nature.

The source used in the early experiments was atomic calcium excited by laser light, which subsequently emitted two correlated photons (see e.g. Aspect et al., 1981, 1982a,b). This source was rather difficult to set up and use, but recently, a source simpler in construction and usage was devised which consists of a pulsed laser and a nonlinear crystal. The group that devised this source (Kwiat et al., 1995) reports that using a 150 mW argon laser and a beta-barium-borate crystal,

\[ \ldots \text{we demonstrated a violation of Bell’s inequality by over 100 standard deviations in less than 5 min.} \]

**5.1 The Bell inequality and its proof**

Perhaps first a small note on terminology here. In some presentations, the Bell inequality and the Bell theorem are two different statements. The Bell inequality is then only the statistical inequality, without any reference to QM or the QM violation of the inequality, and the Bell theorem is the statement that it is not possible to have a Local Realistic model that yields the QM statistics. Here, the Bell inequality is stated in Theorem 5.1, while the QM violation of the inequality is presented at the end of this section (but not in the form of a theorem).
Theorem 5.1 (The Bell inequality)

The following four prerequisites are assumed to hold except at a null set:

(i) **Realism.** Measurement results can be described by probability theory, using (two families of) random variables.

\[
\begin{align*}
A(\phi_A, \phi_B) : \Lambda &\rightarrow V \\
\lambda &\mapsto A(\phi_A, \phi_B, \lambda) \\
B'(\phi_A, \phi_B) : \Lambda &\rightarrow V \\
\lambda &\mapsto B'(\phi_A, \phi_B, \lambda)
\end{align*}
\]

\[\forall \phi_A, \phi_B.\]

(ii) **Locality.** A measurement result should be independent of the detector orientation at the other site,

\[
A(\phi_A, \lambda) \overset{\text{def}}{=} A(\phi_A, \phi_B, \lambda), \text{ independently of } \phi_B
\]

\[
B'(\phi_B, \lambda) \overset{\text{def}}{=} B'(\phi_A, \phi_B, \lambda), \text{ independently of } \phi_A
\]

(iii) **Measurement result restriction.** Only the results $+1$ and $-1$ should be possible:

\[V = \{-1, +1\}.
\]

(iv) **Perfect correlation.** A measurement with equally oriented detectors must yield equal results at the two detectors,

\[\phi_A = \phi_B \implies A(\phi_A, \phi_B, \lambda) = B'(\phi_A, \phi_B, \lambda).
\]

Then

\[|E(AB') - E(AC')| \leq 1 - E(BC').\]

The last expression is shortened somewhat by adopting the notation $A = A(\phi_A, \lambda)$, $B = A(\phi_B, \lambda)$, $B' = B'(\phi_B, \lambda)$, and $C' = B'(\phi_C, \lambda)$, so that the direction is indicated by the label of the random variable itself rather than the index of the detector orientation.

**Proof:** By pairing the two random variables corresponding to the same settings, e.g., $AB' = A(\phi_A, \phi_B, \lambda)B'(\phi_A, \phi_B, \lambda)$, $AC' = A(\phi_A, \phi_C, \lambda)C'(\phi_A, \phi_C, \lambda)$, the shortened notation can be retained in the proof while still making explicit the usage of the prerequisites:
\[ E(AB') - E(AC') \]
\[
\left( i \right) \int AB' \, dP - \int AC' \, dP \left\| \overset{\text{+f}}{=} \right\| \int AB' - AC' \, dP \\
\left( ii \right) \int AB' - AC'B^2 \, dP \left\| \overset{\text{+f}}{=} \right\| \int AB' - AC'B^2 \, dP \\
\left( iii \right) \int AB' - AB'BC \, dP \overset{\text{Distr.}}{=} \int AB' \left( 1 - BC' \right) \, dP \\
\overset{\Delta}{=} \int \left| AB' \left( 1 - BC' \right) \right| \, dP \overset{\text{+f}}{=} \int 1 - BC' \, dP \\
\overset{\text{+f}}{=} \int 1 \, dP - \int BC' \, dP \overset{\text{+f}}{=} 1 - E(BC').
\]

The proof as written in Bell (1964) is shorter than this, but spelling it out has the advantage that it is clear exactly how the different prerequisites are used.

As has been said earlier, the derivation of the ideal QM statistics of the source used in the Bell inequality is available in many standard textbooks on quantum mechanics (see e.g., Peres, 1993), and will not be repeated here. However, the statistics have the sinusoidal form obtained in Chapter 3 where equation (5.2a) can be found. The other coincidence probabilities can be obtained in the same way:

\[
P_{QM}(A = +1 \cap B' = +1) = \frac{\cos^2(\phi_B - \phi_A)}{2} \tag{5.2a}
\]

\[
P_{QM}(A = +1 \cap B' = -1) = \frac{\sin^2(\phi_B - \phi_A)}{2} \tag{5.2b}
\]

\[
P_{QM}(A = -1 \cap B' = +1) = \frac{\sin^2(\phi_B - \phi_A)}{2} \tag{5.2c}
\]

\[
P_{QM}(A = -1 \cap B' = -1) = \frac{\cos^2(\phi_B - \phi_A)}{2} \tag{5.2d}
\]

and they sum to 1. From this, the QM correlation is obtained as

\[
E_{QM}(AB') = P_{QM}(A = +1 \cap B' = +1) - P_{QM}(A = +1 \cap B' = -1)
- P_{QM}(A = -1 \cap B' = +1) + P_{QM}(A = -1 \cap B' = -1) \tag{5.3}
= \cos(2(\phi_B - \phi_A)).
\]

With the choices \( \phi_A = 0, \phi_B = \pi/6 \), and \( \phi_C = \pi/3 \), we have

\[
|E_{QM}(AB') - E_{QM}(AC')| = |\cos(\pi/3) - \cos(2\pi/3)| = |1/2 - (-1/2)| = 1 \tag{5.4a}
\]

\[
1 - E_{QM}(BC') = 1 - \cos(\pi/3) = 1 - 1/2 = 1/2, \tag{5.4b}
\]

that is,

\[
|E_{QM}(AB') - E_{QM}(AB')| = 1 > 1/2 = 1 - E_{QM}(BC'). \tag{5.5}
\]
This is in essence the proof that a Local Realistic model cannot yield QM results. With these choices of $\phi_A$, $\phi_B$, and $\phi_C$, the violation is at its largest, so one could see that the constant 1 in the right-hand side would need to be raised to at least $\frac{3}{2}$ to avoid the QM violation.

### 5.2 “Violation” of the Bell inequality

Let us again use the simple “pie-chart” construction in Section 3.4, where the correlation is easy to calculate (see Fig. 5.2), and does not violate the Bell inequality.

![Figure 5.2: Correlation from the model in Fig. 3.13 (Dashed line: QM correlation).](image)

A small change will now be made in this simple model. In an experiment there are always some measurement errors, and one example of such an error is when the photon remains undetected, i.e., when the detectors are inefficient. Let us now include this in the model as in the construction in Fig. 5.3.

![Figure 5.3: A nonstandard example of a LHV model.](image)

The regions marked “h” and “v” behave as in the previous LHV model. In the regions marked “$\beta$” however, the photons are not detected (in Fig. 5.3, the angle $\theta$ is...
chosen to be \( \pi/4 - 1/2 \), and this will be motivated below). The polarization direction is still decided at the source, but in addition a non-detection error is introduced by the above model. This introduction of errors is local, since it depends only on the polarization vector \( \lambda \) of the photon and the orientation of the detector. There is no additional dependence on the other detector orientation, so the model is still a Local Realistic model.

![Figure 5.4: One of the two detectors is turned the angle \( \phi \).](image)

When the experiment is analyzed using this model, the result is different from the ordinary LHV model. At first, when the detectors are equally oriented, nothing unusual appears: all detected pairs have the same polarization. When the orientation difference is small but nonzero, there are some photon pairs where only one photon is detected, and the number of “valid” measurements is lowered somewhat (see Fig. 5.4). But all of the detected pairs still yield perfectly correlated data for small angles.

![Figure 5.5: Correlation from the model in Fig. 5.3 (Dashed line: QM correlation).](image)

Because of this, the correlation function is constant at small angle differences \( \phi \), and the remaining behavior of the correlation function may be easily obtained from Fig. 5.4. The result is given by the solid line in Fig. 5.5. The reason for the choice...
of $\theta = \pi/4 - 1/2$ is that the correlation from the model is then always greater (to its modulus) than the QM correlation. A check of the data from the detected pairs in the Bell inequality, with the choice of angles $\phi_A = 0$, $\phi_B = \pi/4 - 1/2$, and $\phi_C = \pi/4 + 1/2$, yields $E(BC') \approx -0.43 > -1/2$, and

$$|E(AB') - E(AC')| = |1 - (-1)| = 2,$$

$$1 - E(BC') < 1 - \left(-\frac{1}{2}\right) = \frac{3}{2},$$

so that

$$|E(AB') - E(AC')| = 2 > \frac{3}{2} > 1 - E(BC').$$

This is clearly a violation of the Bell inequality.

### 5.3 The need for a generalized Bell inequality

The problem above has been known for quite some time, and the reason for this problem is that the Bell inequality is simply not valid in the inefficient case. A number of people has noticed and subsequently worked on this problem, e.g., Clauser, Horne, Shimony, and Holt (1969); Clauser and Horne (1974); Mermin and Schwarz (1982); Marshall et al. (1983); Garg and Mermin (1987); Maudlin (1994); De Caro and Garuccio (1996), to mention a few.

Detector inefficiency has previously been included in the Bell inequality using two different approaches. The first is to use probabilities instead of correlations and derive an inequality on the probabilities which is called the Clauser-Horne inequality (see Clauser and Horne, 1974), but this will not be explained further here (but see, e.g., Larsson and Semitecolos, 2000). The second is to assign the measurement result 0 (zero) to an undetected particle, which makes the original Bell inequality inappropriate because of prerequisite (iii) of Theorem 5.1. The proof would fail at the point when that prerequisite is used, so that one cannot insert the product $1 = B^2$ at the appropriate place. An alternative to the Bell inequality which does not require prerequisite (iii) of Theorem 5.1 was derived by Clauser, Horne, Shimony, and Holt (1969):

**Theorem 5.2 (The Clauser-Horne-Shimony-Holt (CHSH) inequality)**

The following three prerequisites are assumed to hold except at a null set:

(i) **Realism.** As in Th.5.1(i)

(ii) **Locality.** As in Th.5.1(ii)

(iii) **Measurement result restriction.** The results may only range from $-1$ to $+1$.

$$V = \{x \in \mathbb{R}; -1 \leq x \leq +1\}$$

Then

$$|E(AC') - E(AD')| + |E(BC') + E(BD')| \leq 2.$$
**Proof:** The proof is in the same spirit as that of Theorem 5.1, but slightly longer:

\[
|E(AC') - E(AD')| \leq \left| \int AC' dP - \int AD' dP \right| \\
\leq | \int AC' dP \pm \int AC' BD' dP - (\int AD' dP \pm \int AC' BD' dP) | \\
\leq \left| \int AC' \pm AC' BD' dP - \int AD' \pm AC' BD' dP \right| \\
\leq | \int AC' \pm AC' BD' dP - \int AD' \pm AD' BC' dP| \\
\leq D_{\text{dist}} | \int AC'(1 \pm BD') dP - \int AD'(1 \pm BC') dP| \\
\leq | \int AC'(1 \pm BD') dP| + | \int AD'(1 \pm BC') dP| \\
\leq \int |AC'(1 \pm BD')| dP + \int |AD'(1 \pm BC')| dP \\
\leq \int 1 \pm BD' dP + \int 1 \pm BC' dP \\
\leq \int 2 \pm dP + \int BD' dP + \int BC' dP \\
\leq 2 \pm (E(BC') + E(BD')) \\
\leq 2 \pm \left| E(AC') - E(AD') \right|, \tag{5.8}
\]

which implies

\[
|E(AC') - E(AD')| \leq 2 - |E(BC') + E(BD')|. \tag{5.9}
\]

\(\square\)

The QM correlation \(E^\text{QM}(AC')\) violates the CHSH inequality as well, and the largest value of the left-hand side in QM is \(2\sqrt{2}\). In this inequality one could use the second approach and insert the zeros at the appropriate place. The problem is that the number of undetected pairs (double zeros) is not known. There is simply no way of getting it out of the coincidence data, so one would need an estimate of the probability of an undetected pair. Such an estimate is derived in Garg and Mermin (1987), by using the assumptions of constant error rate and independent errors, yielding a well-defined detector efficiency \(\eta\). The result is (in their notation)

\[
|E_{13} \pm E_{23}| + |E_{14} \mp E_{24}| \leq 4\eta^{-1} - 2
\]

(Garg and Mermin, 1987, ineq. 1.9). The right-hand side is increased from 2 to \(\frac{4}{\eta} - 2\), and the expectation values are such that the pairs where at least one photon remains undetected are removed. When \(\eta\) is one, the right-hand side is 2 so that this is equivalent to the CHSH inequality. QM still violates the inequality when \(\eta\) is lowered, until
the constant on the right-hand side is $2\sqrt{2}$, i.e., when $\eta = 2(\sqrt{2} - 1) \approx 0.8283$. So to retain a violation from the QM correlation, an efficiency $\eta > 82.83\%$ is needed.

Also, previously a generalization of the original Bell inequality in itself has not been found. The above treatment does not give any hint about how to do this, since the proof of the original Bell inequality fails at a point where it seems nothing can be done to avoid the failure. But the reason for the proof failing at this point is not fundamental; we put in the zeros by hand. There is nothing to say that the zeros should be there, rather, if we were to perform a spin-1 experiment (with results $+1, 0, \text{and} -1$) the insertion of zeros would seem quite unnatural and also bias the result towards zero.

### 5.4 Generalization of the Bell inequality

The solution to this problem is to avoid treating “nondetection” as a measurement result. Thus one should not assign the (artificial) value 0 to an undetected photon, but instead one should treat the missing detection as an *undefined* measurement result. This will make the prerequisite (iii) of Theorem 5.1 again valid as the only measurement results one can get are $+1$ and $-1$. What is instead changed is prerequisite (i) of Theorem 5.1, the notion of Realism. The probabilistic description exists, but at some points of $\Lambda$ the random variable $A(\phi_A, \phi_B, \lambda)$ is *undefined* corresponding to undetected photons. The set of points where $A(\phi_A, \phi_B, \lambda)$ is defined is denoted $\Lambda_A(\phi_A, \phi_B, \lambda)$, as in Chapter 4, and the correlation of $A(\phi_A, \phi_B, \lambda)$ and $B'(\phi_A, \phi_B, \lambda)$ is calculated on the set where both random variables are defined:

$$E_{AB}(AB') = \int_{\Lambda_{AB'}} AB' dP_{AB'}.$$ (5.10)

The proof breaks down in this setting, but the problem is not not prerequisite (iii) anymore; the random variables can only take the values $\pm 1$. Trying the old proof in this setting, the start would look like

$$|E_{AB}(AB') - E_{AC'}(AC')| = \left| \int_{\Lambda_{AB'}} AB' dP_{AB'} - \int_{\Lambda_{AC'}} AC' dP_{AC'} \right|.$$ (5.11)

It is not a trivial operation to add the integrals on the right-hand side anymore, simply because the integration sets are different. Only when

$$\Lambda_{AB'}, \Lambda_{AC'}, \text{and} \Lambda_{BC'} \text{ differ at most by a null set}$$ (5.12)

is it possible to go through the old proof in the manner we have seen above, and this need not generally be the case.

Apparently it is the change of definition-set of the products in the expectation values that is the important concept here, the *change of ensemble* of photon pairs that are seen in the different experiments. The following formal definition of this concept will be used.
Definition 5.3 (Change of ensemble)

\[ \delta_{A_2} = \inf_{\phi_A, \phi_B} P_{AB}(\Lambda_{CD}). \]

The term “change of ensemble” is used because \( \delta_{A_2} \) as defined above measures the minimum size of the common part of the subsets \( \Lambda_{AB} \) and \( \Lambda_{CD} \), the sets from which the correlation is obtained for two different measurement setups. This concept will be frequently used in the following sections, and a conceptual investigation is made in Chapter 9. One motivation to use this concept is the following observation:

Observation 5.4 (Equal and totally different ensembles)

\[ \forall \phi_A, \phi_B, \phi_C, \phi_D : P(\Lambda_{AB} \cap \Lambda_{CD}) = P(\Lambda_{AB}) \Rightarrow \delta_{A_2} = 1 \]
\[ \exists \phi_A, \phi_B, \phi_C, \phi_D : P(\Lambda_{AB} \cap \Lambda_{CD}) = 0 \Rightarrow \delta_{A_2} = 0 \]

We can see that the first line says that \( \delta_{A_2} = 1 \) if and only if (5.12) is satisfied, and then we would expect to arrive at a similar inequality to the original Bell inequality. On the other hand, the latter implication represents the opposite; when \( \delta_{A_2} = 0 \) we cannot expect any connection between the results obtained from measuring \( AB \) and \( CD \), since the product \( AB \) is defined on a completely different set of \( \Lambda \)'s than the product \( CD \) (except for a null set). In this case, the generalization would be expected to be trivial in that it should yield no constraint on the correlations. This is precisely what is obtained in the following generalization of the original Bell inequality.

Theorem 5.5 (Bell inequality with ensemble change)

The following four prerequisites are assumed to hold except at a \( P \)-null set:

(i) Realism. Measurement results can be described by probability theory, using (two families of) random variables, which may be undefined on some part of \( \Lambda \), corresponding to measurement inefficiency:

\[
\begin{align*}
A(\phi_A, \phi_B) : & \Lambda_A(\phi_A, \phi_B) \rightarrow V \\
& \lambda \mapsto A(\phi_A, \phi_B, \lambda) \\
B(\phi_A, \phi_B) : & \Lambda_{AB}(\phi_A, \phi_B) \rightarrow V \\
& \lambda \mapsto B(\phi_A, \phi_B, \lambda)
\end{align*}
\]

(ii) Locality. A measurement result should be independent of the detector orientation at the other site,

\[
\begin{align*}
A(\phi_A, \lambda) & \equiv A(\phi_A, \phi_B, \lambda) \quad \text{on} \quad \Lambda_{A}(\phi_A) \equiv \Lambda_{A}(\phi_A, \phi_B) \\
\quad \text{independently of} \quad \phi_B. \\
B'(\phi_B, \lambda) & \equiv B'(\phi_A, \phi_B, \lambda) \quad \text{on} \quad \Lambda_{B'}(\phi_B) \equiv \Lambda_{B'}(\phi_A, \phi_B) \\
\quad \text{independently of} \quad \phi_A.
\end{align*}
\]
The second step is to translate this into an expression with \( \delta \). The proof consists of two steps. The first part is similar to the proof of Theorem 5.1, using the ensemble \( \Lambda_{AB'BC'} \), on which all the random variables \( A, B, B' \) and \( C' \) are defined. This is to avoid the problem of adding integrals mentioned above. This ensemble may be empty, but only when \( \delta_{4.2} = 0 \) and then the inequality is trivial, so \( \delta_{4.2} > 0 \) can be assumed in the rest of the proof. Now (i)–(iv) yields

\[
|E_{AB'BC'}(AB') - E_{AB'BC'}(AC')| \leq 1 - E_{AB'BC'}(BC'). \tag{5.13}
\]

The second step is to translate this into an expression with \( E_{AB'}(AB') \) and so on, a translation provided by the following inequality

\[
|E_{AB'}(AB') - \delta_{4.2}E_{AB'BC'}(AB')| \leq |P_{AB'}(\Lambda^C_{BC'})E_{AB'}(AB'|\Lambda^C_{BC'})| + |P_{AB'}(\Lambda_{BC'})E_{AB'}(AB'|\Lambda_{BC'}) - \delta_{4.2}E_{AB'BC'}(AB')| \leq P_{AB'}(\Lambda^C_{BC'})E_{AB'}(\Lambda^C_{BC'}) + (P_{AB'}(\Lambda_{BC'}) - \delta_{4.2})E_{AB'BC'}(\Lambda_{BC'}) \\
= 1 - \delta_{4.2}.
\tag{5.14}
\]

We then arrive at

\[
|E_{AB'}(AB') - E_{AC'}(AC')| \leq |E_{AB'}(AB') - \delta_{4.2}E_{AB'BC'}(AB')| + |E_{AC'}(AC') - \delta_{4.2}E_{AB'BC'}(AC')| + \delta_{4.2}|E_{AB'BC'}(AB') - E_{AB'BC'}(AC')| \leq \delta_{4.2}|E_{AB'BC'}(AB') - E_{AB'BC'}(AC')| + 2(1 - \delta_{4.2}) \leq \delta_{4.2}(1 - E_{AB'BC'}(BC')) + 2(1 - \delta_{4.2}) \leq 2 - \delta_{4.2} - E_{BC'}(BC') + (E_{BC'}(BC') - \delta_{4.2}E_{AB'BC'}(BC')) \leq 3 - 2\delta_{4.2} - E_{BC'}(BC'). \tag{5.15}
\]
We saw earlier that if the constant on the right-hand side, \(3/2\), is greater than \(3/2\), no violation is obtained from the QM correlation. So to retain a violation of this inequality, \(\delta_{A,2} > 3/4\) = 75% is needed. However, there is a problem since \(\delta_{A,2}\) is not possible to estimate from experiment, so our next step is to derive a result more directly connected to the experimental data.

### 5.5 Detector-efficiency

The usual way of dealing with detector-efficiency is to make two additions to the pre-requisites of Theorem 5.5, to simplify the argument. One wants to define a number \(\eta\) to be the “detector-efficiency”, and this is usually done by the following two assumptions:

**(v)** Constant detector efficiency. The probability of a detection at any detector at any orientation is \(\eta\).

\[
\eta = P(\Lambda_A) = P(\Lambda_B), \text{ independently of } \phi_A \text{ and } \phi_B.
\]

**(vi)** Independent nondetection errors. The detection errors are probabilistically independent for detection at different detectors at any orientation, e.g.,

\[
P(\Lambda_{A,B}) = P(\Lambda_A)P(\Lambda_B), \text{ independently of } \phi_A \text{ and } \phi_B.
\]

This yields a well-defined detector-efficiency \(\eta\), and it could be argued that on physical grounds the above two assumptions are completely natural and should therefore be used. A word of warning is perhaps appropriate here. On similar grounds it could be argued that the non-detections should not depend upon \(\lambda\), but it can be showed that assuming this, Theorem 5.5 reduces to a similar inequality as that of Theorem 5.1 (the proof is presented in Section 9.3); thus, this assumption is rather strong. It may be the case that the assumptions (v) and (vi) are equally strong, or it may be possible to arrive at the previously derived bounds (e.g., in Garg and Mermin, 1987) without them. This will be investigated below. Another reason not to use them is the existence of models in which they are not valid (see e.g., Maudlin, 1994, Chap. 6, or indeed the model described in Section 5.2). Even if such models are not thought to be very physical, the implication is clear; a result derived using (v) and (vi) is not sufficiently general.

The first assumption is avoided by letting the efficiency be the least probability that a photon is detected, the infimum over all the possible detector orientations and both detectors (infimum is used rather than minimum to eliminate assumptions of the existence of a minima, since the detector orientations are continuous parameters).

**Definition 5.6 (Detector efficiency)**

\[
\eta_1 \overset{\text{def}}{=} \inf_{\phi_A, \phi_B} P(\Lambda_{A,B}).
\]
The second assumption is avoided by looking at the conditional probability that the photon is detected at one detector given that the corresponding photon was detected at the other detector, using infimum in the same manner as above.

**Definition 5.7 (Coincidence efficiency)**

\[
\eta_{2,1} \overset{\text{def}}{=} \inf_{\phi_A, \phi_B, i \neq j} P_A(\Lambda_{B|i}),
\]

Estimation of \( \eta_1 \) from experiment is not as easy as it may seem, since one would need to know the rate of pair emission from the source. Calorimetry is one alternative, but a precise such estimate is quite difficult to achieve. An estimate of \( \eta_{2,1} \) is on the other hand easy to extract from the coincidence data obtained in the experiment. Note that if the assumptions (v) and (vi) above hold, then \( \eta = \eta_1 = \eta_{2,1} \) so that any bound on the latter two then applies to the case of independent errors as well. Now, we can derive the following corollary to the generalized Bell inequality of Theorem 5.5.

**Corollary 5.8 (Bell inequality with measurement inefficiency)**

Assume that Th.5.5(i)–(iv) hold except on a P-null set.

(a) **Detector efficiency.** If \( \eta_1 \geq \frac{3}{4} \), then

\[
|E_{AB'}(AB') - E_{AC'}(AC')| \leq \frac{3 - 2\eta_1}{2\eta_1 - 1} - E_{BC'}(BC'),
\]

(b) **Coincidence efficiency.** If \( \eta_{2,1} \geq \frac{2}{3} \), then

\[
|E_{AB'}(AB') - E_{AC'}(AC')| \leq \frac{4}{\eta_{2,1}} - 3 - E_{BC'}(BC'),
\]

**Proof:** First, to prove (b), use the simple inequality

\[
P_{AC'}(\Lambda_B) = \frac{P_{AC'}(\Lambda_{AB})}{P_{AC'}(\Lambda_A)} = \frac{P_{AC'}(\Lambda_A) + P_{AC'}(\Lambda_B) - P_{AC'}(\Lambda_A \cup \Lambda_B)}{P_{AC'}(\Lambda_A)}
\]

\[
\geq 1 + \frac{\eta_{2,1} - 1}{P_{AC'}(\Lambda_A)} \geq 2 - \frac{1}{\eta_{2,1}}, \quad (5.16)
\]

which gives

\[
P_{AC'}(\Lambda_{BD'}) = P_{AC'}(\Lambda_B) + P_{AC'}(\Lambda_{D'}) - P_{AC'}(\Lambda_B \cup \Lambda_{D'})
\]

\[
\geq 2 \left( 2 - \frac{1}{\eta_{2,1}} \right) - 1 = 3 - \frac{2}{\eta_{2,1}}. \quad (5.17)
\]

This means that

\[
\delta_{a,2} = \inf_{\phi_A, \phi_B} P_{AB'}(\Lambda_{CD'}) \geq 3 - \frac{2}{\eta_{2,1}}, \quad (5.18)
\]
and when the right-hand side is nonnegative ($\eta_2 \geq \frac{2}{3}$), the inequality (b) follows from Theorem 5.5.

Now, to prove (a) the same approach gives

$$P_A(\Lambda_{gf'} = \frac{P(\Lambda_{AB} \cup \Lambda_{gf'})}{P(\Lambda_A)} \geq 1 + \frac{\eta_1 - 1}{P(\Lambda_A)} \geq 2 - \frac{1}{\eta_1},$$

which means that

$$\eta_{2,1} = \inf_{\phi, \phi', i, j} P_A(\Lambda_{B(i)} \geq 2 - \frac{1}{\eta_1}. (5.20)$$

If $\eta_1 \geq \frac{3}{4}$, the above inequality yields $\eta_{2,1} \geq \frac{2}{3}$, and (a) then follows from (b).

A QM violation of these inequalities is obtained when $\eta_1 > \frac{9}{10} = 90\%$ or $\eta_{2,1} > \frac{8}{9} \approx 88.9\%$. Both of these are higher than the $\eta > 82.83\%$ bound derived in Garg and Mermin (1987).

### 5.6 The 82.83% efficiency bound

It seems that QM violates the CHSH inequality more strongly than the Bell inequality in the sense that the efficiency bounds derived from the Bell inequality are higher (less restrictive) than the bound derived from the CHSH inequality in Garg and Mermin (1987). Note that the latter bound is derived using the assumptions (iv) and (v), independent errors at a constant rate. It remains to check whether these two assumptions are necessary, or if the 82.83% bound can be derived in the general situation. Using the above approach in the CHSH inequality, the following is obtained:

**Theorem 5.9 (The CHSH inequality with ensemble change)**

The following three prerequisites are assumed to hold except at a $P$-null set:

(i) **Realism.** As in Th.5.5(i)

(ii) **Locality.** As in Th.5.5(ii)

(iii) **Measurement result restriction.** The results may only range from $-1$ to $+1$.

$$V = \{x \in \mathbb{R}; -1 \leq x \leq +1\}$$

With $\delta_{A,2}$ as in Def. 5.3, this yields

$$|E_{AC'}(AC') - E_{AD'}(AD')| + |E_{BC'}(BC') + E_{BD'}(BD')| \leq 4 - 2\delta_{A,2}.$$
The Bell inequality

Proof: The proof is similar to that of Theorem 5.5. Using the ensemble \( \Lambda_{AC'BD'} \) (when \( \delta_{4,2} > 0 \) in the same manner as in Theorem 5.5), we arrive at

\[
|E_{AC'BD'}(AC') - E_{AC'BD'}(AD')| + |E_{AC'BD'}(BC') + E_{AC'BD'}(BD')| \leq 2
\]

(5.21)

A translation of this inequality into an expression with \( E_{AC'}(AC') \) and so on is provided by the same inequality used in the proof of the generalized Bell inequality, (5.14), which is valid in this case as well, although using Th.5.9(iii) in place of Th.5.5(iii), the last step is an inequality rather than an equality. We have

\[
|E_{AC'}(AC') - \delta_{4,2}E_{AC'BD'}(AC')|
\]

\[
\leq \ldots
\]

\[
\leq P_{AC'}(\Lambda_{BD'})E_{AC'}(|AC'||\Lambda_{BD'}) + (P_{AC'}(\Lambda_{BD'}) - \delta_{4,2})E_{AC'BD'}(|AC'|)
\]

(5.22)

We then arrive at

\[
|E_{AC}(AC') - E_{AD}(AD')| + |E_{BC}(BC') + E_{BD}(BD')|
\]

\[
\leq |E_{AC}(AC') - \delta_{4,2}E_{AC'BD'}(AC')| + |E_{AD}(AD') - \delta_{4,2}E_{AC'BD'}(AD')|
\]

\[
+ \delta_{4,2}|E_{AC'BD'}(AC') - E_{AC'BD'}(AD')|
\]

\[
+ |E_{AC}(BC') - \delta_{4,2}E_{AC'BD'}(BC')| + |E_{AD}(BD') - \delta_{4,2}E_{AC'BD'}(BD')|
\]

\[
+ \delta_{4,2}|E_{AC'BD'}(BC') + E_{AC'BD'}(BD')|
\]

\[
\leq \delta_{4,2}|E_{AC'BD'}(AC') - E_{AC'BD'}(AD')|
\]

\[
+ \delta_{4,2}|E_{AC'BD'}(BC') - E_{AC'BD'}(BD')| + 4(1 - \delta_{4,2})
\]

\[
\leq 2\delta_{4,2} + 4(1 - \delta_{4,2}) = 4 - 2\delta_{4,2},
\]

\( \square \)

A QM violation of this inequality would demand \( \delta_{4,2} > 2 - \sqrt{2} \approx 58.58\% \). This is as expected significantly lower than the 75\% bound obtained from Theorem 5.5. The measurement inefficiency result is

**Corollary 5.10 (The CHSH inequality with measurement inefficiency)**

Assume that Th.5.9(i)–(iii) hold except on a P-null set, and define \( \eta_1 \) and \( \eta_{2,1} \) as in Defs. 5.6 and 5.7. We have

(a) **Detector-efficiency.** If \( \eta_1 \geq \frac{3}{4} \), then

\[
|E_{AC'}(AC') - E_{AD'}(AD')| + |E_{BC'}(BC') + E_{BD'}(BD')| \leq \frac{2}{2\eta_1 - 1}.
\]

(b) **Coincidence efficiency.** If \( \eta_{2,1} \geq \frac{3}{4} \), then

\[
|E_{AC'}(AC') - E_{AD'}(AD')| + |E_{BC'}(BC') + E_{BD'}(BD')| \leq \frac{4}{\eta_{2,1} - 2}.
\]
The proof is simply to use the inequalities used in the Corollary 5.8, (5.18) and (5.20) in Theorem 5.9.

The coincidence efficiency inequality in (b) is similar to the generalization presented in Garg and Mermin (1987) and the bound is of the same size, \( \eta_2 > 2(\sqrt{2} - 1) \approx 82.83\% \). In that paper, the assumptions of independent errors and constant detector efficiency are used, and then \( \eta_2 = \eta_1 = \eta \) and Corollary 5.10 yields the same bound, \( \eta > 2(\sqrt{2} - 1) \approx 82.83\% \). It is now evident that the assumptions (v) and (vi) in Section 5.5 are not essential to the 82.83% bound obtained in Garg and Mermin (1987).
A problem with the model presented in Section 5.2 is that the obtained correlation is not sinusoidal as is obtained from QM, and another is that the coincidence efficiency $P_A(A_B)$ varies with the relative angular difference between the detector settings. This is not present in QM, and is not seen in experiments (not, at least, to the degree as is present in that model). One may then ask if there is a model obeying the QM statistics, while having a “natural” statistics for the measurement errors. In Garg and Mermin (1987), there is a proof that the 82.83% bound is necessary and sufficient; above the bound there is a violation of the CHSH inequality as we have seen in the previous chapter, and below the bound a LHV model exists yielding the QM correlation at the angles used in the CHSH inequality. Unfortunately, the proof is not constructive, and furthermore, there is no hint that this model would yield the full QM statistics at all possible angles of the detectors.

Here, an explicit construction of a LHV model will be presented following Larsson (1999b)∗. The model will exhibit all the QM statistics of the Bell experiment, with independent errors at a constant rate. This latter choice is made because we want a model with “natural” error statistics. Ideally, we would like the model to be usable up to 82.83% efficiency, because this would be a constructive proof that the bound is necessary and sufficient. However, this goal will not quite be reached, as the model will be valid only up to 78.80% (see further discussion at the end of this chapter).

* A reprint of this paper can be found at the end of the thesis.
6.1 QM statistics

The ideal QM statistics has been earlier presented in equations (5.2a)–(5.2d). Since the purpose of this chapter is to construct a LHV model that yields the QM statistics, a repetition is in order:

\[ P_{QM}(A = +1 \cap B' = +1) = \frac{\cos^2(\phi_B - \phi_A)}{2}, \] (6.1a)
\[ P_{QM}(A = +1 \cap B' = -1) = \frac{\sin^2(\phi_B - \phi_A)}{2}, \] (6.1b)
\[ P_{QM}(A = -1 \cap B' = +1) = \frac{\sin^2(\phi_B - \phi_A)}{2}, \] (6.1c)
\[ P_{QM}(A = -1 \cap B' = -1) = \frac{\cos^2(\phi_B - \phi_A)}{2}, \] (6.1d)

and the correlation in (5.3) is obtained as

\[ E_{QM}(AB') = P_{QM}(A = +1 \cap B' = +1) - P_{QM}(A = +1 \cap B' = -1) \]
\[ - P_{QM}(A = -1 \cap B' = +1) + P_{QM}(A = -1 \cap B' = -1) \] (6.2)
\[ = \cos(2(\phi_B - \phi_A)). \]

Locally at one detector, either result is equally probable, i.e.,

\[ P_{QM}(A = +1) = P_{QM}(A = -1) = P_{QM}(B' = +1) = P_{QM}(B' = -1) = \frac{1}{2}. \] (6.3)

We want to add detector inefficiency into the probabilities, in such a way that the error statistics is natural. Independent errors at a constant rate \( \eta \) (i.e., using (v) and (vi) from Chapter 5) yields

\[ P(\Lambda_{AB'}) = P(\Lambda_A)P(\Lambda_{B'}) = \eta^2, \] (6.4)

which modifies our ideal two- and one-photon probabilities slightly,

\[ P_{QM,\eta}(A = +1 \cap B' = +1) = \eta^2 \frac{\cos^2(\phi_B - \phi_A)}{2}, \] (6.5a)
\[ P_{QM,\eta}(A = +1 \cap B' = -1) = \eta^2 \frac{\sin^2(\phi_B - \phi_A)}{2}, \ldots \] (6.5b)

and

\[ P_{QM,\eta}(A = +1) = \ldots = \frac{\eta}{2}. \] (6.6)

Note that this does not change the probabilities on the set where the measurement results are defined,

\[ P_{QM,\eta}^{AB'}(A = +1 \cap B' = +1) = \frac{\cos^2(\phi_B - \phi_A)}{2}, \ldots \] (6.7)
\[ P_{QM,\eta}^{AB'}(A = +1) = \ldots = \frac{1}{2}, \] (6.8)
and neither is the correlation changed (it is calculated on $\Lambda_{AB}$),

$$E_{AB}^{QM,B} (AB') = \cos(2(\phi_B - \phi_A)).$$  
\(6.9\)

We are now ready to construct the model.

### 6.2 Sinusoidal probabilities

The model is an extension of the simple model presented in Section 5.2. There, the hidden variable $\lambda$ is simply an angular coordinate and the measurement results and the detection errors depend only on the orientation of the detector with respect to $\lambda$. In the below construction, $\lambda$ is the pair $\lambda = (\theta, r)$ consisting of an angular variable $\theta \in [0, \pi]$, which may be thought of as the “hidden polarization” as in the previous simple model, along with another variable $r \in [0, 1]$, the “detection parameter”. For each pair, the hidden variable $\lambda$ has a definite value, and the whole ensemble is taken to be an even distribution in the rectangle $(\theta, r) \in [0, \pi] \times [0, 1]$. This will be visualized in rectangular rather than polar coordinates, because it is then easier to see that the probabilities are of the correct form. First, the sinusoidal form of the probabilities will be sought.

The measurement results (and the detection errors) are given in the figures, and the procedure to obtain the measurement result is as follows: at detector 1, the value of $\theta$ is shifted to $\theta' = \theta - \phi_A$ ($r$ is not changed), and the result is read off Fig. 6.1.

At detector 2 the procedure is similar, but in this case the shift is $\theta'' = \theta - \phi_B$. The result is $+1 (-1)$ when $\lambda$ falls in an area marked $+1 (-1)$ in Fig. 6.2. If $\lambda$ should happen to fall in an area marked $\frac{\pi}{2}$, a measurement error (non-detection) occurs and the random variable $A (B')$ is not assigned a value.
The probabilities for the coincident detections are now possible to calculate using Fig. 6.3, where the detector patterns have been shifted and interposed in the $\lambda$-plane. For example, the probability of getting $A = +1$ and $B' = -1$ simultaneously is the size of the area marked "++-".

The size is the area divided by $\pi$, because the total probability is one, whereas the total area is $\pi$:

$$P(A = +1 \cap B' = -1) = \frac{1}{\pi} \int_{\Phi_A}^{\Phi_B} a \sin(2\theta) d\theta = \frac{a \sin^2(\Phi_B - \Phi_A)}{\pi},$$

which has the sinusoidal form we want. Note that the calculation above is only valid if

$$0 \leq a \leq b \leq 1,$$
and it is easy to see that the probability would not be sinusoidal if $a > b$. Thus, the model has the required form of the probabilities and constant efficiency.

### 6.3 Independent errors

An unwanted property is that the efficiencies of the detectors are different:

$$P(\Lambda_{\mu'}) = b, \quad (6.12)$$

while

$$P(\Lambda_A) = \frac{2}{\pi} \int_0^{\pi/2} a \sin(2\theta) d\theta = \frac{2a}{\pi} \leq \frac{2b}{\pi} < b. \quad (6.13)$$

To resolve this, one way is to lower the efficiency of detector 2 by simply inserting random detection errors at that detector (which would yield the model described in Santos (1996)), but there is another way of resolving the problem, by symmetrizing the model (see Fig. 6.4 and 6.5).

![Figure 6.4: The detector pattern for detector 1 in the symmetrized model.](image)

In this model, the parameters $a$ and $b$ are subject to the conditions

$$0 \leq a \leq b \leq \frac{1}{2}. \quad (6.14)$$

By using the equations (6.12) and (6.13), the efficiency is constant at

$$\eta = P(\Lambda_A) = P(\Lambda_{\mu'}) = \frac{2a}{\pi} + b. \quad (6.15)$$

Detections at both detectors occur at the sets where the two patterns overlap; the probability is given by twice that in (6.13), and the errors should be independent, so that

$$\eta^2 = \frac{4a}{\pi}. \quad (6.16)$$
Solving for $a$ and $b$, we have

\[
\begin{align*}
  a &= \frac{\pi \eta^2}{4} \\
  b &= \eta - \frac{\eta^2}{2}.
\end{align*}
\]  

(6.17)

It is easy to see that $b$ obtains its maximal value $\frac{1}{2}$ at $\eta = 1$, but then $a = \frac{\pi}{4} > \frac{1}{2}$ so the model does not work. The requirement in (6.14) that $a \leq b$ for the model to yield sinusoidal probabilities is now

\[
\frac{\pi \eta^2}{4} \leq \eta - \frac{\eta^2}{2},
\]  

(6.18)

and solving for $\eta$,

\[
\eta \leq \frac{4}{2 + \pi} \approx 0.7780,
\]  

(6.19)

i.e., the model described above is usable up to 77.80% efficiency. The probability of “$+$” in the symmetrized model is twice that in (6.10) by the symmetrization, and

\[
P(A = +1 \cap B' = -1) = \frac{2 a \sin^2(\phi_B - \phi_A)}{\pi} = \eta^2 \frac{\sin^2(\phi_B - \phi_A)}{2},
\]  

(6.20)

as prescribed in (6.5b). The other probabilities can be checked in the same manner.

### 6.4 Visibility problems

If there are errors in the measurement results (as opposed to missing measurement results), the condition $AA' = 1$ does not always hold because of the added “noise”. The
result of this is a lowered amplitude of the correlation, a lowered visibility $v$ of the interference pattern,

$$E_{AB}^{\text{QM,} \eta, v}(A'B') = v \cos(2(\phi_B - \phi_A)).$$

(6.21)

This of course lowers the violation of the CHSH inequality. Even though present experiments regularly have a high visibility (e.g. 97% in Weihs et al. (1998)) it would be interesting to include this effect in the model presented in the previous section. Noting that

$$\cos^2(\phi_B - \phi_A) = \frac{1 + \cos(2(\phi_B - \phi_A))}{2}$$

(6.22a)

$$\sin^2(\phi_B - \phi_A) = \frac{1 - \cos(2(\phi_B - \phi_A))}{2}$$

(6.22b)

the probabilities can be written

$$P_{\text{QM,} \eta, v}(A = +1 \cap B' = +1) = \eta^2 \frac{1 + v \cos(2(\phi_B - \phi_A))}{4}$$

(6.23a)

$$P_{\text{QM,} \eta, v}(A = +1 \cap B' = -1) = \eta^2 \frac{1 - v \cos(2(\phi_B - \phi_A))}{4}$$

(6.23b)

$$P_{\text{QM,} \eta, v}(A = -1 \cap B' = +1) = \eta^2 \frac{1 - v \cos(2(\phi_B - \phi_A))}{4}$$

(6.23c)

$$P_{\text{QM,} \eta, v}(A = -1 \cap B' = -1) = \eta^2 \frac{1 + v \cos(2(\phi_B - \phi_A))}{4}$$

(6.23d)

Of course, this means that

$$P_{AB}^{\text{QM,} \eta, v}(A = +1 \cap B' = +1) = \frac{1 + v \cos(2(\phi_B - \phi_A))}{4},$$

(6.24)

Intuitively, when the probability was 0 before it will now be $\frac{1 + v}{4} > 0$ due to the added noise, and likewise, when the probability was $\frac{1}{2}$ before it will now be $\frac{1 + v}{4} < \frac{1}{2}$ for the same reason.

In the model from the previous section, we will introduce lowered visibility by adding some random $\pm 1$ results where otherwise no detection would have occurred. The errors may be added as random errors on the whole of the area marked $A$ in Fig. 6.4, but to write the model as a “deterministic” model and to simplify the calculations, the errors are added in the manner as in Fig. 6.6. There is no effect on the statistical properties of the errors by this choice, i.e., the errors are independent and the probability of an error is constant.

Also in this model, the parameters $a$ and $b$ are subject to the conditions

$$0 \leq a \leq b \leq \frac{1}{2}.$$  

(6.25)

The parameter $c$ is the amount of errors added in the model, so that when $c = 0$ we have full visibility; quite naturally, $0 \leq c \leq 1$ in this model (and then $a \leq bc + a(1 - c) \leq b$).
By using the formulas for the efficiencies of the first simple model from (6.12) and (6.13) with slightly different coefficients, the efficiency is constant at

\[ \eta = b + (bc + \frac{2a(1-c)}{\pi}). \]  

(6.26)

while the probability for a double detection is

\[ \eta^2 = 2(b + \frac{2a(1-c)}{\pi}). \]  

(6.27)

The visibility is the amount of “correct results” at equal detector orientations:

\[ v = \frac{4a}{\pi} / \eta^2, \]  

(6.28)

and after a simple calculation

\[
\begin{align*}
    a &= \frac{\pi v \eta^2}{4} \\
    b &= \eta - \frac{\eta^2}{2} \\
    c &= \frac{\eta (1-v)}{2 - \eta (1+v)}
\end{align*}
\]  

(6.29)

The parameter \( b \) is not changed from the case of full visibility in the previous section, while the change in \( a \) is an extra factor \( v \). Using \( a \leq b \) from (6.25), we have

\[ v \leq \frac{1}{\pi} \left( \frac{4}{\eta} - 2 \right). \]  

(6.30)
This excludes the point \( \eta = 1, v = 1 \), where \( c = 0 \) is on the form \( 0 \), and for all other values of \( \eta \) and \( v \), \( 0 \leq c \leq 1 \). One may especially note that if \( \eta = 1 \), the bound on \( v \) is

\[
v \leq \frac{2}{\pi} \approx 0.6366. \tag{6.32}\]

The probability of the result “+−” is obtained by the expression

\[
P(A = +1 \cap B' = -1) = \frac{2}{\pi} \int_0^{\pi/4} \left( bc + a(1-c) \sin(2\theta) \right) - a \sin(2\theta) d\theta + \frac{2}{\pi} \int_0^{\phi_B-\phi_A} a \sin(2\theta) d\theta
\]

\[
= \frac{1}{2} \left( bc + \frac{2a(1-c)}{\pi} \right) - \frac{a}{\pi} \cos(2(\phi_B - \phi_A)) = \frac{\eta^2}{4} - \frac{\eta^2}{4} \cos(2(\phi_B - \phi_A)),
\]

where the limits of the first integral is obtained by the symmetry of the model. This expression is precisely that of (6.23b), and also here, the remaining probabilities are easily checked in the same manner.

### 6.5 The search for necessary and sufficient conditions

In the CHSH inequality, the 82.83% is obtained by choosing detector orientations so that the violation of the inequality is maximized:

\[
\phi_A = 0, \phi_B = \pi/4, \phi_C = \pi/8, \text{ and } \phi_D = 3\pi/8. \tag{6.34}\]

With this choice of angles in the CHSH inequality, and using \( E_{AB}^{\eta, v} \) from (6.21), we obtain

\[
|v \cos(\pi/4) + v \cos(\pi/4)| + |v \cos(\pi/4) - v \cos(3\pi/4)| = \frac{4v}{\sqrt{2}} \leq \frac{4}{\eta} - 2, \tag{6.35}\]

or

\[
v \leq \frac{1}{2\sqrt{2}} \left( \frac{4}{\eta} - 2 \right), \tag{6.36}\]

In Fig. 6.7, there is a region where the LHV model does not work which is not ruled out by the CHSH inequality. In the CHSH inequality, the correlation is only tested at
No local variables by the CHSH ineq.

Figure 6.7: The region where the model is usable as compared to the region where no local hidden-variable model is possible.

Figure 6.8: The points where the correlation is tested in the CHSH inequality. Note the small difference between the sinusoidal curve and the dashed curve consisting of line segments.
certain points, and at maximum violation, these points are as given in Fig. 6.8. Perhaps is it possible to model the straight-line correlation (the dashed line in Fig. 6.8) in the empty region in Fig. 6.7.

To construct a model for this case, one uses the procedure outlined in the previous sections, where the derivative of the correlation is used to determine the general outline of the pattern. The correlation is composed of straight lines, so therefore the pattern should be staircased (i.e., constant derivative). The derivative should be low from $0$ to $\pi/8$, then somewhat higher from $\pi/8$ to $3\pi/8$, and then again low from $3\pi/8$ to $\pi/2$. The relation between the derivatives is

$$\frac{1 - \frac{1}{\sqrt{2}}}{\frac{\pi}{8}} = \sqrt{2} - 1. \quad (6.37)$$

The procedure continues, and eventually yields (after addition of random errors to lower the visibility) the construction in Fig. 6.9.

Here, our equations are

$$\eta = b + \left( bc + \frac{a(1-c)}{\sqrt{2}} \right), \quad (6.38)$$

$$\eta^2 = 2 \left( bc + \frac{a(1-c)}{\sqrt{2}} \right), \quad (6.39)$$

and

$$v = \frac{\sqrt{2}a}{\eta^2}, \quad (6.40)$$
and we arrive at

\[
\begin{align*}
    a &= \frac{v \eta^2}{\sqrt{2}} \left( = \frac{2 \sqrt{2} v \eta^2}{4} \right) \\
    b &= \eta - \frac{\eta^2}{2} \\
    c &= \frac{\eta (1 - \eta)}{2 - \eta (1 + \eta)}.
\end{align*}
\]

Note that the only change from is that the constant \( \pi \) in the expression for \( a \) is changed to \( 2 \sqrt{2} \). This yields the allowed range as

\[
v \leq \frac{1}{2 \sqrt{2}}\left( \frac{4}{\eta} - 2 \right),
\]

and this is exactly the bound from the CHSH inequality (see Fig. 6.10).

In essence, we have an explicit model proving the necessity of having \( \eta > 82.83\% \) to contradict Local Realism in the CHSH inequality. The bound was shown to be necessary and sufficient already in Garg and Mermin (1987), but here an explicit counterexample is obtained below the bound. In addition, the effect of lowered visibility is included in this construction.

The same construction for the straight-line case allows a higher visibility at a given efficiency than for the sinusoidal case. It seems natural that the sinusoidal form of the correlation is somewhat more difficult to model than the straight-line form. This may be the case, but recent results (Zukowski, Kaszlikowski, Baturo, and Larsson, 1999, 2000) indicates that it is not impossible; therein it is conjectured that a sinusoidal model is possible at 82.83\%. 

Figure 6.10: The region where the straight-line model is usable, as compared to the region where no LHV model is possible.
Chapter 7

The Greenberger-Horne-Zeilinger (GHZ) paradox

So farewell elements of reality! And farewell in a hurry.

(N. David Mermin, 1990d)

In the Bell setting, a pair of particles is used to derive a statistical contradiction; QM violates an inequality that must be valid for correlations from a Local Realistic model. Since there already is a contradiction, there would seem to be no point in continuing, but there are two reasons to continue. The first is simplicity, as it seems inconvenient to compare correlations when it would be much simpler to have a direct contradiction without inequalities. The second reason is of course the detector-efficiency problem. If one had a “stronger contradiction” (whatever that is) it would possibly yield a lower bound on the efficiency.

An approach to getting a stronger contradiction is to use a triple of particles instead of a pair in a setup as in Fig. 7.1, the approach of the GHZ paradox which first appeared in somewhat other form in Greenberger, Horne, and Zeilinger (1989). This paradox is often used in popular presentations such as Mermin (1990c,d), because the contradiction in the GHZ paradox can be expressed in the form of a direct contradiction rather than a statistical inequality.

By using results from Larsson (1998b, 1999a)*, the GHZ analysis will be extended to the inefficient case, yielding the desired stronger contradiction; the detector-efficiency bound is lowered. The notation is visible in Fig. 7.1. In this experimental setup,

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*Reprints of these two papers can be found at the end of the thesis.
two settings of $\phi_A$, $\phi_B$, and $\phi_C$ are used, below labeled $\phi_X = 0$ and $\phi_Y = \pi/4$. The results will be labeled $\pm 1$ as in the previous chapters, and the source is such that when looking at data from all three detectors,

1) if the three detectors use the $\phi_X$ orientation, there is an odd number of $-1$'s ($- - -$, $- + +$, $+ - +$, $+ + -$),

2) if two detectors use the $\phi_Y$ orientation, while the third uses $\phi_X$, there is an even number of $-1$'s ($- + +$, $- - -$, $+ - -$, $+ + +$).

For other combinations (or when using data from only one or two detectors), the data is random and the results are equally probable.

Again an important comment is that there exists a source with the desired properties, and it is only recently that such a source has become available in practice.
(Zeilinger et al., 1997). The source setup is essentially based on the Bell source mentioned in Chapter 5.1, using not one but two “Bell pairs”. A triple is generated out of these two pairs by the following procedure: correlate one particle out of each pair with each other in such a way that it is no longer possible to determine in which “Bell pair” the particles originated. This implies that we now have a correlated quadruple of particles. Reduce this quadruple to a triple by performing a measurement on one of the two particles mentioned above. By arranging the correlation procedure and the measurement properly, half of the triples emerge in the “right” state with the properties described below, and these triples may be identified by looking at the measurement result at the fourth particle. What remains now is to simply delete the triples in the “wrong” state, and we have our source.

7.1 The GHZ paradox and its proof

Theorem 7.1 (The GHZ paradox)

A model with the following four properties except on a null set contains a contradiction:

(i) Realism. Measurement results can be described by probability theory, using (three families of) random variables.

\[
\begin{align*}
A(\phi_A, \phi_B, \phi_C) & : \Lambda \rightarrow V \\
\lambda & \mapsto A(\phi_A, \phi_B, \phi_C, \lambda) \\
B(\phi_A, \phi_B, \phi_C) & : \Lambda \rightarrow V \\
\lambda & \mapsto B(\phi_A, \phi_B, \phi_C, \lambda) \\
C(\phi_A, \phi_B, \phi_C) & : \Lambda \rightarrow V \\
\lambda & \mapsto C(\phi_A, \phi_B, \phi_C, \lambda)
\end{align*}
\]

(ii) Locality. A measurement result should be independent of the detector orientation at the other particles.

\[
\begin{align*}
A(\phi_A, \lambda) & \overset{\text{def}}{=} A(\phi_A, \phi_B, \phi_C, \lambda) \text{ independently of } \phi_B \text{ and } \phi_C. \\
B(\phi_B, \lambda) & \overset{\text{def}}{=} B(\phi_A, \phi_B, \phi_C, \lambda) \text{ independently of } \phi_A \text{ and } \phi_C. \\
C(\phi_C, \lambda) & \overset{\text{def}}{=} C(\phi_A, \phi_B, \phi_C, \lambda) \text{ independently of } \phi_A \text{ and } \phi_B.
\end{align*}
\]

(iii) Measurement result restriction. Only the results ±1 should be possible.

\[V = \{-1, +1\}.
\]

Only the settings \(\phi_X\) and \(\phi_Y\) are to be used, and by the notation \(X = A(\phi_X, \lambda), Y = A(\phi_Y, \lambda), X' = B(\phi_X, \lambda)\) and so on, the following will be shortened considerably. There are then only six different random variables to be used, \(X, X', X'', Y, Y', Y''\). These should satisfy the following.
(iv) **GHZ requirement.** The following measurement results should be obtained:

\[
XY'Y'' = 1 \\
YX'Y'' = 1 \\
YY'X'' = 1 \\
\text{and} \\
XX'X'' = -1
\]

**Proof:** By using in order from left to right: (iv), (iii), (ii), and (iv),

\[-1 = XX'X'' = XX'X'' \cdot Y^2 \cdot Y'^2 = XY'Y'' \cdot YX'Y'' \cdot YY'X'' = 1, \tag{7.1}\]

with probability one. This is clearly a contradiction. \(\square\)

## 7.2 Generalization of the GHZ paradox

A statistical inequality for the GHZ paradox can be derived (see e.g., Mermin, 1990b), but that approach will not be used here. The reason for this is to retain the original formulation of a direct contradiction as opposed to a statistical inequality. The intent is to show that, contrary to popular belief, the GHZ experiment is not a “single shot” experiment even in this formulation; it has been argued that once a GHZ experiment is set up, one measurement suffices to show a contradiction. However, as will be shown below, if the experimental efficiency is below a certain bound, there is no contradiction.

The effect of detector inefficiency on the GHZ paradox has previously been studied in Greenberger et al. (1990), and more recently another paper (Belinski˘ı, 1997) has lowered the bound to the 75% level derived below. In that paper the standard technique of assigning 0 to the undetected photons has been used, and via an inequality for the GHZ experiment similar to the Bell approach, the bound is obtained. However, in that paper the bound is derived under the assumptions of independent errors and constant detector efficiency (as described in Section 5.5), and moreover, the bound presented is a sufficient bound (above it we have a contradiction). A question is if this is also a necessary bound, or in other words, is this bound indeed the lowest possible to obtain from the GHZ paradox?

In the same way as in Chapter 5, we will use the concept of “change of ensemble” to obtain the result, and Definition 5.3 generalizes in the following way to the three-detector setup.

**Definition 7.2 (Change of ensemble)**

\[
\delta_{b,3} = \min_{\phi_A,\phi_B,\phi_C, \phi_F, \phi_G, \phi_H} \ P_{\Lambda_{FG'H}}(\Lambda_{FG'H})
\]
Because there is in this setting only two settings of each detector instead of the continuous parameter used in the Bell case, one may use “min” instead of “inf”. In a similar way as in the Bell case, the following observation applies.

**Observation 7.3 (Equal and totally different ensembles)**

\[
\forall \phi_A, \phi_B, \phi_C, \phi_F, \phi_G, \phi_H : P(A_{ABCE} \land \Lambda_{FGHI}) = P(A_{ABCE}) \iff \delta_{b,3} = 1
\]

\[
\exists \phi_A, \phi_B, \phi_C, \phi_F, \phi_G, \phi_H : P(A_{ABCE} \land \Lambda_{FGHI}) = 0 \iff \delta_{b,3} = 0
\]

This follows easily from the definition (note that the latter expression is an equivalence), and it is clear that the definition has the properties we want in this case as well. However, because of the low number of random variables that are used in this setting (there are only six of them), a much simpler form of the latter expression is available.

**Observation 7.4 (Total change of ensemble)**

\[
P(\Lambda_{XX'X''YY'}) = 0 \iff \delta_{b,3} = 0
\]

To prove \(\iff\) (\(\Rightarrow\) is trivial), simply use that there are only six possible random variables \(X, X', X'', Y, Y', \) and \(Y''\) so that

\[
\Lambda_{XX'X''YY''} \subseteq \Lambda_{ABCEFGHI}, \quad \forall \phi_A, \phi_B, \phi_C, \phi_F, \phi_G, \text{ and } \phi_H,
\]

which in turn implies

\[
P(\Lambda_{XX'X''YY''}) \leq P(\Lambda_{ABCEFGHI}), \quad \forall \phi_A, \phi_B, \phi_C, \phi_F, \phi_G, \text{ and } \phi_H
\]

This is of course precisely what was to be expected, because it says that if \(\delta_{b,3} = 0\), the ensemble of the measurement setup \(XX'X''\) would change completely when the measurement setup is changed completely (to \(YY''Y''\)).

This observation will be important in the proof of the following generalization of the GHZ paradox.

**Theorem 7.5 (The GHZ paradox with ensemble change)**

Assume that a model has the following four properties except on a \(P\)-null set:

1. **Realism.** Measurement results can be described by probability theory, using (three families of) random variables.

\[
\begin{align*}
\Lambda(\phi_A, \phi_B, \phi_C) : \Lambda_A(\phi_A, \phi_B, \phi_C) & \to V \\
\lambda \mapsto \Lambda(\phi_A, \phi_B, \phi_C, \lambda) \\
B'(\phi_A, \phi_B, \phi_C) : \Lambda_B(\phi_A, \phi_B, \phi_C) & \to V \\
\lambda \mapsto B'(\phi_A, \phi_B, \phi_C, \lambda) \\
C''(\phi_A, \phi_B, \phi_C) : \Lambda_{C''}(\phi_A, \phi_B, \phi_C) & \to V \\
\lambda \mapsto C''(\phi_A, \phi_B, \phi_C, \lambda)
\end{align*}
\]

\[
\forall \phi_A, \phi_B, \phi_C.
\]
(ii) Locality. A measurement result should be independent of the detector orientation at the other particles.

\[ A(\phi_A, \lambda) \overset{\text{def}}{=} A(\phi_A, \phi_B, \phi_C, \lambda) \text{ on } \Lambda_A(\phi_A) \overset{\text{def}}{=} \Lambda_A(\phi_A, \phi_B, \phi_C) \]

independently of \( \phi_B \) and \( \phi_C \).

\[ B'(\phi_B, \lambda) \overset{\text{def}}{=} B'(\phi_A, \phi_B, \phi_C, \lambda) \text{ on } \Lambda_B(\phi_B) \overset{\text{def}}{=} \Lambda_B(\phi_A, \phi_B, \phi_C) \]

independently of \( \phi_A \) and \( \phi_C \).

\[ C'(\phi_C, \lambda) \overset{\text{def}}{=} C'(\phi_A, \phi_B, \phi_C, \lambda) \text{ on } \Lambda_C(\phi_C) \overset{\text{def}}{=} \Lambda_C(\phi_A, \phi_B, \phi_C) \]

independently of \( \phi_A \) and \( \phi_B \).

(iii) Measurement result restriction. Only the results \( \pm 1 \) should be possible.

\[ V = \{-1, +1\}. \]

Only the settings \( \phi_X \) and \( \phi_Y \) are to be used, and by the notation \( X = A(\phi_X, \lambda) \), \( Y = A(\phi_Y, \lambda) \), \( X' = B'(\phi_X, \lambda) \) and so on, the following will be shortened considerably. There are then only six different random variables to be used,

\[ X, X', X'', Y, Y', \text{ and } Y'' \]

defined on

\[ \Lambda_X, \Lambda_{X'}, \Lambda_{X''}, \Lambda_Y, \Lambda_{Y'}, \text{ and } \Lambda_{Y''}, \text{ respectively.} \]

These must then satisfy

(iv) GHZ requirement. The following measurement results should be obtained:

\[ XX'X'' = -1 \text{ on } \Lambda_{XX'X''} \]

This model contains a contradiction if and only if

\[ \delta_{0,3} > 0 \]

Proof: The proof is simply an observation in (7.1), that all of the random variables \( X \), \( X' \), \( X'' \), \( Y \), \( Y' \), and \( Y'' \) must be defined for the equality to be valid, and the set at which this is true is \( \Lambda_{XX'X''YY'Y''} \).

If \( \delta_{0,3} = 0 \), this is a null set, and on such a set (iv) need not be satisfied. Then (7.1) is no longer valid at any point of the probability space \( \Lambda \) and there is no contradiction from
the prerequisites. Indeed, it is easy to construct a model satisfying (iv) on $\Lambda^c_{XX'X''Y'Y''}$ (see e.g. below).

If $\delta_{b,3} > 0$, the set has positive probability and the contradiction remains. \qed

Again the concept of ensemble change is the fundamental concept, in the same way as in the Bell theorem, only that the violation is stronger here. In the Bell theorem $\delta_{4,2} > 75\%$ was needed for the QM correlation to violate the inequality ($\delta_{4,2} > 58.58\%$ in the CHSH inequality). In the GHZ paradox on the other hand, $\delta_{b,3} > 0$ directly yields a contradiction in the model. This is the sense that the GHZ paradox yields a “stronger contradiction” than Bell’s inequality (and the CHSH inequality). We have now what we need to look at detector-efficiency.

### 7.3 Necessary and sufficient detector-efficiency conditions

We want to derive bounds for the detector-efficiency, for which we need a formal definition of this concept, and we may use Defs. 5.6 and 5.7 just as in the Bell case. Remember that there are only two settings of each detector in this case so that “min” can be used instead of “inf”. In addition, since there are three detectors, it is possible to define three-detector coincidence efficiencies.

**Definition 7.6 (Three-detector coincidence efficiencies)**

\[
\eta_{3,2} \overset{\text{def}}{=} \min_{\phi_A, \phi_B, \phi_C} \phi_{\neq i}, \phi_{\neq j}, \phi_{\neq k} P_{A(0)B(0)C(0)} (\Lambda_{C(0)})
\]

\[
\eta_{3,1} \overset{\text{def}}{=} \min_{\phi_A, \phi_B, \phi_C} \phi_{\neq i}, \phi_{\neq j}, \phi_{\neq k} P_{A(0)} (\Lambda_{B(0)C(0)})
\]

Here, $\eta_{3,2}$ is the least probability of detecting a photon at one detector given that a photon was detected at the two other detectors, or in other words $\eta_{3,2}$ is “the efficiency when adding the last detector”. The $\eta_{3,1}$ is the least probability of detecting photons at two detectors given that a photon was detected at the third. The two assumptions of constant detector-efficiency and independent errors are avoided in the same way as in the Bell case.

Of course the list of possible efficiency measures may be made longer, but only the four mentioned here will be used in the following because of two different reasons.

1) $\eta_1$ is the intuitive efficiency measure, and can be obtained from experimental data if the assumptions of independent errors and constant detector efficiency hold.

2) $\eta_{2,1}$, $\eta_{3,2}$, and $\eta_{3,1}$ are useful when not assuming independent errors and constant detector efficiency. They are also obtainable from experimental data without these assumptions.
Bounds for these four efficiency measures may now be obtained by using Theorem 7.5, and we arrive at the theorem below. The necessity of the four bounds is shown by counterexample (an explicit construction is given in the proof of the theorem).

**Theorem 7.7 (Necessary and sufficient detector-efficiency conditions for the GHZ paradox)**

A model satisfying (i)–(iv) of Theorem 7.5 except on a $P$-null set contains a contradiction if at least one of the following is satisfied:

\[
\eta_1 > \frac{5}{6} \approx 83.33\% \quad (a)
\]
\[
\eta_{2,1} > \frac{4}{5} = 80\% \quad (b)
\]
\[
\eta_{3,2} > \frac{3}{4} = 75\% \quad (c)
\]
\[
\eta_{3,1} > \frac{3}{5} = 60\% \quad (d)
\]

Furthermore, if none of (a)–(d) is satisfied there exists a LHV model satisfying (i)–(iv) which yields the QM statistics of the GHZ source for the $\phi_X$ and $\phi_Y$ orientations except for detector inefficiency.

**Proof:** The proof begins with the first statement.

(a) Using $\eta_1$, a simple derivation yields

\[
P(\Lambda_{XX'Y'Y'}) = 1 - \sum_{\Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}}}
\]
\[
\geq 1 - \sum_{\Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}} \cup \Lambda_{XX'}^{\text{C}}}
\]
\[
= P(\Lambda_X) + P(\Lambda_{XX'}) + P(\Lambda_{XX'}) - 5
\]
\[
\geq 6\eta_1 - 5.
\]

(b) The same approach as above on $P_X(\Lambda_{XX'Y'Y'})$ is quite useless since $\eta_{2,1}$ does not yield any estimate on probabilities where the same particle occurs twice, e.g. $P_X(\Lambda_Y)$. One could perhaps be tempted to change the definition so that this problem would not occur, but then an important property of $\eta_{2,1}$ would be lost. It would not be easy to extract an estimate from experiment anymore. Instead, a slightly more sophisticated approach must be used to avoid probabilities with
two occurrences of the same particle:

\[
P_{XX'}(\Lambda_Y) = \frac{P_{XX'}(\Lambda_{XY})}{P_{X'}(\Lambda_X)} = \frac{P_{XX'}(\Lambda_X) + P_{XX'}(\Lambda_Y) - P_{XX'}(\Lambda_X \cup \Lambda_Y)}{P_{X'}(\Lambda_X)} \geq 1 + \frac{\eta_{2,1} - 1}{P_{X'}(\Lambda_X)} \geq 1 + \frac{\eta_{2,1} - 1}{\eta_{2,1}} = 2 - \frac{1}{\eta_{2,1}}. \tag{7.5}
\]

If the set in the parenthesis in the left-hand side is \(\Lambda_{\chi^a}\) or \(\Lambda_{\gamma^a}\), the same inequality holds. It also holds if the set is \(\Lambda_{\gamma^a}\) (exchange \(X\) with \(X'\) throughout the equation). Now the approach from (a) yields

\[
P_{XX'}(\Lambda_{\chi^aYY^a}) = 1 - P_{XX'}(\Lambda_{\chi^a} \cup \Lambda_{Y^a} \cup \Lambda_Y \cup \Lambda_{\gamma^a}) \geq 1 - P_{XX'}(\Lambda_{\chi^a}) - P_{XX'}(\Lambda_{Y^a}) - P_{XX'}(\Lambda_Y) - P_{XX'}(\Lambda_{\gamma^a}) = P_{XX'}(\Lambda_{\chi^a}) + P_{XX'}(\Lambda_{Y^a}) + P_{XX'}(\Lambda_Y) + P_{XX'}(\Lambda_{\gamma^a}) - 3 \geq 4(2 - \frac{1}{\eta_{2,1}}) - 3 = 5 - \frac{4}{\eta_{2,1}},
\]

and then,

\[
P(\Lambda_{XX'YY'Y^a}) = P_{XX'}(\Lambda_{\chi^aYY^a})P_X(\Lambda_{X'})P(\Lambda_X) \geq P_{XX'}(\Lambda_{\chi^aYY^a})\eta_{2,1}\eta_1 \geq (5\eta_{2,1} - 4)\eta_1. \tag{7.7}
\]

(c) Using \(\eta_{3,2}\) and almost the same approach as in (b),

\[
P_{XX'X'}(\Lambda_Y) = \frac{P_{XX'X'}(\Lambda_X \cap \Lambda_Y)}{P_{XX'}(\Lambda_X)} = \frac{P_{XX'X'}(\Lambda_X) + P_{XX'X'}(\Lambda_Y) - P_{XX'X'}(\Lambda_X \cup \Lambda_Y)}{P_{XX'}(\Lambda_X)} \geq 1 + \frac{\eta_{3,2} - 1}{P_{XX'}(\Lambda_X)} \geq 1 + \frac{\eta_{3,2} - 1}{\eta_{3,2}} = 2 - \frac{1}{\eta_{3,2}}. \tag{7.8}
\]

The result is the same if the set inside the parentheses on the left-hand side is \(\Lambda_{\gamma^a}\).
or \( \Lambda_{yy} \), (exchange \( X \) with \( X' \) or \( X \) with \( X'' \), resp.). This yields

\[
P_{XXX''}(\Lambda_{YY'Y''}) = 1 - P_{XX'X''}(\Lambda_Y^c \cup \Lambda_{Y'}^c \cup \Lambda_{Y''}^c) \\
\geq 1 - P_{XX'X''}(\Lambda_Y^c) - P_{XYY''}(\Lambda_{Y'}^c) - P_{XX''X''}(\Lambda_{Y''}^c) \\
= P_{XX'X''}(\Lambda_Y) + P_{XYY''}(\Lambda_{Y'}) + P_{XX''X''}(\Lambda_{Y''}) - 2 \\
\geq 3(2 - \frac{1}{\eta_{3,2}}) - 2 = 4 - \frac{3}{\eta_{3,2}}.
\]  
(7.9)

and then,

\[
P(\Lambda_{XXX''YY'Y''}) = P_{XX'X''}(\Lambda_{YYY''})P_{XX'}(\Lambda_{X'})P_X(\Lambda_X)P(\Lambda_X) \\
\geq P_{XX'X''}(\Lambda_{YYY''})\eta_{3,2}\eta_1 \geq (4\eta_{3,2} - 3)\eta_{2,1}\eta_1. \tag{7.10}
\]

(d) Using \( \eta_{3,1} \) the result is

\[
P_{XX'X''}(\Lambda_{YY''}) = \frac{P_X(\Lambda_{XXX''YY'Y''})}{P_X(\Lambda_{XXX'})} \\
= \frac{P_X(\Lambda_{XXX'}) + P_X(\Lambda_{YY''}) - P_X(\Lambda_{XXX'} \cup \Lambda_{YY''})}{P_X(\Lambda_{XXX'})} \\
\geq 1 + \frac{\eta_{3,1} - 1}{P_X(\Lambda_{XXX'})} \\
\geq 1 + \frac{\eta_{3,1} - 1}{\eta_{3,1}} = 2 - \frac{1}{\eta_{3,1}}. \tag{7.11}
\]

The same of course holds if the set is \( \Lambda_{YY} \) or \( \Lambda_{YY'Y''} \) (exchange \( X \) with \( X' \) or \( X \) with \( X'' \), resp.), and using another probabilistic identity,

\[
2P_{XX'X''}(\Lambda_{YY'Y''}) = P_{XX'X''}(\Lambda_{YY'}) + P_{XX'X''}(\Lambda_{YY''}) + P_{XX'X''}(\Lambda_{YY'Y''}) \\
- P_{XX'X''}(\Lambda_{YY'} \cup \Lambda_{YY'} \cup \Lambda_{YY''}) \\
\geq P_{XX'X''}(\Lambda_{YY'}) + P_{XX'X''}(\Lambda_{YY''}) + P_{XX'X''}(\Lambda_{YY'Y''}) - 1 \\
\geq 3(2 - \frac{1}{\eta_{3,1}}) - 1 = 5 - \frac{3}{\eta_{3,1}},
\]  
(7.12)

which yields

\[
P(\Lambda_{XXX'YY'Y''}) = P_{XX'X''}(\Lambda_{YY'Y''})P_X(\Lambda_{X'})P_X(\Lambda_X) \\
\geq P_{XX'X''}(\Lambda_{YY'Y''})\eta_{3,1}\eta_1 \geq (5\eta_{3,1} - 3)\eta_1. \tag{7.13}
\]
Collecting the inequalities (7.4), (7.7), (7.10) and (7.13) we have

\[
P(\Lambda_{X'X''Y''Y'}) \geq 6\eta_1 - 5 \quad \text{(7.14a)}
\]

\[
P(\Lambda_{X'X''Y''Y'}) \geq (5\eta_{2,1} - 4)\eta_1 \quad \text{(7.14b)}
\]

\[
P(\Lambda_{X'X''Y''Y'}) \geq (4\eta_{3,2} - 3)\eta_{2,1}\eta_1 \quad \text{(7.14c)}
\]

\[
P(\Lambda_{X'X''Y''Y'}) \geq (5\eta_{3,1} - 3)\frac{\eta_1}{2}. \quad \text{(7.14d)}
\]

Then \(P(\Lambda_{X'X''Y''Y'}) > 0\) if at least one of (a)–(d) is satisfied (and both \(\eta_{2,1}\) and \(\eta_1\) are strictly positive). Using Theorem 7.5 (and Observation 7.4) the first statement follows.

The second statement will be proved by construction; the model will of course be contradiction-free. Let \(\Lambda\) (the sample space) be a point set with 48 different points, and let the points have equal probability (\(\frac{1}{48}\)). At the first 16 points the random variables are obtained by rather simple checks: \(\eta\) corresponds to a reordering of the rows in the table. By construction the model is then contradiction-free. Let \(\eta\) be a point set with 48 different points, and let the points have equal probability (\(\frac{1}{48}\)). At the first 16 points the random variables are obtained by rather simple checks: \(\eta\) corresponds to a reordering of the rows in the table. By construction the model is then contradiction-free. Let \(\Lambda\) be the sample space. To obtain the model we will of course be construction-free. Let \(\Lambda\) be the sample space. To obtain the model we will of course be construction-free. Let \(\Lambda\) be the sample space. To obtain the model we will of course be construction-free. Let \(\Lambda\) be the sample space. To obtain the model we will of course be construction-free.

\[
\begin{array}{cccccc}
 X & X' & X'' & Y & Y' & Y'' \\
\hline
- & - & - & \# & + & - \\
- & - & - & \# & - & + \\
- & + & + & \# & + & - \\
+ & - & + & \# & + & - \\
+ & + & - & \# & + & + \\
+ & + & - & \# & - & - \\
\end{array}
\]  

\[
\begin{array}{cccccc}
 X & X' & X'' & Y & Y' & Y'' \\
\hline
\# & - & - & + & - & - \\
\# & - & - & - & + & + \\
\# & + & + & + & + & + \\
\# & - & + & + & - & - \\
\# & + & - & + & - & + \\
\# & + & - & - & - & + \\
\end{array}
\]

The values at the remaining 32 points is obtained by interchange of the first and second particle \((X \doteq X', Y \doteq Y')\) and by interchange of the first and third particle \((X \doteq X'', Y \doteq Y'')\). Interchange of the second and third particle \((X' \doteq X'', Y' \doteq Y'')\) corresponds to a reordering of the rows in the table. By construction the model is then symmetric with respect to interchange of the particles in the sense that the probabilities are the same. \(\Lambda\) consists of six disjoint sets of eight points each, at which one of the random variables is undefined.

This yields \(P(\Lambda_{A}) = \frac{5}{6}\) for all random variables, so that \(\eta_1 = \frac{5}{6}\), and the others are obtained by rather simple checks: \(\eta_{2,1} = \frac{4}{5}\), \(\eta_{3,2} = \frac{3}{4}\), and \(\eta_{3,1} = \frac{3}{5}\), e.g.,

\[
P(\Lambda_{YY''Y''}) = P(\Lambda_{X}) + P(\Lambda_{XX'}) + P(\Lambda_{XX''}) = \frac{3}{6},
\]

\[
P(\Lambda_{YY''}) = P(\Lambda_{YY''Y''}) + P(\Lambda_{XX''}) = \frac{4}{6}
\]

\[
\Rightarrow P_{YY''}(\Lambda_{Y}) = \frac{3}{4}.
\]
By inspection $XY''Y' = 1 = -XX'X''$ in $\Lambda_x^b$ and $YY'X'' = 1 = YY'X''$ in $\Lambda_x^b$, and by using the symmetry (iv) follows. Indeed, all the QM statistics for the $\phi_x$ and $\phi_y$ orientations for the GHZ source (obeying (iv)) are possible to derive, except of course for detector inefficiency. This completes the proof.

There are four bounds (a)–(d) in the theorem above, but which is most important? The lowest bound seems to be $\eta_{3,1} > 60\%$, but one must remember that this corresponds to adding two detectors, so it is only natural that this bound is low. One would naively expect $\eta_{3,1} = \eta_{3,2}\eta_{2,1}$, but this need not be generally true given only the Defs 5.7 and 7.6, since no additional assumptions are used. The bounds, however, does seem to follow this rule as $60\%$ is $75\%$ of $80\%$, and in the model given in the latter part of the proof of Theorem 7.7, the above equation does hold.

The important bound is instead the lowest when adding one detector, which is $\eta_{3,2} > 75\%$, “adding the last detector”. An interesting observation is that in the Bell case the lowest bound was on $\eta_{2,1}$, “adding the last detector”.

### 7.4 Independent errors

In the above construction, the nondetection errors are dependent, because the model is constructed to provide a counterexample in a theorem where no assumptions are made on the properties of the nondetection errors. There is a possibility that the bound would be lowered for the case of independent errors and constant efficiency. Let us examine Theorem 7.7, using the two following additions (compare with Section 5.5):

(v) Constant detector efficiency. The probability of a detection at any detector at any orientation is $\eta$.

$$\eta = P(\Lambda_x) = P(\Lambda_{x'}) = P(\Lambda_{y'})$$

$$= P(\Lambda_y) = P(\Lambda_{y'}) = P(\Lambda_{y''}).$$

(vi) Independent nondetection errors. The detection errors are probabilistically independent for detection at different detectors at any orientation, e.g.,

$$P(\Lambda_{y''x'}) = P(\Lambda_y)P(\Lambda_{y''}),$$

$$P(\Lambda_{x'y''}) = P(\Lambda_x)P(\Lambda_{y''}).$$

With these assumptions, the four different efficiency measures above reduce to one ($\eta$) because of the following:

$$\eta_1 = \eta_{2,1} = \eta_{3,2} = \eta, \quad \eta_{3,1} = \eta^2.$$  \hspace{1cm} \text{(7.17a)}

This is a trivial implication of (v) and (vi), and using these assumptions, an immediate corollary of Theorem 7.7 is the following.

---

1Note that this is not a complete model of the QM state as only the $\phi_x$ and $\phi_y$ orientations are used.
Corollary 7.8  (Necessary and sufficient detector-efficiency conditions for the GHZ paradox with independent errors)

A model satisfying (i)–(iv) of Theorem 7.5 except on a P-null set, and in addition satisfying (v) and (vi), contains a contradiction if the following is satisfied:

\[ \eta > \frac{3}{4} = 75\%. \]

Furthermore, if the above inequality is not satisfied there exists a LHV model satisfying (i)–(vi) which yields the QM statistics of the GHZ source for the \( \Phi_X \) and \( \Phi_Y \) orientations except for detector inefficiency.

**Proof:** The first statement is easily proven using the inequalities (a)–(d) of Theorem 7.7 and equations (7.17a) and (7.17b). There is a violation if one of

\[
\eta > \frac{5}{6} \approx 83.33\%, \quad (7.18a) \\
\eta > \frac{4}{5} = 80\%, \quad (7.18b) \\
\eta > \frac{3}{4} = 75\%, \quad (7.18c) \\
\eta^2 > \frac{3}{5} = 60\% \quad (7.18d)
\]

is satisfied, i.e., as soon as the lowest bound is. This is equation (7.18c) since the square root of \( \frac{3}{5} \) is greater than \( \frac{3}{4} \).

The second statement is, as in Theorem 7.7, proved by construction. There, the model consists of a sample space \( \Lambda \) of 48 points with equal probability, and these points are assigned measurement results according to two tables.

The model yields the required results and has constant efficiency, but the non-detection errors are dependent. The probability of a triple coincidence (e.g., at \( XX'X'' \) orientation) is \( \frac{1}{2} \), which is too large, but even so, the model is useful and will be used as a basic building block. This set of 48 points is used as a subset of the full sample space in the construction below, denoted \( \Lambda_{2/3} \), the set of double or triple coincidences. The probability of \( \Lambda_{2/3} \) should be such that

\[
P(\Lambda_{XX'X''}) = \left( \frac{3}{4} \right)^3 = \frac{27}{64} \left( \frac{1}{2} = \frac{32}{64} \right), \quad (7.19)
\]

and that is achieved by giving the subset a total probability of \( \frac{54}{64} \) (the model is symmetric in such a way that the other triple coincidences have the same probability). In this subset, the value assignment is done as in the tables (see Fig. 7.7).

The probability of two or more detections at a particular setup should be the square of \( \frac{3}{4} \), and indeed, using the above subset construction (which is symmetric for different setups),

\[
P(\Lambda_{XX'}) = \frac{2}{3} \times \frac{54}{64} \times \frac{9}{16} = \left( \frac{3}{4} \right)^2. \quad (7.20)
\]
Thus, the total probability of \( \Lambda \) is
\[
P(\Lambda) = \frac{3}{4} + \frac{1}{4} = 1.
\]

In addition, there are no purely single detections yet, only double and triple coincidences. Let it consist of six points, because we want two points for each of the three particles (allowing for the two results ±1). Assign equal probability to the points so that
\[
P(\Lambda_1 \cap \Lambda_2/3) = \frac{3}{4} \left( 1 - \frac{3}{4} \right)^2 = \frac{3}{64},
\]
which yields
\[
P(\Lambda_1) = P(\Lambda_1 \cap \Lambda_2/3) + P(\Lambda_1 \cap \Lambda_{2/3}) = \frac{3}{4}.
\]

Thus, the total probability of \( \Lambda_1 \) should be \( \frac{3}{64} \).

Finally, the full sample space should include the set of no detection (\( \Lambda_0 \)) at a probability of
\[
P(\Lambda_0) = \left( 1 - \frac{3}{4} \right)^3 = \frac{1}{64},
\]
and adding such a subset (consisting of one point), the total probability of the sample space adds to one.

<table>
<thead>
<tr>
<th>Group</th>
<th>Points</th>
<th>$P(\text{point})$</th>
<th>$\sum P$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda_2/3$ (double/triple coinc.)</td>
<td>48</td>
<td>$\frac{9}{64}$</td>
<td>$\frac{54}{64}$</td>
</tr>
<tr>
<td>$\Lambda_1$ (purely single detections)</td>
<td>6</td>
<td>$\frac{3}{8} \times \frac{1}{64}$</td>
<td>$\frac{9}{64}$</td>
</tr>
<tr>
<td>$\Lambda_0$ (no detection)</td>
<td>1</td>
<td>$\frac{1}{64}$</td>
<td>$\frac{1}{64}$</td>
</tr>
</tbody>
</table>

The resulting model is visible in Fig. 7.2. The statistics are precisely that of the GHZ source using 75% (constant) efficiency and independent errors:

1) One particle. Detection probability: $\frac{3}{4}$. Measurement at one site of $X^{(i)}$ or $Y^{(i)}$ (six different possibilities) yields two equally probable results ($\pm$).

2) Two particles. Probability of pair detection: $\left(\frac{3}{4}\right)^2$. Measurement at two different sites, of $X^{(i)}X^{(j)}$, $X^{(i)}Y^{(j)}$, or $Y^{(i)}Y^{(j)}$ ($i \neq j$, 12 different possibilities) yields four equally probable results: $++$, $+-$, $-+$, and $--$.

3) Three particles. Probability of triple detection: $\left(\frac{3}{4}\right)^3$.
   
   (a) Measurement at three sites of $YY'Y''$, $XX'X''$, $XY'X''$, or $YX'X''$ yields eight equally probable results: $++$, $++$, $+-$, $+-$, $-+$, $-+$, $--$, and $--$.

   (b) On measurement at three sites of $XX'X''$, only four results appear. These four results are equally probable, and are $--$, $++$, $+-$, and $-+$, each with an odd number of minus signs as in (iv).

   (c) On measurement at three sites of $XY'Y''$, $YX'Y''$, or $YY'X''$, again only four results appear. In this case, the four equally probable results are $++$, $++$, $+-$, $-+$, and $--$, each with an even number of minus signs as in (iv).

The above statements are easily checked in Fig. 7.2, where the symmetry of the model simplifies the check significantly. This completes the proof.

We have seen that the dependent errors of the model in the proof of Theorem 7.7 are not an important feature of a LHV model of the GHZ experiment. There can be no test below 75% that can rule out local variables on the basis that the nondetection errors should be independent, and thus, a GHZ experiment with independent errors and constant efficiency refutes local variables if and only if the efficiency is higher than 75%. An experiment at a lower efficiency would not be conclusive, since the model presented in this paper is valid at 75%, and it is easily extended to lower efficiency.

In particular, this means that a “single-shot” GHZ experiment cannot be said to violate Local Realism, even if the paradox is formulated as a direct contradiction rather
than a statistical inequality. It is important that the efficiency of the measurement setup is better than the mentioned 75%, and that can only be tested by a statistical test on several runs of the experiment. Of course, one also needs to perform the experiment at the four mentioned setups \((XX'X'', XY'Y'', YX'Y'', and YY'X'')\) to verify that the experiment really yields the QM results.
A new experimental test of local hidden-variable theories based on optical interference is proposed.

(Franson, 1989)

In the previous pages we have been considering two established examples of QM contradictions of Local Realism. There are many more available in the literature, but in this thesis only one additional example will be investigated, commonly known as “the Franson experiment” (Franson, 1989). This experimental setup is interesting because it is much simpler to set up in practice since no polarization measurement is needed; at each side of the experiment a simple optical interferometric device is used (see Fig. 8.1). This makes it interesting to use for commercial applications of these fundamental relations, for instance Quantum Cryptography (see e.g., Ekert, 1991).

From our point of view it is even more interesting, but for a more theoretical reason; as we will see below, this experimental data contains unexpected subtleties. This is because half of the measurement data is discarded in the process of generating the correlation function to be tested in the Bell inequality, corresponding to an effective
efficiency of 50% even if the detectors used are 100% efficient (the considerations in this chapter will be restricted to this ideal situation). By now, the reader will have seen that there is, in such a situation, no contradiction arising from the appropriately modified Bell inequality. Following Aerts, Kwiat, Larsson, and Żukowski (1999)*, we will here emphasize this by constructing a LHV model of the experiment. The construction will be similar to (and is in fact inspired by) the model in Chapter 6. However, it will also be shown that no such model exists if an additional constraint is put on the rate of change of the detector settings $\phi_A$ and $\phi_B$.

### 8.1 One-particle interferometry

To introduce the concept of interferometry, a device commonly used in quantum-optical setups will here be presented: the Mach-Zehnder interferometer. The presentation will be made in the same style as the polarization setting of Chapter 3 and is again intended as a help to readers unfamiliar with this concept, rather than as a formal derivation of the properties of such an interferometer.

A Mach-Zehnder interferometer is an optical device consisting of two mirrors, a phase-shifter and two “beam-splitters” as depicted in Fig. 8.2.

![A Mach-Zehnder interferometer.](image)

An interferometer like this is easiest to explain in the context of continuous electromagnetic waves, just as was the case in Chapter 3. The beam-splitters work in the following way (see Fig. 8.3): the ingoing beam is split, and the intensity of each outgoing beam is halved (the amplitude is divided by $\sqrt{2}$). The phase of the transmitted beam is not changed, but in the reflected beam, a $\pi/2$ phase-shift is introduced.

* A reprint of this paper can be found at the end of the thesis.
The two internal beams are subsequently mixed at the second beam-splitter (to simplify, the phase-shifter is here assumed to be absent). The two beams are affected in the same way by the second beam-splitter as the ingoing beam was by the first. The beam that was reflected at the first beam-splitter will be split in two, introducing another phase-shift in the beam going upward in the picture. The beam that was transmitted through the first beam-splitter, also yields an upward beam at the second beam-splitter, a beam which is transmitted in the second beam-splitter too. Therefore, there will be no phase-shift of the twice transmitted beam and a phase-shift $\pi$ in the twice reflected beam (see Fig. 8.4).

The recombination is visible in Fig. 8.5, where the two upward beams emerging from the second beam-splitter will have a relative phase-shift of $\pi$ radians, and thus will cancel. In the other out-beam, the parts are both phase-shifted $\pi/2$, being reflected
at one beam-splitter each. These two will then add up to a wave as intense as the original ingoing beam.

![Figure 8.5: The result of the recombination at the second beam-splitter.](image)

When this device is applied to individual photons, the result is that all the photons going into the device exit at the beam marked $+1$.

![Figure 8.6: The Mach-Zehnder interferometer used on individual photons.](image)

Note that there are no photons drawn in the interior of the interferometer; this is because it is not possible to know what path the photon took inside the interferometer and still get photons only in the $+1$ beam. The path taken inside the interferometer and the path taken after the interferometer are *complementary* properties (similarly to
This will not be discussed further here, but is an example of the QM properties of this device.

Using a phase-shifting element at the indicated position in the interferometer (see Fig. 8.7), it is possible to control the amount of light escaping at each output of the interferometer.

We are adding two sinusoidal waves with a relative phase-shift, and the examples in the figure are 0, $\pi/2$, and $\pi$. We have already seen what will happen when using the phase-shift 0. With the phase-shift $\pi$, the addition in the rightward output will cancel, and all the light will exit in the upward beam. For a $\pi/2$ phase-shift, the outputs will share the intensity equally, which means that the amplitudes both will be $1/\sqrt{2}$ of that of the input beam.

Performing a more complete calculation it is possible to show that the intensity of the upward and rightward outputs are proportional to $\sin^2(\phi)$ and $\cos^2(\phi)$ (the amplitudes of the upward and rightward outputs are proportional to $\sin(\phi)$ and $\cos(\phi)$, respectively). As expected, these intensities sum to 1, so that all the light entering the interferometer also exits it.

For the case of looking at one photon, the probability of that photon exiting at either port will now follow the above formulas. In other words:

$$P(+1) = \cos^2(\phi),$$  \hspace{1cm} (8.1a)

$$P(-1) = \sin^2(\phi).$$  \hspace{1cm} (8.1b)

### 8.2 Two-particle interferometry

In the Franson experiment a modified device is used; the Unbalanced Mach-Zehnder interferometer, depicted in Fig. 8.8.

Unfortunately, there is no good classical description of the internals in this interferometer (see e.g., Ou and Mandel, 1990; Franson, 1991), but an attempt will be made to explain what the interferometer does. This is a difficult task, and the description here will be (too) brief, but the interested reader may find a more complete derivation
of the properties in the original paper (Franson, 1989) or textbooks of quantum optics (see e.g., Scully and Zubairy, 1997), where second-quantized QM operators are used to derive the below QM probabilities.

The reason that the electromagnetic waves could be used in the previous section is that in that case, a photon can be approximated by a short classical electromagnetic pulse that is split, phase-shifted and recombined in the interferometer. In an unbalanced interferometer, such an approximation yields an electromagnetic pulse that is shorter than the length-difference of the paths. Thus there will be no overlap at the second beam-splitter and indeed, there is no interference when looking at only one interferometer, and

$$P^\text{QM}(+1) = P^\text{QM}(-1) = \frac{1}{2}, \quad (8.2)$$

independently of the setting of $\phi$. The classical approximation of a photon works to the degree that there is no one-particle interference in the experiment (even in the QM description).

Unfortunately, this classical picture breaks down in the next step. Using the full experimental setup schematically shown in Fig. 8.1 with both interferometers present, it is possible to establish two-particle interference. The aim is to arrange the experiment so that two “beams” arrive at the second beam-splitter simultaneously, so that interference occurs. By setting up the experiment so that the moment of emission is unknown, there are two possibilities for a photon to arrive at the second beam-splitter. Either, the photon was emitted early but took the long path through the unbalanced interferometer, or it was emitted later and took the short path. Now, interference may occur, and this will be visible in the data as a sinusoidal variation of the correlation between $A$ (on the
left side) and \( B' \) (on the right) when the detections are coincident:

\[
P_{\text{QM}}(A = +1 \cap B' = +1 \cap \text{coinc.}) = \frac{1 + \cos(\phi_A + \phi_B)}{8}, \tag{8.3a}
\]

\[
P_{\text{QM}}(A = +1 \cap B' = -1 \cap \text{coinc.}) = \frac{1 - \cos(\phi_A + \phi_B)}{8}, \tag{8.3b}
\]

\[
P_{\text{QM}}(A = -1 \cap B' = +1 \cap \text{coinc.}) = \frac{1 - \cos(\phi_A + \phi_B)}{8}, \tag{8.3c}
\]

\[
P_{\text{QM}}(A = -1 \cap B' = -1 \cap \text{coinc.}) = \frac{1 + \cos(\phi_A + \phi_B)}{8}, \tag{8.3d}
\]

This is only because the emission moment is unknown; in fact, measurement of the moment of emission is complementary to measurement of the product \( AB' \) (compare with the complementarity present in one-particle interferometry in Section 8.1). Of course, the settings \( \phi_A \) and \( \phi_B \) in the above formulas are those present when a photon passes through the corresponding phase-shifter, i.e., the phase setting at the actual detection time \( t_d \), minus the time \( t_{ret} \) it takes a light-signal (at the speed \( c \)) to reach the detector from the location of the phase shifter by the optical paths available within the interferometer.

For the 50\% two-photon events that are not coincident, one photon takes the short path while the other takes the long path, and the moment of emission can be determined from the moment of detection of the early photon (that took the short path). In this case, there is no interference:

\[
P_{\text{QM}}(A = +1_L \cap B' = +1_E) = P_{\text{QM}}(A = +1_L \cap B' = -1_E) = P_{\text{QM}}(A = -1_L \cap B' = +1_E) = P_{\text{QM}}(A = -1_L \cap B' = -1_E) = P_{\text{QM}}(A = +1_E \cap B' = +1_L) = P_{\text{QM}}(A = +1_E \cap B' = -1_L) = P_{\text{QM}}(A = -1_E \cap B' = +1_L) = P_{\text{QM}}(A = -1_E \cap B' = -1_L) = 1/16, \tag{8.4}
\]

independently of the setting of \( \phi_A \) and \( \phi_B \). Here, \( E \) denotes the earlier detection, and \( L \) denotes the later detection. The time-difference of these two detections is equal to the delay in the interferometer, \( d/c \).

Initially, the experiment is assumed to enforce locality by the same method as is common in polarization experiments intended to violate the Bell inequality (see Chapter 5), i.e., by switching the local phase settings on the time scale \( d/c \), where \( D \) is the source–interferometer distance, and it is assumed that this distance is much longer than the path-length difference \( d \) within an interferometer, i.e., \( D \gg d \). The two experimenters (one at each side) record the \( \pm 1 \) counts, the detection times, and the appropriate values of the local phase settings. After the experiment is completed they perform an analysis of their recorded data, rejecting all pairs of events whose registration times differ by \( d/c \), since there is no interference present in those events.

When restricting the analysis to the data from the coincident detection, the resulting correlation is

\[
E_{\text{coinc.}}^{\text{QM}}(AB') = \cos(\phi_A + \phi_B). \tag{8.5}
\]
This correlation has the same sinusoidal form as obtained from the Bell experiment (although, signs and frequencies differ), but the violation remains the same; the CHSH inequality is violated by this correlation. In Franson (1989) it is argued that this alone suffices to make this experimental setup a test of Local Realism.

However, it is not shown in that paper that the inequality is applicable in this situation. We have seen that if the Bell experiment is inefficient, the inequality changes and may not be violated anymore. Some skepticism has been expressed (De Caro and Garuccio, 1994; Kwiat, 1995) that a true, unambiguous test of a Bell inequality is possible in the Franson experiment, even in principle, since even the ideal gedanken-experiment requires a postselection procedure in which 50% of the events are discarded when computing the correlation.

The situation is further obscured by similar claims concerning certain other two-photon polarization experiments (Ou and Mandel, 1988; Shih and Alley, 1988) where the problem of discarding 50% of the events also appears. This was initially treated on equal footing with the problems of Franson-type experiments, but a recent analysis in Popescu et al. (1997) reestablishes the possibility of violating Local Realism. Unfortunately, that analysis cannot be adapted to the Franson experiment.

8.3 A LHV model for the Franson experiment

Following the example from Chapter 6, a construction of a LHV model will be presented. Similarly to that model, the hidden variables are chosen to be an angular coordinate \( \theta \in [0, 2\pi] \) and an additional coordinate \( r \in [0, 1] \), uniformly distributed. Each pair of particles is then described by a definite value of \( \lambda = (\theta, r) \), defined at the source at the moment of emission.

The moment of emission should also be part of the hidden variables because it is an element of reality as described in Chapter 2. The reason is that if, say, the right interferometer was removed, the photon traveling to the right would be detected by the detector +1, and the detection time \( t_d \) would indicate the moment of emission of that photon, establishing the moment of emission of the other photon as an element of reality. However, the emission time will be left out of the formalism; it is enough to know the time of detection \( t_d \) and whether the detection was early or late, which will be given by the hidden variables \( (\theta, r) \). Note, however, that in the QM description there is no well-defined moment of emission; only when the emission time is not known (not measured) will there be interference.

Thus, we have four possible results (early detection: \(+1_E, -1_E\), and late detection: \(+1_L, \text{ and } -1_L\)), and the LHV model will determine what the result will be. The measurement setup is such that when determining the moment of emission (by removing the right interferometer), the events at the left interferometer yields information on the ratio early/late events. Half of the events on the left side are early (E) and half are late (L). With the right interferometer in place, \( \frac{1}{4} \) of the events are early on the left and late on the right (EL), \( \frac{1}{4} \) are late on the left and early on the right (LE), and \( \frac{1}{2} \) are coincident. These coincident events must then consist of equal parts early-early (EE)
and late-late (LL) events. Note that no such distinction exists in the QM description.

At the left detector station, the measurement result is decided by the hidden variables \( \theta, r \) and the local setting \( \phi_A \) of the apparatus. When a photon arrives at the detection station, the variable \( \theta \) is shifted by the current setting of the local phase shifter (i.e., \( \theta' = \theta - \phi_A \)), and the result is read off Fig. 8.9.

![Figure 8.9: LHV model for detections at the left station.](image)

At the right detector station, a similar procedure is followed. In this case, the shift is to the value \( \theta'' = \theta + \phi_B \), and the result is obtained in Fig. 8.10 in the same manner as before.

The single-particle detection probabilities straightforwardly follow the quantum predictions, because in both Figs. 8.9 and 8.10, the total areas corresponding to +1_E, -1_E, +1_L, and -1_L are all equal. The particle is equally likely to arrive early or late, and equally likely to go to the +1 or -1 output port of the interferometer. The coincidence probabilities are determined by interposing the two figures with the proper shifts.

For example, the probability of having \( A = +1_E \) and \( B' = -1_E \) simultaneously is the area of the set indicated in Fig. 8.11 divided by \( 2\pi \) (the total area is \( 2\pi \) whereas the total probability is 1). The net coincidence probability is

\[
P(A = +1 \cap B' = -1 \cap \text{coinc.}) = P(A = +1_E \cap B' = -1_E) + P(A = +1_L \cap B' = -1_L) = \frac{2}{2\pi} \int_0^{\phi_A + \phi_B} \frac{\pi}{8} \sin(\theta) d\theta = \frac{1}{8} \left( 1 - \cos(\phi_A + \phi_B) \right).
\]
Figure 8.10: The measurement result at the right station given by the shifted hidden variables. The symbols have the same meaning as in Fig. 8.9.

Figure 8.11: The shaded regions give the values for the initial hidden variables for which $A = +1_E$ or $B = -1_E$ are obtained (note that $\theta' = \theta - \phi_A$ while $\theta'' = \theta + \phi_B$). The overlap region of length $\phi_A + \phi_B$ represents the hidden variables for which both $A = +1_E$ and $B = -1_E$ are obtained.
It is easy to verify that this model also gives the correct prediction for the other detection events. Somewhat remarkably, this construction implies that the Franson experiment does not and cannot violate Local Realism if it is assumed that the switching is done on the time scale \( D/c \).

A number of objections to this model may be raised. One is that the model is asymmetric with respect to the detector patterns, but this can be easily remedied in the same way as in Chapter 6. Another is that the model does not take into account several simple modifications that can be performed on the interferometers, like blocking a path, removing a mirror and so on. It should be underlined that the model above is not intended to provide a complete physical description of all possible phenomena that can be observed with the type of source which is used in the Franson experiment; it is merely provided to show that the original formulation of the experiment cannot be used to exhibit a violation of Local Realism. The existence of a LHV model like the above removes any possibility of a violation. It is not difficult to model other phenomena provided that the changes are made on the time scale \( D/c \). In case an interferometer is dismantled, the detection is always +1\(_E\). If one path is blocked, the events are randomly chosen from +1 or −1 each with probability \( 1/4 \) (early or late as appropriate), or “no detection” with probability \( 1/2 \). Any such local change of the measurement setup at either detector site is possible to model; it is a simple matter to modify the Local Realistic detector patterns to remove the interference, as indicated above (whereas it is rather more difficult to establish interference in a LHV model for the original setting).

### 8.4 Fast switching

The model presented above does not take the time-delay \( d/c \) into account, other than predicting the presence of it based on the phase setting (assumed to be constant during the passage). This is reasonable, provided the state of the interferometer does not undergo rapid changes, i.e., the rate of changes is \( \ll c/d \). If the phase switching is performed at the time scale \( d/c \) typical for retardations within the interferometers, a more detailed analysis can be performed on the timing of events at one of the interferometers, using the fact that the moment of emission is an element of reality.

![Figure 8.12: The timing of events as seen from the source for the E/L decision.](image)

This description may only be used in a LHV setting, because in QM, there is no time of emission.

In the interferometer, the decision of a detection to occur early (at \( t_E \)) or late (at \( t_L = t_E + d/c \)) cannot be made later than the time \( t_E \). This decision is based on the local
variables and the properly retarded phase setting. No phase setting after \( t_E - t_{\text{ret}} \) can causally affect this E/L choice\(^{\dagger}\).

The choice \( \pm 1 \) is also based on the local variables and the properly retarded phase setting at the interferometer in question, but this choice may be made as late as the detection time \( t_d \) (\( t_d = t_E \) for early events or \( t_d = t_L = t_E + \frac{d}{c} \) for late).

\[ \text{emission} \]
\[ t_E \]
\[ t_d \]
\[ t_L \]

Latest possible \( \pm 1 \) decision for \( E \) events

Latest possible \( \pm 1 \) decision for \( L \) events

Figure 8.13: Again, the timing of events as seen from the source, here for the \( \pm 1 \) decision. The early photon is detected at \( t_E \), while the late photon is detected at \( t_L \).

Therefore, in the case of a late detection, the choice E/L is made earlier than the \( \pm 1 \) choice (the choices are made at \( t_E \) and \( t_L \), respectively) based on possibly different phase settings. From the point of view of an experimenter at one site, it is not possible to discern early detections from late ones; only the result \( \pm 1 \), the detection time \( t_d \), and two possibly different phase settings at \( t_d = \frac{d}{c} - t_{\text{ret}} \) and \( t_d - t_{\text{ret}} \) are available.

\[ \text{L emiss.} \quad \text{E emiss.} \]
\[ \frac{d}{c} \]
\[ t_d \]

Latest possible \( L \) and \( E \) decision

Latest possible \( \pm 1 \) decision

Figure 8.14: The timing of events as seen from the detector, with the different latest decision times. Note that this picture of events only applies to the LHV case; there is not a well-defined emission time in the QM case.

An experimenter (at, e.g., the left detector station) knows that for the events that are late, the later of these two phase settings cannot causally have affected the E/L decision, so the late subensemble (in the LHV model) does not depend on the phase setting at \( t_d - t_{\text{ret}} \) but only on the phase setting at the earlier time \( t_d = \frac{d}{c} - t_{\text{ret}} \). By rejecting events where the phase setting at \( t_d = \frac{d}{c} - t_{\text{ret}} \) does not have a certain value (\( \phi_l \), say), she ensures that the late subensemble does not change at all. To allow for settings other than \( \phi_l \) at the later decision time, a device which switches fast (on the time scale \( \frac{d}{c} \)) and randomly between phase settings is needed.

\(^{\dagger}\)The proper retardation for the E/L choice is really along the shortest phase-switch-detector path (not the optical path). It is possible to arrange the experiment so the two are roughly of the same size, and our argument still holds.
Thus, in the modified full experiment both detector sites should use fast devices that randomly switch between the phase settings $\phi_l$, $\phi_A$, and $\phi_B$ on the left side and $\phi_l$, $\phi_C$, and $\phi_D$ on the right. They record the appropriate data and reject (a) pairs of events whose registration times differ by $\frac{d}{c}$ and (b) pairs of events which do not have the feature that the phase setting at $t_d - \frac{d}{c} - t_{ret}$ was $\phi_l$ on the left and $\phi_r$ on the right. The latter event rejection ensures that the hypothetical LL subensemble within the remaining data is fixed with respect to the phase settings at $t_d - \frac{d}{c} - t_{ret}$. Unfortunately, the remaining EE subensemble may still depend on the phase setting at $t_d - \frac{d}{c} - t_{ret}$ even after this selection, and since the coincident ensemble consists of 50% LL and 50% EE events, $\delta_{LL} = \frac{1}{2}$ in the notation of Chapter 5. Denoting the correlation on the remaining data $E_{coinc}(AB|\phi_l, \phi_r)$ and using the generalized Bell inequality (Theorem 5.5), we have

**Corollary 8.1 (A CHSH inequality for fixed LL subensemble)**

Assume that the prerequisites of the CHSH inequality with ensemble change (Th.5.9(i)–(iii)) hold except at a null set, and that the coincident ensemble consists of an equal amount of EE and LL events. If the LL subensemble is fixed with respect to the phase-switch settings, then $\delta_{LL} = \frac{1}{2}$ and

$$|E_{coinc}(AC|\phi_l, \phi_r) - E_{coinc}(AD|\phi_l, \phi_r)|$$

$$+ |E_{coinc}(BC|\phi_l, \phi_r) + E_{coinc}(BD|\phi_l, \phi_r)| \leq 4 - \frac{2}{2} = 3.$$

Unfortunately, this inequality is not violated by the conditional quantum correlation function $E_{coinc}(AC|\phi_l, \phi_r) = \cos(\phi_A + \phi_C)$ which yields a maximum of $2\sqrt{2}$.

### 8.5 A “chained” Bell inequality

To solve this problem, a “chained” extension of the CHSH inequality (see Garuccio and Selleri, 1980; Braunstein and Caves, 1989) will be presented. The reason for the term “chained” is that the sequence of random variables in the below inequality resembles a chain; $A$ is the link between the two first correlations, $F'$ is the link between the next two and so on up to the link $H'$ from the last correlation to the first, closing the chain. In the below inequality, three settings are used at each site: $\phi_A$, $\phi_B$, $\phi_C$ and $\phi_F$, $\phi_G$, $\phi_H$, respectively. The inequality corresponding to the CHSH inequality is

**Theorem 8.2 (Chained Bell inequality)**

Assume that the prerequisites of the CHSH inequality (Th.5.2(i)–(iii)), hold except at a null set. Then

$$|E(AH') + E(AF')| + |E(BF') + E(BG')| + |E(CG') - E(CH')| \leq 4.$$
Proof:

\[
|E(AH') + E(AF')| + |E(BF') + E(BG')| + |E(CG') - E(CH')| \\
\leq |E(AH') + E(AF')| + |E(BF') + E(BG')| \\
+ |E(CG') - E(CF')| + |E(CF') - E(CH')| \leq 4. \tag{8.7}
\]

The QM correlation from the Bell experiment violates this inequality, and the maximum violation is at (e.g.) \( \phi_A = 0, \phi_B = \pi/3, \phi_C = 2\pi/3, \phi_F = \pi/6, \phi_G = \pi/2, \) and \( \phi_H = -\pi/6 \). There, we obtain

\[
5\cos(\pi/6) - \cos(5\pi/6) = 6\cos(\pi/6) = 3\sqrt{3} \approx 5.1961 > 4. \tag{8.8}
\]

In the case with ensemble change, the above inequality is valid on the subset of \( \Lambda \) where all six random variables are defined, \( \Lambda_{AF^*BGCHP} \). Because there are more random variables in this setting, we need to modify the measure of ensemble change slightly:

**Definition 8.3 (Change of ensemble)**

\[
\delta_{\text{AE}} = \inf_{\delta_A, \delta_B, \delta_C, \delta_F, \delta_G, \delta_H} P_{AF^*} (\Lambda_{BGCHP}).
\]

Including ensemble change, the theorem becomes

**Theorem 8.4 (Chained Bell inequality with ensemble change)**

Assume that the prerequisites of the CHSH inequality with ensemble change (Th.5.9(i)–(iii)) hold except at a null set. Then

\[
|E_{AF^*}(AH') + E_{AF^*}(AF')| + |E_{BF^*}(BF') + E_{BG^*}(BG')| \\
+ |E_{CG^*}(CG') - E_{CH^*}(CH')| \leq 6 - 2\delta_{\text{AE}}.
\]

**Proof:** The proof is similar to that of Theorem 5.5. Using the ensemble \( \Lambda_{AF^*BGCHP} \) (when \( \delta_{\text{AE}} > 0 \), we arrive at

\[
|E_{AF^*BGCHP}(AH') + E_{AF^*BGCHP}(AF')| \\
+ |E_{AF^*BGCHP}(BF') + E_{AF^*BGCHP}(BG')| \\
+ |E_{AF^*BGCHP}(CG') - E_{AF^*BGCHP}(CH')| \leq 4 \tag{8.9}
\]

The technique of translating this inequality into an expression with \( E_{AF^*}(AF') \) is the same as in the generalized Bell inequality (where inequality (5.14) is used). The same approach yields

\[
|E_{AF^*}(AF') - \delta_{\text{AE}} E_{AF^*BGCHP}(AF')| \leq 1 - \delta_{\text{AE}}. \tag{8.10}
\]

It is now a simple matter to obtain the result. \( \Box \)
To obtain a QM violation from this inequality, we would need

\[
\delta_{6,2} \approx 40.19\%.
\]

It would seem that the QM violation of this inequality is greater than the violation of the generalized CHSH inequality, but this is not the case. Recall that the measure of change of ensemble \(\delta_{6,2}\) in the above inequality is different from the one used in the CHSH inequality \((\delta_{4,2})\). They are connected by

\[
P_{AF'}(\Lambda_{BG'CH'}) = P_{AF'}(\Lambda_{BG'}) + P_{AF'}(\Lambda_{CH'}) - P_{AF'}(\Lambda_{BG'} \cup \Lambda_{CH'}),
\]

so that

\[
\delta_{6,2} \geq 2\delta_{4,2} - 1,
\]

Thus, a violation is obtained for \(\delta_{4,2} > 2 - \frac{3\sqrt{3}}{4} \approx 70.10\%\), which is a higher bound than obtained from the CHSH inequality \((\delta_{4,2} > 2 - \sqrt{2} \approx 58.58\%)\). This is also visible in the proof of Theorem 8.2, since the chained inequality is derived from the CHSH inequality; if the chained inequality is to be violated, the CHSH inequality must also be violated.

In the case of ideal Franson interferometry, however, the filtering described above yields \(\delta_{6,2} = \frac{1}{2}\) and the inequality is

**Corollary 8.5 (Chained Bell inequality for fixed LL subensemble)**

Assume that the prerequisites of the CHSH inequality with ensemble change (Th.5.9(iii)) hold except at a null set, and that the coincident ensemble consists of an equal amount of EE and LL events. If the LL subensemble is fixed with respect to the phase-switch settings, then \(\delta_{6,2} = \frac{1}{2}\) and

\[
\left| E_{\text{coinc}}(AF' | \phi_r, \phi_l) + E_{\text{coinc}}(AF' | \phi_r, \phi_l) \right|
\]

\[
+ \left| E_{\text{coinc}}(BF' | \phi_r, \phi_l) + E_{\text{coinc}}(BG' | \phi_r, \phi_l) \right|
\]

\[
+ \left| E_{\text{coinc}}(CG' | \phi_r, \phi_l) - E_{\text{coinc}}(CH' | \phi_r, \phi_l) \right| \leq 5.
\]

This inequality is violated by the QM prediction used in (8.8), since \(3\sqrt{3} \approx 5.1961 > 5\).

Let us now assume that the switching is fast (on the time scale \(d/c\)), random, and that two subsequent settings are independent. Then, the LL ensemble-selecting settings \(\phi_r\) and \(\phi_l\) are random and independent of the settings \(\phi_A\) and \(\phi_F\), and

\[
E_{\text{coinc}}(AF') = E \left( E_{\text{coinc}}(AF' | \phi_r, \phi_l) \right)
\]

Thus, when these last assumptions hold, the filtering is not really necessary, and we have
Corollary 8.6 (Chained Bell inequality for independent LL subensemble)
Assume that the prerequisites of the CHSH inequality with ensemble change (Th.5.9(i)–(iii)) hold except at a null set, and that the coincident ensemble consists of an equal amount of EE and LL events. If the LL subensemble changes randomly, but is independent of the phase-switch settings, then \[ \delta^6_{\text{LL}} = \frac{1}{2} \] and
\[
\left| E_{\text{coinc.}}(AH') + E_{\text{coinc.}}(AF') \right| + \left| E_{\text{coinc.}}(BF') + E_{\text{coinc.}}(BG') \right| + \left| E_{\text{coinc.}}(CG') - E_{\text{coinc.}}(CH') \right| \leq 5.
\]

This sets higher demands on experiments than previously expected; switching on the time-scale \( d/c \) is significantly more difficult to achieve than switching on the time-scale \( D/c \). Furthermore, the amount of noise is also constrained, since the inequality is violated only if the visibility (the amplitude of the correlation, see Chapter 6) is more than \( \frac{5}{\sqrt{3}} \approx 96.22\% \), which is significantly higher than the usual 71\% bound used in the reported experiments (see e.g., Kwiat et al., 1993; Tapster et al., 1994; Tittel et al., 1998). As for the question of a bound on efficiency, a derivation of such a bound will be more complicated than for the pure Bell inequality, and will not be done here.

As a final note, there is a claim in a recent paper (Franson, 2000) that fast switching (on the time-scale \( d/c \)) is not needed to obtain a test of Local Realism. However, as is clearly seen in Section 8.3, this is not true. Unless fast switching is used, there is no test of Local Realism because there is a LHV model. But the paper does motivate further excursions into the subtleties of Franson interferometry.
Chapter 9

Interpretation

… if you can demonstrate that quantum mechanics imposes some limit on the degree to which the ideal experiment can be approached, I will be very interested. I will also be very surprised!

(John S. Bell, 1986)

Concluding this thesis we will make a slight digression into interpretational issues, based on Larsson (2000)*. The kind of LHV models presented in this thesis will be set as an interpretation of QM, assuming that QM statistics hold in experiments (up to detector efficiency). This interpretation will be compared to the ones in Chapter 2 and 3, which will be restated in the language of probability theory, to allow such a comparison.

In Chapter 2, it is obvious that the two presented QM interpretations have very different properties. The most notable difference is the notions of complementarity in the Copenhagen interpretation and realism in the the Bohm interpretation. These two notions are incompatible in the form used in the two mentioned interpretations, and cannot be used simultaneously. Nevertheless, the presented interpretation of LHV models will contain both notions (in a slightly modified form).

But let us first look at the difference between QM chance and probability theory.

9.1 QM chance

Probabilities and expectation values are obtained in QM in a fashion fundamentally different from standard Kolmogorovian probability theory, and only the basics will be mentioned here, using polarization as example. In QM, a system is described by a normalized vector \[ |\psi\rangle \], the “quantum state”, in a Hilbert space \( \mathcal{H} \), the space of all

* A reprint of this paper can be found at the end of the thesis.
states. An object that has a certain property \( \alpha \), e.g., “the photon is horizontally polarized,” corresponds to a vector that lies in the subspace \( \mathcal{H}_\alpha \) associated with that property, while an object not having the property corresponds to a vector in the orthogonal complement \( \mathcal{H}_\alpha^\perp \) of that subspace. Formally, an object for which \( |\psi\rangle \) lies in \( \mathcal{H}_\alpha \) has \( P(\alpha) = 1 \), whereas when \( |\psi\rangle \) lies in \( \mathcal{H}_\alpha^\perp \), \( P(\alpha) = 0 \). More generally, the probability of the system having the property \( \alpha \) is calculated as the inner product of the vector \( |\psi\rangle \) and its projection onto \( \mathcal{H}_\alpha \), normally denoted

\[
P(\alpha) = \langle \psi | \hat{W}_\alpha | \psi \rangle,
\]

where \( \hat{W}_\alpha \) is the projection operator onto \( \mathcal{H}_\alpha \).

To include measurement results in our QM description, let us look at our example, where we associate the numerical value \(+1\) to horizontal polarization and the value \(-1\) to vertical polarization, as in the previous chapters. There are thus two results: \( A = \pm 1 \), and these values are encoded into the measurement operator

\[
\hat{A} = +\hat{W}_\alpha - \hat{W}_\alpha^\perp.
\]

More generally, for a set of different outcomes \( \alpha_i \) and their corresponding measurement outcomes \( a_i \),

\[
\hat{A} = \sum_i a_i \hat{W}_{\alpha_i}.
\]

It is therefore important to know the eigenvalues of our measurement operator; they encode the possible measurement results. More generally it is important to know the spectrum of our operator, but it is here assumed that \( \{a_i\} \) is finite and that all \( a_i \) are different for simplicity. The expectation value of the measurement results is given by

\[
E(A) = \sum_i a_i P(\alpha_i) = \sum_i a_i \langle \psi | \hat{W}_{\alpha_i} | \psi \rangle = \langle \psi | \hat{A} | \psi \rangle.
\]

Let us compare the description above with Kolmogorovian probability theory, to see the differences (compare with Chapter 4). In this description, the properties of the system are represented by a “sample” which encodes a pre-required existence of the properties of the system. This sample is represented as a point \( \lambda \) in a sample space \( \Lambda \). An object that has a certain property \( \alpha \) (e.g., “the photon is horizontally polarized”) would in this description have a sample lying in \( \Lambda_\alpha \), the subset of all samples corresponding to this property, whereas a system not having the property would have a sample lying in the complement set \( \Lambda_\alpha^\complement \). In probability theory such a subset is called an “event”, and on the collection of all events \( \mathcal{F} \) (a collection of sets) we have a probability measure \( P \), allowing us to calculate the probability as

\[
P(\alpha) = \int_{\Lambda_\alpha} dP.
\]

Again, we introduce measurement results into the formalism. Here, they are encoded into random variables which are functions from the sample space into (e.g.) the
real numbers. In our case, the measurement result $A$ is described by a random variable (see Fig. 9.1)

$$A(\lambda) = \begin{cases} +1, & \text{if } \lambda \in \Lambda_{\alpha} \\ -1, & \text{if } \lambda \in \Lambda_{\beta} \end{cases}$$  \hspace{1cm} (9.6)\]

In the general case, $A$ is

$$A(\lambda) = a_i, \text{ if } \lambda \in \Lambda_{\alpha_i},$$  \hspace{1cm} (9.7)\]

and given this, the expectation value is

$$E(A) = \sum_i a_i P(\alpha_i) = \sum_i a_i \int_{\lambda_{\alpha_i}} d\lambda = \int_A A(\lambda) d\lambda.$$  \hspace{1cm} (9.8)\]

The formal differences between QM and Kolmogorovian probability theory are visible here; compare (9.1) with (9.5) and (9.4) with (9.8).

## 9.2 Two probabilistic descriptions of QM

We will here again examine the two interpretations briefly touched upon in Chapter 2 and 3 and their differences, this time from the perspective of a probabilistic description. As example we will use the Bell experiment as presented in Chapter 5, where two measurement results $A$ and $B$ are obtained at two different sites, each having an orientation setting $\phi_A$ and $\phi_B$, choosing which direction is “horizontal” and which is “vertical”.

The Copenhagen interpretation takes the stand that the studied object has no properties other than its QM description (“wave-function”) until we have made a measurement. Granted, the probability of the object having a certain property is given by QM, but the property itself does not exist until we measure it. For example, a statement like “This photon is horizontally polarized” is very much dependent upon the
measurement device used. A measurement device constructed to discern horizontally polarized photons from other photons is needed to ascertain any validity of the statement. If the measurement device measures, e.g., polarization in a π/4 rotated frame, the photon cannot be said to have the property horizontal or vertical polarization in the original frame. The statement is neither true nor untrue, it is meaningless, because in this interpretation, polarization in two frames rotated π/4 with respect to each other are complementary; if one is measured, the other is undefined.

We have a description allowing us to calculate probabilities (9.1) and expectation values (9.4), but there is no probability space or probability measure in the Kolmogorovian sense. When the measurement is made\(^\dagger\), there emerges a probabilistic description: a sample space, a probability measure, and two random variables. Notable is that if we were to make a different measurement, we would get a completely different sample space with a completely different probability measure and completely different random variables. This is where complementarity emerges in the probabilistic description; \(\Lambda\), \(P\), \(A\), and \(B\) change completely for different experimental setups.

I: The Copenhagen interpretation:

Given \(\phi_A\) and \(\phi_B\), there exists a probabilistic description \((\Lambda, \mathcal{F}, P)\) with

\[
A : \Lambda \rightarrow \pm 1 \\
B' : \Lambda \rightarrow \pm 1.
\]

The description does not exist unless \(\phi_A\) and \(\phi_B\) are given\(^\ddagger\).

In the Bohm interpretation, on the other hand, an object indeed possesses properties which have well-defined values; elements of reality. In our example, the statement “This photon is horizontally polarized” is either true or untrue; it is always a meaningful statement. These properties follow the laws of ordinary mechanics with the modification that the behavior is affected by an additional potential, a quantum potential originating in the QM description. This potential permeates the whole system,\(^\ddagger\)

\(\dagger\) If preferred, one could say that the probabilistic description appears when the measurement device is chosen.

\(\ddagger\) Another way of describing this is known as quantum probability (see e.g., Accardi, 1984), which would in the notation presented here be

I': Quantum probability

There always exist probabilistic descriptions

\[\{A(\phi_A, \phi_B), \mathcal{F}(\phi_A, \phi_B), P_A(\phi_A, \phi_B)\}\]

for all setups with

\[
A(\phi_A, \phi_B) : \Lambda(\phi_A, \phi_B) \rightarrow \pm 1 \\
B'(\phi_A, \phi_B) : \Lambda(\phi_A, \phi_B) \rightarrow \pm 1.
\]

A probabilistic description always exists, but different experimental settings are difficult to compare, because there is no common probability space to compare the results in. Furthermore, the description changes abruptly even for small changes of the parameters \(\phi_A\) and \(\phi_B\). Here, the concept of realism is taken to contain less dramatic changes.
causing it to behave in classically unexpected ways, and it depends heavily on the measurement setup we have chosen; for different setups, the potential is different. Two different measurement devices designed to measure spin along different axes would yield different quantum potentials, causing the object’s properties to behave differently. This in itself is not strange, but what may seem strange is that changes in the measurement setup affects the quantum potential instantaneously throughout the system, and thus, a change of the setting $\phi_A$ changes the quantum potential at the second site instantaneously, so that the result $B'$ is affected.

The Bohm interpretation says that the object does have a definite value of the property, but it is part of an ensemble in which the values may be different; probabilities and expectations are given by (9.1) and (9.4)**. In the Bohm interpretation the probabilistic description always exists, but it is nonlocal as the result at one detector (e.g., $B'$) is affected by both settings $\phi_A$ and $\phi_B$.

**II: The Bohm interpretation:

There always exists a probabilistic description $(\Lambda, \mathcal{F}, P)$ with

\[ A(\phi_A, \phi_B) : \Lambda \rightarrow \pm 1 \]
\[ B'(\phi_A, \phi_B) : \Lambda \rightarrow \pm 1. \]

In this way the two key pieces of these two interpretations are exhibited: complementarity in the Copenhagen interpretation by the change in the probabilistic description and realism in the Bohm interpretation by the permanent existence of one such description.

**II: The Bohm interpretation: $(\Lambda, \mathcal{F}, P)$ always exists, but random variables are nonlocal.

---

**Figure 9.2: Probabilistic descriptions of the two interpretations**

**To clarify, these two interpretations agree on the formulas (9.1) and (9.4), and when the object is described by a vector that lies entirely in some $\mathcal{H}_q$, also the Copenhagen interpretation agrees that the relevant property exists (but not if the vector is not in $\mathcal{H}_q$).
9.3 Change of ensemble

Let us now look at LHV models with change of ensemble, starting with a conceptual examination of the “efficiency loophole”. A common picture of non-detection measurement errors is schematically shown in Fig. 9.3. The errors are thought of as put “on top” of some underlying “ideal” measurement result, i.e., the measurement results are there, but because of detector errors, some photons are not registered in the measuring equipment.

![Figure 9.3: Intuitive picture of a non-detection error; the photon has a definite polarization, but is not detected due to an error.](image)

As can be seen in the picture above, the non-detection error is often thought of as independent from the hidden variables. Or more formally, it is assumed that there are two different hidden variables; one deciding the measurement result and the other whether a photon is detected or not\(^\dagger\). We can see this more clearly by recasting the picture above mathematically as in Fig. 9.4.

![Figure 9.4: Probabilistic description of Fig. 9.3. The random variable \(A(\phi_A)\) is a mapping from \(\Lambda = \Omega \times [0, \eta] \subset \Lambda = \Omega \times [0, 1]\) to the two possible results. If \(\lambda\) is not in \(\Lambda_A\), \(A(\phi_A)\) is undefined and the photon is undetected.](image)

\(^\dagger\)That is, an additional assumption is used, the no-enhancement assumption. This is often presented as the requirement that certain probabilities are less than 1, but in the formalism used here that statement is void since we are using the Kolmogorov axioms. The no-enhancement assumption is instead transformed into the assumption that the measurement result (±1) always exists in the model, independently of whether the photon is detected or not. Hence the two separate hidden variables.
Here, the non-detection errors are governed by a different hidden variable than the measurement results, and this probabilistic model has precisely the properties reflected in the intuitive picture in Fig. 9.3. By using the results in Chapter 5 we can see that this puts severe restrictions on the model, because in Fig. 9.4, \( \Lambda_A \) is independent of \( \phi_A \). Thus, in this model,

\[
\forall \phi_A, \phi_B, \phi_C, \phi_D : \Lambda_{AB} = \Lambda_{CD}, \tag{9.9}
\]
even when \( \eta < 1 \), and by Observation 5.4, we have \( \delta_{\phi_1} = 1 \) which in turn would yield

\[
|E_{AB}(A'B') - E_{AC}(AC')| \leq 1 - E_{BC}(BC'). \tag{9.10}
\]

This inequality is very similar to the original Bell inequality, and is violated by QM. Furthermore, the CHSH version of it is violated by experimental results available in the literature (e.g. Aspect et al., 1981, 1982a,b). A similar argument applies for the generalization of the GHZ paradox in Chapter 7.

So, to allow for the model to mimic the QM statistics, the errors cannot be put “on top” in this way, but must be included at a more fundamental level in the model, as in Fig. 9.5. Note that the set \( \Lambda_A(\phi_A) \) depends on the detector orientation \( \phi_A \), and that this is an important property as observed above. Thus, the concept of change of ensemble is more fundamental in this context than detector-efficiency.

![Figure 9.5: A random variable which is only defined on a subset \( \Lambda_A(\phi_A) \) of the full sample space \( \Lambda \); here the efficiency is less than 1.](image)

### 9.4 Interpretation of LHV models

Using the above observation we can include LHV models into this interpretational picture; the statement corresponding to I and II would be

**IIIa:** Local Realistic interpretation:

There always exists a probabilistic description \((\Lambda, \mathcal{F}, P)\) with

\[
\begin{align*}
A(\phi_A) : & \Lambda_A(\phi_A) \rightarrow \pm 1 \\
B'(\phi_B) & : \Lambda_B(\phi_B) \rightarrow \pm 1.
\end{align*}
\]
This is a probabilistic description which includes *elements of reality* in the spirit of the Bohm interpretation; *one* probabilistic description of what the result will (or will not) be. However, the ensemble changes upon change of our detector settings because of the dependence of $\Lambda_A$ (say) on $\phi_A$. This is where *complementarity* arises in this interpretation.

The probability of the event $\alpha$ (in the frequency interpretation of probability) is the many-experiment limit of the ratio of $\alpha$-events to the total number of events. Since the undetected events are just that; all we see is events having their sample in $\Lambda_A / \phi_A$ (compare with Def. 4.8):

$$P_A(\alpha) = P(\alpha | \Lambda_A(\phi_A)) = \frac{P(\alpha \cap \Lambda_A(\phi_A))}{P(\Lambda_A(\phi_A))}. \quad (9.11)$$

In a two-particle experiment, i.e., when correlating results from *both* detectors, single events are quite naturally discarded as measurement errors. The remaining results come from a smaller subset in the sample space, the subset where *both* $A$ and $B$ are defined: $\Lambda_{AB}(\phi_A, \phi_B) = \Lambda_A(\phi_A) \cap \Lambda_B(\phi_B)$ (see Fig. 9.6). We then have

**IIIb**: Local Realistic interpretation (cont.):

For two-particle experiments, the *product* is

$$A(\phi_A)B(\phi_B) : \Lambda_{AB}(\phi_A, \phi_B) \rightarrow \pm 1.$$

![Figure 9.6: A two-particle experiment in this description.](image)

Our sample space is now restricted to $\Lambda_{AB}(\phi_A, \phi_B)$, which again changes when the settings change. Our probability measure is also affected by this, changing with the settings. The probability measure to use here is

$$P_{AB}(\alpha) = P(\alpha | \Lambda_{AB}(\phi_A, \phi_B)). \quad (9.12)$$

There seems to be an introduction of nonlocality in IIIb because of the presence of both parameters in the set $\Lambda_{AB}(\phi_A, \phi_B)$; this is only apparently so, since the result and
detection in, say, the $A$ experiment only depends on $\phi_A$. Only because we restrict our resulting data to events where both $A$ and $B'$ are defined does the apparent nonlocality in IIIb appear. The model in Chapter 6 fits well into this picture, and this is also true for other models in the literature (see e.g., Santos, 1996; Gisin and Gisin, 1999).

In conclusion, Local Realistic models may yield the QM statistics by using the “efficiency loophole” in the Bell inequality, or more precisely, using a changing ensemble. This allows for the QM predictions to hold, but the efficiency problem is then fundamental rather than due to inexact measurements. Whether or not an approach of this type yields a good description of Nature, well, that remains to be seen.
Papers

The papers associated with this thesis have been removed for copyright reasons. For more details about these see:

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