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Energy Tensors: Quadratic, Phase Invariant Image Operators^{*}

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Abstract. In this paper we briefly review a not so well known quadratic, phase invariant image processing operator, the energy operator, and describe its tensor-valued generalization, the energy tensor. We present relations to the real-valued and the complex valued energy operators and discuss properties of the three operators. We then focus on the discrete implementation for estimating the tensor based on Teager's algorithm and frame theory. The kernels of the real-valued and the tensor-valued operators are formally derived. In a simple experiment we compare the energy tensor to other operators for orientation estimation. The paper is concluded with a short outlook to future work.

1 Introduction

Quadratic image processing operators are an important extension of linear systems theory for various purposes, as e.g. corner detection [1–3]. Phase-based approaches are an important field when it comes to intensity invariant processing and feature detection, see e.g. quadrature filters [4], phase-based disparity and motion estimation [5], and phase congruency for edge detection [6].

Although the phase-based techniques are based on the Hilbert transform and are therefore supposed to be linear, the extraction of the phase and the computation of phase invariant features requires non-linear operations as modulus or arcustangent. This essential non-linearity gives rise to the field of non-linear phase invariant operators: If we use non-linear operations anyway, there is little reason to restrict our basis filters to linear operators.

One of the most popular operators in context of quadratic, phase invariant signal processing is the energy operator [7, 8], which bears its name because it responds with the energy of a single oscillation. For images, this operator can be extended in various ways: as a real-valued operator [9], as a complex-valued operator [10], or as a tensor-valued operator [11], the energy tensor. The latter can be further generalized to higher order derivatives in terms of the gradient energy tensor [12].

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The purpose of this paper is to investigate the background of the energy-operator-like approaches more in detail. In particular we will derive some relations between the operators and their kernels, i.e., those functions which map to zero. In most theoretic considerations we will stick to the continuous case, but we will also propose an algorithm for discrete signals.

2 Energy-Operator-Like Operators

2.1 Definitions

According to [9], the energy operator is defined for continuous 1D signals $s(t)$ as

$$\Psi_1[s(t)] = [\dot{s}(t)]^2 - s(t)\ddot{s}(t) . \quad (1)$$

The real-valued energy operator for continuous 2D signals $s(\mathbf{x})$, $\mathbf{x} = [x \ y]^T$, is defined as [9]

$$\Psi_r[s(\mathbf{x})] = [s_x(\mathbf{x})]^2 - s(\mathbf{x})s_{xx}(\mathbf{x}) + [s_y(\mathbf{x})]^2 - s(\mathbf{x})s_{yy}(\mathbf{x}) . \quad (2)$$

The complex-valued energy operator for continuous 2D signals is defined as [10]

$$\Psi_c[s(\mathbf{x})] = [D\{s\}(\mathbf{x})]^2 - s(\mathbf{x})D^2\{s\}(\mathbf{x}) , \quad (3)$$

where $D\{\cdot\} = \frac{\partial}{\partial x} + i\frac{\partial}{\partial y}$. The tensor-valued energy operator, the energy tensor, is defined as [11]

$$\Psi_t[s(\mathbf{x})] = [\nabla s(\mathbf{x})][\nabla s(\mathbf{x})]^T - s(\mathbf{x})\mathbf{H}s(\mathbf{x}) , \quad (4)$$

where $\nabla s = [s_x \ s_y]^T$ and $\mathbf{H} = \nabla\nabla^T$.

2.2 Basic Relations

Obviously, the operators (2)-(4) are direct generalizations of (1) since all are of the same structure. However, the three different generalizations also show some mutual dependencies. The real-valued 2D operator consists simply of the 1D operator applied to both coordinates and summed up. Furthermore, it is obtained as the trace of (4):

$$\Psi_r[s(\mathbf{x})] = \text{trace}\{\Psi_t[s(\mathbf{x})]\} , \quad (5)$$

i.e., it is equal to the sum of the eigenvalues of Ψ_t . Following the proof in [13], the complex-valued operator corresponds to the double-angle representation of Ψ_t . More concrete: if $\mathbf{n} = [n_1 \ n_2]^T$ denotes the eigenvector corresponding to the larger eigenvalue λ_1 of Ψ_t and if we decompose the tensor-valued operator as

$$\Psi_t[s(\mathbf{x})] = \lambda_1\mathbf{n}\mathbf{n}^T + \lambda_2\mathbf{n}^\perp\mathbf{n}^{\perp T} = \begin{bmatrix} \Psi_{11} & \Psi_{12} \\ \Psi_{12} & \Psi_{22} \end{bmatrix} , \quad (6)$$

we obtain the complex-valued operator as

$$\Psi_c = (\lambda_1 - \lambda_2)(n_1 + in_2)^2 = \Psi_{11} - \Psi_{22} + i2\Psi_{12} . \quad (7)$$

To draw a parallel to better known operators, the three energy operators have the same mutual dependencies as gradient magnitude squared, complex gradient squared ($[D\{s\}]^2$), and outer product of gradients.

2.3 Basic Properties

As stated in [9] for the 1D case and the real-valued 2D case, all energy operators share the property

$$\Psi[s_1 s_2] = s_1^2 \Psi[s_2] + s_2^2 \Psi[s_1] . \quad (8)$$

Due to the fact that the real-valued and the complex-valued operators can be derived from the tensor-valued one, it is sufficient to show (8) for Ψ_t .

Proof.

$$\begin{aligned} \Psi_t[s_1 s_2] &= [\nabla(s_1 s_2)][\nabla(s_1 s_2)]^T - (s_1 s_2) \mathbf{H}(s_1 s_2) \\ &= [s_2 \nabla s_1 + s_1 \nabla s_2][s_2 \nabla s_1 + s_1 \nabla s_2]^T - (s_1 s_2) \nabla[s_2 \nabla s_1 + s_1 \nabla s_2]^T \\ &= s_1^2 [\nabla s_2][\nabla s_2]^T + s_2^2 [\nabla s_1][\nabla s_1]^T + (s_1 s_2)([\nabla s_1][\nabla s_2]^T + [\nabla s_2][\nabla s_1]^T) \\ &\quad - (s_1 s_2)([\nabla s_2][\nabla s_1]^T + [\nabla s_1][\nabla s_2]^T) - (s_1 s_2)(s_2 \mathbf{H} s_1 + s_1 \mathbf{H} s_2) \\ &= s_1^2 \Psi_t[s_2] + s_2^2 \Psi_t[s_1] . \end{aligned}$$

As also stated in [9] for the 1D case and the real-valued 2D case, all energy operators respond identically zero to exponential signals:

$$\Psi[A \exp(ax + by)] = 0 , \quad (9)$$

where $A, a, b \in \mathbb{C}$. With the same argument as above it is sufficient to show (9) for Ψ_t .

Proof. Let $s(\mathbf{x}) = A \exp(ax + by)$.

$$\Psi_t[s] = \begin{bmatrix} a \\ b \end{bmatrix} s \begin{bmatrix} a & b \end{bmatrix} s - s \begin{bmatrix} aa & ab \\ ab & bb \end{bmatrix} s = 0 .$$

As mentioned above, the energy operator bears its name because of the fact that it tracks the energy of a single oscialltion. For general signals however, positivity of the 'energy' operators is not always given. In [14] the authors give a proof for a necessary and sufficient condition for positivity: The logarithm of the signal magnitude must be concave between every two consecutive zeros of the signal. The relation to the signal logarithm will be considered in more detail further below.

3 Discrete Implementation of the Energy Tensor

3.1 Teager's Algorithm

Teager's algorithm [7] is a very efficient way to compute the energy operator in 1D. The discrete 1D energy operator is computed as

$$\Psi_1[s_k] = s_k^2 - s_{k+1} s_{k-1} . \quad (10)$$

Teager's algorithm can also be used for the real-valued 2D operator by adding the results of the algorithm in x - and y -direction [9]. For the 2D complex-valued and tensor-valued operator, the implementation based on central differences or modified Sobel operators [15] is straightforward. This changes if one tries to use differences of neighbored samples in order to end up with a 3×3 operator.

3.2 A 2D Teager's Algorithm

It is straightforward that we can apply the 1D algorithm in four different directions:

$$t_1[s_{k,l}] = s_{k,l}^2 - s_{k+1,l} s_{k-1,l} \quad (11)$$

$$t_2[s_{k,l}] = s_{k,l}^2 - s_{k+1,l+1} s_{k-1,l-1} \quad (12)$$

$$t_3[s_{k,l}] = s_{k,l}^2 - s_{k,l+1} s_{k,l-1} \quad (13)$$

$$t_4[s_{k,l}] = s_{k,l}^2 - s_{k-1,l+1} s_{k+1,l-1} \quad (14)$$

These four responses can be combined to a tensor in a similar way as described for the quadrature filter in [4].

As a first step, we consider the series expansion of the four responses for a pure oscillation with frequency vector $[u \ v]^T$:

$$T_1 \propto \sin^2 u = cu^2 + \mathcal{O}(u^4) \quad (15)$$

$$T_2 \propto \sin^2(u+v) = c(u+v)^2 + \mathcal{O}((u+v)^4) \quad (16)$$

$$T_3 \propto \sin^2 v = cv^2 + \mathcal{O}(v^4) \quad (17)$$

$$T_4 \propto \sin^2(u-v) = c(u-v)^2 + \mathcal{O}((u-v)^4), \quad (18)$$

i.e., the four responses correspond to the frame tensors

$$B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad (19)$$

$$B_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B_4 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad (20)$$

The dual frame with minimum norm is obtained as [16]

$$\tilde{B}_1 = \begin{bmatrix} 0.6 & 0 \\ 0 & -0.4 \end{bmatrix} \quad \tilde{B}_2 = \begin{bmatrix} 0.2 & 0.25 \\ 0.25 & 0.2 \end{bmatrix} \quad (21)$$

$$\tilde{B}_3 = \begin{bmatrix} -0.4 & 0 \\ 0 & 0.6 \end{bmatrix} \quad \tilde{B}_4 = \begin{bmatrix} 0.2 & -0.25 \\ -0.25 & 0.2 \end{bmatrix} \quad (22)$$

We can therefore compute the discrete energy tensor as

$$\Psi_t[s_{k,l}] = \sum_{n=1}^4 t_n[s_{k,l}] \tilde{B}_n \quad (23)$$

Note that the quadratic term is the same in (11–14), so that we can speed-up the calculation by setting

$$t'_n[s_{k,l}] = t_n[s_{k,l}] - s_{k,l}^2 \quad (24)$$

and

$$\Psi_t[s_{k,l}] = s_{k,l}^2 \begin{bmatrix} 0.6 & 0 \\ 0 & 0.6 \end{bmatrix} + \sum_{n=1}^4 t'_n[s_{k,l}] \tilde{B}_n \quad (25)$$

3.3 Regularization

As pointed out in [11], images typically contain local DC components which cause problems for energy operators [7]. Since we are interested in a simple operator, more advanced methods for suppressing DC components are out of scope and we stick to a simple 3×3 bandpass which subtracts the DC component from the signal.

4 Kernels of Energy Operators

In this section we derive the kernels of the energy operators. Since the operators are quadratic expression of second order derivatives, the equation

$$\Psi[s] = 0 \quad (26)$$

is not at all simple to solve. We start with the 1D case.

4.1 Kernel of the 1D Operator

We start with the observation that

$$\Psi_1[s(t)] = -s(t)^2 \frac{d}{dt} \frac{\dot{s}(t)}{s(t)} . \quad (27)$$

Assuming that $s(t) \neq 0$, we therefore get instead of (26)

$$\dot{s}(t) = a s(t) \quad (28)$$

for a suitable constant $a \in \mathbb{C}$. As it is well known, the solution to this equation is $s(t) = A \exp(at)$, and assuming s being continuous, (9) is not only sufficient but also necessary to obtain $\Psi_1 = 0$.

Remembering the undergraduate math lessons one might come up with the quick-and-dirty solution of (28), namely to 'multiply' with dt and to integrate. As a result, one obtains the logarithm of $s(t)$, which directly leads to the positivity statement in [14]:

$$\Psi_1[s(t)] = -s(t)^2 \frac{d^2}{dt^2} \log(|s(t)|) . \quad (29)$$

4.2 Kernel of the Tensor-Valued Operator

We will now show that the 2D separable exponential (9) is the kernel of the tensor-valued operator $\Psi_t[s(\mathbf{x})]$. At the end of Sect. 2.3 we have already shown that (9) is in the kernel of Ψ_t .

What is left to show is that all functions in the kernel of Ψ_t are of the form (9). The two diagonal elements of the tensor require $s(x, \cdot) = A \exp(ax)$ respectively $s(\cdot, y) = A \exp(by)$. Hence, all functions in the kernel of Ψ_t are of the form (9).

4.3 Kernel of the 2D Real-Valued Operator

The class of real-valued functions where Ψ_r becomes zero can be extended beyond (9). In particular, the real-valued operator Ψ_r becomes zero for and only for

$$|s(\mathbf{x})| = \exp(u(x + iy)) , \quad (30)$$

where $u(x + iy)$ is a harmonic function.

Proof. Likewise as in the 1D case, we assume $s \neq 0$ and

$$0 = \frac{\Psi_t}{s^2} = \nabla^T \frac{\nabla s}{s} = \Delta \log(|s(\mathbf{x})|) , \quad (31)$$

where $\Delta = \nabla^T \nabla$ denotes the Laplacian. The constraint (31) means that $\log(|s(\mathbf{x})|)$ is harmonic. Hence, $|s|$ must be the exponential of a harmonic function.

Note that u might be generated as the real part of an analytic function [17]. Furthermore, negative s are obtained by adding $i\pi$ to u .

5 Experiments

In this section we show some experiments which relate the performance of the energy tensor to other operators for orientation estimation. In this context it is important to notice that the energy tensor does not only estimate the local orientation but also the local coherence, i.e., it provides a full structure tensor from a 3x3 region.

We therefore compared the pure energy tensor with the standard gradient operator from Matlab and an improved 3x3 Sobel operator [15]. Since it is also common to use regularized operators, we compared the regularized energy tensor (Gaussian post-filtering with $\sigma^2 = 1$, size 7x7) with a comparable Gabor filter set and the structure tensor (same Gaussian post-filter).

The signal which is used in the experiment is a 2D Chirp-like signal with Gaussian noise (29dB SNR), cf. Fig. 5. The orientation error was calculated according to [16] as

$$\Delta\theta = \cos^{-1} \left(\sqrt{\sum \cos^2(\theta_m(\mathbf{x}) - \theta_t(\mathbf{x}))} \right) , \quad (32)$$

where θ_m and θ_t indicate the estimated orientation and the ground truth, respectively. The results are plotted in Fig. 5 as functions of the radius, which is reciprocal to the frequency.

Obviously, the energy tensor outperforms the other approaches for the 3x3 filter size. For 9x9 filters, it is still much better than the Gabor filter, but slightly worse than the structure tensor. The slightly poorer performance is caused by the approximation error for the calculation of the dual frame, resulting in a small bias for orientations which are no multiple of $\pi/4$.

Fig. 1. Left: Test image with 29 dB SNR. Right: Orientation error (in radians) depending on the radius aka frequency. Top: 3x3 operators: ET (energy tensor), grad (Matlab gradient), Sobel. Bottom: 9x9 operators: ETR (regularized ET), Gabor, STR (structure tensor).

6 Outlook

The energy operator can easily be extended to higher dimensions. Both, the continuous operator and the nD Teager's algorithm are straightforward to generalize. What might turn out to be more complicated are the kernels of the discrete operators and the higher dimensional continuous operators as well as the kernel of the complex operator.

For practical applications real-time implementations of the 2D and 3D Teager's algorithm are planned. Experimental comparisons for several applications as corner detection and optical flow estimation will be done.

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