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ON THE CONNECTEDNESS OF THE BRANCH LOCUS OF THE MODULI SPACE OF RIEMANN SURFACES OF GENUS 4

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Abstract. Using uniformization of Riemann surfaces by Fuchsian groups and the equisymmetric stratification of the branch locus of the moduli space of surfaces of genus 4, we prove its connectedness. As a consequence, one can deform a surface of genus 4 with automorphisms, i.e. symmetric, to any other symmetric genus 4 surface through a path consisting entirely of symmetric surfaces.

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1. Introduction. Two closed Riemann surfaces X, Y of genus g are called *equisymmetric* if their automorphism groups determine conjugate finite subgroups in the modular group of genus g.

Harvey [**9**] alluded to the existence of the *equisymmetric stratification* of the moduli space $\mathcal{M}_g$ of Riemann surfaces of genus g, each strata consists in the points of the moduli space corresponding to equisymmetric surfaces. The branch locus $\mathcal{B}_g$ of $\mathcal{M}_g$ is formed by the strata corresponding to surfaces of genus g admitting non-trivial automorphisms (or admitting other automorphisms that are the identity and the hyperelliptic involution if $g = 2$). Broughton [**5**] showed that the equisymmetric stratification is indeed a stratification of $\mathcal{M}_g$ by irreducible algebraic subvarieties whose interior, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in the stratum. In this way we can equip the moduli space with a structure of complex of groups.

It is well known that $\mathcal{B}_1$ consists of two points and $\mathcal{B}_2$ is not connected, since R. Kulkarni (see [**11**]) showed that the curve $w^2 = z^5 - 1$ is isolated in $\mathcal{B}_2$, i.e. this single surface is a connected component of $\mathcal{B}_2$. More precisely $\mathcal{B}_2$ has exactly two connected components (see [**1**]). On the contrary the branch locus $\mathcal{B}_3$ is connected (see also [**1**]).

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Now we focus our attention to the case of genus 4. Each equisymmetric stratum of $\mathcal{M}_4$ corresponds with a conjugacy class of finite subgroups of the modular group represented as the full group of automorphisms of some compact Riemann surfaces of genus 4. Using the list of finite maximal signatures for Fuchsian groups in [**15**], we find in [**7**] such classes of full groups of automorphisms (related results are presented in [**3**] and [**4**])

In this paper we shall show the connectedness of the branch locus of the moduli space of genus 4 by means of its equisymmetric stratification. Moreover, we show that the stratum formed by the surfaces X_4 admitting an involution τ such that $X_4/\langle\tau\rangle$ is an orbifold with two cone points and underlying topological space a surface of genus 2 intersects to several equisymmetric strata that cover all the branch loci, i.e. such stratum plays a rôle similar to a spine for the connectedness of the branch loci.

The results of this work have been announced in [**1**].

2. Riemann surfaces and Fuchsian groups. Given a Fuchsian group Γ, the algebraic structure of Γ and the geometric structure of the orbifold $\mathcal{D}/\Gamma$ are given by the signature of Γ

$$s(\Gamma) = (g;\ m_1, \ldots, m_r). \tag{1}$$

The group with the signature (1) has a canonical presentation given by generators

$$\begin{array}{lll} \text{(a)} & x_i,\ i = 1, \ldots, r & \text{(elliptic transformations)} \\ \text{(b)} & a_i, b_i,\ i = 1, \ldots g & \text{(hyperbolic translations)} \end{array}$$

and relations

$$\begin{array}{ll} (1) & x_i^{m_i} = 1,\ i = 1, \ldots, r, \\ (2) & x_1 \ldots x_r a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} = 1. \end{array} \tag{2}$$

Given a subgroup Γ' of index N in a Fuchsian group Γ, one can calculate the signature of Γ' by

THEOREM 1. *([**14**]) Let Γ be a Fuchsian group with signature (1) and canonical presentation (2). Then Γ contains a subgroup Γ' of index N with signature*

$$s(\Gamma') = (h; m'_{11}, m'_{12}, \ldots, m'_{1s_1}, \ldots, m'_{r1}, \ldots, m'_{rs_r}),$$

if and only if there exists a transitive permutation representation $\theta : \Gamma \to \Sigma_N$ satisfying the following conditions:

(1) The permutation $\theta(x_i)$ has precisely s_i cycles of lengths less than m_i, the lengths of these cycles being $m_i/m'_{i1}, \ldots, m_i/m'_{is_i}$.

(2) The Riemann–Hurwitz formula

$$\mu(\Gamma')/\mu(\Gamma) = N.$$

where $\mu(\Gamma)$ and $\mu(\Gamma')$ are the hyperbolic areas of the surfaces $\mathcal{D}/\Gamma$ and $\mathcal{D}/\Gamma'$.

Given a Riemann surface $X = \mathcal{D}/\Gamma$, with $\mathcal{D}$ the unit disc and Γ a surface Fuchsian group, a finite group G is a group of automorphisms of X if and only if there exists a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$ with $\ker(\theta) = \Gamma$.

We shall use the following terminology.

DEFINITION 2. A closed Riemann surface X which can be realized as a p-sheeted covering of the Riemann sphere is said to be p-gonal, and such a covering will be called a p-gonal *morphism*. When $p = 2$, the surface will be called *hyperelliptic*.

Let Γ be a Fuchsian group with signature (1). Then the Teichmüller space $T(\Gamma)$ of Γ is homeomorphic to a complex ball of dimension $d(\Gamma) = 3g - 3 + r$ (see [**13**]). Let $\Gamma' \leq \Gamma$ be Fuchsian groups, the inclusion mapping $\alpha : \Gamma \to \Gamma'$ induces an embedding $T(\alpha) : T(\Gamma) \to T(\Gamma')$ defined by $[r] \mapsto [r\alpha]$. See [**13**] and [**15**]. The modular group $\mathrm{Mod}(\Gamma)$ of Γ is the quotient $\mathrm{Mod}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$. The *moduli space* of Γ is the quotient $\mathcal{M}(\Gamma) = T(\Gamma)/\mathrm{Mod}(\Gamma)$ endowed with the quotient topology.

DEFINITION 3. As an application of Nielsen realisation theorem, one can identify the *branch locus* of the action of $\mathrm{Mod}(\Gamma)$ on $T(\Gamma)$ as the set $\mathcal{B}_g = \{X \in \mathcal{M}_g : \mathrm{Aut(X)} \neq 1_{\mathrm{d}}\}$, for $g \geq 3$.

A Fuchsian group Γ such that there does not exist any other Fuchsian group containing it with finite index is called a *finite maximal* Fuchsian group. To decide whether a given finite group can be the full group of automorphism of some compact Riemann surface, we will need all pairs of signatures $s(\Gamma)$ and $s(\Gamma')$ for some Fuchsian groups Γ and Γ' such that $\Gamma' \leq \Gamma$ and $d(\Gamma) = d(\Gamma')$. The full list of such pairs of signatures was obtained by Singerman in [**15**].

An *(effective and orientable) action* of a finite group G on a Riemann surface X is a representation $\epsilon : G \to \mathrm{Aut}(X)$. Two actions ϵ and ϵ' of G on a Riemann surface X are *(weakly) topologically equivalent* if there is an $w \in \mathrm{Aut}(G)$ and an $h \in Hom^+(X)$ such that $\epsilon'(g) = h\epsilon w(g)h^{-1}$. The equisymmetric strata are in correspondence with topological equivalence classes of orientation preserving actions of a finite group G on a surface X. See [**5**]. Let $\mathcal{M}^G$ denote the stratum of surfaces with full automorphism group the conjugacy class of the finite group G in the modular group and let $\overline{\mathcal{M}}^G$ denote the set of surfaces such that the automorphisms group contains a subgroup in the class defined by G.

We have the following theorem.

THEOREM 4. *([5]) Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, G a finite subgroup of the corresponding modular group Mod_g. Then*

(1) $\overline{\mathcal{M}}^G$ is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$.

(2) $\mathcal{M}^G$, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in $\overline{\mathcal{M}}^G$.

Each stratum corresponds with a finite subgroup of the modular group represented as the full group of automorphisms of some compact Riemann surface. To find such full automorphisms groups, we need to use the list of finite maximal signatures for Fuchsian groups in [**15**].

Each action of a finite group G on a surface X_4 is determined by an epimorphism $\theta : \Delta \to G$ from a Fuchsian group Δ such that $\ker(\theta) = \Gamma$, where $X_4 = \mathcal{D}/\Gamma$ and Γ is a **surface** Fuchsian group. The condition Γ to be a surface Fuchsian group imposes that the order of the image under θ of an elliptic generator x_i of Δ is the same as the order of x_i. Two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define two topologically equivalent actions of G on X if and only if there exist automorphisms $\phi : \Delta \to \Delta$, $w : G \to G$ such that $\theta_2 = w \circ \theta_1 \circ \phi^{-1}$. See [**6**, Proposition 2.2] and [**16**, Proposition 2.2].

Let $\mathcal{B}$ be the subgroup of $\mathrm{Aut}(\Delta)$ induced by orientation preserving homeomorphisms of the orbifold $\mathcal{D}/\Delta$. Then two different epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define the same class of G-actions if and only if they lie in the same $\mathcal{B} \times \mathrm{Aut}(G)$-class.

3. The connectedness of the branch locus in the moduli space of Riemann surfaces of genus 4. The equisymetric stratification given in [**7**] provides the structure of the branch locus of the moduli space $\mathcal{M}_4$. In order to establish the connectedness of the branch locus, we can consider a covering using some of the connected strata.

THEOREM 5. *The branch locus is contained in*

$$\overline{\mathcal{M}}^{2,0} \bigcup \overline{\mathcal{M}}^{2,1} \bigcup \overline{\mathcal{M}}^{2,2} \bigcup \overline{\mathcal{M}}^{3,01} \bigcup \overline{\mathcal{M}}^{3,02} \bigcup \overline{\mathcal{M}}^{3,1} \bigcup \overline{\mathcal{M}}^{5,1}$$

where $\overline{\mathcal{M}}^{p,i}$ *or* $\overline{\mathcal{M}}^{p,0i}$ *are the equisymmetric strata determined by classes of group actions of prime order p.*

Proof. Since every finite group G contains an element of prime order p, where p divides the order of $|G|$, by Theorem 4, the branch locus is the union of closed subvarieties $\overline{\mathcal{M}}^{p,i}$, determined by a class of actions of a cyclic group of prime order p. By [**7**, Theorem 2] (see also [**3**] and [**11**]), these subvarieties are the following:

(a) $\overline{\mathcal{M}}^{2,2}$, $\overline{\mathcal{M}}^{2,1}$ and $\overline{\mathcal{M}}^{2,0}$ corresponding to epimorphisms $\theta : \Delta \to C_2$ with signatures $s(\Delta_1) = (2; 2, 2)$, $s(\Delta_2) = (1; 2, \overset{6}{\ldots}, 2)$ and $s(\Delta_3) = (0; 2, \overset{10}{\ldots}, 2)$ respectively. Observe that the Fuchsian groups Δ_3 provide the *hyperelliptic locus*.

(b) $\overline{\mathcal{M}}^{3,2}$, $\overline{\mathcal{M}}^{3,1}$, $\overline{\mathcal{M}}^{3,01}$ and $\overline{\mathcal{M}}^{3,02}$ corresponding to epimorphisms $\theta : \Delta \to C_3 = \langle a : a^3 = 1\rangle$: one class for the non-maximal groups with signature $s(\Delta_1) = (2; -)$, one class for the signature $s(\Delta_2) = (1; 3, 3, 3)$ and two classes for $s(\Delta_3) = (0; 3, 3, 3, 3, 3, 3)$ respectively. The last two classes of epimorphisms $\theta : \Delta_3 \to C_3$ are defined by $\theta_{01}(x_{2i}) = a$ and $\theta_{01}(x_{2i-1}) = a^{-1}$, $1 \leq i \leq 3$, and $\theta_{02}(x_i) = a$, $1 \leq i \leq 6$, yielding the *cyclic trigonal locus*.

(c) The *cyclic pentagonal locus*. $\overline{\mathcal{M}}^{5,1}$, $\overline{\mathcal{M}}^{5,2}$ and $\overline{\mathcal{M}}^{5,3}$ correponding to epimorphisms $\overline{\theta} : \Delta \to C_5 = \langle a : a^5 = 1\rangle$, with $s(\Delta) = (0; 5, 5, 5, 5)$. One subvariety is provided by epimorphisms $\overline{\theta}_1(x_1) = \overline{\theta}_1(x_2) = \overline{\theta}_1(x_3) = a$, $\overline{\theta}_1(x_4) = a^2$. The second one is given by epimorphisms $\overline{\theta}_2(x_1) = a$, $\overline{\theta}_2(x_2) = a^2$, $\overline{\theta}_2(x_3) = a^3$ and $\overline{\theta}_2(x_4) = a^4$. The third subvariety is given by $\overline{\theta}_3(x_1) = \overline{\theta}_3(x_3) = a$ and $\overline{\theta}_3(x_2) = \theta_3(x_4) = a^4$. The groups Δ here inducing the strata $\mathcal{M}^{5,i}$ are non-maximal.

Then we have that the branch loci is contained in

$$\bigcup_{i=0}^{2} \overline{\mathcal{M}}^{2,i} \bigcup_{i=1}^{2} \overline{\mathcal{M}}^{3,0i} \bigcup_{i=1}^{2} \overline{\mathcal{M}}^{3,i} \bigcup_{i=1}^{3} \overline{\mathcal{M}}^{5,i}.$$

Now we shall show that we can delete of the above union the strata $\mathcal{M}^{3,2}$, $\mathcal{M}^{5,2}$ and $\mathcal{M}^{5,3}$.

Each surface in $\mathcal{M}^{3,2}$ is uniformized by the kernel of an epimorphism $\theta : \Delta_1 \to C_3 = \langle a : a^3 = 1\rangle$, with $s(\Delta_1) = (2; -)$. Since the signature $s(\Delta_1)$ is not maximal, the group Δ_1 is contained in a group Δ with signature $(0; 2, \overset{6}{\ldots}, 2)$. Now the epimorphism θ can be extended to an epimorphism $\theta' : \Delta \to D_3 = \langle a, s : a^3 = s^2 = (sa)^2 = 1\rangle$ defined as $\theta(x_1) = s$, $\theta(x_2) = sa$, $\theta(x_3) = s$, $\theta(x_4) = s$ and $\theta(x_5) = s$, in such a way that $\ker\theta = \ker\theta'$. Applying Theorem 1 to $\ker\theta = \ker\theta' \leq \theta^{-1}(\langle s\rangle)$ and using the representation

$\omega : \Delta \to \Sigma_3$ given by the action of Δ via θ' on the $\langle s\rangle$-cosets of D_3, we see that $\omega(x_i) = (a, b)(c)$, so the signature of $\theta^{-1}(\langle s\rangle)$ is $(1; 2, \overset{6}{\ldots}, 2)$. We have obtained that every surface in this stratum has an involution with six fixed points. Thus $\mathcal{M}^{3,2} \subset \overline{\mathcal{M}}^{2,1}$.

In the same way, for the stratum $\mathcal{M}^{5,2}$ we can construct the epimorphisms $\theta_2 : \Delta \to D_5 = \langle a, s : a^5 = s^2 = (sa)^2 = 1\rangle$ defined by $\theta_2(x_1) = s$, $\theta_2(x_2) = sa$, $\theta_2(x_3) = a$ and $\theta_2(x_4) = a^3$, with $s(\Delta) = (0; 2, 2, 5, 5)$. Applying Theorem 1, using the action on the $\langle s\rangle$-cosets of D_5, we see that $s(\theta_2^{-1}(\langle s\rangle)) = (2; 2, 2)$. Thus $\mathcal{M}^{5,2} \subset \overline{\mathcal{M}}^{2,2}$. Observe that epimorphisms θ_2 are extensions of epimorphisms $\overline{\theta}_2$ in part (c).

Again for the stratum $\mathcal{M}^{5,3}$, we can consider the epimorphisms $\theta_3 : \Delta \to D_{10} = \langle a, s : a^{10} = s^2 = sa^2 = 1\rangle$ defined by $\theta_3(x_1) = a^5$, $\theta_3(x_2) = sa^5$ and $\theta_3(x_3) = sa^2$, with $s(\Delta) = (0; 2, 2, 2, 5)$. Applying Theorem 1, using the action on the $\langle a^5\rangle$- and $\langle s\rangle$-cosets, we see that $s(\theta_3^{-1}(\langle a^5\rangle)) = (0; 2, \overset{10}{\ldots}, 2)$ and $s(\theta_3^{-1}(\langle s\rangle)) = (2; 2, 2)$. Thus $\mathcal{M}^{5,3} \subset \overline{\mathcal{M}}^{2,0} \bigcap \overline{\mathcal{M}}^{2,2}$. □

REMARK 1. At the end of the above proof we have established that $\mathcal{M}^{5,3} \subset \overline{\mathcal{M}}^{2,0} \bigcap \overline{\mathcal{M}}^{2,2}$, this fact will be used in the proof of Theorem 7.

Notice that the epimorphisms θ_3 are extensions of epimorphisms $\varphi : \Delta \to D_5$, with $s(\Delta) = (0; 2, 2, 5, 5)$, defined as $\varphi(x_1) = s$, $\varphi(x_2) = s$, $\varphi(x_3) = a$ and $\varphi(x_4) = a^{-1}$. Observe that epimorphisms φ are extensions of epimorphisms $\overline{\theta}_3$ in part (c).

Now we shall study how the strata in Theorem 5 intersect between them. From now onwards, we will use the notation and numbering as in [**7**, Theorem 2]. First, we have the following inclusions for cyclic trigonal surfaces (see [**10**, Theorem 7]).

THEOREM 6. *There exist the following inclusions for strata containing cyclic trigonal Riemann surfaces of genus 4:*

(1) *The stratum* $\mathcal{M}^{C_6\times C_2}$ *belongs to* $\overline{\mathcal{M}}^{3,02} \bigcap \overline{\mathcal{M}}^{2,2} \bigcap \overline{\mathcal{M}}^{2,1}$.
(2) *The stratum* $\mathcal{M}^{D_6}$ *belongs to* $\overline{\mathcal{M}}^{3,01} \bigcap \overline{\mathcal{M}}^{2,2}$.
(3) *The trigonal, pentagonal surface* T_4 *with* $\mathrm{Aut}(T_4) = C_{15}$ *belongs to* $\overline{\mathcal{M}}^{3,02} \bigcap \overline{\mathcal{M}}^{5,1}$.
(4) *The stratum* $\mathcal{M}^{D_3\times C_3}$ *formed by the cyclic trigonal surfaces uniformized by the kernel of an epimorphism from maximal Fuchsian groups with signature* $(0; 2, 2, 3, 3)$ *and* $\mathrm{Aut}(X) = D_3 \times C_3$ *is contained in* $\overline{\mathcal{M}}^{3,02} \bigcap \overline{\mathcal{M}}^{3,2} \bigcap \overline{\mathcal{M}}^{3,1}$.
(5) *The stratum* $\mathcal{M}^{D_3\times D_3}$ *formed by the cyclic trigonal surfaces uniformized by the kernel of an epimorphism from maximal Fuchsian groups with signature* $(0; 2, 2, 2, 3)$ *and* $\mathrm{Aut}(X) = D_3 \times D_3$ *is in* $\overline{\mathcal{M}}^{3,01} \bigcap \overline{\mathcal{M}}^{3,2}$.

Proof. (1) The surfaces in the stratum $\mathcal{M}^{C_6\times C_2}$ are uniformized by the kernel of the epimorphisms $\theta : \Delta \to C_6 \times C_2 = \langle a, s : a^6 = s^2 = [a, s] = 1\rangle$, $\theta(x_1) = s$, $\theta(x_2) = sa^3$ and $\theta(x_3) = a^2$, where $s(\Delta) = (0; 2, 2, 3, 6)$. Applying Theorem 1 using the action on the $\langle a^2\rangle$-, $\langle a^3\rangle$- and $\langle s\rangle$-cosets, we obtain the required inclusion $\mathcal{M}^{C_6\times C_2} \subset \overline{\mathcal{M}}^{3,02} \bigcap \overline{\mathcal{M}}^{2,2} \bigcap \overline{\mathcal{M}}^{2,1}$.

(2) The surfaces in the stratum $\mathcal{M}^{D_6}$ are uniformized by the kernel of the epimorphisms $\theta : \Delta \to D_6 = \langle a, s : a^6 = s^2 = (sa)^2 = 1\rangle$ defined by $\theta(x_1) = s$, $\theta(x_2) = sa^3$ and $\theta(x_3) = a^2$, and also in this case, $s(\Delta) = (0; 2, 2, 3, 6)$. Applying Theorem 1 to the $\langle a^2\rangle$-, $\langle a^3\rangle$- and $\langle s\rangle$-cosets, we obtain the required inclusion $\mathcal{M}^{D_6} \subset \overline{\mathcal{M}}^{3,01} \bigcap \overline{\mathcal{M}}^{2,2}$.

(3) The surface T_4 is determined by the epimorphism $\theta : \Delta \to C_{15} = \langle a : a^{15} = 1\rangle$ defined by $\theta(x_1) = a^5$, $\theta(x_2) = a^3$ and $s(\Delta) = (0; 3, 5, 15)$. Now $\theta(x_1)$ leaves five

$\langle a^3\rangle$-cosets fixed and $\theta(x_3)$ acts on the $\langle a^3\rangle$-cosets as a 5-cycle. In the same way $\theta(x_2)$ acts as the identity and $\theta(x_3)$ acts as a 3-cycle on the $\langle a^5\rangle$-cosets. Moreover θ restricts to the epimorphism $\varphi : \Lambda_3 = \theta^{-1}(\langle a^5\rangle) \to C_3$ with $\varphi(y_i) = a^5,\ 1 \le i \le 5$, and $\phi : \Lambda_5 = \theta^{-1}(\langle a^3\rangle) \to C_5$ with $\phi(z_1) = \phi(z_2) = \phi(z_3) = a^3$. By Theorem 1, the groups Λ_3 and Λ_5 have signatures $s(\Lambda_3) = (0; 3, 3, 3, 3, 3, 3)$ and $s(\Lambda_5) = (0; 5, 5, 5, 5)$ respectively. Then $T_4 \subset \overline{\mathcal{M}}^{3,02} \bigcap \overline{\mathcal{M}}^{5,1}$.

(4) The stratum $\mathcal{M}^{D_3\times C_3}$ is determined by the epimorphisms $\theta_1 : \Delta \to \mathrm{C}_3 \times \mathrm{D}_3 = \langle b : b^3 = 1\rangle \times \langle a, s : a^3 = s^2 = (sa)^2 = 1\rangle$ with maximal Fuchsian groups Δ with signature $s(\Delta) = (0; 2, 2, 3, 3)$. The epimorphisms $\theta_1 : \Delta_4 \to \mathrm{C}_3 \times \mathrm{D}_3$ are defined by $\theta_1(x_1) = s,\ \theta_1(x_2) = sa,\ \theta_1(x_3) = a^{-1}b$ and $\theta_1(x_4) = b^{-1}$. Now $\theta_1(x_3)$ produces two 3-cycles when acting on the $\langle b\rangle$- or $\langle a\rangle$-cosets and $\theta_1(x_3)$ leaves three fixed points when acting on the $\langle ab\rangle$- or $\langle a^2b\rangle$-cosets. Again $\theta_1(x_4)$ leaves six fixed points when acting on the $\langle b\rangle$-cosets and no fixed points on the $\langle a\rangle$-, $\langle ab\rangle$- or $\langle a^2b\rangle$-cosets. By Theorem 1, we have $s(\theta_1^{-1}(\langle a\rangle)) = (2; -)$, $s(\theta_1^{-1}(\langle ab\rangle)) = s(\theta_1^{-1}(\langle a^2b\rangle)) = (1; 3, 3, 3)$ and $s(\theta_1^{-1}(\langle b\rangle)) = (0; 3, 3, 3, 3, 3, 3)$. Furthermore the order three action determined by $\mathcal{D}/\Delta_4 \to \mathcal{D}/\theta_1^{-1}(\langle b\rangle)$ has the same rotation angles for all the fixed points, since b^2 is central in $\mathrm{C}_3 \times \mathrm{D}_3$. Therefore $\mathcal{M}^{D_3\times C_3} \subset \overline{\mathcal{M}}^{3,02} \bigcap \overline{\mathcal{M}}^{3,2} \bigcap \overline{\mathcal{M}}^{3,1}$.

(5) The stratum $\mathcal{M}^{D_3\times D_3}$ is determined by epimorphism $\theta : \Delta \to \mathrm{D}_3 \times \mathrm{D}_3 = \langle a, s : a^3 = s^2 = (sa)^2 = 1\rangle \times \langle b, t : b^3 = t^2 = (tb)^2 = 1\rangle$ defined by $\theta(x_1) = s$, $\theta(x_2) = tb$, $\theta(x_3) = sta$ and $\theta(x_4) = a^2b$, with $s(\Delta) = (0; 2, 2, 2, 3)$. The action of $\theta(x_4) = a^2b$ on the $(\langle ab\rangle)$- and $(\langle a^2b\rangle)$-cosets leaves six fixed points in both cases. The action of $\theta(x_4) = a^2b$ on the $(\langle a\rangle)$- and $(\langle b\rangle)$-cosets leaves no fixed points. According to Theorem 1, $s(\theta_1^{-1}(\langle a\rangle)) = s(\theta_1^{-1}(\langle b\rangle)) = (2; -)$ and $s(\theta_1^{-1}(\langle ab\rangle)) = s(\theta_1^{-1}(\langle a^2b\rangle)) = (0; 3, 3, 3, 3, 3, 3)$, in the last case, the rotation angles for half of set of the fixed points is of angle $2\pi/3$ and for the other half is $-2\pi/3$. Therefore $\mathcal{M}^{D_3\times D_3} \subset \overline{\mathcal{M}}^{3,01} \bigcap \overline{\mathcal{M}}^{3,2}$. □

The stratum $\mathcal{M}^{D_3\times D_3}$ in part 5 of Theorem 6 was studied in [**8**]: the family of cyclic trigonal Riemann surfaces of genus 4 admitting two trigonal morphisms.

As a consequence of Theorems 5 and 6 we have the following.

THEOREM 7. *The branch locus of the moduli space of Riemann surfaces of genus 4 is connected. Moreover the subvariety $\overline{\mathcal{M}}^{2,2}$ has non-empty intersection with all the other subvarieties of the branch locus determined by symmetry classes of cyclic groups of order of a prime integer.*

Proof. By Remark 1 and Theorem 6, the branch locus of the moduli space of Riemann surfaces is connected. The subvariety $\overline{\mathcal{M}}^{2,2}$ has non-empty intersection with the subvarieties $\overline{\mathcal{M}}^{2,0}$, $\overline{\mathcal{M}}^{2,1}$, $\overline{\mathcal{M}}^{3,01}$ and $\overline{\mathcal{M}}^{3,02}$. The subvariety $\overline{\mathcal{M}}^{3,02}$ has non-empty intersection with $\overline{\mathcal{M}}^{3,1}$ and $\overline{\mathcal{M}}^{5,1}$.

We show now that $\overline{\mathcal{M}}^{2,2}$ has non-empty intersection with $\overline{\mathcal{M}}^{3,1}$ and $\overline{\mathcal{M}}^{5,1}$.

To do that, consider first the stratum $\mathcal{M}^{A_4}$ in [**7**, Theorem 2] determined by Fuchsian groups with signature $s(\Delta_2) = (0; 2, 3, 3, 3)$ and epimorphisms $\theta : \Delta_2 \to A_4 = \langle a, s | a^3 = s^2 = (as)^3 = 1\rangle$ defined as $\theta(x_1) = s$, $\theta(x_2) = a$, $\theta(x_3) = as$ and $\theta(x_4) = sas$. By Theorem 1, any element of order 3 in A_4 leaves just one coset fixed when acting on the $\langle a\rangle$, $\langle sa\rangle$, $\langle as\rangle$ or the $\langle sas\rangle$ cosets, since all of them are conjugated. Then $\theta^{-1}(C_3)$ has signature $(1; 3, 3, 3)$ and the corresponding surfaces belong to $\overline{\mathcal{M}}^{3,1}$. On the other hand $s(\theta^{-1}(\langle s\rangle)) = (2; 2, 2)$. So $\mathcal{M}^{A_4} \subset \overline{\mathcal{M}}^{3,1} \bigcap \overline{\mathcal{M}}^{2,2}$.

Finally, consider the surface $Q_4 = \mathcal{M}^{C_{10}}$ in [**7**, Theorem 2] determined by Fuchsian group with signature $s(\Delta) = (0; 5, 10, 10)$ and epimorphisms $\theta : \Delta \to \mathrm{C}_{10}$, $\theta(x_1) = a^2$, $\theta(x_2) = a$ and $\theta(x_2) = a^7$. Applying Theorem 1 to the $\langle a^2 \rangle$- and $\langle a^5 \rangle$-cosets we see that $s(\theta^{-1}(\langle a^5 \rangle)) = (2; 2, 2)$ and $s(\theta^{-1}(\langle a^2 \rangle)) = (0; 5, 5, 5, 5)$, where the three stabilizers induced by x_1 and x_2 rotate the same angle. Thus $Q_4 \subset \overline{\mathcal{M}}^{5,1} \bigcap \overline{\mathcal{M}}^{2,2}$. An algebraic equation for the Riemann surface Q_4 is given in [**17**]. $\square$

The surface U_4 in [**7**, Theorem 2] is known as *Bring's curve* and is the only cyclic pentagonal surface in $\mathcal{M}_4$ admitting several, indeed six, pentagonal morphisms. The Bring's curve U_4 is determined by the "natural" epimorphism $\theta : \Delta \to \Sigma_5$, with $s(\Delta) = (0; 2, 4, 5)$. Again $U_4 \subset \overline{\mathcal{M}}^{5,2} \bigcap \overline{\mathcal{M}}^{3,2}$.

Kulkarni [**12**] showed the existence of isolated points in the branch locus of $\mathcal{M}_g$ if and only if $2g + 1$ is a prime integer, not 7. In [**2**] it is shown that the branch locus in genus 7 is connected, and the branch loci in genera 5 and 6 are connected except for the existence of one isolated point in each case. Bartolini and Izquierdo [**2**] also showed that for every genus g, all the strata of the branch locus determined by actions of C_2 and C_3 belong to the same connected component. These strata have large dimension.

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