Linköping Studies in Science and Technology.
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On the Branch Loci of Moduli Spaces of Riemann Surfaces of Low Genera

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Abstract

Compact Riemann surfaces of genus greater than 1 can be realized as quotient spaces of the hyperbolic plane by the action of Fuchsian groups. The Teichmüller space is the set of all complex structures of Riemann surfaces and the moduli space the set of conformal equivalence classes of Riemann surfaces. For genus greater than two the branch locus of the covering of the moduli space by the Teichmüller space can be identified with the set of Riemann surfaces admitting non-trivial automorphisms. Here we give the orbifold structure of the branch locus of surfaces of genus 5 by studying the equisymmetric stratification of the branch locus. This gives the orbifold structure of the moduli space.

We also show that the strata corresponding to surfaces with automorphisms of order 2 and 3 belong to the same connected component for every genus. Further we show that the branch locus is connected with the exception of one isolated point for genera 5 and 6, it is connected for genus 7 and it is connected with the exception of two isolated points for genus 8.

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Introduction

The first appearance of Riemann surfaces was in Riemann's dissertation *Foundations for a general theory of functions of a complex variable* in 1851. Riemann used the surfaces as a tool to study many-valued complex functions. The first abstract definition was introduced by Weyl in 1913. Another perspective of Riemann surfaces is the uniformization theory of Poincaré, Klein and Koebe. The theory states that every Riemann surface admits a Riemann metric of constant curvature. There are three types of geometries, elliptic geometry with curvature 1, euclidean geometry with curvature 0 and hyperbolic geometry with curvature -1.

An Automorphism of a Riemann surface is a biholomorphic self-map. At the end of the 19th century different properties of automorphisms where studied by Klein, Poincaré, Hurwitz, Clebsch and others. Hurwitz [14] showed that the total number of automorphisms for a surface of genus $g \geq 1$ is bounded by $84(g-1)$. Wiman [24] found that the order of a single automorphism is bounded by $2(2g+1)$.

More recently automorphisms of Riemann surfaces as been studied due to their relation to moduli spaces of Riemann surfaces. Harvey [13] alluded to the equisymmetric stratification of the branch locus and Broughton [5] proved it. Broughton also studied the structure of branch locus for Riemann surfaces of genus 3. For genus 4 the structure was found by Costa and Milagros.

For genus 5 the automorphism groups has been studied by Kimura and Kuribayashi classifying them up to $GL(5, \mathbb{C})$-conjugacy. Breuer [3] generalized those ideas and classified in this way all the automorphism groups for $2 \leq g \leq 48$. Here we will study the automorphism groups of Riemann surfaces of genus 5 up to topological equivalence and determine the complete structure of the equisymmetric stratification of the branch locus. This can be divided into the following steps.

1. Determine all pairs of signatures s and finite groups G such that G has an s-generating vector. This has been done [3, 17].

2. Determine the $Aut(G)$ classes of the generating vectors. Then determine the $\mathcal{B}$-orbits of the classes. This has been done with the use of GAP [11].

3. Remove non-maximal actions. This has been done with the use of Singerman's list [23].

4. Compute actions determined by maximal subgroups of G. This has been done with the use of GAP [11].

Outline of the thesis

Chapter 1

Here we introduce the basic concepts for hyperbolic geometry, Fuchsian groups, Riemann surfaces and Teichmüller theory. Further we introduce the concept of equisymmetric stratification.

Chapter 2

Here we consider surfaces of genus 5. We give all classes of actions of finite groups and determine which are maximal.

Chapter 3

With the results in chapter 2 we study how the induced strata are related to each other. The calculations in chapter 2 and 3 has been aided by the use of GAP.

Chapter 4

In this paper we show that the strata corresponding to actions of order 2 and 3 belong to the same connected component for arbitrary genera. Further we show that the branch locus is connected with the exception of one isolated point for genera 5 and 6, it is connected for genus 7 and it is connected with the exception of two isolated points for genus 8.

1

Preliminaries

Riemann surfaces can be studied in several different ways, for instance as manifolds, complex curves or quotient spaces. When studying automorphisms of Riemann surfaces of genus 2 or greater it is convenient to represent them as quotient spaces of the hyperbolic plane by the action of Fuchsian groups. For details on those topics see [2, 15], for basic groups theory see [10] and for further details on Teichmüller theory see [21].

1.1 Hyperbolic Geometry

To work with Riemann surfaces as quotient spaces we need the concept of hyperbolic geometry. Two common models of the hyperbolic plane which we will use are *the Poincaré half-plane model* $\mathcal{U} = \{z \in \hat{\mathbb{C}} | Im(z) > 0\}$ with metric given by

$$ds = \frac{|dz|}{Im(z)}$$

and *the Poincaré disc model* $\Delta = \{z \in \hat{\mathbb{C}} | |z| < 1\}$ with metric given by

$$ds = \frac{2|dz|}{1 - |z|^2}.$$

The *hyperbolic lines* on the Poincaré models correspond to circles on the extended complex plane perpendicular to the boundary of $\mathcal{U}$ or Δ respectively. Onwards we will denote the hyperbolic plane, regardless of which model we use, by $\mathcal{U}$. The group of conformal isometries of $\mathcal{U}$ is the group of Möbius transformations

$$\frac{az + b}{cz + d}, \quad a, b, c, d \in \mathbb{R},\ ad - bc = 1$$

which is isomorphic to the group $PSL(2,\mathbb{R})$. The anti-conformal isometries are similarly given by

$$\frac{a\bar{z}+b}{c\bar{z}+d}, \quad a,b,c,d \in \mathbb{R},\ ad-bc=-1.$$

The isometries can be divided into classes depending on the nature of their fixed points. Those classes can be characterized by their traces as follows. A conformal isometry is

parabolic if $|a+d|=2$. It has one fixed point $\alpha \in \mathbb{R} \cup \{\infty\}$ and is conjugate to $z+1$ *or* $z-1$.

hyperbolic if $|a+d|>2$. It has one fixed set $\{\alpha,\beta\} \in \mathbb{R} \cup \{\infty\}$ and is conjugate to λz $(\lambda > 1)$.

elliptic if $|a+d|<2$. It has one fixed point $\xi \in \mathcal{U}$ and is conjugate to a rotation of $\mathcal{U}$.

We are interested in subgroups acting *properly discontinuously* on the hyperbolic plane. This means that for any compact subset U we have that $g(U) \cap U = \emptyset$ except for a finite number of elements g.

1.2 Fuchsian Groups

A group of conformal Möbius transformations is a *Fuchsian group* if it leaves a disc invariant on which it acts properly discontinuously. Here we will assume that this disc is $\mathcal{U}$, then a Fuchsian group is a discrete subgroup of $PSL(2,\mathbb{R})$. We are only interested in Fuchsian groups Γ such that the quotient space $\mathcal{U}/\Gamma$ is a compact space. Then Γ has the following presentation

$$\left\langle a_1,b_1,\ldots,a_g,b_g,x_1\ldots x_k | x_1^{m_1}=\cdots=x_k^{m_k}=\prod x_i \prod [a_i,b_i]=1 \right\rangle \tag{1.1}$$

where x_i is a elliptic element and a_i and b_i are hyperbolic elements. If a group Γ has presentation 1.1 then we say that Γ has *signature*

$$s(\Gamma)=(g;m_1,\ldots,m_k) \tag{1.2}$$

where g is called the *genus* of the topological surface $\mathcal{U}/\Gamma$ and $m_i,\ i=1\ldots k$ are the *orders* of the stabilizers of the cone points of the surface. The signature (1.2) gives us the algebraic structure of Γ and the geometrical structure of $\mathcal{U}/\Gamma$.

A Fuchsian group without elliptic elements is called a *surface group*. Later we will see why this is important.

Example 1.1

Let Δ be a hyperbolic triangle with angles $\pi/m_1, \pi/m_2, \pi/m_3$ and let $r_i,\ i=1,2,3$ be the reflections in the sides of Δ. If Γ^* is the group generated by r_1,r_2,r_3 then $\Gamma = \Gamma^* \cap PSL(2,\mathbb{R})$ is a Fuchsian group called a *triangle group* and is generated by the

elliptic elements r_1r_2, r_2r_3, r_3r_1 where $(r_1r_2)^{m_3} = (r_2r_3)^{m_1} = (r_3r_1)^{m_2} = Id$. Γ has the following presentation

$$\langle x_1, x_2, x_3 | x_1^{m_1} = x_2^{m_2} = x_3^{m_3} = x_1x_2x_3 = 1 \rangle .$$

A *fundamental domain* F of Fuchsian group is a closed subset of $\mathcal{U}$ such that

(i) $\mathcal{U} = \bigcup_{g\in\Gamma} g(F)$.

(ii) if $p \in F$ and $g(p) \in F$, where $g \neq Id$, then $p \in \delta F$.

(iii) $\mu(\delta F) = 0$, where μ is the hyperbolic measure.

If Γ has signature (1.2) then we can choose a fundamental domain F for Γ such that δF is a hyperbolic polygon. Further, two sides α, α' of the polygon are *congruent* if there exists $g \in \Gamma$ such that $g(\alpha) = \alpha'$. Then $g(F) \cap F = \alpha'$. Each elliptic element x_i pairs sides γ_i, γ_i' such that the sides form an angle $2\pi/m_i$, a_i pairs sides α, α' and b_i pairs sides β, β'. The hyperbolic structure of the fundamental domain under the side pairings, $F/\sim$, is the same as the hyperbolic structure of $\mathcal{U}/\Gamma$. We can also go the opposite way, formulated in the following theorem by Poincaré:

Theorem 1.1. *Let F be a hyperbolic polygon with side pairings as above with non-zero area then there exists a Fuchsian group with F as fundamental domain.*

The hyperbolic area of a Fuchsian group corresponds to the hyperbolic area of any of its fundamental domains and is given by

$$\mu(\Gamma) = 2\pi(2g - 2 + \sum_i \left(1 - \frac{1}{m_i}\right)).$$

Now assume that Γ' is a subgroup of Γ of index N. If F is a fundamental domain of Γ then $\bigcup g_i(F)$ is a fundamental domain of Γ', where $\{g_i\}$ is a transversal for Γ'. From this the Riemann-Hurwitz formula follows:

$$[\Gamma : \Gamma'] = N = \frac{\mu(\Gamma')}{\mu(\Gamma)}. \tag{1.3}$$

The possible signatures and indices of subgroups of a given Fuchsian group is given by the following theorem due to Singerman:

Theorem 1.2. *[22] Let Γ have signature (*1.2*) and presentation (*1.1*). Then Γ contains a subgroup Γ' of index n and signature $(g'; m_{11}, \ldots, m_{ks_k})$ if and only if*

(i) there exist a transitive permutation representation $\theta : \Gamma \to \Sigma_n$ satisfying $\theta(x_i)$ has exactly s_i cycles of length less than m_i, the lengths being $m_i/m_{i1} \ldots m_i/m_{is_i}$.

*(ii) the Riemann-Hurwitz formula (*1.3*) is satisfied.*

1.3 Riemann Surfaces

A *Riemann surface* is a Haussdorf space X that is locally homeomorphic to the complex plane. This means that each point in X has an open neighbourhood U_i such that there exists a homeomorphism $\Phi_i : U_i \to V_i$, where V_i is an open subset of $\mathbb{C}$. We call the pair (Φ_i, U_i) a *chart*. An *atlas* is a set of charts A covering X and if $U_i \cap U_j \neq \emptyset$ then the *transition function*

$$\Phi_i \circ \Phi_j^{-1} : \Phi_j(U_i \cap U_j) \to \Phi_i(U_i \cap U_j)$$

is analytic. Further we call two analytic atlases A, B *compatible* if all the transition functions of charts $(\Phi, U) \in A, (\Psi, V) \in B$ are analytic. Such atlases form an equivalence class called a *complex structure*. Different complex structures on the same topological surface yield different Riemann surfaces. With atlases we can define holomorphic maps between Riemann surfaces. Let X and Y be Riemann surfaces with atlases $\{(\Phi_i, U_i)\}$ and $\{(\Psi_j, V_j)\}$ respectively. A map $f : X \to Y$ is called *holomorphic (or meromorphic)* if the maps

$$\Psi_j \circ f \circ \Phi_i^{-1} : \Phi_i(U_i \cap f^{-1}(V_j)) \to \mathbb{C}$$

are analytic (or meromorphic), f is a homomorphism of Riemann surfaces. Further if f is bijective and f^{-1} is also holomorphic then we call f *biholomorphic*. Two Riemann surfaces are *conformally equivalent* if there exists a biholomorphic map between them. We do not distinguish between conformally equivalent surfaces.

Example 1.2

(i) $\mathbb{C}$ with an atlas consisting of the identity map.

(ii) $\hat{\mathbb{C}}$ with an atlas consisting of the identity map of $\mathbb{C}$, Id, together with the map $\phi : \hat{\mathbb{C}} \setminus \{0\} \to \mathbb{C}$ defined as $\phi(z) = 1/z$ and $\phi(\infty) = 0$. This is indeed an atlas since $Id \circ \phi^{-1} = 1/z$ and $\phi \circ Id^{-1} = 1/z$ are analytic on $\mathbb{C} \cap \hat{\mathbb{C}} \setminus \{0\} = \mathbb{C} \setminus \{0\}$.

A *covering* is a surjective continuous map $f : X \to Y$ between two Riemann surfaces such that for any point $p \in Y$ the preimage of some neighbourhood V is a disjoint union of open sets of X each mapping homeomorphically to V. The set $f^{-1}(p) \subset X$ is called the *fiber* of p. Each fiber has the same cardinality, if it is a finite number n we say that the covering is *n-sheeted*.

Theorem 1.3. *[19] Let $f : X \to Y$ be a covering, with Y path-wise connected. For each $p \in Y$, the fundamental group $\pi_1(Y, p)$ acts transitively on the right in the fiber $f^{-1}(p)$. The stabilizer of each point $x \in f^{-1}(p)$ is $H(x) = f_{\#}\pi_1(X, x)$.*

The permutations of a fiber $f^{-1}(p)$ by $\in \pi_1(Y, p)$ forms a group called the *monodromy group* of f at p. For Riemann surfaces all fibers are permuted in the same way so we might talk about the monodromy group of f, in this case it is sometimes called the *deck transformation group*.

Now for an arbitrary surjective continuous map $f : X \to Y$ there might exist points where we can not find neighborhoods with preimages consisting of disjoints union of open sets mapping homeomorphically to the neighborhood. Such points are called *branch points*. A *branched covering* is a surjective continuous map $f : X \to Y$ such that the map

$f : X^* \to Y^*$ is a covering where Y^* is the surface Y with the branch points removed and X^* is the surface X with the preimages of the branch points removed.

A branched covering : $X \to Y$ between Riemann surfaces is a morphism of Riemann surfaces.

Example 1.3

Consider the map $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ defined by $z \mapsto z^3$. f is a branched covering with branch points 0 and ∞. We note that close to the branch points f is three-to-one.

Riemann Surfaces as Quotient Spaces

In order to study moduli spaces of Riemann surfaces we consider Riemann surfaces as quotient spaces of the hyperbolic plane. In fact every Riemann surface is conformally equivalent to a quotient space of a simply connected Riemann surface. This follows from the uniformization theorems.

Theorem 1.4. *[15] (Uniformization theorem) Any simply connected Riemann surface is isomorphic to one of the following spaces:*

(i) $\mathbb{C}$

(ii) $\hat{\mathbb{C}}$

(iii) $\mathcal{U}$

Theorem 1.5. *[15] Any Riemann surface X of genus $g \geq 2$ is birationally equivalent to the orbit space $\mathcal{U}/\Gamma$ for a Fuchsian group Γ with orbit genus g. Furthermore, $\mathcal{U}$ is the universal covering of X.*

From the theorem it follows that Γ is the lift of the fundamental group, $\pi_1(X, x)$, to the universal covering $\mathcal{U}$. Now assume that Γ' is a subgroup of Γ of index N. Then we have a branched covering $f : \mathcal{U}/\Gamma' \to \mathcal{U}/\Gamma$. Let $\{g_i\}$ be a transversal of Γ' in Γ then each coset of Γ induces a sheet of the covering f. We note that Γ permutes the sheets and generates the monodromy group of the covering. The *monodromy* of the covering is the transitive permutation representation $\theta : \Gamma \to \Sigma_N$ given by the action of Γ on the cosets of Γ'. On the other hand, assume we have an epimorphism $\theta : \Gamma \to G$ for some finite group G. Let G' be a subgroup of G of index N, then the images of the elements of Γ permutes the cosets of G' inducing a permutation representation of Γ. By Theorem 1.2 Γ has a subgroup Γ' of index N and the induced permutation representation of Γ is the monodromy of the covering $\mathcal{U}/\Gamma' \to \mathcal{U}/\Gamma$.

We prefer to work with Riemann surfaces as quotient spaces of surface groups due to properties we will see later on. We can consider an arbitrary Riemann surface of genus greater than 1 as a quotient space of a surface group as we see in the following theorem:

Theorem 1.6. *[15] Any compact Riemann surface X of genus $g \geq 2$ is conformally equivalent to the orbit space $\mathcal{U}/\Gamma$ for a Fuchsian surface group Γ with orbit genus g. Furthermore, $\mathcal{U}$ is the universal covering of X, and the fundamental group $\Gamma = Aut_X(\mathcal{U})$ is unique up to conjugacy in $Aut(\mathcal{U})$.*

By the use of surface Fuchsian groups we obtain several useful properties. The next theorem shows one which is important when working with Teichmüller spaces.

Theorem 1.7. *[15] Let Γ_1 and Γ_2 be two surface Fuchsian groups. Then the Riemann surfaces $\mathcal{U}/\Gamma_1$ and $\mathcal{U}/\Gamma_2$ are conformally equivalent if and only if there exists $t \in PSL(2,\mathbb{R})$ such that $t\Gamma_1 t^{-1} = \Gamma_2$.*

1.4 Automorphisms of Riemann Surfaces

An automorphism of a Riemann surface X is a conformal homeomorphism $f : X \to X$. We denote the group of automorphisms of a Riemann surface $Aut(X)$.

Example 1.4

(i) $Aut(\mathbb{C}) = \{az + b | a, b \in \mathbb{C}, a \neq 0\}$.

(ii) $Aut(\hat{\mathbb{C}}) = PSL(2,\mathbb{C})$.

(iii) $Aut(\mathcal{U}) = PSL(2,\mathbb{R})$.

There is a well known upper bound of the cardinality of the automorphism group of a compact Riemann surface of genus greater than one:

Theorem 1.8. *[14] Let X be a compact Riemann surface of genus $g \geq 2$. Then $|Aut(X)| \leq 84(g-1)$.*

A Riemann surface X such that $|Aut(X)| = 84(g-1)$ is called a *Hurwitz surface* and $Aut(X)$ a *Hurwitz group.*

Example 1.5

The most well-known Hurwitz surface is the Klein quartic of genus 3. It can be defined as a complex curve by the equation

$$x^3y + y^3z + z^3x = 0.$$

The Automorphism group of the Klein quartic is isomorphic to $PSL(2,7)$ and $|PSL(2,7)| = 168 = 84(3-1)$.

Let $X = \mathcal{U}/\Gamma$ be a Riemann surface uniformized by a surface Fuchsian group Γ. Then a group G is a group of automorphisms of X if and only if there exists a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$ such that $ker(\theta) = \Gamma$ [15]. In this case Δ is the lift of the automorphism group of X to the universal covering.

Given a Fuchsian group Δ with signature $s = (g; m_1, \ldots, m_k)$ and a finite group G, then the epimorphism $\theta : \Delta \to G$ is uniquely associated to an *s-generating vector* of G is a vector $(a_1, b_1, \ldots, a_g, b_g; c_1, \ldots, c_k)$, where $a_i, b_i, c_i \in G$ and such that c_i has order m_i and the long relation is satisfied. So by studying generating vectors we can determine if G is a group of automorphisms of a surface of genus g.

Theorem 1.9. *[4] (Riemann's Existence Theorem) The group G acts on the surface X, of genus g, with branching data $(g; m_1, \ldots, m_k)$ if and only if the Riemann-Hurwitz formula (1.3) is satisfied, and G has a $(g; m_1, \ldots, m_k)$-generating vector.*

Example 1.6

Let G be isomorphic to the cyclic group C_6 and consider the signature $s = (0, 2, 2, 3, 6, 6)$. By Theorem 1.9 C_6 acts on a surface of genus 5 since with the Riemann-Hurwitz formula (1.3) we find that $2g - 2 = 6(-2 + 1/2 + 1/2 + 2/3 + 5/6 + 5/6) = 8$ and (a^3, a^3, a^4, a, a) is an s-generating vector. Thus there exist a Fuchsian group Δ with signature $s(\Delta) = (0, 2, 2, 3, 6, 6)$ and a presentation

$$\left\langle x_1, x_2, x_3, x_4, x_5 | x_1^2 = x_2^2 = x_3^3 = x_4^6 = x_5^6 = x_1x_2x_3x_4x_5 = 1 \right\rangle$$

such that there exists an epimorphism $\theta : \Delta \to C_6$ by $\theta(x_1) = a^3$, $\theta(x_2) = a^3$, $\theta(x_3) = a^4$, $\theta(x_4) = a$ and $\theta(x_5) = a$. Note that any elliptic element x of Δ is conjugate to some x_i thus $\theta(x) = \theta(y^{-1}x_iy) = \theta(x_i) \neq 1$. So $ker(\theta)$ is a surface group of genus g. Also, θ is the monodromy of the covering $X = \mathcal{U}/\Gamma \to \mathcal{U}/\Delta = X/G$.

Note that in Theorem 1.9 we only need to consider a signature $(g; m_1, \ldots, m_k)$, that means the algebraic structure of some group. Further on, we will denote an abstract group with presentation (1.1) and signature (1.2) by $\Delta(g; m_1, \ldots, m_k)$.

Maximal Fuchsian Groups

We are interested in determining when an action of a finite group yields the full group of automorphisms of some Riemann surface. If it is the case we call it a *maximal* action due to the following properties. A Fuchsian group Δ is called a *maximal* Fuchsian group if there is no other Fuchsian group Δ' containing Δ with finite index and such that $d(\Delta) = d(\Delta')$. We also call a signature non-maximal if it is the signature of some non-maximal Fuchsian group. The full list of pairs of signatures $s(\Delta)$, $s(\Delta')$ as above was obtained by Singerman in [23].

$s(\Delta)$	$s(\Delta')$	$\lvert\Delta' : \Delta\rvert$
$(2; -)$	$(0; 2, 2, 2, 2, 2, 2)$	2
$(1; t, t)$	$(0; 2, 2, 2, 2, t)$	2
$(1; t)$	$(0; 2, 2, 2, 2t)$	2
$(0; t, t, t, t)$	$(0; 2, 2, 2, t)$	4
$(0; t, t, u, u)$	$(0; 2, 2, t, u)$	2
$(0; t, t, t)$	$(0; 3, 3, t)$	3
$(0; t, t, t)$	$(0; 2, 3, 2t)$	6
$(0; t, t, u)$	$(0; 2, t, 2u)$	2
$(0; 7, 7, 7)$	$(0; 2, 3, 7)$	24
$(0; 2, 7, 7)$	$(0; 2, 3, 7)$	9
$(0; 3, 3, 7)$	$(0; 2, 3, 7)$	8
$(0; 4, 8, 8)$	$(0; 2, 3, 8)$	12
$(0; 3, 8, 8)$	$(0; 2, 3, 8)$	10
$(0; 9, 9, 9)$	$(0; 2, 3, 9)$	12
$(0; 4, 4, 5)$	$(0; 2, 4, 5)$	6
$(0; n, 4n, 4n)$	$(0; 2, 3, 4n)$	6
$(0; n, 2n, 2n)$	$(0; 2, 4, 2n)$	4
$(0; 3, n, 3n)$	$(0; 2, 3, 3n)$	4
$(0; 2, n, 2n)$	$(0; 2, 3, 2n)$	3

Table 1.4: Pairs of non-maximal signatures.

When an action is induced by a non-maximal signature it can still be maximal as we see in the following proposition:

Proposition 1.1. *Let Γ be a surface Fuchsian group and $X = \mathcal{U}/\Gamma$ a Riemann surface. Assume that there exists a surface kernel epimorphism $\theta : \Delta \to G$ such that $ker(\theta) = \Gamma$ where $s(\Delta)$ is a non-maximal signature. Further assume that there exists another surface kernel epimorphism $\theta' : \Delta' \to G'$ such that $ker(\theta') = \Gamma$ where $\Delta' \geq \Delta$, $G' \geq G$ and $\theta'^{-1}(G) = \Delta$. Then $G = \Delta/\Gamma$ is the full group of automorphisms $Aut(X)$ if and only if $\theta'|_\Delta$ is not equivalent under automorphisms of Δ and G to θ for any such extensions $\theta' : \Delta' \to G'$.*

Example 1.7

Let Δ be a group with signature $s(\Delta) = (0; 4, 4, 4, 4)$ such that there exists an epimorphism $\theta : \Delta \to C_4 \times C_2$ defined by $\theta(x_1) = a$, $\theta(x_2) = a$, $\theta(x_3) = ab$ and $\theta(x_4) = ab$. With Riemann-Hurwitz formula we find that θ induces a group of automorphisms of a surface of genus 5. This group is indeed not the full group of automorphisms. Consider a group Δ' with signature $s(\Delta') = (0; 2, 2, 2, 4)$ such that there exists an epimorphism $\theta' : \Delta' \to (C_4 \times C_2 \times C_2) \rtimes C_2$ defined by $\theta'(y_1) = ba$, $\theta'(y_2) = b$, $\theta'(y_3) = c$ and $\theta'(y_4) = ac$ acting on a surface of genus 5. Consider the following commutative diagram

$$\begin{array}{ccc}
(x_1, x_2, x_3, x_4) & \xrightarrow{\theta} & (a, a, ab, ab) \\
\downarrow & & \downarrow \\
(y_3y_2y_4^{-1}y_2y_4y_2y_4y_2y_3, y_4, y_2y_4y_2, y_3y_2y_4y_2y_3) & \xrightarrow{\theta'} & (ac, ac, (ac)^3(bc)^2, (ac)^3(bc)^2)
\end{array}$$

where $C_4 \times C_2 \simeq \langle ac, (ac)^2(bc)^2 \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$. As seen in the diagram the action θ can be extended to θ'.

To find out if an action $\theta : \Delta \to G$ extends to an action $\theta' : \Delta' \to G'$ we consider the monodromy of the covering $\mathcal{U}/\theta'^{-1}(G) \to \mathcal{U}/\Delta'$. We show the calculations by an example.

Example 1.8

We return to the actions of Example 1.7. Let $\Delta_1 = \theta'^{-1} \langle ac, (ac)^2(bc)^2 \rangle$. To determine the monodromy group of the induced covering $\mathcal{U}/\Delta_1 \to \mathcal{U}/\Delta'$ we consider the cosets of $\langle ac, (ac)^2(bc)^2 \rangle$ and by enumerating the cosets we have the following permutation representation of the monodromy:

$\theta'(y_i)$	$\langle ac, (ac)^2(bc)^2 \rangle$
$ba \to$	$(1\ 4)(2\ 3)$
$b \to$	$(1\ 2)(3\ 4)$
$c \to$	$(1\ 3)(2\ 4)$
$ac \to$	$(1)(2)(3)(4)$

Sufficient conditions for non-maximal actions has been studied in [6].

1.5 Teichmüller Theory

On page 6 we constructed Riemann surfaces on a topological surface X and classified them up to conformal equivalence. We are interested in the space of those classes; the moduli space. However this space is not a manifold in its natural topology, it is an orbifold. To overcome this we consider Teichmüller spaces of Riemann surfaces. The Teichmüller space is constructed by classes of Riemann surfaces marked by quasi-conformal maps. However, when a Riemann surface is uniformized by a surface Fuchsian group we can construct the Teichmüller space by classes of surface Fuchsian groups. One benefit of the second approach is that we do not have to explicitly use quasi-conformal maps, see [21].

Let Γ be an abstract group with presentation (1.1) and let $R(\Gamma)$ be the set of monomorphisms $r : \Gamma \to PSL(2, \mathbb{R})$. $R(\Gamma)$ has a natural topology given by

$$r \mapsto (r(x_1), \ldots, r(x_k), r(a_1), \ldots, r(b_g)).$$

The group $PSL(2, \mathbb{R})$ acts on $R(\Gamma)$ by conjugation; for $t \in PSL(2, \mathbb{R})$ and $x \in \Gamma$ we denote by $t^*(r) \to PSL(2, \mathbb{R})$ the isomorphism $x \to t^{-1}r(x)t$. The quotient space $T(\Gamma) = R(\Gamma)/PSL(2, \mathbb{R})^*$ is called the *extended Teichmüller space* of Γ and is homeomorphic to two copies of a complex space of dimension $d(\Gamma) = 3g - 3 + k$. If Γ is a surface group then we denote $T(\Gamma)$ by T_g. Teichmüller spaces for different Fuchsian groups are related as follows.

Theorem 1.10. *[12] Let Γ and Γ' be two Fuchsian groups such that there exists a monomorphism $i : \Gamma \to \Gamma'$ then the induced map*

$$i^* : T(\Gamma') \to T(\Gamma),\ [r] \mapsto [r \circ i],$$

is an isometric embedding.

Let $M(\Gamma)$ denote the group of outer automorphisms of Γ. $M(\Gamma)$, which is also called *the Teichmüller modular group* or the *mapping class group* of Γ, acts on $T(\Gamma)$ as $[r] \to [r \circ \alpha]$ where $\alpha \in M(\Gamma)$. The moduli space of Γ is the quotient space $\mathcal{M}(\Gamma) = T(\Gamma)/M(\Gamma)$. Similarly as before we denote by $\mathcal{M}_g$ the moduli space of a surface group of genus g. We are interested in the *branch locus* $\mathcal{B}_g$ of the covering $T_g \to \mathcal{M}_g$. As an application of Nielsen Realization Theorem [5, 13] one can identify the branch locus of the action of $M(\Gamma)$ as the set $\mathcal{B}_g = \{X \in \mathcal{M}_g | Aut(X) \neq 1\}$, for $g \geq 3$.

The Branch Locus

Consider the automorphism group $Aut(X)$ of a Riemann surface $X = \mathcal{U}/\Gamma$. $G = Aut(X)$ determines a conjugacy class of subgroups, $\bar{G}$, of $M(\Gamma)$ [5]. We call $\bar{G}$, or just G, the *symmetry type* of X. Two surfaces are *equisymmetric* is they have the same symmetry type, that is their automorphism groups are conjugate in $\mathcal{M}_g$. Conformally equivalent surfaces clearly have the same symmetry type so we can talk about the symmetry type of points in the moduli space. In terms of actions of finite groups on surfaces equisymmetry corresponds to topologically equivalent actions. Two actions θ, θ' of G on a surface X

are *topologically equivalent* if there is an $w \in \mathrm{Aut}(G)$ and an $h \in Hom^+(X)$ such that $\theta'(g) = h\theta w(g)h^{-1}$. We can formulate this for Riemann surfaces as follows.

Lemma 1.1. *[4] Two epimorphisms $\theta_1, \theta_2 : \Delta \to G$ define two topologically equivalent actions of G on X if there exists two automorphisms $\phi : \Delta \to \Delta$ and $w : G \to G$ such that the following diagram commutes:*

$$\begin{tikzcd} \Delta \arrow[r, "\theta_1"] & G \arrow[d, "w"] \\ \Delta \arrow[u, "\phi"] \arrow[r, "\theta_2"] & G \end{tikzcd}$$

ϕ is the induced automorphism by the lifting h^* of h to the universal orbifold covering. Let $\mathcal{B}$ be the subgroup of $Aut(\Delta)$ composed of such automorphisms. While it is possible to find a finite generating set for $\mathcal{B}$ we will only produce the elements in an ad hoc fashion, using the following proposition.

Proposition 1.2. *[4] Suppose that $\gamma = 0$, or 1, let $a = a_1$, $b = b_1$ and let other notation be as above. Consider the automorphisms of Δ defined by the formulae below, where the action of an automorphism on a generator of Δ is written down only if it actually moves the generator. A product of these automorphisms defines an element of $\mathcal{B}$ if and only if the induced action preserves branching orders.*

I.a $a \to ab^n$, $n \in \mathbb{Z}$,

I.b $b \to ba^n$, $n \in \mathbb{Z}$,

II $x_j \to x_{j+1}$, $x_{j+1} \to x_{j+1}^{-1} x_j x_{j+1}$

III.a $a \to xa$, $x_j \to y x_j y^{-1}$, *where* $x = b^{-1}wz$, $y = zb^{-1}w$, $w = x_1 \dots x_{j-1}$, $z = x_{j+1} \dots x_r$.

III.b $b \to xb$, $x_j \to y x_j y^{-1}$, *where* $x = wza$, $y = zaw$, $w = x_1 \dots x_{j-1}$, $z = x_{j+1} \dots x_r$.

We define the *G-equisymmetric stratum* $\mathcal{M}_g^G$ as the set of classes of surfaces with symmetry type G. By $\overline{\mathcal{M}}_g^G$ we denote the set of surfaces whose automorphism group contains a subgroup in the class defined by G. We will denote the set of all surfaces admitting automorphisms of order p by $\overline{\mathcal{M}}_g^p$. We have the following theorem by Broughton:

Theorem 1.11. *([5]) Let $\mathcal{M}_g$ be the moduli space of Riemann surfaces of genus g, G a finite subgroup of the corresponding modular group M_g. Then:*

(1) $\overline{\mathcal{M}}_g^G$ is a closed, irreducible algebraic subvariety of $\mathcal{M}_g$.

(2) $\mathcal{M}_g^G$, if it is non-empty, is a smooth, connected, locally closed algebraic subvariety of $\mathcal{M}_g$, Zariski dense in $\overline{\mathcal{M}}_g^G$.

We also have the following theorem:

Theorem 1.12. *[8] Let $\overline{\mathcal{M}}_g^{C_p,i}$ be the stratum corresponding to an action i of the cyclic group C_p of prime order p. Then*

$$\mathcal{B}_g = \bigcup_{p \in \mathbb{P}} \overline{\mathcal{M}}_g^{C_p,i}. \tag{1.4}$$

2

The Equisymmetric Stratification of the Branch Locus of Riemann Surfaces of Genus 5

The equisymmetric stratification of the branch locus, $\mathcal{B}_g$, provides the orbifold structure of the moduli space, $\mathcal{M}_g$. Costa and Izquierdo [9] found the orbifold structure of $\mathcal{M}_4$. Kimura [16] found all the actions of finite groups on surfaces of genus 4. Magaard and others [20] studied the strata corresponding to large automorphism groups for genera up to 10.

Here we calculate the orbifold structure of $\mathcal{M}_5$. To determine the structure of the strata we need representations for the automorphism groups of Riemann surfaces of genus 5. We classify the actions by topological equivalence. Kuribayashi and Kimura [17] has classified them up to $GL(5, \mathbb{C})$-conjugacy. The classification in [17] coincides with the topological classification with the exception of one case, studied in Lemma 2.9 and Remark 2.1. The topological classification here has been done by calculating all actions with GAP [11] and reducing the number of equivalence classes accordingly. Examples of GAP programs appear in Appendix B. The same algorithm is used through Lemma 2.2 to Lemma 2.13. We exemplify this with detailed calculations in the proofs of Lemma 2.4 and Lemma 2.13.

Remember that a finite group G is a group of automorphisms of a Riemann surface $X = \mathcal{U}/\Gamma$, where Γ is Fuchsian surface group, if there exists a Fuchsian group Δ and an epimorphism $\theta : \Delta \to G$ such that $\Gamma = ker(\theta)$. Observe that the equisymmetric strata are associated with maximal actions of finite groups. Non-maximal actions extend to maximal actions as in the following diagram

$$\begin{array}{ccc} \Delta' & \xrightarrow{\theta'} & G' \\ \uparrow & & \uparrow \\ \Delta & \xrightarrow{\theta} & G \\ \uparrow & & \uparrow \\ \Gamma & \longrightarrow & Id \end{array}$$

where θ is a non-maximal action and θ' is a maximal action. We prove non-maximality by finding extensions Δ' of Δ and G' of G such that $\theta'|_{\Delta} = \theta$. To do this we use Theorem 1.2. We begin with a lemma calculating the possible signatures of Fuchsian groups inducing actions of finite groups.

Lemma 2.1. *[17] Let Δ be a Fuchsian group and G a finite group. If there exist an epimorphism $\theta : \Delta \to G$ such that $ker(\theta)$ is a surface group of genus 5. then Δ has one of the following signatures.*

$\|G\|$	*signature*	$\|G\|$	*signature*	$\|G\|$	*signature*
2	$(0;2,\overset{12}{\ldots},2)$	10	$(0;2,2,2,2,5)$		$(0;2,12,12)^*$
	$(1;2,\overset{8}{\ldots},2)$		$(0;2,2,10,10)^*$	30	$(0;2,6,15)$
	$(2;2,2,2,2)$		$(1;5)$	32	$(0;2,2,2,4)$
	$(3;-)$	11	$(0;11,11,11)$		$(0;4,4,4)^*$
3	$(0;3,\overset{7}{\ldots},3)$	12	$(0;2,2,2,2,3)$		$(0;2,8,8)^*$
	$(1;3,3,3,3)$		$(0;3,3,3,3)^*$	40	$(0;2,4,20)$
4	$(0;2,\overset{8}{\ldots},2)$		$(0;2,3,4,4)$	48	$(0;2,2,2,3)$
	$(0;2,\overset{5}{\ldots},2,4,4)$		$(0;2,2,6,6)^*$		$(0;3,4,4)^*$
	$(0;2,2,4,4,4,4)$		$(0;6,12,12)^*$		$(0;2,6,6)^*$
	$(1;2,2,2,2)$		$(1;3)^*$		$(0;2,4,12)$
	$(1;2,4,4)$	15	$(0;3,15,15)^*$	60	$(0;3,3,5)^*$
	$(2;-)^*$	16	$(0;2,2,2,2,2)$	64	$(0;2,4,8)^*$
5	$(1;5,5)^*$		$(0;2,2,4,4)^*$	80	$(0;2,5,5)^*$
6	$(0;2,2,2,2,3,3)$		$(0;4,8,8)^*$	96	$(0;3,3,4)^*$
	$(0;2,3,3,3,6)$		$(1;2)^*$		$(0;2,4,6)$
	$(0;2,2,3,6,6)$	20	$(0;2,2,2,10)$	120	$(0;2,3,10)$
	$(0;6,6,6,6)^*$		$(0;4,4,10)^*$	160	$(0;2,4,5)$
	$(1;3,3)^*$		$(0;2,20,20)^*$	192	$(0;2,3,8)$
8	$(0;2,\overset{6}{\ldots},2)$	22	$0;2,11,22$		
	$(0;2,2,2,4,4)$	24	$(0;2,2,3,3)^*$		
	$(0;4,4,4,4)^*$		$(0;2,2,2,6)$		
	$(0;2,4,8,8)$		$(0;4,4,6)^*$		
	$(1;2,2)^*$		$(0;3,6,6)^*$		

* *Non-maximal*

Proof: From Riemann's Existence Theorem 1.9 we see that the existence of an epimorphism $\theta : \Delta \to G$ such that $ker(\theta)$ is a surface group of genus 5 depends on the signature

of Δ and the properties of the group G. The signatures must satisfy the Riemann-Hurwitz formula with $s(\Gamma) = (5; -)$. This gives us the equation

$$8 = |G|(2g - 2 + \sum \left(1 - \frac{1}{m_i}\right)) \tag{2.1}$$

and the signatures above are the solutions to this equation such that there exists G with an s-generating vector. □

First we will list for each group G the actions of G corresponding to a given signature by generating vectors with elements the images of the generators of a Fuchsian group with the given signature, $(\theta(a_1), \ldots, \theta(b_g); \theta(x_1), \ldots, \theta(x_k))$. We also show which actions are non-maximal and which are maximal, giving the symmetry of the surfaces in the stratum.

Lemma 2.2. *The actions of order* 2 *are*

$$\begin{array}{ll}
\theta_{2,0} : \Delta(0; 2, \overset{12}{\ldots}, 2) \to C_2 & (a, a, a, a, a, a, a, a, a, a, a, a) \\
\theta_{2,1} : \Delta(1; 2, \overset{8}{\ldots}, 2) \to C_2 & (1, 1; a, a, a, a, a, a, a, a) \\
\theta_{2,2} : \Delta(2; 2, 2, 2, 2) \to C_2 & (1, 1, 1, 1; a, a, a, a) \\
\theta_{2,3} : \Delta(3; -) \to C_2 & (a, 1, 1, 1, 1, 1; -)
\end{array}$$

Lemma 2.3. *The actions of order* 4 *are*

$$\begin{array}{ll}
\theta_{4,1} : \Delta(0; 2, 2, 4, 4, 4, 4) \to C_4 & (a^2, a^2, a, a, a, a) \\
\theta_{4,2} : \Delta(0; 2, 2, 4, 4, 4, 4) \to C_4 & (a^2, a^2, a, a, a^{-1}, a^{-1}) \\
\theta_{4,3} : \Delta(0; 2, 2, 2, 2, 2, 4, 4) \to C_4 & (a^2, a^2, a^2, a^2, a^2, a, a) \\
\theta_{4,4} : \Delta(1; 2, 4, 4) \to C_4 & (1, 1; a^2, a, a) \\
\theta_{4,5} : \Delta(1; 2, 2, 2, 2) \to C_4 & (a, 1; a^2, a^2, a^2, a^2) \\
\theta_{4,6} : \Delta(2; -) \to C_4 & (a, 1, 1, 1; -) \\
\theta_{4,7} : \Delta(0; 2, \overset{8}{\ldots}, 2) \to C_2 \times C_2 & (a, \overset{6}{\ldots}, a, b, b) \\
\theta_{4,8} : \Delta(0; 2, \overset{8}{\ldots}, 2) \to C_2 \times C_2 & (a, a, a, a, b, b, b, b) \\
\theta_{4,9} : \Delta(0; 2, \overset{8}{\ldots}, 2) \to C_2 \times C_2 & (a, a, a, a, b, b, ab, ab) \\
\theta_{4,10} : \Delta(1; 2, 2, 2, 2) \to C_2 \times C_2 & (1, 1; a, a, b, b) \\
\theta_{4,11} : \Delta(1; 2, 2, 2, 2) \to C_2 \times C_2 & (b, 1; a, a, a, a) \\
\theta_{4,12} : \Delta(2; -) \to C_2 \times C_2 & (a, b, 1, 1; -) \\
\theta_{4,13} : \Delta(2; -) \to C_2 \times C_2 & (a, 1, b, 1; -)
\end{array}$$

where $\theta_{4,6}$, $\theta_{4,12}$ *and* $\theta_{4,13}$ *are non-maximal actions.*

Proof: Here we will prove non-maximality.

(1) $\theta_{4,6}$ extends to $\theta_{8,17}$ in Lemma 2.4 with the isomorphism $C_4 \simeq \langle a \rangle \subset D_4$.

(2) $\theta_{4,12}$ extends to $\theta_{8,13}$ in Lemma 2.4 with the isomorphism $C_2 \times C_2 \simeq \langle b, abc \rangle \subset C_2 \times C_2 \times C_2$.

(3) $\theta_{4,13}$ extends to $\theta_{8,14}$ in Lemma 2.4 with the isomorphism $C_2 \times C_2 \simeq \langle ab, bc \rangle \subset C_2 \times C_2 \times C_2$. □

Lemma 2.4. *The actions of order* 8 *are*

$$
\begin{array}{ll}
\theta_{8,1} : \Delta(0;2,4,8,8) \to C_8 & (a^4, a^2, a, a) \\
\theta_{8,2} : \Delta(0;2,4,8,8) \to C_8 & (a^4, a^2, a^3, a^{-1}) \\
\theta_{8,3} : \Delta(1;2,2) \to C_8 & (a, 1; a^4, a^4) \\
\theta_{8,4} : \Delta(0;2,2,2,4,4) \to C_4 \times C_2 & (b, b, b, ba^3, a) \\
\theta_{8,5} : \Delta(0;2,2,2,4,4) \to C_4 \times C_2 & (b, b, a^2b, ab, a) \\
\theta_{8,6} : \Delta(0;2,2,2,4,4) \to C_4 \times C_2 & (b, b, a^2, a, a) \\
\theta_{8,7} : \Delta(0;2,2,2,4,4) \to C_4 \times C_2 & (a^2, a^2b, b, a, a^3) \\
\theta_{8,8} : \Delta(0;2,2,2,4,4) \to C_4 \times C_2 & (a^2, a^2, b, a^3b, a) \\
\theta_{8,9} : \Delta(0;4,4,4,4) \to C_4 \times C_2 & (a, a, ab, ab) \\
\theta_{8,10} : \Delta(0;4,4,4,4) \to C_4 \times C_2 & (a, a^3, ab, a^3b) \\
\theta_{8,11} : \Delta(1;2,2) \to C_4 \times C_2 & (a, 1; b, b) \\
\theta_{8,12} : \Delta(1;2,2) \to C_4 \times C_2 & (a, b; a^2, a^2) \\
\theta_{8,13} : \Delta(0;2,.^6.,2) \to C_2 \times C_2 \times C_2 & (a, a, a, ab, bc, c) \\
\theta_{8,14} : \Delta(0;2,.^6.,2) \to C_2 \times C_2 \times C_2 & (a, a, b, b, c, c) \\
\theta_{8,15} : \Delta(0;2,.^6.,2) \to C_2 \times C_2 \times C_2 & (a, a, b, ab, bc, abc) \\
\theta_{8,16} : \Delta(1;2,2) \to C_2 \times C_2 \times C_2 & (b, c; a) \\
\theta_{8,17} : \Delta(0;2,.^6.,2) \to D_4 & (s, s, s, s, sa, sa) \\
\theta_{8,18} : \Delta(0;2,.^6.,2) \to D_4 & (s, s, sa, sa, a^2, a^2) \\
\theta_{8,19} : \Delta(0;2,2,2,4,4) \to D_4 & (s, s, a^2, a, a) \\
\theta_{8,20} : \Delta(1;2,2) \to D_4 & (t, 1; s, s) \\
\theta_{8,21} : \Delta(0;4,4,4,4) \to Q & (i, i, j, j)
\end{array}
$$

where $\theta_{8,3}$, $\theta_{8,9}$, $\theta_{8,10}$, $\theta_{8,11}$, $\theta_{8,12}$, $\theta_{8,16}$, $\theta_{8,20}$ and $\theta_{8,21}$ are non-maximal actions.

Proof: We show the detailed calculations to find actions of Q as an example of the algorithm. There are six classes of epimorphisms up to the action of $Aut(Q) \simeq S_4$. Those give the following generating vectors, (i,i,j,j), $(i,j,-i,j)$, (i,j,j,i), $(i,j,i,-j)$, $(i,j,-j,-i)$ and $(i,-i,-j,j)$. Now let B_i be the elements of type II in Proposition 1.2 defined by $x_i \mapsto x_j$, $x_j \mapsto x_j^{-1}x_ix_j$. We see that $B_2(i,i,j,j) = (i,j,-jij,j) = (i,j,-i,j)$, $B_3(i,j,-i,j) = (i,j,j,-j-ij) = (i,j,j,i)$, $B_3(i,j,j,i) = (i,j,i,-iji) = (i,j,i,-j)$, $B_3(i,j,i,-j) = (i,j,-j,ji-j) = (i,j,-j,-i)$ and $B_2(i,j,-i,j) = (i,-i,ij-i,j) = (i,-i,-j,j)$. Thus there is one class of epimorphisms $\theta_{8,21} : \Delta(0;4,4,4,4) \to Q$.

Now we will prove non-maximality starting by considering actions induced by the signature $(0;4,4,4,4)$.

(1) For $\theta_{8,9}$ consider the isomorphism $C_4 \times C_2 \simeq \langle ac, (ac)^2(bc)^2 \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$. Enumerating the cosets we have the permutation representation of the monodromy of the desired covering $\mathcal{U}/\Delta \to \mathcal{U}/\Delta'$.

$\theta_{32,2}(y_i)$	$\langle ac, (bc)^2 \rangle$
$ba \to$	$(1\ 4)(2\ 3)$
$b \to$	$(1\ 2)(3\ 4)$
$c \to$	$(1\ 3)(2\ 4)$
$ac \to$	$(1)(2)(3)(4)$

(2) Similarly for $\theta_{8,10}$ consider the isomorphism $C_4 \times C_2 \simeq \langle ac, (ab)^2 \rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$. This gives the following coset permutation representation producing the desired covering $\mathcal{U}/\Delta \to \mathcal{U}/\Delta'$.

$\theta_{32,1}(y_i)$	$\langle ac,(bc)^2\rangle$
$cac\to$	$(1\ 3)(2\ 4)$
$b\to$	$(1\ 2)(3\ 4)$
$bc\to$	$(1\ 4)(2\ 3)$
$ac\to$	$(1)(2)(3)(4)$

inducing the generating vector $(ac,(ac)^3,ac(ab)^2,(ac)^3(ab)^2)$.

(3) For $\theta_{8,21}$ consider the isomorphism $Q\simeq\langle ac,(ab)^2\rangle\subset(D_4\times C_2)\rtimes C_2$. We have the following permutation representation of the covering $\mathcal{U}/\Delta\to\mathcal{U}/\Delta'$ is

$abac\to$	$(1\ 4)(2\ 3)$
$a\to$	$(1\ 3)(2\ 4)$
$b\to$	$(1\ 3)(2\ 4)$
$ac\to$	$(1)(2)(3)(4)$

thus $s(\theta_{32,3}^{-1}\langle(ac,ab)^2\rangle)=(0;4,4,4,4)$.

(4) With similar calculations to the ones above we see that $\theta_{8,3}$ extends to $\theta_{16,4}$ by $C_8\simeq\langle a\rangle\subset D_8$. $\theta_{8,11}$ and $\theta_{8,12}$ extends to $\theta_{16,5}$ and $\theta_{16,6}$ respectively by $C_4\times C_2\simeq\langle a,b\rangle\subset D_4\times C_2$. $\theta_{8,16}$ extends to $\theta_{16,3}$ by $C_2\times C_2\times C_2\simeq\langle a,bc,cd\rangle\subset C_2^4$. $\theta_{8,20}$ extends to $\theta_{16,5}$ by $C_4\times C_2\simeq\langle a,sb\rangle\subset D_4\times C_2$. □

Lemma 2.5. *The actions of order* 16 *are*

$\theta_{16,1}:\Delta(0;4,8,8)\to C_8\times C_2$	(a^6b,a,ab)
$\theta_{16,2}:\Delta(0;2,2,4,4)\to C_4\times C_2\times C_2$	(b,c,a,a^3bc)
$\theta_{16,3}:\Delta(0;2,2,2,2,2)\to C_2^4$	$(a,b,c,d,abcd)$
$\theta_{16,4}:\Delta(0;2,2,2,2,2)\to D_8$	$(s,sa,sa^7,sa^2,(a)^4)$
$\theta_{16,5}:\Delta(0;2,2,2,2,2)\to D_4\times C_2$	(s,s,sa,sab,b)
$\theta_{16,6}:\Delta(0;2,2,2,2,2)\to D_4\times C_2$	$(s,sa,sb,sab,(a)^2)$
$\theta_{16,7}:\Delta(0;2,2,4,4)\to D_4\times C_2$	$(s,sb,ab,(a)^3)$
$\theta_{16,8}:\Delta(0;2,2,4,4)\to C_8\rtimes_3 C_2$	$(s,s,as,(as)^3)$
$\theta_{16,9}:\Delta(0;2,2,4,4)\to(C_4\times C_2)\rtimes C_2$	(b,c,ab,ac)
$\theta_{16,10}:\Delta(0;4,8,8)\to C_8\rtimes_5 C_2$	(asa,a,as)
$\theta_{16,11}:\Delta(1;2)\to C_8\rtimes_5 C_2$	$(a,s;a^4)$
$\theta_{16,12}:\Delta(1;2)\to C_4\rtimes C_4$	$(s,t;s^2)$
$\theta_{16,13}:\Delta(0;2,2,4,4)\to G_{4,4}$	(b,a^2,a,ab)
$\theta_{16,14}:\Delta(0;2,2,4,4)\to G_{4,4}$	(a^2b,b,a,a)
$\theta_{16,15}:\Delta(0;2,2,4,4)\to G_{4,4}$	(b,b,a,a^{-1})
$\theta_{16,16}:\Delta(0;2,2,4,4)\to G_{4,4}$	$(aba,aba^{-1}b,a,ab)$
$\theta_{16,17}:\Delta(1;2)\to G_{4,4}$	$(a,b;aba^{-1}b)$

where $\theta_{16,1}$, $\theta_{16,2}$, $\theta_{16,7}$, $\theta_{16,8}$, $\theta_{16,9}$, $\theta_{16,10}$, $\theta_{16,11}$, $\theta_{16,12}$, $\theta_{16,14}$, $\theta_{16,15}$, *and* $\theta_{16,17}$ *are non-maximal actions.*

Proof: Here we will prove non-maximality.

(1) $\theta_{16,1}$ extends to θ_{192} in Lemma 2.11 with the isomorphism $C_8\times C_2\simeq\langle t,(st^7s)^2\rangle\subset(((C_4\times C_2)\rtimes C_4)\rtimes C_3)\rtimes C_2$.

(2) $\theta_{16,2}$ extends to $\theta_{32,2}$ in Lemma 2.6 with the isomorphism $C_4\times C_2\times C_2\simeq\langle a,bcb,c\rangle\subset(C_4\times C_2\times C_2)\rtimes C_2$.

(3) $\theta_{16,7}$ extends to $\theta_{32,1}$ with the isomorphism $D_4 \times C_2 \simeq \langle ac, c, (ab)^2 \rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$.

(4) $\theta_{16,8}$ extends to $\theta_{32,3}$ with the isomorphism $C_8 \rtimes_3 C_2 \simeq \langle ab, bc \rangle \subset (D_4 \times C_2) \rtimes C_2$.

(5) $\theta_{16,9}$ extends to $\theta_{32,3}$ with the isomorphism $(C_4 \times C_2) \rtimes C_2 \simeq \langle (ab)^2, a, c \rangle \subset (D_4 \times C_2) \rtimes C_2$.

(6) $\theta_{16,10}$ extends to $\theta_{64,2}$ in Lemma 2.6 with the isomorphism $C_8 \rtimes_5 C_2 \simeq \langle a, [a, s] \rangle \subset ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$.

(7) $\theta_{16,11}$ extends to $\theta_{32,3}$ with the isomorphism $C_8 \rtimes_5 C_2 \simeq \langle ab, c \rangle \subset (D_4 \times C_2) \rtimes C_2$.

(8) $\theta_{16,12}$ extends to $\theta_{32,2}$ with the isomorphism $C_4 \rtimes C_4 \simeq \langle a, bc \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$.

(9) $\theta_{16,14}$ extends to $\theta_{32,2}$ with the isomorphism $G_{4,4} \simeq \langle ac, b \rangle \subset (C_4 \times C_2 \times C_2) \rtimes C_2$.

(10) $\theta_{16,15}$ extends to $\theta_{32,1}$ with the isomorphism $G_{4,4} \simeq \langle ac, b \rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$.

(11) $\theta_{16,17}$ extends to $\theta_{32,1}$ with the isomorphism $G_{4,4} \simeq \langle ab, c \rangle \subset (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$. □

Lemma 2.6. *The actions of order* 32 *and order* 64 *are*

$$\begin{array}{ll}
\theta_{32,1} : \Delta(0; 2, 2, 2, 4) \to C_2^4 \rtimes C_2 & (cac, b, bc, ac) \\
\theta_{32,2} : \Delta(0; 2, 2, 2, 4) \to (C_4 \times C_2 \times C_2) \rtimes C_2 & (ba, b, c, ac) \\
\theta_{32,3} : \Delta(0; 2, 2, 2, 4) \to (D_4 \times C_2) \rtimes C_2 & (abac, a, b, ac) \\
\theta_{32,4} : \Delta(0; 4, 4, 4) \to (C_4 \times C_2) \rtimes C_4 & (s, t, t^3 s^3) \\
\theta_{32,5} : \Delta(0; 4, 4, 4) \to ((C_4 \times C_2) \rtimes C_2) \rtimes C_2 & (s, st, ts^2) \\
\theta_{32,6} : \Delta(0; 2, 8, 8) \to (C_8 \times C_2) \rtimes C_2 & (t, s, s^{-1}t) \\
\theta_{32,7} : \Delta(0; 2, 8, 8) \to (C_8 \rtimes C_2) \rtimes C_2 & (t, s, s^{-1}t) \\
\theta_{64,1} : \Delta(0; 2, 4, 8) \to ((C_8 \times C_2) \rtimes C_2) \rtimes C_2 & (t, ta^{-1}, a) \\
\theta_{64,2} : \Delta(0; 2, 4, 8) \to ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2 & (t, ta^{-1}, a)
\end{array}$$

where $\theta_{32,4}$, $\theta_{32,5}$, $\theta_{32,6}$, $\theta_{32,7}$ *and* $\theta_{64,1}$ *are non-maximal actions.*

Proof: Here we will prove non-maximality.

(1) $\theta_{32,4}$ extends to θ_{192} in Lemma 2.11 with the isomorphism $(C_4 \times C_2) \rtimes C_4 \simeq \langle (t^7 s)^2, (st^7)^2 \rangle \subset (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$.

(2) $\theta_{32,5}$ extends to $\theta_{64,2}$ define above with the isomorphism $((C_4 \times C_2) \rtimes C_2) \simeq \langle at, s \rangle \subset ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$.

(3) $\theta_{32,6}$ extends to θ_{192} in Lemma 2.11 with the isomorphism $(C_8 \times C_2) \rtimes C_2 \simeq \langle t, (st)^2 t^4 \rangle \subset (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$.

(4) $\theta_{32,7}$ extends to $\theta_{64,2}$ defined above with the isomorphism $(C_8 \rtimes C_2) \rtimes C_2 \simeq \langle a^2 s, a \rangle \subset ((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$.

(5) $\theta_{64,1}$ extends to θ_{192} in Lemma 2.11 with the isomorphism $((C_8 \times C_2) \rtimes C_2) \rtimes C_2 \simeq \langle st^2 s^2 t, st^6 s^2 \rangle \subset (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$. □

Lemma 2.7. *The actions of order* 3 *are*

$$\begin{array}{ll}
\theta_{3,0} : \Delta(0; 3, 3, 3, 3, 3, 3, 3) \to C_3 & (a, a, a, a, a, a^2, a^2) \\
\theta_{3,1} : \Delta(1; 3, 3, 3, 3) \to C_3 & (1, 1; a, a, a^2, a^2)
\end{array}$$

Lemma 2.8. *The actions of order* 6 *are*

$$\begin{array}{ll}
\theta_{6,1} : \Delta(0;2,2,2,2,3,3) \to C_6 & (a^3,a^3,a^3,a^3,a^2,a^4) \\
\theta_{6,2} : \Delta(0;2,3,3,3,6) \to C_6 & (a^3,a^2,a^2,a^4,a) \\
\theta_{6,3} : \Delta(0;2,2,3,6,6) \to C_6 & (a^3,a^3,a^4,a,a) \\
\theta_{6,4} : \Delta(0;6,6,6,6) \to C_6 & (a,a,a^5,a^5) \\
\theta_{6,5} : \Delta(1;3,3) \to C_6 & (a,1;a^2,a^4) \\
\theta_{6,6} : \Delta(0;2,2,2,2,3,3) \to D_3 & (s,s,s,s,a,a^2) \\
\theta_{6,7} : \Delta(1;3,3) \to D_3 & (s,1;a,a^2)
\end{array}$$

where $\theta_{6,4}$, $\theta_{6,5}$ *and* $\theta_{6,7}$ *are non-maximal actions.*

Proof: Here we will prove non-maximality.

(1) $\theta_{6,4}$ extends to $\theta_{24,6}$ in Lemma 2.10 with the isomorphism $C_6 \simeq \langle a^5 b\rangle \subset D_6 \times C_2$.

(2) $\theta_{6,5}$ extends to $\theta_{12,4}$ with the isomorphism $C_6 \simeq \langle a\rangle \subset D_6$.

(3) $\theta_{6,7}$ extends to $\theta_{12,5}$ with the isomorphism $D_3 \simeq \langle a^2, sa\rangle \subset D_6$. □

Lemma 2.9. *The actions of order* 12 *are*

$$\begin{array}{ll}
\theta_{12,1} : \Delta(0;6,12,12) \to C_{12} & (a^{10},a,a) \\
\theta_{12,2} : \Delta(0;2,2,6,6) \to C_6 \times C_2 & (b,b,a,a^{-1}) \\
\theta_{12,3} : \Delta(0;2,2,6,6) \to C_6 \times C_2 & (b,a^3,a,ba^2) \\
\theta_{12,4} : \Delta(0;2,2,2,2,3) \to D_6 & (s,s,sa^3,sa,a^2) \\
\theta_{12,5} : \Delta(0;2,2,2,2,3) \to D_6 & (a^3,a^3,s,sa^4,a^2) \\
\theta_{12,6} : \Delta(0;2,2,6,6) \to D_6 & (sa^2,s,a,a) \\
\theta_{12,7} : \Delta(1;3) \to D_6 & (s,a;a^2) \\
\theta_{12,8} : \Delta(0;3,3,3,3) \to A_4 & (a,a^2,b,b^2) \\
\theta_{12,9} : \Delta(0;3,3,3,3) \to A_4 & (a,a,ab,b^2) \\
\theta_{12,10} : \Delta(0;2,3,4,4) \to C_3 \rtimes C_4 & (s^2,t,s,st^2) \\
\theta_{12,11} : \Delta(1;3) \to C_3 \rtimes C_4 & (t,s;t)
\end{array}$$

where $\theta_{12,1}$, $\theta_{12,2}$, $\theta_{12,3}$, $\theta_{12,6}$, $\theta_{12,7}$, $\theta_{12,8}$, $\theta_{12,9}$ *and* $\theta_{12,11}$ *are non-maximal actions.*

Remark 2.1. We note that there are two non-equivalent actions induced by the signature $(0;3,3,3,3)$. Indeed by considering all actions $\mathcal{B}$ on (a,a^2,b,b^2), 218 in total, we note that none of the resulting generating vectors contain a pair of equal elements. These groups have the same representation up to $GL(5,\mathbb{C})$-conjugacy, see [17]. See appendix B for further information.

Proof: Here we will prove non-maximality.

(1) $\theta_{12,1}$ extends to $\theta_{48,1}$ in Lemma 2.11 with the isomorphism $C_{12} \simeq \langle a\rangle \subset (C_{12} \times C_2) \rtimes C_2$.

(2) $\theta_{12,2}$ extends to $\theta_{24,6}$ with the isomorphism $C_6 \times C_2 \simeq \langle a,b\rangle \subset D_6 \times C_2$.

(3) $\theta_{12,3}$ extends to $\theta_{24,7}$ with the isomorphism $C_6 \times C_2 \simeq \langle b,(ab)^2\rangle \subset (C_6 \times C_2) \rtimes C_2$.

(4) $\theta_{12,6}$ extends to $\theta_{24,6}$ with the isomorphism $D_6 \simeq \langle ab, sb\rangle \subset D_6 \times C_2$.

(5) $\theta_{12,7}$ extends to $\theta_{24,6}$ with the isomorphism $D_6 \simeq \langle a^2b, sa^3\rangle \subset D_6 \times C_2$.

(6) $\theta_{12,8}$ extends to $\theta_{48,5}$ in Lemma 2.11 with the isomorphism $A_4 \simeq \langle b^3ab^3, a\rangle \subset S_4 \times C_2$.

(7) $\theta_{12,9}$ extends to $\theta_{24,2}$ with the isomorphism $A_4 \simeq \langle (ab)^2, b^2 \rangle \subset S_4 \times C_2$.

(8) $\theta_{12,11}$ extends to $\theta_{24,7}$ with the isomorphism $C_3 \rtimes C_4 \simeq \langle ab^3, b^2 \rangle \subset (C_6 \times C_2) \rtimes C_2$. □

Lemma 2.10. *The actions of order* 24 *are*

$$\begin{array}{ll}
\theta_{24,1} : \Delta(0;2,12,12) \to C_{12} \times C_2 & (b, a, a^{-1}b) \\
\theta_{24,2} : \Delta(0;2,2,3,3) \to A_4 \times C_2 & (ab^3, b^3, b^2, b^4a) \\
\theta_{24,3} : \Delta(0;2,2,3,3) \to A_4 \times C_2 & (ab^3, ab^3, b^2, b^4) \\
\theta_{24,4} : \Delta(0;3,6,6) \to A_4 \times C_2 & (b^4, ba, ab) \\
\theta_{24,5} : \Delta(0;2,2,3,3) \to S_4 & (ab, ab, a, a^{-1}) \\
\theta_{24,6} : \Delta(0;2,2,2,6) \to D_6 \times C_2 & (s, a^3, sa^4b, a^5b) \\
\theta_{24,7} : \Delta(0;2,2,2,6) \to (C_6 \times C_2) \rtimes C_2 & (a, b^3, ab^4, aba) \\
\theta_{24,8} : \Delta(0;4,4,6) \to (C_3 \rtimes C_4) \times C_2 & (s, st, s^2t^5)
\end{array}$$

where $\theta_{24,1}$, $\theta_{24,3}$, $\theta_{24,4}$, $\theta_{24,5}$, *and* $\theta_{24,8}$ *are non-maximal actions.*

Proof: Here we will prove non-maximality.

(1) $\theta_{24,1}$ extends to $\theta_{48,1}$ with the isomorphism $C_{12} \times C_2 \simeq \langle a, (ab)^2 \rangle \subset (C_{12} \times C_2) \rtimes C_2$.

(2) $\theta_{24,3}$ extends to $\theta_{48,5}$ with the isomorphism $A_4 \times C_2 \simeq \langle ab^2a^2, ac \rangle \subset S_4 \times C_2$.

(3) $\theta_{24,4}$ extends to $\theta_{96,1}$ with the isomorphism $A_4 \times C_2 \simeq \langle [a,b], a \rangle \subset A_4 \times C_2 \times C_2$

(4) $\theta_{24,5}$ extends to $\theta_{48,5}$ with the isomorphism $S_4 \simeq \langle a, b \rangle \subset S_4 \times C_2$.

(5) $\theta_{24,1}$ extends to $\theta_{48,1}$ with the isomorphism $(C_3 \rtimes C_4) \times C_2 \simeq \langle ab, a^2 \rangle \subset (C_{12} \times C_2) \rtimes C_2$. □

Lemma 2.11. *The actions of order* 48, 96 *and* 192 *are*

$$\begin{array}{ll}
\theta_{48,1} : \Delta(0;2,4,12) \to (C_{12} \times C_2) \rtimes C_2 & (b, ba, a^{11}) \\
\theta_{48,2} : \Delta(0;2,6,6) \to A_4 \times C_2 \times C_2 & (b, a, a^5b) \\
\theta_{48,3} : \Delta(0;3,4,4) \to A_4 \rtimes C_4 & (a, b, b^3a^2) \\
\theta_{48,4} : \Delta(0;3,4,4) \to A_4 \rtimes C_4 & (a, b^3, ba^2) \\
\theta_{48,5} : \Delta(0;2,2,2,3) \to S_4 \times C_2 & (ab, b^2c, ca^2b^3, a) \\
\theta_{96,1} : \Delta(0;2,4,6) \to (A_4 \times C_2 \times C_2) \rtimes C_2 & (b, bac, ca^5) \\
\theta_{96,2} : \Delta(0;3,3,4) \to ((C_4 \times C_2) \rtimes C_4) \rtimes C_3 & (s, s^2t^7, t) \\
\theta_{192} : \Delta(0;2,3,8) \to (((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2 & (st^5, s, t^5)
\end{array}$$

where $\theta_{48,2}$, $\theta_{48,3}$ *and* $\theta_{96,2}$ *are non-maximal actions.*

Proof: Here we will prove non-maximality.

(1) $\theta_{48,2}$ extends to $\theta_{96,1}$ since $A_4 \times C_2 \times C_2 \subset (A_4 \times C_2 \times C_2) \rtimes C_2$

(2) $\theta_{48,3}$ extends to $\theta_{96,1}$ with the isomorphism $A_4 \rtimes C_4 \simeq \langle a^4, a^3cb \rangle \subset (A_4 \times C_2 \times C_2) \rtimes C_2$.

(3) $\theta_{96,2}$ extends to θ_{192}. □

Lemma 2.12. *The actions of order* 5, 10, 15, 20, 30, 40, 60, 80, 120 *and* 160 *are*

$$\begin{array}{ll}
\theta_5 : \Delta(1;5,5) \to C_5 & (1,1;a,a^4) \\
\theta_{10,1} : \Delta(0;2,2,10,10) \to C_{10} & (a^5,a^5,a,a^9) \\
\theta_{10,2} : \Delta(0;2,2,2,2,5) \to D_5 & (s,s,s,sa^5,a) \\
\theta_{10,3} : \Delta(1;5) \to D_5 & (s,a;a^2) \\
\theta_{15} : \Delta(0;3,15,15) \to C_{15} & (a^5,a^4,a^8) \\
\theta_{20,1} : \Delta(0;2,20,20) \to C_{20} & (a^{10},a,a^9) \\
\theta_{20,2} : \Delta(0;2,2,2,10) \to D_{10} & (a^5,s,sa^2,a^3) \\
\theta_{20,3} : \Delta(0;4,4,10) \to C_5 \rtimes C_4 & (ab,a,a^2b) \\
\theta_{30} : \Delta(0;2,6,15) \to D_5 \times C_3 & (b^3,ba,bab) \\
\theta_{40} : \Delta(0;2,4,20) \to D_5 \times C_4 & (b,ba,a^{-1}) \\
\theta_{60} : \Delta(0;3,3,5) \to A_5 & (b,bab,b^2a^4b) \\
\theta_{80} : \Delta(0;2,5,5) \to (C_2^4) \rtimes C_5 & (b,a,a^{-1}b) \\
\theta_{120} : \Delta(0;2,3,10) \to A_5 \times C_2 & (ba^2c,b,a^2c) \\
\theta_{160} : \Delta(0;2,4,5) \to ((C_2^4) \rtimes C_5) \rtimes C_2 & (ab,b^{-1},a^{-1})
\end{array}$$

where θ_5, $\theta_{10,1}$, $\theta_{10,3}$, θ_{15}, $\theta_{20,1}$, $\theta_{20,3}$, θ_{60} and θ_{80} are non-maximal actions.

Proof: For $\theta_{10,3}$ consider the subgroup $\langle a^2, sa\rangle \subset D_{10}$, clearly $\theta_{20,2}$ induces $\theta_{10,3}$. Using Table 1.4 it is easy to see that θ_5, $\theta_{10,1}$, θ_{15}, $\theta_{20,1}$, $\theta_{20,3}$, θ_{60} and θ_{80} are non-maximal actions due to the fact there is only one action for each group and the actions θ_{30}, θ_{40}, θ_{120} and θ_{160} do exist. □

Lemma 2.13. *The actions of order* 11 *and* 22 *are*

$$\begin{array}{ll}
\theta_{11,1} : \Delta(0;11,11,11) \to C_{11} & (a,a^2,a^8) \\
\theta_{11,2} : \Delta(0;11,11,11) \to C_{11} & (a,a,a^9) \\
\theta_{22} : \Delta(0;2,11,22) \to C_{22} & (a^{11},a^{10},a)
\end{array}$$

where $\theta_{11,2}$ is a non-maximal action.

Proof: Here we will prove non-maximality.

There are two classes of actions of C_{11} with representatives $\theta_{11,1} : \Delta \to C_{11}$, defined by (a,a^2,a^8) and $\theta_{11,2} : \Delta \to C_{11}$, defined by (a,a,a^9). Now $\theta_{11,2}$ extends to $\phi : \Delta(0;2,11,22) \to C_{22}$ defined by (b^{11},b^{10},b). By Theorem 1.2 $\phi^{-1}\langle b^2\rangle$ is a group with signature $(0;11,11,11)$ and the images of the elliptic generators by ϕ (with the isomorphism $b^2 \to a$) are a, a and a^9 inducing $\theta_{11,2}$, thus $\overline{\mathcal{M}}_5^{11,2} \equiv \overline{\mathcal{M}}_5^{22}$. By Theorem 1.2 $s(\phi_2^{-1}\langle b^{11}\rangle) = (0;2,\overset{12}{\ldots},2)$ thus $\overline{\mathcal{M}}_5^{22} \subset \overline{\mathcal{M}}_5^{C_2,0}$. The epimorphism $\theta_{11,1}$ yield a maximal action of C_{11} in $\mathcal{M}_5$ producing an isolated point $\overline{\mathcal{M}}_5^{11,1}$. □

Now each maximal action corresponds to an equisymmetric strata [5]. Letting the strata corresponding to a maximal action $\theta_{i,j}$ be denoted $\mathcal{M}_5^{i,j}$ we have from Lemma 2.2 through Lemma 2.13 above the following theorem.

Theorem 2.1. *The equisymmetric strata of the branch locus $\mathcal{B}_5$ are the following.*

Strata	*dim.*	*Strata*	*dim.*	*Strata*	*dim.*	*Strata*	*dim.*
$\mathcal{M}_5^{2,0}$	9	$\mathcal{M}_5^{6,1}$	3	$\mathcal{M}_5^{8,19}$	2	$\mathcal{M}_5^{24,7}$	1
$\mathcal{M}_5^{2,1}$	8	$\mathcal{M}_5^{6,2}$	2	$\mathcal{M}_5^{10,2}$	2	$\mathcal{M}_5^{30}$	0
$\mathcal{M}_5^{2,2}$	7	$\mathcal{M}_5^{6,3}$	2	$\mathcal{M}_5^{11,1}$	0	$\mathcal{M}_5^{32,1}$	1
$\mathcal{M}_5^{2,3}$	6	$\mathcal{M}_5^{6,6}$	3	$\mathcal{M}_5^{12,4}$	2	$\mathcal{M}_5^{32,2}$	1
$\mathcal{M}_5^{3,0}$	4	$\mathcal{M}_5^{8,1}$	1	$\mathcal{M}_5^{12,5}$	2	$\mathcal{M}_5^{32,3}$	1
$\mathcal{M}_5^{3,1}$	4	$\mathcal{M}_5^{8,2}$	1	$\mathcal{M}_5^{12,10}$	1	$\mathcal{M}_5^{40}$	0
$\mathcal{M}_5^{4,1}$	3	$\mathcal{M}_5^{8,4}$	2	$\mathcal{M}_5^{16,3}$	2	$\mathcal{M}_5^{48,1}$	0
$\mathcal{M}_5^{4,2}$	3	$\mathcal{M}_5^{8,5}$	2	$\mathcal{M}_5^{16,4}$	2	$\mathcal{M}_5^{48,4}$	0
$\mathcal{M}_5^{4,3}$	4	$\mathcal{M}_5^{8,6}$	2	$\mathcal{M}_5^{16,5}$	2	$\mathcal{M}_5^{48,5}$	1
$\mathcal{M}_5^{4,4}$	3	$\mathcal{M}_5^{8,7}$	2	$\mathcal{M}_5^{16,6}$	2	$\mathcal{M}_5^{64,2}$	0
$\mathcal{M}_5^{4,5}$	4	$\mathcal{M}_5^{8,8}$	2	$\mathcal{M}_5^{16,13}$	1	$\mathcal{M}_5^{120}$	0
$\mathcal{M}_5^{4,7}$	5	$\mathcal{M}_5^{8,13}$	3	$\mathcal{M}_5^{16,16}$	1	$\mathcal{M}_5^{96,1}$	0
$\mathcal{M}_5^{4,8}$	5	$\mathcal{M}_5^{8,14}$	3	$\mathcal{M}_5^{20,2}$	1	$\mathcal{M}_5^{160}$	0
$\mathcal{M}_5^{4,9}$	5	$\mathcal{M}_5^{8,15}$	3	$\mathcal{M}_5^{22}$	0	$\mathcal{M}_5^{196}$	0
$\mathcal{M}_5^{4,10}$	4	$\mathcal{M}_5^{8,17}$	3	$\mathcal{M}_5^{24,2}$	1		
$\mathcal{M}_5^{4,11}$	4	$\mathcal{M}_5^{8,18}$	3	$\mathcal{M}_5^{24,6}$	1		

3

The Orbifold Structure of the Moduli Space

Here we will compute the structure of the equisymmetric stratification of the branch locus. Remember that $G = Aut(X)$ determines a conjugacy class of subgroups of $M(\Gamma)$, defining a symmetry type of X [5]. $\mathcal{M}_g^G$ is the set of classes of surfaces with full automorphism group inducing the symmetry G and $\overline{\mathcal{M}}_g^G$ is the set of surfaces such that the automorphisms group contains a subgroup in the class defined by G. If G contains a subgroup G' then $\overline{\mathcal{M}}_g^G \subset \overline{\mathcal{M}}_g^{G'}$. The action of G determines a Fuchsian group Δ' and an epimorphism $\theta' : \Delta' \to G$. Letting $\Delta' = \theta'^{-1}(G')$ and examining the monodromy of the covering

$$\mathcal{U}/\Delta \to \mathcal{U}/\Delta'$$

we can find the induced action $\theta : \Delta \to H$ by the use of Theorem 1.2. Now as we have a one to one correspondence of the cosets of H and the cosets of Δ it is enough to study the permutations in the finite group G. As in Chapter 2, all the group symbolic calculations in finding the induced actions have been done with GAP.

We begin with a theorem giving the structure of the branch locus in terms of strata corresponding to actions of prime order.

Theorem 3.1. *[1] The branch locus is contained in*

$$\overline{\mathcal{M}}_5^{2,0} \cup \overline{\mathcal{M}}_5^{2,1} \cup \overline{\mathcal{M}}_5^{2,2} \cup \overline{\mathcal{M}}_5^{2,3} \cup \overline{\mathcal{M}}_5^{3,0} \cup \overline{\mathcal{M}}_5^{3,1} \cup \overline{\mathcal{M}}_5^{11,1}.$$

Proof: We use Theorem 1.12 to prove the statement.

(1) $\mathcal{M}_5^{2,0}$, $\mathcal{M}_5^{2,1}$, $\mathcal{M}_5^{2,2}$ and $\mathcal{M}_5^{2,3}$ correspond to epimorphisms $\theta : \Delta \to C_2$ with signatures $s(\Delta_0) = (0; 2, \overset{12}{\ldots}, 2)$, $s(\Delta_1) = (1; 2, \overset{8}{\ldots}, 2)$, $s(\Delta_2) = (2; 2, 2, 2, 2)$ and $s(\Delta_3) = (3; -)$ respectively.

(2) The strata $\mathcal{M}_5^{3,0}$ and $\mathcal{M}_5^{3,1}$ correspond to epimorphisms $\theta : \Delta \to C_3$ where $s(\Delta_0) = (0; 3, \overset{7}{\ldots}, 3)$ and $s(\Delta_1) = (1; 3, 3, 3, 3)$ respectively.

(3) $\overline{\mathcal{M}}_5^5$ is induced by non-maximal epimorphisms $\theta : \Delta \to C_5$, $s(\Delta) = (1; 5, 5)$. They extend to surface kernel epimorphism $\phi : \Delta' \to D_5 = \langle a, s | a^5 = s^2 = (sa)^2 = 1 \rangle$,

$s(\Delta') = (0; 2, 2, 2, 2, 5)$, defined by (s, s, s, sa, a^{-1}). We see that $s(\phi^{-1}\langle a\rangle) = (1; 5, 5)$ and $s(\phi^{-1}\langle s\rangle) = (2; 2, 2, 2, 2)$. Thus $\overline{\mathcal{M}}_5^5 \equiv \overline{\mathcal{M}}_5^{D_5,\theta} \subset \overline{\mathcal{M}}_5^{2,2}$.

(4) Signature $(0; 11, 11, 11)$. There are two classes of actions of C_{11} with representatives $\theta_{11,1} : \Delta \to C_{11}$, defined by (a, a^2, a^{-3}) and $\theta_{11,2} : \Delta \to C_{11}$, defined by (a, a, a^{-2}). Now $\theta_{11,2}$ extends to $\phi : \Delta(0; 2, 11, 22) \to C_{22}$ defined by (b^{11}, b^{10}, b). By Theorem 1.2 $\phi^{-1}\left\langle b^2\right\rangle$ is a group with signature $(0; 11, 11, 11)$ and the images of the elliptic generators by ϕ (with the isomorphism $b^2 \to a$) are a, a and a^{-2}. So $\overline{\mathcal{M}}_5^{11,2} \equiv \overline{\mathcal{M}}_5^{22}$. By Theorem 1.2 $s(\phi_2^{-1}\left\langle b^{11}\right\rangle) = (0; 2, \overset{12}{\ldots}, 2)$, thus $\overline{\mathcal{M}}_5^{22} \subset \overline{\mathcal{M}}_5^{C_2,0}$. The epimorphism $\theta_{11,1}$ yield a maximal action of C_{11} in $\mathcal{M}_5$ producing an isolated point $\overline{\mathcal{M}}_5^{11,1}$. □

3.1 Surfaces with Automorphisms of Order a Power of 2

We begin with the strata corresponding to actions of 2-groups. We show the inclusion relations of the strata in decreasing order of the number of automorphisms of the surfaces. From [1] it follows that those strata belong to the same connected component of the branch locus.

Theorem 3.2.

$$\overline{\mathcal{M}}_5^{64,2} \subset \overline{\mathcal{M}}_5^{32,1} \cap \overline{\mathcal{M}}_5^{16,16} \cap \overline{\mathcal{M}}_5^{8,2} \cap \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,5}$$

Proof: Maximal subgroups of $((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2$ are; $\left\langle t, s, a^2 s\right\rangle \simeq (C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$, $\langle at, s\rangle \simeq ((C_4 \times C_2) \rtimes C_2) \rtimes C_2$ and $\left\langle a^2 s, a\right\rangle \simeq ((C_8 \rtimes C_2) \rtimes C_2)$. Now $\theta_{64,2}$ induces actions $\theta_{32,1}$, $\theta_{32,5}$ and $\theta_{32,7}$, where actions $\theta_{32,5}$ and $\theta_{32,7}$ are non-maximal.

(1) Consider the action $\theta_{32,5} : \Delta(0; 4, 4, 4) \to ((C_4 \times C_2) \rtimes C_2) \rtimes C_2$. Maximal subgroups of $((C_4 \times C_2) \rtimes C_2) \rtimes C_2$ are $\left\langle s, (st)^2\right\rangle, \left\langle st, s^2\right\rangle \simeq G_{4,4}$ and $\left\langle s^2 t, t, [t, s^{-1}]\right\rangle \simeq D_4 \times C_2$. The permutation representations of the monodromy of the coverings induced are given by the following table:

	$\left\langle s, (st)^2\right\rangle$	$\left\langle st, s^2\right\rangle$	$\left\langle s^2 t, t, [t, s^{-1}]\right\rangle$
$s \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$st \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$ts^2 \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$

From the table above we see that $s(\theta_{32,5}^{-1}\left\langle s, (st)^2\right\rangle) = (0; 2, 2, 4, 4)$ and $((st)^2, (ts^2)^2, tst, s) \simeq (s(st)^2 s, s(st)^2 s^{-1}(st)^2, s, s(st)^2)$ which corresponds to the action $\theta_{16,16} : \Delta(0; 2, 2, 4, 4) \to G_{4,4}$ as in lemma 2.5. We also see that $s(\theta_{32,5}^{-1}\left\langle st, s^2\right\rangle) = (0; 2, 2, 4, 4)$ and $(s^2, (ts^2)^2, ts, st) \simeq ((st)s^2(st), (st)s^2(st)^{-1}s^2, st, (st)s^2)$ which also corresponds to the action $\theta_{16,16}$. Finally $s(\theta_{32,5}^{-1}\left\langle s^2 t, t, [t, s^{-1}]\right\rangle) = (0; 2, 2, 4, 4)$, inducing the action $\theta_{16,7} : \Delta(0; 2, 2, 4, 4) \to D_4 \times C_2$ which is non-maximal.

(2) Consider the action $\theta_{16,7} : \Delta(0; 2, 2, 4, 4) \to D_4 \times C_2$. Maximal Subgroups of $D_4 \times C_2$ are $\langle a, b\rangle \simeq C_4 \times C_2$, $\left\langle a^2, s, b\right\rangle$, $\left\langle a^2, sa, b\right\rangle \simeq C_2 \times C_2 \times C_2$ and $\langle a, s\rangle$, $\langle ab, s\rangle$, $\langle a, sb\rangle$, $\langle ab, sb\rangle \simeq D_4$. The permutation representations of the monodromy of the coverings induced are given by the following table:

	$\langle a,b\rangle$	$\langle a^2,s,b\rangle$	$\langle a^2,sa,b\rangle$
$s\rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$sb\rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$ab\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$a^3\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$

Now $s(\theta_{16,7}^{-1}\langle a,b\rangle)=(0;4,4,4,4)$ and (ab,a^3b,a^3,a) corresponds to the action $\theta_{8,10}:\Delta(0;4,4,4,4)\rightarrow C_4\times C_2$ in Lemma 2.4. Further $s(\theta_{16,7}^{-1}\langle a^2,s,b\rangle)=(0;2,2,2,2,2,2,2)$ and (s,s,sb,sb,a^2,a^2) corresponds to the action $\theta_{8,14}:\Delta(0;2,\overset{6}{\ldots},2)\rightarrow C_2\times C_2\times C_2$. Finally $s(\theta_{16,7}^{-1}\langle a^2,sa,b\rangle)=(1;2,2)$, inducing the action $\theta_{8,16}:\Delta(1;2,2)\rightarrow C_2\times C_2\times C_2$. $\theta_{8,10}$ and $\theta_{8,16}$ are non-maximal, for $\theta_{8,16}$ see Theorem 3.6. For the rest of the subgroups the permutation representations of the monodromy of the coverings induced are given by the following table:

	$\langle a,s\rangle$	$\langle ab,s\rangle$	$\langle a,sb\rangle$	$\langle ab,sb\rangle$
$s\rightarrow$	$(1)(2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$sb\rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$
$ab\rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$
$a^3\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$

From the table above we see that $s(\theta_{16,7}^{-1}\langle a,s\rangle)=s(\theta_{16,7}^{-1}\langle ab,s\rangle)=s(\theta_{16,7}^{-1}\langle a,sb\rangle)=s(\theta_{16,7}^{-1}\langle ab,sb\rangle)=(0;2,2,2,4,4)$, inducing the action $\theta_{8,19}:\Delta(0;2,2,2,4,4)\rightarrow D_4$.

(3) Consider the action $\theta_{8,10}:\Delta(0;4,4,4,4)\rightarrow C_4\times C_2$. Maximal subgroups of $C_4\times C_2$ are $\langle a\rangle,\langle ab\rangle\simeq C_4$ and $\langle a^2,b\rangle,\langle ab\rangle\simeq C_2\times C_2$. The permutation representations of the monodromy of the coverings induced are given by the following table:

	$\langle a\rangle$	$\langle ab\rangle$	$\langle a^2,b\rangle$
$a\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$a^3\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$ab\rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$a^3b\rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$

By Theorem 1.2 we see that $s(\theta_{8,10}^{-1}\langle a\rangle)=s(\theta_{8,10}^{-1}\langle ab\rangle)=(0;2,2,4,4,4,4)$, inducing the action $\theta_{4,2}$. Also, $s(\theta_{8,10}^{-1}\langle a^2,b\rangle)=(1;2,2,2,2)$, inducing the action $\theta_{4,5}$.

(4) Consider the action $\theta_{32,7}:\Delta(0;2,8,8)\rightarrow(C_8\rtimes C_2)\rtimes C_2$. Maximal subgroups of $((C_8\rtimes C_2)\rtimes C_2)$ are $\langle s,[s,t]\rangle,\langle st,[s,t]\rangle\simeq C_8\rtimes_5 C_2$ and $\langle s^2,t,[s,t]\rangle\simeq D_4\times C_2$. The coset permutations are given in the following table:

	$\langle s,[s,t]\rangle$	$\langle st,[s,t]\rangle$	$\langle s^2,t,[s,t]\rangle$
$t\rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$s\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$s^{-1}t\rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$

Now $s(\theta_{32,7}^{-1}\langle s,[s,t]\rangle)=s(\theta_{32,7}^{-1}\langle st,[s,t]\rangle)=(0;4,8,8)$ inducing the action $\theta_{16,10}:\Delta(0;4,8,8)\rightarrow C_8\rtimes_5 C_2$ and $s(\theta_{32,7}^{-1}\langle s^2,t,[s,t]\rangle)=(0;2,2,4,4)$ inducing the action $\theta_{16,7}\Delta(0;2,2,4,4)\rightarrow D_4\times C_2$ in Lemma 2.5, where $\theta_{16,7}$ and $\theta_{16,10}$ are non-maximal, see Theorem 3.4.

(5) Consider the action $\theta_{16,10}$. Maximal subgroups of $C_8 \rtimes_5 C_2$ are $\langle a\rangle, \langle as\rangle \simeq C_8$ and $\langle a^2, s\rangle \simeq C_4 \times C_2$. (a^4, a^6, a^5, a) corresponds to the action $\theta_{8,2}$, (a^2, a^3, sa^2, sa^6) corresponds to the action $\theta_{8,10}$.

□

Theorem 3.3.

$$\overline{\mathcal{M}}_5^{32,1} \subset \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{16,5} \cap \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{8,19} \cap \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,5}$$

Proof: Maximal subgroups of $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$ are; $\langle b, c, (ab)^2, (ac)^2\rangle \simeq C_2 \times C_2 \times C_2 \times C_2$, $\langle a, b, (ac)^2\rangle$, $\langle ac, c, (ab)^2\rangle$, $\langle a, bc, (ac)^2\rangle \simeq D_4 \times C_2$ and $\langle ab, c\rangle$,$\langle ac, b\rangle$, $\langle ac, bc\rangle \simeq G_{4,4}$. For subgroups G of $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$ the monodromies of the coverings $\mathcal{U}/\theta_{32,1}^{-1}(G) \to \mathcal{U}/\Delta(0; 2, 2, 2, 4)$ are given by:

	$\langle a, b, (ac)^2\rangle$	$\langle a, c, (ab)^2\rangle$	$\langle a, bc, (ac)^2\rangle$
$cac \to$	$(1)(2)$	$(1)(2)$	$(1)(2)$
$b \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$bc \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$ac \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$

From the table above we see that $s(\theta_{32,1}^{-1}\langle a, b, (ac)^2\rangle) = (0; 2, 2, 2, 2, 2)$ and $(b, b, a, cac, (ac)^2) = (b, b, a, a(ac)^2, (ac)^2)$ corresponds to the action $\theta_{16,5} : \Delta(0; 2, 2, 2, 2, 2) \to D_4 \times C_2$. Also, $s(\theta_{32,1}^{-1}\langle ac, c, (ab)^2\rangle) = (0; 2, 2, 4, 4)$ inducing the non-maximal action $\theta_{16,7} : \Delta(0; 2, 2, 4, 4) \to D_4 \times C_2$, see Theorem 3.2 for calculations. Finally $s(\theta_{32,1}^{-1}\langle a, bc, (ac)^2\rangle) = (0; 2, 2, 2, 2, 2)$ and the generating vector $(bc, bc, a, cac, (ac)^2) = (bc, bc, a, a(ac)^2, (ac)^2)$ corresponds to the action $\theta_{16,3} : \Delta(0; 2, 2, 2, 2, 2) \to C_2 \times C_2 \times C_2 \times C_2$. The permutations of the cosets of the rest of the subgroups are given in the following table:

	$\langle ab, c\rangle$	$\langle ac, b\rangle$	$\langle ac, bc\rangle$
$cac \to$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$
$b \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$bc \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$ac \to$	$(1\ 2)$	$(1)(2)$	$(1)(2)$

Now $s(\theta_{32,1}^{-1}\langle ab, c\rangle) = (1; 2)$, inducing the action $\theta_{16,17} : \Delta(1; 2) \to G_{4,4}$. We also see that $s(\theta_{32,7}^{-1}\langle ac, b\rangle) = (0; 2, 2, 4, 4)$ and the generating vector (b, b, ac, ca) corresponds to the action $\theta_{16,15} : \Delta(0; 2, 2, 4, 4) \to G_{4,4}$, similarly $s(\theta_{32,1}^{-1}\langle ac, bc\rangle) = (0; 2, 2, 4, 4)$ and the generating vector (bc, bc, ac, ca) corresponds to the action $\theta_{16,15}$. Both $\theta_{16,15}$ and $\theta_{16,17}$ are non-maximal.

(1) Consider the actions $\theta_{16,15}$ and $\theta_{16,17}$. Maximal subgroups of $G_{4,4}$ are $\langle a, [a, b]\rangle$, $\langle ab, a^2\rangle \simeq C_4 \times C_2$ and $\langle a^2, b, [a, b]\rangle \simeq C_2 \times C_2 \times C_2$. The permutations of the cosets by the actions $\theta_{16,15}$ and $\theta_{16,17}$ are given in the following table:

	$\langle a,[a,b]\rangle$	$\langle ab,a^2\rangle$	$\langle a^2,b,[a,b]\rangle$
$b\rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$b\rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$a\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$a^3\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$[a,b]$	$(1)(2)$	$(1)(2)$	$(1)(2)$

From the table above we see that $s(\theta_{16,15}^{-1}\langle a,[a,b]\rangle)=(0;4,4,4,4)$ and the generating vector (a,a^3,bab,ba^3b) corresponds to the action $\theta_{8,10}:\Delta(0;4,4,4,4)\rightarrow C_4\times C_2$ which is non-maximal, for calculations, see Theorem 3.2. $s(\theta_{16,15}^{-1}\langle ab,a^2\rangle)=(1;2,2)$ and (a^2,a^2) corresponds to the non-maximal action $\theta_{8,12}:\Delta(1;2,2)\rightarrow C_4\times C_2$, see Theorem 3.9. Also, $s(\theta_{16,15}^{-1}\langle a^2,b,[a,b]\rangle)=(0;2,2,2,2,2,2)$ and (b,b,a^2,a^2,aba,aba) corresponds to the action $\theta_{8,14}$. For $\theta_{16,17}$ we see from the table that $s(\theta_{16,17}^{-1}\langle a,[a,b]\rangle)=s(\theta_{16,17}^{-1}\langle ab,a^2\rangle)=s(\theta_{16,17}^{-1}\langle a^2,b,[a,b]\rangle)=(1;2,2)$ and $([a,b],[a,b])$ correspond to the actions $\theta_{8,11}:\Delta(1;2,2)\rightarrow C_4\times C_2$ and $\theta_{8,16}:\Delta(1;2,2)\rightarrow C_2\times C_2\times C_2$. $\theta_{8,11}$ and $\theta_{8,16}$ are non-maximal, see Theorems 3.8 and 3.6. □

Theorem 3.4.

$$\overline{\mathcal{M}}_5^{32,2}\subset\overline{\mathcal{M}}_5^{16,5}\cap\overline{\mathcal{M}}_5^{16,6}\cap\overline{\mathcal{M}}_5^{8,6}\cap\overline{\mathcal{M}}_5^{8,14}\cap\overline{\mathcal{M}}_5^{8,15}\cap\overline{\mathcal{M}}_5^{4,2}\cap\overline{\mathcal{M}}_5^{4,5}$$

Proof: Maximal subgroups of $(C_4\times C_2\times C_2)\rtimes C_2$ are; $\langle a,c,(bc)^2\rangle\simeq C_4\times C_2\times C_2$, $\langle a,b,(bc)^2\rangle$, $\langle bc,c,a^2\rangle$, $\langle bac,c,a^2\rangle\simeq D_4\times C_2$, $\langle a,bc\rangle\simeq C_4\rtimes C_4$, $\langle ac,b\rangle$ and $\langle ac,ba\rangle\simeq G_{4,4}$. For subgroups G of $(C_4\times C_2\times C_2)\rtimes C_2$ the monodromies of the coverings $\mathcal{U}/\theta_{32,2}^{-1}(G)\rightarrow\mathcal{U}/\Delta(0;2,2,2,4)$ are given by:

	$\langle a,c,(bc)^2\rangle$	$\langle a,bc\rangle$	$\langle ac,b\rangle$	$\langle ac,ba\rangle$
$ba\rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$b\rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$c\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$
$ac\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$

From the table above we see that $s(\theta_{32,2}\langle a,c,(bc)^2\rangle)=(0;2,2,4,4)$, inducing the action $\theta_{16,2}$. $s(\theta_{32,2}\langle a,bc\rangle)=(1;2)$, inducing the action $\theta_{16,12}$. $s(\theta_{32,2}\langle ac,b\rangle)=(0;2,2,4,4)$ and the generating vectors $(a^{-1}ba,b,ac,ac)=((ac)^2b,b,ac,ac)$ corresponds to the action $\theta_{16,14}$. $s(\theta_{32,2}\langle ac,ba\rangle)=(0;2,2,4,4)$ and the generating vector $(a^{-1}ba^2,ba,ac,ac)=((ac)^2ba,ba,ac,ac)$ corresponds to the action $\theta_{16,14}$. $\theta_{16,2}$, $\theta_{16,12}$ and $\theta_{16,14}$ are non-maximal. For the last three subgroups we have:

	$\langle a,b,(bc)^2\rangle$	$\langle bc,c,a^2\rangle$	$\langle bac,c,a^2\rangle$
$ba\rightarrow$	$(1)(2)$	$(1\ 2)$	$(1)(2)$
$b\rightarrow$	$(1)(2)$	$(1)(2)$	$(1\ 2)$
$c\rightarrow$	$(1\ 2)$	$(1)(2)$	$(1)(2)$
$ac\rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$

We have the signature $s(\theta_{32,2}\langle a,b,(bc)^2\rangle)=(0;2,2,2,2,2)$, by Theorem 1.2, and the generating vector $(b,ba,cbc,cbac,a^2)=(b,ba,b(bc)^2,ba(bc)^2,a^2)$ corresponds to the

action $\theta_{16,6} : \Delta(0;2,2,2,2,2) \to D_4 \times C_2$. We do also see that $s(\theta_{32,2}\left\langle bc, c, (bc)^2\right\rangle) = (0;2,2,2,2,2)$ and the generating vector $(b, b, c, a^{-1}ca, a^2) = (b, b, c, ca^2, a^2)$ corresponds to the action $\theta_{16,5} : \Delta(0;2,2,2,2,2) \to D_4 \times C_2$. $s(\theta_{32,2}\left\langle bac, c, (bc)^2\right\rangle) = (0;2,2,2,2,2)$, which also induce the action $\theta_{16,5}$.

(1) Consider the action $\theta_{16,2} : \Delta(0;2,2,4,4) \to C_4 \times C_2 \times C_2$. Maximal subgroups of $C_4 \times C_2 \times C_2$ are: $\langle a, b\rangle, \langle a, c\rangle, \langle a, bc\rangle, \langle ac, b\rangle, \langle ab, c\rangle, \langle ab, bc\rangle \simeq C_4 \times C_2$ and $\left\langle a^2, b, c\right\rangle \simeq C_2 \times C_2 \times C_2$. $\theta_{16,2}$ is represented by the generating vector (b, c, a, a^3bc) and since the group is abelian it is easy to see that the subgroups induce the action $\theta_{8,6} : \Delta(0;2,2,2,4,4) \to C_4 \times C_2$, the non-maximal action $\theta_{8,9} : \Delta(0;4,4,4,4) \to C_4 \times C_2$, the non-maximal action $\theta_{8,12} : \Delta(1;2,2) \to C_4 \times C_2$, see Theorem 3.9, and the action $\theta_{8,14}$.

(2) Consider the action $\theta_{8,9}$. Maximal subgroups of $C_4 \times C_2$ are $\langle a\rangle, \langle ab\rangle \simeq C_4$ and $\left\langle a^2, b\right\rangle \simeq C_2 \times C_2$. We see that $s(\theta_{8,9}^{-1}\langle a\rangle) = s(\theta_{8,9}^{-1}\langle ab\rangle) = (0;2,2,4,4,4,4)$, inducing the action $\theta_{4,2}$ and that $s(\theta_{8,9}^{-1}\left\langle a^2, b\right\rangle) = (1;2,2,2,2)$, inducing the action $\theta_{4,5}$.

(3) Consider the action $\theta_{16,12} : \Delta(1;2) \to C_4 \rtimes C_4$. Maximal subgroups of $C_4 \rtimes C_4$ are $\left\langle s, t^2\right\rangle$, $\left\langle t, s^2\right\rangle$, $\left\langle st, s^2\right\rangle \simeq C_4 \times C_2$. (s^2, s^2) corresponds to the action $\theta_{8,12}$,$\theta_{8,11}$ and $\theta_{8,11}$ respectively. $\theta_{8,11}$ and $\theta_{8,12}$ are non-maximal, see Theorems 3.8 and 3.9 for calculations.

(4) Consider the action $\theta_{16,14} : \Delta(0;2,2,4,4) \to G_{4,4}$. Maximal subgroups of $G_{4,4}$ are $\langle a, [a, b]\rangle, \left\langle ab, a^2\right\rangle \simeq C_4 \times C_2$ and $\left\langle a^2, b, [a, b]\right\rangle \simeq C_2 \times C_2 \times C_2$. The permutations of the cosets are:

	$\langle a, [a, b]\rangle$	$\left\langle ab, a^2\right\rangle$	$\left\langle a^2, b, [a, b]\right\rangle$
$a^2b \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$b \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$a \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$a \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$

From the table we see that $s(\theta_{16,14}^{-1}\langle a, [a, b]\rangle) = (0;4,4,4,4)$ and (a, a, bab, bab) corresponds to the action $\theta_{8,9} : \Delta(0;4,4,4,4) \to C_4 \times C_2$. We also see that $s(\theta_{16,14}^{-1}\left\langle ab, a^2\right\rangle) = (1;2,2)$ and (a^2, a^2) corresponds to the non-maximal action $\theta_{8,11} : \Delta(1;2,2) \to C_4 \times C_2$, see Theorem 3.8 for calculations. Finally $s(\theta_{16,14}^{-1}\left\langle a^2, b, [a, b]\right\rangle) = (0;2,2,2,2,2,2,2)$ and $(a^2, a^2, b, a^2b, aba^3, a^3ba^3)$ corresponds to the action $\theta_{8,15} : \Delta(0;2,\overset{6}{\ldots},2) \to C_2 \times C_2 \times C_2$. $\square$

Theorem 3.5.

$$\overline{\mathcal{M}}_5^{32,3} \subset \overline{\mathcal{M}}_5^{16,6} \cap \overline{\mathcal{M}}_5^{8,7} \cap \overline{\mathcal{M}}_5^{8,18} \cap \overline{\mathcal{M}}_5^{8,19} \cap \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,5}$$

Proof: Maximal Subgroups of $(D_4 \times C_2) \rtimes C_2$ are; $\langle ab, a\rangle, \langle abc, a\rangle \simeq D_8$, $\left\langle (ab)^2, b, c\right\rangle \simeq D_4 \times C_2$, $\langle acb, b\rangle, \langle ab, bc\rangle \simeq C_8 \rtimes_3 C_2$, $\left\langle (ab)^2, a, c\right\rangle \simeq (C_4 \times C_2) \rtimes C_2$ and $\langle ab, c\rangle \simeq C_8 \times_5 C_2$. For subgroups G of $(D_4 \times C_2) \rtimes C_2$ the monodromies of the coverings $\mathcal{U}/\theta_{32,3}^{-1}(G) \to \mathcal{U}/\Delta(0;2,2,2,4)$ are given by:

	$\langle ab, a\rangle$	$\langle abc, a\rangle$	$\left\langle (ab)^2, b, c\right\rangle$
$abac$	$(1\ 2)$	$(1)(2)$	$(1)(2)$
a	$(1)(2)$	$(1)(2)$	$(1\ 2)$
b	$(1)(2)$	$(1\ 2)$	$(1)(2)$
ac	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$

From the table above we see that $s(\theta_{32,3}^{-1}\langle ab, a\rangle) = s(\theta_{32,3}^{-1}\langle abc, a\rangle) = (0; 2, 2, 2, 2, 2)$, inducing the action $\theta_{16,4}$. We also see that $s(\theta_{32,3}^{-1}\left\langle (ab)^2, b, c\right\rangle) = (0; 2, 2, 2, 2, 2)$ and the generating vector $(b, aba, abac, acab, (ac)^2) = (b, b(ab)^6, b(ab)^6c, b(ab)^4c, (ab)^4)$ corresponds to the action $\theta_{16,6} : \Delta(0; 2, 2, 2, 2, 2) \to D_4 \times C_2$ in Lemma 2.5. For the rest of the subgroups the monodromies are:

	$\langle acb, b\rangle$	$\langle ab, bc\rangle$	$\left\langle (ab)^2, a, c\right\rangle$	$\langle ab, c\rangle$
$abac$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
a	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
b	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$
ac	$(1)(2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$

From the table above we see that $s(\theta_{32,3}^{-1}\langle acb, b\rangle) = s(\theta_{32,3}^{-1}\langle ab, bc\rangle) = (0; 2, 2, 4, 4)$, inducing the action $\theta_{16,8} : \Delta(0; 2, 2, 4, 4) \to C_8 \rtimes_3 C_2$ in Lemma 2.5. Further we see $s(\theta_{32,3}^{-1}\left\langle (ab)^2, a, c\right\rangle) = (0; 2, 2, 4, 4)$, inducing the action $\theta_{16,9}$. $s(\theta_{32,3}^{-1}\langle ab, c\rangle) = (1; 2)$, inducing the action $\theta_{16,11}$. $\theta_{16,8}$,$\theta_{16,9}$ and $\theta_{16,11}$ are non-maximal.

(1) Consider the action $\theta_{16,8} : \Delta(0; 2, 2, 4, 4) \to C_8 \rtimes_3 C_2$. Maximal subgroups of $C_8 \rtimes_3 C_2$ are $\langle a\rangle \simeq C_8$, $\left\langle a^2, s\right\rangle \simeq D_4$ and $\left\langle a^2, as\right\rangle \simeq Q$. The coset permutations are:

	$\langle a\rangle$	$\left\langle a^2, s\right\rangle$	$\left\langle a^2, as\right\rangle$
s	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
s	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
as	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$(as)^3$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$

Now we see that $s(\theta_{16,8}^{-1}\langle a\rangle) = (1; 2, 2)$, inducing the action $\theta_{8,3} : \Delta(1; 2, 2) \to C_8$, which is non-maximal. For calculations, see Theorem 3.7. We also see that $s(\theta_{16,8}^{-1}\left\langle a^2, s\right\rangle)$ $= (0; 2, 2, 2, 2, 2, 2)$ and $(s, s, sa^2, sa^2, a^4, a^4)$ corresponds to the action $\theta_{8,18} : \Delta(0; 2, \overset{6}{\ldots}, 2) \to D_4$. $s(\theta_{16,8}^{-1}\left\langle a^2, as\right\rangle) = (0; 4, 4, 4, 4)$, inducing the action $\theta_{8,21} : \Delta(0; 4, 4, 4, 4)$ $\to Q$, which is non-maximal.

(2) Consider the action $\theta_{8,21}$. Maximal subgroups of Q are $\langle i\rangle, \langle j\rangle, \langle k\rangle \simeq C_4$. We find that $s(\theta_{8,21}^{-1}\langle i\rangle) = s(\theta_{8,21}^{-1}\langle i\rangle) = (0; 2, 2, 4, 4, 4, 4)$, inducing the action $\theta_{4,2}$ and $s(\theta_{8,21}\langle k\rangle) = (1; 2, 2, 2, 2)$, inducing the action $\theta_{4,5}$.

(3) Consider the action $\theta_{16,9} : \Delta(0; 2, 2, 4, 4) \to (C_4 \times C_2) \rtimes C_2$. Maximal subgroups of $(C_4 \times C_2) \rtimes C_2$ are; $\langle a, b\rangle, \langle a, c\rangle, \langle a, abc\rangle \simeq C_4 \times C_2$, $\langle ab, c\rangle, \langle ac, b\rangle, \langle bc, b\rangle \simeq D_4$ and $\langle ab, ac\rangle \simeq Q$. The permutations of the cosets are:

	$\langle a, b\rangle$	$\langle a, c\rangle$	$\langle a, abc\rangle$	$\langle ab, c\rangle$	$\langle ac, b\rangle$	$\langle bc, b\rangle$	$\langle ab, ac\rangle$
b	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$
c	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
ab	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
ac	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$

From the table above we see that $s(\theta_{16,9}^{-1}\langle a,b\rangle) = s(\theta_{16,9}^{-1}\langle a,c\rangle) = (0;2,2,2,4,4)$ and (a^2,a^2b,b,ab,a^3b) corresponds to the action $\theta_{8,7}$. We also see that $s(\theta_{16,9}^{-1}\langle a,abc\rangle) = (1;2,2)$, inducing the non-maximal action $\theta_{8,12}$, see Theorem 3.9 for calculations. Now $s(\theta_{16,9}^{-1}\langle ab,c\rangle) = s(\theta_{16,9}^{-1}\langle ab,c\rangle) = (0;2,2,2,4,4)$, inducing the action $\theta_{8,19}$, $s(\theta_{16,9}^{-1}\langle bc,b\rangle) = (0;2,2,2,2,2,2)$ and $(b,b,c,c,(bc)^2,(bc)^2)$ corresponds to the action $\theta_{8,18}$ and for the last subgroup we have $s(\theta_{16,9}^{-1}\langle ab,ac\rangle) = (0;4,4,4,4)$, inducing the action $\theta_{8,21}$.

(3) Consider the action $\theta_{16,11} : \Delta(1;2) \to C_8 \rtimes_5 C_2$. Maximal subgroups of $C_8 \rtimes_5 C_3$ are $\langle a\rangle, \langle as\rangle \simeq C_8$ and $\langle a^2,s\rangle \simeq C_4 \times C_2$. The monodromies of the desired coverings are

	$\langle a\rangle$	$\langle as\rangle$	$\langle a^2,s\rangle$
a^4	$(1)(2)$	$(1)(2)$	$(1)(2)$

and $\theta_{16,11}$ induces $\theta_{8,3}$ and $\theta_{8,12}$, both are non-maximal, see Theorems 3.7 and 3.9 for calculations. □

Theorem 3.6.

$$\overline{\mathcal{M}}_5^{16,3} \subset \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{4,11}$$

Proof: There are 15 maximal subgroups of $C_2 \times C_2 \times C_2 \times C_2$, all isomorphic to $C_2 \times C_2 \times C_2$. Since the groups are abelian, the only possible actions are $\theta_{8,14}$ and $\theta_{8,16}$. $s(\theta_{16,3}^{-1}\langle a,b,c\rangle) = (0;2,2,2,2,2,2)$ and $s(\theta_{16,3}^{-1}\langle ad,bd,cd\rangle) = (1;2,2)$ so both are induced and $\theta_{8,16}$ is non-maximal.

(1) Consider the action $\theta_{8,16}$. All maximal subgroups are isomorphic to $C_2 \times C_2$. By commutativity the only possible actions are $\theta_{4,11}$ and $\theta_{4,12}$. $s(\theta_{8,16}^{-1}\langle a,b\rangle) = (1;2,2,2,2)$ and $s(\theta_{8,16}^{-1}\langle b,c\rangle) = (2;-)$ so both are induced. □

Theorem 3.7.

$$\overline{\mathcal{M}}_5^{16,4} \subset \overline{\mathcal{M}}_5^{8,18} \cap \overline{\mathcal{M}}_5^{4,5}$$

Proof: Maximal subgroups of D_8 are $\langle a\rangle \simeq C_8$ and $\langle a^2,s\rangle, \langle a^2,sa\rangle \simeq D_4$. $s(\theta_{16,4}^{-1}\langle a\rangle) = (1;2,2)$, inducing the action $\theta_{8,3}$. $s(\theta_{16,4}^{-1}\langle a^2,s\rangle) = s(\theta_{16,4}^{-1}\langle a^2,sa\rangle) = (0;2,2,2,2,2,2)$, inducing the action $\theta_{8,18}$. $\theta_{8,3}$ is non-maximal.

(1) Consider the action $\theta_{8,3}$. The only maximal subgroup of C_8 is $\langle a^2\rangle \simeq C_4$ and $\theta_{8,3}$ induces $\theta_{4,5}$. □

Theorem 3.8.

$$\overline{\mathcal{M}}_5^{16,5} \subset \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{8,17} \cap \overline{\mathcal{M}}_5^{4,11}$$

Proof: Maximal subgroups of $D_4 \times C_2$ are; $\langle a,b\rangle \simeq C_4 \times C_2$, $\langle a^2,s,b\rangle, \langle a^2,sa,b\rangle \simeq C_2 \times C_2 \times C_2$ and $\langle a,s\rangle, \langle ab,s\rangle, \langle a,sb\rangle, \langle ab,sb\rangle \simeq D_4$. For subgroups G of $D_4 \times C_2$ the monodromies of the coverings $\mathcal{U}/\theta_{16,5}^{-1}(G) \to \mathcal{U}/\Delta(0;2,2,2,2,2)$ are given by:

	$\langle a,b\rangle$	$\langle a^2,s,b\rangle$	$\langle a^2,sa,b\rangle$	$\langle a,s\rangle$	$\langle ab,s\rangle$	$\langle a,sb\rangle$	$\langle ab,sb\rangle$
s	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
s	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
sa	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
sab	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$
b	$(1)(2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$

From the table above we see that $s(\theta_{16,5}^{-1}\langle a,b\rangle)=(1;2,2)$, inducing the non-maximal action $\theta_{8,11}$. We do also see that $s(\theta_{16,5}^{-1}\left\langle a^2,s,b\right\rangle)=s(\theta_{16,5}^{-1}\left\langle a^2,sa,b\right\rangle)=(0;2,2,2,2,2,2,2)$ and (s,s,sa^2,sa^2,b,b) and (sa,sa,sa^3,sa^2,b,b) correspond to $\theta_{8,14}$. $s(\theta_{16,5}^{-1}\langle a,s\rangle)=s(\theta_{16,5}^{-1}\langle ab,s\rangle)=(0;2,2,2,2,2,2,2)$, inducing the action $\theta_{8,17}$. Further, $s(\theta_{16,5}^{-1}\langle a,sb\rangle)=s(\theta_{16,5}^{-1}\langle ab,sb\rangle)=(1;2,2)$, inducing the action $\theta_{8,20}$ which is non-maximal.

(1) Consider the action $\theta_{8,11}$. Maximal subgroups are $\langle a\rangle,\langle ab\rangle\simeq C_4$ and $\left\langle a^2,b\right\rangle\simeq C_2\times C_2$. Thus $\theta_{8,11}$ induces $\theta_{4,6}$ and $\theta_{4,11}$.

(2) Consider the action $\theta_{8,20}$. Maximal subgroups of D_4 are $\langle a\rangle\simeq C_4$, $\left\langle a^2,s\right\rangle$ and $\left\langle a^2,sa\right\rangle\simeq C_2\times C_2$.Thus $\theta_{8,20}$ induces $\theta_{4,6}$, $\theta_{4,11}$ and $\theta_{4,12}$. □

Theorem 3.9.

$$\overline{\mathcal{M}}_5^{16,6}\subset\overline{\mathcal{M}}_5^{8,14}\cap\overline{\mathcal{M}}_5^{8,18}\cap\overline{\mathcal{M}}_5^{4,5}\cap\overline{\mathcal{M}}_5^{4,11}$$

Proof: Maximal subgroups of $D_4\times C_2$ are; $\langle a,b\rangle\simeq C_4\times C_2$, $\left\langle a^2,s,b\right\rangle,\left\langle a^2,sa,b\right\rangle\simeq C_2\times C_2\times C_2$ and $\langle a,s\rangle,\langle ab,s\rangle,\langle a,sb\rangle,\langle ab,sb\rangle\simeq D_4$. By considering the actions on the cosets of the subgroups we see that $s(\theta_{16,6}^{-1}\langle a,b\rangle)=(1;2,2)$, inducing the action $\theta_{8,12}$, $s(\theta_{16,6}^{-1}\left\langle a^2,s,b\right\rangle)=s(\theta_{16,6}^{-1}\left\langle a^2,sa,b\right\rangle)=(0;2,2,2,2,2,2,2)$ and the generating vectors (s,s,sb,sb,a^2,a^2) and (sa,sa,sab,sab,a^2,a^2) correspond to $\theta_{8,14}$. The action $\theta_{8,12}$ is non-maximal. Similarly $s(\theta_{16,6}^{-1}\langle a,s\rangle)=s(\theta_{16,6}^{-1}\langle ab,s\rangle)=s(\theta_{16,6}^{-1}\langle a,sb\rangle)=s(\theta_{16,6}^{-1}\langle ab,sb\rangle)=(0;2,2,2,2,2,2,2)$, inducing the action $\theta_{8,18}$.

(1) Consider the action $\theta_{8,12}$. Maximal subgroups are $\langle a\rangle,\langle ab\rangle\simeq C_4$ and $\left\langle a^2,b\right\rangle\simeq C_2\times C_2$. Thus $\theta_{8,12}$ induces $\theta_{4,5}$ and $\theta_{4,11}$. □

Theorem 3.10.

$$\overline{\mathcal{M}}_5^{16,13}\subset\overline{\mathcal{M}}_5^{8,4}\cap\overline{\mathcal{M}}_5^{8,8}\cap\overline{\mathcal{M}}_5^{8,13}$$

Proof: Maximal subgroups of $G_{4,4}$ are; $\langle a,[a,b]\rangle,\left\langle ab,a^2\right\rangle\simeq C_4\times C_2$ and $\left\langle a^2,b,[a,b]\right\rangle\simeq C_2\times C_2\times C_2$. For subgroups G of $G_{4,4}$ the monodromies of the coverings $\mathcal{U}/\theta_{16,13}^{-1}(G)\to\mathcal{U}/\Delta(0;2,2,4,4)$ are given by:

	$\langle a,[a,b]\rangle$	$\left\langle ab,a^2\right\rangle$	$\left\langle a^2,b,[a,b]\right\rangle$
$b\to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$a^2\to$	$(1)(2)$	$(1)(2)$	$(1)(2)$
$a\to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$ab\to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$

From the table we see that $s(\theta_{16,13}^{-1}\langle a,[a,b]\rangle)=(0;2,2,2,4,4)$ and the generating vector $(a^2,a^2,(ab)^2,bab,a)=(a^2,a^2,a^2[a,b],a[a,b],a)$ corresponds to $\theta_{8,8}$. We also see that $s(\theta_{16,13}^{-1}\left\langle ab,a^2\right\rangle)=(0;2,2,2,2,4,4)$ and $(a^2,a^2,a^2,ba,ab)=(a^2,a^2,a^2,(ab)^3a^2,ab)$ corresponds to $\theta_{8,4}$. Finally, $s(\theta_{16,13}^{-1}\left\langle a^2,b,[a,b]\right\rangle)=(0;2,2,2,2,2,2,2)$ and $(a^2,a^2,a^2,(ab)^2,aba^{-1},b)=(a^2,a^2,a^2,a^2[a,b],[a,b]b,b)$ corresponds to $\theta_{8,13}$. □

Theorem 3.11.

$$\overline{\mathcal{M}}_5^{16,16}\subset\overline{\mathcal{M}}_5^{8,5}\cap\overline{\mathcal{M}}_5^{8,15}$$

Proof: Maximal subgroups of $G_{4,4}$ are $\langle a,[a,b]\rangle,\left\langle ab,a^2\right\rangle\simeq C_4\times C_2$ and $\left\langle a^2,b,[a,b]\right\rangle\simeq C_2\times C_2\times C_2$. For subgroups G of $G_{4,4}$ the monodromies of the coverings $\mathcal{U}/\theta_{16,16}^{-1}(G)\to\mathcal{U}/\Delta(0;2,2,4,4)$ are given by:

	$\langle a,[a,b]\rangle$	$\langle ab,a^2\rangle$	$\langle a^2,b,[a,b]\rangle$
$aba \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$[a,b] \to$	$(1)(2)$	$(1)(2)$	$(1)(2)$
$a \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$ab \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$

From the table above we see that $s(\theta_{16,16}^{-1}\langle a,[a,b]\rangle) = (0;2,2,2,4,4)$ and $([a,b],[a,b],a^2[a,b],a[a,b],a)$ corresponds to the action $\theta_{8,5}$. Also, $s(\theta_{16,16}^{-1}\langle ab,a^2\rangle) = (0;2,2,2,4,4)$ and $([a,b],[a,b],a^2,ba,ab) = ((ab)^2a^2,(ab)^2a^2,a^2,(ab)^3a^2,ab)$ corresponds to the action $\theta_{8,5}$. Finally, we see that $s(\theta_{16,16}^{-1}\langle a^2,b,[a,b]\rangle) = (0;2,2,2,2,2,2)$ and $([a,b],[a,b],a^2,(ab)^2,ba^2,aba) = ([a,b],[a,b],a^2,a^2[a,b],ba^2,a^2b[a,b])$ corresponds to the action $\theta_{8,15}$. □

With the same algorithm used in the previous theorems it is easy to prove the following theorems regarding strata corresponding to actions of low order.

Theorem 3.12. *The subvariety inclusions for strata corresponding to automorphism groups of order* 8 *are:*

$$\overline{\mathcal{M}}_5^{8,1} \subset \overline{\mathcal{M}}_5^{4,1}$$
$$\overline{\mathcal{M}}_5^{8,2} \subset \overline{\mathcal{M}}_5^{4,2}$$
$$\overline{\mathcal{M}}_5^{8,4} \subset \overline{\mathcal{M}}_5^{4,4} \cap \overline{\mathcal{M}}_5^{4,7}$$
$$\overline{\mathcal{M}}_5^{8,5} \subset \overline{\mathcal{M}}_5^{4,4} \cap \overline{\mathcal{M}}_5^{4,9}$$
$$\overline{\mathcal{M}}_5^{8,6} \subset \overline{\mathcal{M}}_5^{4,1} \cap \overline{\mathcal{M}}_5^{4,8}$$
$$\overline{\mathcal{M}}_5^{8,7} \subset \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,9}$$
$$\overline{\mathcal{M}}_5^{8,8} \subset \overline{\mathcal{M}}_5^{4,3} \cap \overline{\mathcal{M}}_5^{4,7}$$
$$\overline{\mathcal{M}}_5^{8,13} \subset \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{2,3}$$
$$\overline{\mathcal{M}}_5^{8,14} \subset \overline{\mathcal{M}}_5^{4,8} \cap \overline{\mathcal{M}}_5^{4,11} \cap \overline{\mathcal{M}}_5^{2,3}$$
$$\overline{\mathcal{M}}_5^{8,15} \subset \overline{\mathcal{M}}_5^{4,9} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{4,11}$$
$$\overline{\mathcal{M}}_5^{8,17} \subset \overline{\mathcal{M}}_5^{4,10}$$
$$\overline{\mathcal{M}}_5^{8,18} \subset \overline{\mathcal{M}}_5^{4,5} \cap \overline{\mathcal{M}}_5^{4,9}$$
$$\overline{\mathcal{M}}_5^{8,19} \subset \overline{\mathcal{M}}_5^{4,2} \cap \overline{\mathcal{M}}_5^{4,9} \cap \overline{\mathcal{M}}_5^{4,11}$$

Theorem 3.13. *The subvariety inclusions for strata corresponding to automorphism groups of order* 4 *are:*

$$\overline{\mathcal{M}}_5^{4,1} \subset \overline{\mathcal{M}}_5^{2,1}$$
$$\overline{\mathcal{M}}_5^{4,2} \subset \overline{\mathcal{M}}_5^{2,1}$$
$$\overline{\mathcal{M}}_5^{4,3} \subset \overline{\mathcal{M}}_5^{2,0}$$
$$\overline{\mathcal{M}}_5^{4,4} \subset \overline{\mathcal{M}}_5^{2,2}$$
$$\overline{\mathcal{M}}_5^{4,5} \subset \overline{\mathcal{M}}_5^{2,1}$$
$$\overline{\mathcal{M}}_5^{4,7} \subset \overline{\mathcal{M}}_5^{2,0} \cap \overline{\mathcal{M}}_5^{2,2} \cap \overline{\mathcal{M}}_5^{2,3}$$
$$\overline{\mathcal{M}}_5^{4,8} \subset \overline{\mathcal{M}}_5^{2,1} \cap \overline{\mathcal{M}}_5^{2,3}$$
$$\overline{\mathcal{M}}_5^{4,9} \subset \overline{\mathcal{M}}_5^{2,1} \cap \overline{\mathcal{M}}_5^{2,2}$$
$$\overline{\mathcal{M}}_5^{4,10} \subset \overline{\mathcal{M}}_5^{2,2} \cap \overline{\mathcal{M}}_5^{2,3}$$
$$\overline{\mathcal{M}}_5^{4,11} \subset \overline{\mathcal{M}}_5^{2,1} \cap \overline{\mathcal{M}}_5^{2,3}$$

3.2 Surfaces with Automorphisms of Orders 2 and 3

Here we will give the structure of the strata of surfaces with automorphisms of orders 2 and 3. In [1] (chapter 4) we proved that all those strata are in the same connected component of the branch locus. We show the inclusion relations of the strata in decreasing order.

Theorem 3.14.

$$\overline{\mathcal{M}}_5^{192} \subset \overline{\mathcal{M}}_5^{48,5} \cap \overline{\mathcal{M}}_5^{32,2} \cap \overline{\mathcal{M}}_5^{8,1}$$

Proof: Maximal Subgroups of $(((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$ are isomorphic to $((C_4 \times C_2) \rtimes C_4) \rtimes C_3$, $((C_8 \times C_2) \rtimes C_2) \rtimes C_2$ or $S_4 \times C_2$. Each one induce exactly one class of actions. The actions $\theta_{96,2}$ and $\theta_{64,1}$ are non-maximal.

(1) Consider the action $\theta_{96,2} : \Delta(0;3,3,4) \to ((C_4 \times C_2) \rtimes C_4) \rtimes C_3$. Maximal Subgroups of $((C_4 \times C_2) \rtimes C_4) \rtimes C_3$ are isomorphic to $(C_4 \times C_2) \rtimes C_4$ and $A_4 \times C_2$. The action $\theta_{32,4}$ is non-maximal. There is one conjugacy class of subgroups isomorphic to $A_4 \times C_2$ represented by $\left\langle stst^3st^3, t^2s \right\rangle$. With cosets represented by the transversal $\{1, t^3, s^2t^3s, s^2t^3st^3\}$ we have the following permutation.

$$\begin{array}{ll} s \to & (1)(2\ 3\ 4) \\ s^2t^3 \to & (1\ 2\ 3)(4) \\ t \to & (1\ 2)(3\ 4) \end{array}$$

From the table we see that $s(\theta_{96,2}^{-1}\left\langle stst^3st^3, t^2s \right\rangle) = (0;2,2,3,3)$ and the generating vector $(b^2ab, b^2ab, ab^2, b^5ab^5) \simeq (ab^3, ab^3, b^2, b^4)$ which is the non-maximal action $\theta_{24,3} : \Delta(0;2,2,3,3) \to A_4 \times C_2$. See Theorem 3.18 for calculations.

(2) Consider the action $\theta_{64,1} : \Delta(2,4,8) \to ((C_8 \times C_2) \rtimes C_2) \rtimes C_2$. Maximal Subgroups of $((C_8 \times C_2) \rtimes C_2) \rtimes C_2$ are $\left\langle sa^6, t, s \right\rangle \simeq (C_4 \times C_2 \times C_2) \rtimes C_2$, $\langle at, ta \rangle \simeq (C_4 \times C_2) \rtimes C_4$ and $\langle a, s \rangle \simeq (C_8 \times C_2) \rtimes C_2$. The induced permutations of the cosets are:

	$\left\langle sa^6, t, s \right\rangle$	$\langle at, ta \rangle$	$\langle a, s \rangle$
$t \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$ta^{-1} \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$a \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$

inducing actions $\theta_{32,2} : \Delta(0;2,2,2,4) \to (C_4 \times C_2 \times C_2) \rtimes C_2$, $\theta_{32,4} : \Delta(0;4,4,4) \to (C_4 \times C_2) \rtimes C_4$ and $\theta_{32,6} : \Delta(0;2,8,8) \to (C_8 \times C_2) \rtimes C_2$. The actions $\theta_{32,4}$ and $\theta_{32,6}$ are non-maximal.

(3) Consider the action $\theta_{32,4} : \Delta(0;4,4,4) \to (C_4 \times C_2) \rtimes C_4$. Maximal Subgroups of $(C_4 \times C_2) \rtimes C_4$ are isomorphic to $C_4 \times C_2 \times C_2$ inducing $\theta_{16,2}$, which is non-maximal, see Theorem 3.4 for calculations.

(4) Consider the action $\theta_{32,6} : \Delta(0;2,8,8) \to (C_8 \times C_2) \rtimes C_2$. Maximal Subgroups of $(C_4 \times C_2) \rtimes C_4$ are $\left\langle s^2, t, [s,t] \right\rangle \simeq C_4 \times C_2 \times C_2$ and $\langle s, [s,t] \rangle$, $\left\langle st, s^3tst \right\rangle \simeq C_8 \times C_2$ inducing $\theta_{16,1}$ and $\theta_{16,2}$. The actions $\theta_{16,1}$ and $\theta_{16,2}$ are non-maximal. See Theorem 3.4 for calculations.

(3) Consider the action $\theta_{16,1} : \Delta(0;4,8,8) \to C_8 \times C_2$. Maximal subgroups of $C_8 \times C_2$ are $\langle a \rangle, \langle ab \rangle \simeq C_8$ and $\left\langle a^2, b \right\rangle \simeq C_4 \times C_2$, inducing $\theta_{8,1}$ and $\theta_{8,9}$, which is non-maximal. For calculations, see Theorem 3.4.

□

Theorem 3.15.

$$\overline{\mathcal{M}}_5^{96,1} \subset \overline{\mathcal{M}}_5^{48,5} \cap \overline{\mathcal{M}}_5^{32,1} \cap \overline{\mathcal{M}}_5^{24,7} \cap \overline{\mathcal{M}}_5^{24,2} \cap \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{16,13} \cap \overline{\mathcal{M}}_5^{12,10} \cap \overline{\mathcal{M}}_5^{6,3}$$

Proof: Maximal subgroups of $(A_4 \times C_2 \times C_2) \rtimes C_2$ are isomorphic to $S_4 \times C_2$, $A_4 \times C_2 \times C_2$, $A_4 \rtimes C_4$ of order 48, $(C_2 \times C_2 \times C_2 \times C_2 \times) \rtimes C_2$ of order 32 and $(C_6 \times C_2) \rtimes C_2$ of order 24. Thus $\theta_{96,1}$ induces $\theta_{48,5}$, $\theta_{48,2}$, $\theta_{32,1}$ and $\theta_{24,7}$. Now $\langle a^4, a^3cb \rangle \simeq A_4 \rtimes C_4$ and the induced permutations of the cosets are:

	$\langle a^4, a^3cb \rangle$
$b \rightarrow$	$(1\ 2)$
$bac \rightarrow$	$(1)(2)$
$ca^5 \rightarrow$	$(1\ 2)$

$((ca^5)^2, bac, a^3baca^3) \simeq (a^4, a^3cb, (a^3cb)^3a^2)$ which is $\theta_{48,3} : \Delta(0;3,4,4) \rightarrow A_4 \rtimes C_4$. The actions $\theta_{48,2}$ and $\theta_{48,3}$ are non-maximal.

(1) Consider the action $\theta_{48,2} : \Delta(0;2,6,6) \rightarrow A_4 \times C_2 \times C_2$. There are three subgroups isomorphic to $A_4 \times C_2$ given by $\langle [a,b], a \rangle$, $\langle [a,b], a^2b \rangle$ and $\langle [a,b], ab \rangle$. The monodromies of the desired covering are:

	$\langle [a,b], a \rangle$	$\langle [a,b], a^2b \rangle$	$\langle [a,b], ab \rangle$
$b \rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$a \rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$
$a^5b \rightarrow$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$

We see that $s(\theta_{48,2}^{-1} \langle [a,b], a \rangle) = s(\theta_{48,2}^{-1} \langle [a,b], ab \rangle) = (0;3,6,6)$ and this gives us the action $\theta_{24,4} : \Delta(0;3,6,6) \rightarrow A_4 \times C_2$. $s(\theta_{48,2}^{-1} \langle [a,b], a \rangle) = (0;2,2,3,3)$ and $(b, aba^5, a^2, (a^5b)^2) \simeq ([a,b](a^2b)^3, (a^2b)^3, [a,b]^2, [a,b]^4(a^2b))$ which is $\theta_{24,2}$. Further one subgroup is isomorphic to $C_2 \times C_2 \times C_2 \times C_2$, inducing $\theta_{16,3}$, and there is one conjugacy class isomorphic to $C_6 \times C_2$. This class is represented by $\langle ab, a^3 \rangle$. With cosets given by the transversal $\{1, a^2(ba^{-1})^2, [a,b], [b,a^{-1}]\}$ we have the following permutations of the cosets:

	$\langle ab, a^3 \rangle$
$b \rightarrow$	$(1\ 2)(3\ 4)$
$a \rightarrow$	$(1\ 2\ 4)(3)$
$a^5b \rightarrow$	$(1\ 3\ 4)(2)$

This gives the generating vector $(a^3, (a^5b)^3, [a,b]^{-1}a[a,b], (a^2(ba^{-1})^2))^{-1}a^5ba^2(ba^{-1})^2)$ $= (a^3, (ab)^3, (ab)^4a^3, (ab)^5)$ corresponds to non -maximal action $\theta_{12,3}$, see Theorem 3.21. The action $\theta_{24,4}$ is non-maximal.

(2) Consider the action $\theta_{24,4} : \Delta(0;3,6,6) \rightarrow A_4 \times C_2$. Maximal subgroups of $A_4 \times C_2$ are $\langle ab^2, b^2 \rangle \simeq A_4$, $\langle a, b^3, b^2ab \rangle \simeq C_2 \times C_2 \times C_2$ and one conjugacy class isomorphic to $\langle b \rangle \simeq C_6$. The coset permutations are:

	$\langle ab^2, b^2\rangle$	$\langle a, b^3, b^2ab\rangle$	$\langle b\rangle$
$b^4 \to$	$(1)(2)$	$(1\ 2\ 3)$	$(2\ 3\ 4)$
$ba \to$	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 2\ 4)$
$ab \to$	$(1\ 2)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$

By Theorem 1.2 we find that $s(\theta_{24,4}^{-1}\langle ab^2, b^2\rangle) = (0; 3, 3, 3, 3)$ and $(b^4, b^4, (ab)^2, (ba)^2) \simeq (ab^2, ab^2, b^2ab^2, ab^4) \simeq (ab^2, ab^2, ab^4, b^4)$ which corresponds to the non-maximal action $\theta_{12,9} : \Delta(0; 3, 3, 3, 3) \to A_4$, see Theorem 3.19. We also find that $s(\theta_{24,4}^{-1}\langle a, b^3, b^2ab\rangle) = (1; 2, 2)$ and the non-maximal action $\theta_{8,16}$ is induced, see Theorem 3.6 for calculations. $s(\theta_{24,4}^{-1}\langle b\rangle) = (0; 2, 2, 3, 6, 6)$ and $\theta_{6,3}$ is induced.

(3) Consider the action $\theta_{48,3} : \Delta(0; 3, 4, 4) \to A_4 \rtimes C_4$. Maximal subgroups are $\langle abab^{-1}, bab\rangle \simeq A_4 \times C_2$, one conjugacy class isomorphic to $\langle ab, abab^{-1}\rangle \simeq G_{4,4}$ and one conjugacy class isomorphic to $\langle b^{-1}ab^2, a\rangle \simeq C_3 \times C_4$. We have the following permutation representation:

	$\langle abab^{-1}, bab\rangle$	$\langle ab, abab^{-1}\rangle$	$\langle b^{-1}ab^2, a\rangle$
$a \to$	$(1)(2)$	$(1\ 2\ 3)$	$(1)(2\ 3\ 4)$
$b \to$	$(1\ 2)$	$(1)(2\ 3)$	$(1\ 2\ 4\ 3)$
$b^3a^2 \to$	$(1\ 2)$	$(1\ 3)(2)$	$(1\ 2)(3)(4)$

Now $s(\theta_{48,3}^{-1}\langle abab^{-1}, bab\rangle) = (0; 2, 2, 3, 3)$ and $(b^2, (ab)^2, a, bab^{-1}) \simeq (abab^{-1}(bab)^3, (bab)^3, (bab)^2, (bab)^4abab^{-1})$ corresponds to $\theta_{24,2} : \Delta(0; 2, 2, 3, 3) \to A_4 \times C_2$. Also, $s(\theta_{48,3}^{-1}\langle ab, abab^{-1}\rangle) = (0; 2, 2, 4, 4)$ and $(a(b^3a^2)^2a^2, b^2, a^2ba) = ((ab)^2, (ab)^2abab^{-1}, (ab)^3, ababab^{-1})$ corresponding to $\theta_{16,13}$. $s(\theta_{48,3}^{-1}\langle b^{-1}ab^2, a\rangle) = (0; 2, 3, 4, 4)$ inducing $\theta_{12,10} : \Delta(0; 2, 3, 4, 4) \to C_3 \rtimes C_4$. □

Theorem 3.16.

$$\overline{\mathcal{M}}_5^{48,1} \subset \overline{\mathcal{M}}_5^{24,6} \cap \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{12,10} \cap \overline{\mathcal{M}}_5^{8,4} \cap \overline{\mathcal{M}}_5^{4,4}$$

Proof: Maximal subgroups of $(C_{12}\times C_2)\rtimes C_2$ are $\langle a, (ab)^2\rangle \simeq C_{12}\times C_2$, $\langle b, (ab)^2, a^2\rangle \simeq S_3 \times C_2 \times C_2$, $\langle ab, a^2\rangle \simeq (C_3 \rtimes C_4) \times C_2$ of order 24 and $\langle a^3, b\rangle, \langle a^3, a^4b\rangle, \langle a^3, ba^4\rangle$ of order 16. The first three subgroups induce $\theta_{24,1}$, $\theta_{24,6}$ and $\theta_{24,8}$ respectively. $\theta_{24,1}$ and $\theta_{24,8}$ are non-maximal. For the last three subgroups we have the following permutations of the cosets:

	$\langle a^3, b\rangle$	$\langle a^3, a^4b\rangle$	$\langle a^3, ba^4\rangle$
$b \to$	$(1)(2\ 3)$	$(1\ 2)(3)$	$(1\ 3)(2)$
$ba \to$	$(1\ 2)(3)$	$(1\ 3)(2)$	$(1)(2\ 3)$
$a^{11} \to$	$(1\ 3\ 2)$	$(1\ 3\ 2)$	$(1\ 3\ 2)$

We see that $s(\theta_{48,1}^{-1}\langle a^3, b\rangle) = (0; 2, 2, 4, 4)$ and $(b, (ba)^2, a^9, a^8ba^5) = (b, (a^3b)^2, a^9, ba^9)$ $\simeq (b, a^6, ba^3, a^3) \simeq (b, a^6, a^3, a^3b)$ corresponds to $\theta_{16,3}$. We also see that $s(\theta_{48,1}^{-1}\langle a^3, a^4b\rangle)$ $= (0; 2, 2, 4, 4)$ and $(a^8ba^4, (ba)^2, a^9, a^4ba^9) = (a^4b, (a^7b)^2, a^9, a^4ba^9) \simeq (a^4b, a^6, a^4ba^3, a^3)$ $\simeq (a^4b, a^6, a^3, a^7b)$ corresponds to $\theta_{16,3}$. Finally, we get the signature $s(\theta_{48,1}^{-1}\langle a^3, ba^4\rangle)$ $= (0; 2, 2, 4, 4)$ and the generating vector $(a^4ba^8, a^4(ba)^2a^8, a^9, ba) = (ba^4, (ab)^2, a^9, ba)$ $\simeq (ba^4, (a)^6, ba^7, a^3) \simeq (ba^4, (a)^6, a^3, a^3ba^4)$ corresponds to $\theta_{16,3}$.

(1) Consider the action $\theta_{24,1} : \Delta(0;2,12,12) \to C_{12} \times C_2$. Maximal subgroups of $C_{12} \times C_2$ are $\langle a \rangle, \langle ab \rangle \simeq C_{12}$, $\langle a^2, b \rangle \simeq C_6 \times C_2$ and $\langle a^3, b \rangle \simeq C_4 \times C_2$. $\langle a \rangle$ and $\langle ab \rangle$ induces $\theta_{12,1}$, $\langle a^2, b \rangle$ induces $\theta_{12,2}$ and $\langle a^3, b \rangle$ induces $\theta_{8,4}$. The actions $\theta_{12,1}$ and $\theta_{12,2}$ are non-maximal, see Theorem 3.20.

(2) Consider the action $\theta_{12,1}$. Maximal subgroups of C_{12} are $\langle a^2 \rangle \simeq C_6$ and $\langle a^3 \rangle \simeq C_4$. $\langle a^2 \rangle$ induces $\theta_{6,4}$, which is non-maximal, see Theorem 3.20, and $\langle a^3 \rangle$ induces $\theta_{4,4}$.

(3) Consider the action $\theta_{24,8} : \Delta(0;4,4,6) \to (C_3 \rtimes C_4) \times C_2$. Maximal subgroups of $(C_3 \rtimes C_4) \times C_2$ are $\langle t, s^2 \rangle \simeq C_6 \times C_2$, $\langle s, t^2 \rangle, \langle st, t^2 \rangle \simeq C_3 \rtimes C_4$ and $\langle s, t^3 \rangle, \langle st^2, t^3 \rangle, \langle st^4, t^3 \rangle \simeq C_4 \times C_2$. The desired monodromies are:

	$\langle t, s^2 \rangle$	$\langle s, t^2 \rangle$	$\langle st, t^2 \rangle$
$s \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$st \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$
$s^2t^5 \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$

From the table we see that $s(\theta_{24,8}^{-1} \langle t, s^2 \rangle) = (0;2,2,6,6)$ and $(s^2, s^2, s^2t^5, s^2, s^2t^5)$ corresponds to the non-maximal action $\theta_{12,2} : \Delta(0;2,2,6,6) \to C_6 \times C_2$, see Theorem 3.20. Also, $s(\theta_{24,8}^{-1} \langle s, t^2 \rangle) = s(\theta_{24,8}^{-1} \langle st, t^2 \rangle) = (0;2,2,3,4)$ inducing $\theta_{12,10} : \Delta(0;2,3,4,4) \to C_3 \rtimes C_4$. □

Theorem 3.17.

$$\overline{\mathcal{M}}_5^{48,4} \subset \overline{\mathcal{M}}_5^{24,2} \cap \overline{\mathcal{M}}_5^{16,13} \cap \overline{\mathcal{M}}_5^{12,10}$$

Proof: Maximal subgroups of $A_4 \rtimes C_4$ are $\langle ab^3ab, ab^2 \rangle \simeq A_4 \times C_2$, one conjugacy class represented by $\langle b, (ab)^2 \rangle \simeq G_{4,4}$ and one conjugacy class represented by $\langle abab^2, a \rangle \simeq C_3 \rtimes C_4$. For subgroups G of $A_4 \rtimes C_4$ the monodromies of the coverings $\mathcal{U}/\theta_{48,4}^{-1}(G) \to \mathcal{U}/\Delta(0;3,4,4)$ are given by:

	$\langle ab^3ab, ab^2 \rangle$	$\langle b, (ab)^2 \rangle$	$\langle abab^2, a \rangle$
$a \to$	$(1)(2)$	$(1\ 2\ 3)$	$(1)(2\ 3\ 4)$
$b^3 \to$	$(1\ 2)$	$(1)(2\ 3)$	$(1\ 2\ 4\ 3)$
$ba^2 \to$	$(1\ 2)$	$(1\ 3)(2)$	$(1\ 2)(3)(4)$

Now $s(\theta_{48,4}^{-1} \langle [a,b], ab^2 \rangle) = (0;2,2,3,3)$ and $(b^2, (ba^2)^2, bab^{-1}, a) = (ab^3ab(ab^2)^3, (ab^2)^3, (ab^2)^2, (ab^2)^4ab^3ab)$ corresponds to $\theta_{24,2}$. Further we see that $s(\theta_{48,4}^{-1} \langle b, (ab)^2 \rangle) = (0;2,2,4,4)$ and $((ba^2)^2, a^{-1}b^2a, b^3, aba) = ((b(ab)^2)^2, (ab)^2, b^3, (ab)^2b) \simeq (b^2, (ab)^2, (ab)^2b, b) \simeq ((ab)^2, b^2, b, b(ab)^2)$ corresponds to $\theta_{16,13}$. $s(\theta_{48,4}^{-1} \langle abab^2, a \rangle) = (0;2,3,4,4)$ inducing $\theta_{12,10}$. □

Theorem 3.18.

$$\overline{\mathcal{M}}_5^{48,5} \subset \overline{\mathcal{M}}_5^{16,5} \cap \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{8,14} \cap \overline{\mathcal{M}}_5^{3,1}$$

Proof: Maximal subgroups of $S_4 \times C_2$ are $\langle ab^2a^2, ac \rangle \simeq A_4 \times C_2$, $\langle a, b \rangle, \langle a, bc \rangle \simeq S_4$, one conjugacy class represented by $\langle b, ab^2a^2b, c \rangle \simeq D_4 \times C_2$ and one conjugacy class represented by $\langle ac, ab^2a^2b \rangle \simeq D_6$. The desired permutation representations of the action $\theta_{48,5}$ are:

	$\langle ab^2a^2, ac\rangle$	$\langle a, b\rangle$	$\langle a, bc\rangle$	$\langle b, ab^2a^2b, c\rangle$	$\langle ac, ab^2a^2b\rangle$
$ab \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1\ 3)(2)$	$(1\ 2)(3)(4)$
$b^2c \to$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 4)(2\ 3)$
$ca^2b^3 \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)(3)$	$(1\ 3)(2)(4)$
$a \to$	$(1)(2)$	$(1)(2)$	$(1)(2)$	$(1\ 2\ 3)$	$(1)(2\ 3\ 4)$

From the table above we see that $s(\theta_{48,5}^{-1}\langle ab^2a^2, ac\rangle) = (0; 2, 2, 3, 3)$ and the generating vector $(b^2c, b^2c, a, (b^2ab^3)a(b^2ab^3)^{-1}) = (b^2c, b^2c, a, a^2) = ((ac)^2ab^2a^2ac, (ac)^2ab^2a^2ac, (ac)^4, (ac)^2)$ corresponds to $\theta_{24,3} : \Delta(0; 2, 2, 3, 3) \to A_4 \times C_2$. $s(\theta_{48,5}^{-1}\langle a, b\rangle) = s(\theta_{48,5}\langle a, bc\rangle) = (0; 2, 2, 3, 3)$ inducing $\theta_{24,5}$. $s(\theta_{48,5}^{-1}\langle b, ab^2a^2b, c\rangle) = (0; 2, 2, 2, 2, 2)$ and $(a^2ba^2, a^2ca^2b^3a, a^2b^2ca, ab^2ca^2, b^2c)$ corresponds to $\theta_{16,5} : \Delta(0; 2, 2, 3, 3) \to S_4$. $s(\theta_{48,5}^{-1}\langle ac, ab^2a^2b\rangle) = (0; 2, 2, 2, 2, 3)$ inducing $\theta_{12,4}$, since $(ac)^3 = c$. The actions $\theta_{24,3}$ and $\theta_{24,5}$ are non-maximal.

(1) Consider the action $\theta_{24,3} : \Delta(0; 2, 2, 3, 3) \to A_4 \times C_2$. Maximal subgroups of $A_4 \times C_2$ are $\langle (ab)^2, b^2\rangle \simeq A_4$, $\langle a, b^3, b^5ab\rangle \simeq C_2 \times C_2 \times C_2$ and one conjugacy class represented by $\langle b\rangle \simeq C_6$. The cosets of the subgroups are permutated as follows:

	$\langle (ab)^2, b^2\rangle$	$\langle a, b^3, b^5ab\rangle$	$\langle b\rangle$
$ab^3 \to$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2)(3\ 4)$
$ab^3 \to$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2)(3\ 4)$
$b^2 \to$	$(1)(2)$	$(1\ 3\ 2)$	$(1)(2\ 4\ 3)$
$b^4 \to$	$(1)(2)$	$(1\ 2\ 3)$	$(1)(2\ 3\ 4)$

From the table above we see that $s(\theta_{48,5}^{-1}\langle (ab)^2, b^2\rangle) = (0; 3, 3, 3, 3)$ and $((ab)b^2(ab)^{-1}, (ab)b^4(ab)^{-1}, b^2, b^4) = ((ab)^2, (ab)^4, b^2, b^4)$ corresponding to $\theta_{12,8}$. $s(\theta_{48,5}^{-1}\langle a, b^3, b^5ab\rangle) = (0; 2, 2, 2, 2, 2, 2)$ and $(ab^3, ab^3, b^2ab, b^2ab, b^4ab^5, b^4ab^5)$ corresponds to $\theta_{8,14}$. Finally, $s(\theta_{48,5}^{-1}\langle b\rangle) = (1; 3, 3)$ inducing the non-maximal action $\theta_{6,5}$, see Theorem 3.22 for calculations. The action $\theta_{12,8}$ is non-maximal.

(2) Consider the action $\theta_{12,8}$. Maximal subgroups of A_4 are $\langle ab^2, a^2b\rangle \simeq C_2 \times C_2$ and one conjugacy class represented by $\langle a\rangle \simeq C_3$. $s(\theta_{12,8}\langle ab^2, a^2b\rangle) = (2; -)$ inducing $\theta_{4,12}$. $s(\theta_{12,8}\langle a\rangle) = (1; 3, 3, 3, 3)$ inducing $\theta_{3,2}$.

(3) Consider the action $\theta_{24,5} : \Delta(0; 2, 2, 3, 3) \to S_4$. Maximal subgroups of S_4 are $\langle a, b^2a\rangle \simeq A_4$, one conjugacy class represented by $\langle b, ab^3a\rangle \simeq D_4$ and one conjugacy class represented by $\langle a, b^2ab^3\rangle \simeq D_3$. The desired monodromies are:

	$\langle a, b^2a\rangle$	$\langle b, ab^3a\rangle$	$\langle a, b^2ab^3\rangle$
$ab \to$	$(1\ 2)$	$(1\ 3)(2)$	$(1\ 2)(3\ 4)$
$ab \to$	$(1\ 2)$	$(1\ 3)(2)$	$(1\ 2)(3\ 4)$
$a \to$	$(1)(2)$	$(1\ 2\ 3)$	$(1)(2\ 3\ 4)$
$a^2 \to$	$(1)(2)$	$(1\ 3\ 2)$	$(1)(2\ 4\ 3)$

Now $s(\theta_{24,5}^{-1}\langle a, b^2a\rangle) = (0; 3, 3, 3, 3)$ and $(a, a^2, ba^2b^3, bab^3) = (a, a^2, b^2a, (b^2a)^2)$ corresponds to $\theta_{12,8} : \Delta(0; 3, 3, 3, 3) \to A_4$. $s(\theta_{24,5}^{-1}\langle b, ab^3a\rangle) = (1; 2, 2)$, inducing the non-maximal action $\theta_{8,20}$, see Theorem 3.6. $s(\theta_{24,5}\langle a, b^2ab^3\rangle) = (1; 3, 3)$, inducing the non-maximal action $\theta_{6,7}$, see Theorem 3.22.

□

Theorem 3.19.

$$\overline{\mathcal{M}}_5^{24,2} \subset \overline{\mathcal{M}}_5^{8,13} \cap \overline{\mathcal{M}}_5^{6,1} \cap \overline{\mathcal{M}}_5^{3,1}$$

Proof: Maximal subgroups of $A_4 \times C_2$ are $\left\langle (ab)^2, b^2 \right\rangle \simeq A_4$, $\left\langle a, b^3, b^5ab \right\rangle \simeq C_2 \times C_2 \times C_2$ and one conjugacy class represented by $\langle b \rangle \simeq C_6$. For the subgroups G of $A_4 \times C_2$ the monodromies for the coverings $\mathcal{U}/\theta_{24,2}^{-1}(G) \to \mathcal{U}/\Delta(0;2,2,3,3)$ are given by:

	$\left\langle (ab)^2, b^2 \right\rangle$	$\left\langle a, b^3, b^5ab \right\rangle$	$\langle b \rangle$
$ab^3 \to$	$(1\ 2)$	$(1)(2)(3)$	$(1\ 2)(3\ 4)$
$b^3 \to$	$(1\ 2)$	$(1)(2)(3)$	$(1)(2)(3)(4)$
$b^2 \to$	$(1)(2)$	$(1\ 3\ 2)$	$(1)(2\ 4\ 3)$
$b^4a \to$	$(1)(2)$	$(1\ 2\ 3)$	$(1\ 2\ 4)(3)$

From the table above we see that $s(\theta_{24,2}^{-1}\left\langle (ab)^2, b^2 \right\rangle) = (0;3,3,3,3)$ and $(b^2, b^2, b^4a, b^5b^4ab)$ corresponds to $\theta_{12,9}$. We also find that $s(\theta_{24,2}^{-1}\left\langle a, b^3, b^5ab \right\rangle) = (0;2,2,2,2,2,2,2)$ and $(b^3, b^3, b^3, ab^3, bab^2, b^5ab^4)$ corresponds to $\theta_{8,13}$. $s(\theta_{24,2}^{-1}\langle b \rangle) = (0;2,2,2,2,3,3,3)$, inducing $\theta_{6,1}$. The action $\theta_{12,9}$ is non-maximal.

(1) Consider the action $\theta_{12,9}$. Maximal subgroups of A_4 are $\left\langle ab^2, a^2b \right\rangle \simeq C_2 \times C_2$ and one conjugacy class represented by $\langle a \rangle \simeq C_3$. $s(\theta_{12,9}\left\langle ab^2, a^2b \right\rangle) = (2;-)$ inducing $\theta_{4,12}$. $s(\theta_{12,9}\langle a \rangle) = (1;3,3,3,3)$ inducing $\theta_{3,2}$.

□

Theorem 3.20.

$$\overline{\mathcal{M}}_5^{24,6} \subset \overline{\mathcal{M}}_5^{12,5} \cap \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{8,13} \cap \overline{\mathcal{M}}_5^{6,6} \cap \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,2}$$

Proof: Maximal subgroups of $D_6 \times C_2$ are $\langle a, s \rangle, \langle a, sb \rangle, \langle ab, s \rangle, \langle ab, sb \rangle, \left\langle a^2b, s \right\rangle$, $\left\langle a^2b, sa^3 \right\rangle \simeq D_6$, $\langle a, b \rangle \simeq C_6 \times C_2$ and one conjugacy class represented by $\left\langle a^3, s, b \right\rangle \simeq C_2 \times C_2 \times C_2$. For the subgroups G of $D_6 \times C_2$ the monodromies for the coverings $\mathcal{U}/\theta_{24,6}^{-1}(G) \to \mathcal{U}/\Delta(0;2,2,2,6)$ are given by:

	$\langle a, s \rangle$	$\langle a, sb \rangle$	$\langle ab, s \rangle$	$\langle ab, sb \rangle$	$\left\langle a^2b, s \right\rangle$	$\left\langle a^2b, sa^3 \right\rangle$
$s \to$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$
$a^3 \to$	$(1)(2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 2)$
$sa^4b \to$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$
$a^5b \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$

$\theta_{24,6}$ has generating vector (s, a^3, sa^4b, a^5b) and from the table above we see that we see that it induces the action $\theta_{12,5} : \Delta(0;2,2,2,2,3) \to D_6$, the non-maximal action $\theta_{12,6} : \Delta(0;2,2,6,6) \to D_6$, the action $\theta_{12,4} : \Delta(0;2,2,2,2,3) \to D_6$ and the non-maximal action $\theta_{12,7} : \Delta(1;3) \to D_6$. The last two subgroups induce the non-maximal action $\theta_{12,2} : \Delta(0;2,2,6,6) \to C_6 \times C_2$ and the action $\theta_{8,13}$.

(1) Consider the action $\theta_{12,2}$. Maximal subgroups of $C_6 \times C_2$ are $\langle a \rangle, \langle ab \rangle, \left\langle a^2b \right\rangle \simeq C_6$ and $\left\langle a^3, b \right\rangle \simeq C_2 \times C_2$. We find that $s(\theta_{12,2}^{-1}\langle a \rangle) = (0;6,6,6,6)$, inducing $\theta_{6,4}$, $s(\theta_{12,2}^{-1}\langle ab \rangle) = s(\theta_{12,2}^{-1}\left\langle a^2b \right\rangle) = (1;3,3)$, inducing the non-maximal action $\theta_{6,5}$, see Theorem 3.22, and $s(\theta_{12,2}^{-1}\left\langle a^3, b \right\rangle) = (0;2,\overset{8}{\ldots},2)$, inducing $\theta_{4,7}$. The action $\theta_{6,4}$ is non-maximal.

(2) Consider the action $\theta_{6,4}$. By Theorem 1.2 we see that $s(\theta_{6,4}^{-1}\left\langle a^2\right\rangle) = (1;3,3,3,3)$, inducing $\theta_{3,1}$ and $s(\theta_{6,4}^{-1}\left\langle a^3\right\rangle) = (2;2,2,2,2)$, inducing $\theta_{2,2}$.

(3) Consider the action $\theta_{12,6}$. Maximal subgroups of D_6 are $\langle a\rangle \simeq C_6$, $\left\langle a^2, s\right\rangle, \left\langle a^2, sa^3\right\rangle \simeq D_3$ and one class represented by $\left\langle a^3, s\right\rangle \simeq C_2 \times C_2$. We find that $s(\theta_{12,6}^{-1}\langle a\rangle) = (0;6,6,6,6)$, inducing $\theta_{6,4}$, $s(\theta_{12,6}^{-1}\left\langle a^2, s\right\rangle) = (0;2,2,2,2,3,3)$, inducing $\theta_{6,6}$. Now $s(\theta_{12,6}^{-1}\left\langle a^2, sa^3\right\rangle) = (1;3,3)$, inducing non-maximal action $\theta_{6,7}$ and that $s(\theta_{12,6}^{-1}\left\langle a^3, s\right\rangle) = (1;2,2,2,2)$, inducing $\theta_{4,10}$.

(4) Consider the action $\theta_{12,7}$. We see that $s(\theta_{12,7}^{-1}\langle a\rangle) = (1;3,3)$, inducing the non-maximal action $\theta_{6,5}$, $s(\theta_{12,7}^{-1}\left\langle a^2, s\right\rangle) = s(\theta_{12,7}\left\langle a^2, sa^3\right\rangle) = (1;3,3)$, inducing $\theta_{6,7}$ and $s(\theta_{12,7}^{-1}\left\langle a^3, s\right\rangle) = (2;-)$, inducing $\theta_{4,10}$. □

Theorem 3.21.

$$\overline{\mathcal{M}}_5^{24,7} \subset \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{8,17} \cap \overline{\mathcal{M}}_5^{6,3} \cap \overline{\mathcal{M}}_5^{4,8}$$

Proof: Maximal subgroups of $(C_6 \times C_2) \rtimes C_2$ are $\left\langle b, (ab)^2\right\rangle \simeq C_6 \times C_2$, $\langle [a,b], a\rangle \simeq D_6$, $\left\langle ab^3, b^2\right\rangle \simeq C_3 \rtimes C_4$ and one conjugacy class represented by $\left\langle ab^3, a\right\rangle \simeq D_4$. The coset permutations are:

	$\left\langle b,(ab)^2\right\rangle$	$\langle [a,b],a\rangle$	$\left\langle ab^3,b^2\right\rangle$	$\left\langle ab^3,a\right\rangle$
$a \rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1)(2\ 3)$
$b^3 \rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1)(2)(3)$
$ab^4 \rightarrow$	$(1\ 2)$	$(1)(2)$	$(1\ 2)$	$(1\ 2)(3)$
$aba \rightarrow$	$(1)(2)$	$(1\ 2)$	$(1\ 2)$	$(1\ 3\ 2)$

From the table above we see that $s(\theta_{24,7}^{-1}\left\langle b,(ab)^2\right\rangle) = (0;2,2,6,6)$ and $(ab^3a, b^3, b, aba) = ((ab)^2b^3, b^3, b, (ab)^2b^5)$ corresponds to $\theta_{12,3} : \Delta(0;2,2,6,6) \rightarrow C_6 \times C_2$. We also see that $s(\theta_{24,7}^{-1}\langle [a,b],a\rangle) = (0;2,2,2,2,3)$ and the generating vector $(a, bab^{-1}, ab^4, bab^3, ab^2a) = (a, a[a,b], [a,b]^2a, [a,b]a, [a,b]^4)$ corresponds to $\theta_{12,4} : \Delta(0;2,2,2,2,3) \rightarrow D_6$. $s(\theta_{24,7}^{-1}\left\langle ab^3, b^2\right\rangle) = (1;3)$, inducing $\theta_{12,11} : \Delta(1;3) \rightarrow C_3 \rtimes C_4$. $s(\theta_{24,7}^{-1}\left\langle a, ab^3\right\rangle) = (0;2,2,2,2,2,2)$ and $(b^3, b^3, b^3, a, (aba)^3, a) = (a(ab^3), a(ab^3), a(ab^3), a, (ab^3)a, a)$ corresponds to $\theta_{8,17}$. The actions $\theta_{12,3}$ and $\theta_{12,11}$ are non-maximal.

(1) Consider the action $\theta_{12,3}$. Maximal subgroups of $C_6 \times C_2$ are $\langle a\rangle, \langle ab\rangle, \left\langle a^2b\right\rangle \simeq C_6$ and $\left\langle a^3, b\right\rangle \simeq C_2 \times C_2$. $s(\theta_{12,3}^{-1}\langle a\rangle) = (0;2,2,3,6,6)$, inducing $\theta_{6,3}$. $s(\theta_{12,3}^{-1}\langle ab\rangle) = (1;3,3)$, inducing The non-maximal action $\theta_{6,5}$. See Theorem 3.22. $s(\theta_{12,3}^{-1}\left\langle a^2b\right\rangle) = (0;2,2,3,6,6)$, inducing $\theta_{6,3}$. $s(\theta_{12,3}^{-1}\left\langle a^3, b\right\rangle) = (0;2,.^8.,2)$, inducing $\theta_{4,8}$.

(2) Consider the action $\theta_{12,11}$. Maximal subgroups of $C_3 \rtimes C_4$ are $\left\langle s^2t\right\rangle \simeq C_6$ and one conjugacy class represented by $\langle s\rangle \simeq C_4$. $s(\theta_{12,11}^{-1}\left\langle s^2t\right\rangle) = (1;3,3)$, inducing $\theta_{6,5}$. $s(\theta_{12,11}^{-1}\langle s\rangle) = (2;-)$, inducing $\theta_{4,6}$. □

Theorem 3.22.

$$\begin{array}{ll} \overline{\mathcal{M}}_5^{12,4} & \subset \overline{\mathcal{M}}_5^{6,6} \cap \overline{\mathcal{M}}_5^{4,10} \cap \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,3} \\ \overline{\mathcal{M}}_5^{12,5} & \subset \overline{\mathcal{M}}_5^{6,1} \cap \overline{\mathcal{M}}_5^{6,6} \cap \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,3} \end{array}$$

Proof: Maximal subgroups of D_6 are $\langle a\rangle \simeq C_6$, $\langle a^2, s\rangle$, $\langle a^2, sa^3\rangle \simeq D_3$ and of class represented by $\langle a^3, s\rangle$. By considering the induced permutations of the cosets of the maximal subgroups, by the generating vector (s, s, sa^3, sa, a^2) of $\theta_{12,4}$, we find that $\theta_{12,4}$ induces the non-maximal action $\theta_{6,5}$, $\theta_{6,6}$ and $\theta_{4,10}$.

$\theta_{12,5}$ induces, via the generating vector (a^3, a^3, s, sa^4, a^2), the actions $\theta_{6,1}$, $\theta_{6,6}$, the non-maximal action $\theta_{6,7}$ and the action $\theta_{4,7}$.

(1) The actions $\theta_{6,5}$ and $\theta_{6,7}$ induce both actions $\theta_{3,1}$ and $\theta_{2,3}$. □

Theorem 3.23.

$$\overline{\mathcal{M}}_5^{12,10} \subset \overline{\mathcal{M}}_5^{6,1} \cap \overline{\mathcal{M}}_5^{4,3}$$

Proof: Maximal subgroups of $C_3 \rtimes C_4$ are $\langle s^2t\rangle \simeq C_6$ and one conjugacy class represented by $\langle t\rangle \simeq C_4$. We have the following subgroup signatures. $s(\theta_{12,10}^{-1}\langle s^2t\rangle) = (0; 2, 2, 2, 2, 3, 3)$, inducing $\theta_{6,1}$. $s(\theta_{12,10}^{-1}\langle t\rangle) = (0; 2, 2, 2, 2, 2, 4, 4)$, inducing $\theta_{4,3}$. □

With calculations similar to the ones above it is easy to prove the following theorem.

Theorem 3.24. *The subvariety inclusions for strata corresponding to automorphism groups of order* 6 *are:*

$$\overline{\mathcal{M}}_5^{6,1} \subset \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,0} \qquad \overline{\mathcal{M}}_5^{6,3} \subset \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,1}$$

$$\overline{\mathcal{M}}_5^{6,2} \subset \overline{\mathcal{M}}_5^{3,0} \cap \overline{\mathcal{M}}_5^{2,2} \qquad \overline{\mathcal{M}}_5^{6,6} \subset \overline{\mathcal{M}}_5^{3,1} \cap \overline{\mathcal{M}}_5^{2,2}$$

3.3 Surfaces with Automorphisms of Order 5 or 11

Here we give the structure of the strata corresponding to surfaces admitting automorphisms of order 5 or 11. We show the inclusion relations of the strata in decreasing order.

Theorem 3.25.

$$\overline{\mathcal{M}}_5^{160} \subset \overline{\mathcal{M}}_5^{32,1} \cap \overline{\mathcal{M}}_5^{16,3} \cap \overline{\mathcal{M}}_5^{10,2}$$

Proof: Maximal subgroups of $((C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5) \rtimes C_2$ are isomorphic to $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$, $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_2$ and D_5. $\langle a, b(a^2b^2)^2\rangle \simeq D_5$ and $s(\theta_{160}^{-1}\langle a, b(a^2b^2)^2\rangle) = (0; 2, 2, 2, 2, 5)$, inducing $\theta_{10,2}$. Further θ_{80} and $\theta_{32,1}$ are induced. θ_{80} is non-maximal.

(1) Consider the action $\theta_{80} : \Delta(0; 2, 5, 5) \to (C_2^4) \rtimes C_5$. Maximal subgroups of $(C_2 \times C_2 \times C_2 \times C_2) \rtimes C_5$ are isomorphic to $C_2 \times C_2 \times C_2 \times C_2$ and C_5, inducing $\theta_{16,3}$ and θ_5. □

Theorem 3.26.

$$\overline{\mathcal{M}}_5^{120} \subset \overline{\mathcal{M}}_5^{24,2} \cap \overline{\mathcal{M}}_5^{12,4} \cap \overline{\mathcal{M}}_5^{20,2}$$

Proof: Maximal subgroups of $A_5 \times C_2$ are $\langle a, b\rangle \simeq A_5$, one conjugacy class represented by $\langle a^4bab, bc\rangle \simeq A_4 \times C_2$, one conjugacy class represented by $\langle bac, a^2b\rangle \simeq D_{10}$ and one conjugacy class represented by $\langle bc, ab^2a^4ba\rangle \simeq D_6$. We have the following permutations of the cosets:

	$\langle a^4bab, bc\rangle$	$\langle bc, ab^2a^4ba\rangle$
$ba^2c \to$	$(1\ 5)(2\ 3)(4)$	$(1\ 10)(2\ 6)(3\ 9)(5\ 7)(4)(8)$
$b \to$	$(1)(3)(2\ 5\ 4)$	$(1)(2\ 7\ 4)(3\ 8\ 5)(6\ 9\ 10)$
$a^2c \to$	$(1\ 5\ 3\ 2\ 4)$	$(1\ 10\ 3\ 7\ 6)(2\ 4\ 5\ 8\ 9)$

From the table we see that $s(\theta_{120}^{-1}\langle a^4bab, bc\rangle) = (0; 2, 2, 3, 3)$, inducing the action $\theta_{24,2} : \Delta(0; 2, 2, 3, 3) \to A_4 \times C_2$ since $\theta_{24,3}$ is non-maximal. We see that $s(\theta_{120}^{-1}\langle bc, ab^2a^4ba\rangle) = (0; 2, 2, 2, 2, 3)$ and $((a^2c)^5, (a^2c)^5, (a^{-1}ba)ba^2c(a^{-1}ba)^{-1}, (a^{-1}b)ba^2c(a^{-1}b)^{-1}, b) = (c, c, bcab^2a^4ba, ab^2a^4babc, (bc)^4)$ corresponds to $\theta_{12,4} : \Delta(0; 2, 2, 2, 2, 3) \to D_6$. We also have that $s(\theta_{120}^{-1}\langle a, b\rangle) = (0; 3, 3, 5)$, inducing θ_{60} which is non-maximal and $s(\theta_{120}^{-1}\langle bac, a^2b\rangle) = (0; 2, 2, 2, 10)$, inducing $\theta_{20,2} : \Delta(0; 2, 2, 2, 10) \to D_{10}$.

(1) Consider the action $\theta_{60} : \Delta(0; 3, 3, 5) \to A_5$. Maximal subgroups of A_5 are one conjugacy class represented by $\langle bab, ba^2\rangle \simeq A_4$, one conjugacy class represented by $\langle ba, a^2b\rangle \simeq D_5$ and one conjugacy class represented by $\langle b, [a, b]a\rangle \simeq D_3$. Thus θ_{60} induces $\theta_{12,9}$, due to results above, $\theta_{10,3}$, and $\theta_{6,5}$, since there are no elements of order 2. $\theta_{12,9}$, $\theta_{10,3}$ and $\theta_{6,5}$ are non-maximal, see Theorems 3.19, 3.22 and 3.29. □

Theorem 3.27.

$$\overline{\mathcal{M}}_5^{40} \subset \overline{\mathcal{M}}_5^{20,2} \cap \overline{\mathcal{M}}_5^{8,8} \cap \overline{\mathcal{M}}_5^{4,3}$$

Proof: Maximal subgroups of $D_5 \times C_4$ are $\langle a\rangle \simeq C_{20}$, $\langle a^2, b\rangle \simeq D_{10}$, $\langle ab, a^4\rangle \simeq C_5 \rtimes C_4$ and one conjugacy class represented by $\langle a^5, b\rangle \simeq C_4 \times C_2$. The cosets are given by the transversal $\{1, a^{16}, [a, b], [b, a], a^4\}$. We have the following coset permutations:

	$\langle a^5, b\rangle$
$b \to$	$(1)(2\ 5)(3\ 4)$
$ba \to$	$(1\ 2)(3\ 5)(4)$
$a^{19} \to$	$(1\ 5\ 4\ 3\ 2)$

From the permutations above $s(\theta_{40}^{-1}\langle a^5, b\rangle) = (0; 2, 2, 2, 4, 4)$ and the generating vector $((ba)^2, a^4(ba)^2a^{16}, b, [b, a]ba[b, a]^{-1}, a^{15}) = (a^{10}, a^{10}, b, a^5b, a^{15})$ corresponds to $\theta_{8,8} : \Delta(0; 2, 2, 2, 4, 4) \to C_4 \times C_2$. For the other subgroups we find that $s(\theta_{40}^{-1}\langle a\rangle) = (0; 2, 20, 20)$, inducing $\theta_{20,1} : \Delta(0; 2, 20, 20) \to C_{20}$, $s(\theta_{40}^{-1}\langle a^2, b\rangle) = (0; 2, 2, 2, 10)$, inducing $\theta_{20,2} : \Delta(0; 2, 2, 2, 10) \to D_{10}$ and $s(\theta_{40}^{-1}\langle ab, a^4\rangle) = (0; 4, 4, 10)$, inducing $\theta_{20,3} : \Delta(0; 4, 4, 10) \to C_5 \rtimes C_4$. $\theta_{20,1}$ and $\theta_{20,3}$ are non-maximal.

(1) Consider the action $\theta_{20,1}$. Maximal subgroups of C_{20} are isomorphic to $\langle a^2\rangle \simeq C_{10}$ or $\langle a^5\rangle \simeq C_4$, inducing $\theta_{10,1}$ and $\theta_{4,3}$.

(2) Consider the action $\theta_{20,3}$. Maximal subgroups of $C_5 \rtimes C_4$ are isomorphic to $\langle a^2b\rangle \simeq C_{10}$ or $\langle a\rangle \simeq C_4$, inducing $\theta_{10,1}$ and $\theta_{4,3}$. □

Theorem 3.28.

$$\overline{\mathcal{M}}_5^{30} \subset \overline{\mathcal{M}}_5^{10,2} \cap \overline{\mathcal{M}}_5^{6,2} \cap \overline{\mathcal{M}}_5^{3,0}$$

Proof: Maximal subgroups of $D_5 \times C_3$ are $\langle ab^2\rangle \simeq C_{15}$, $\langle a, b^3\rangle \simeq D_5$ and one conjugacy class represented by $\langle b\rangle \simeq C_6$. For $\langle b\rangle$ we have the following coset permutations:

	$\langle b\rangle$
$b^3 \to$	$(1)(2\ 5)(3\ 4)$
$ba \to$	$(1\ 2)(3\ 5)(4)$
$bab \to$	$(1\ 5\ 4\ 3\ 2)$

and we see that $s(\theta_{30}^{-1}\langle b\rangle) = (0;2,3,3,3,6)$, inducing $\theta_{6,2}$. For the other subgroups we find that $s(\theta_{30}^{-1}\langle ab^2\rangle) = (0;3,15,15)$, inducing $\theta_{15} : \Delta(0;3,15,15) \to C_{15}$ and $s(\theta_{30}^{-1}\langle a,b^3\rangle) = (0;2,2,2,2,5)$, inducing $\theta_{10,2} : \Delta(0;2,2,2,2,5) \to D_5$.

(1) Consider the action θ_{15}. Maximal subgroups of C_{15} are isomorphic to $\langle a^3\rangle \simeq C_5$ or $\langle a^5\rangle \simeq C_3$, inducing θ_5 and $\theta_{3,0}$. □

Theorem 3.29.

$$\overline{\mathcal{M}}_5^{20,2} \subset \overline{\mathcal{M}}_5^{10,2} \cap \overline{\mathcal{M}}_5^{4,7} \cap \overline{\mathcal{M}}_5^{2,0} \cap \overline{\mathcal{M}}_5^{2,3}$$

Proof: Maximal subgroups of D_{10} are $\langle a^2,s\rangle$, $\langle a^2,a^5s\rangle \simeq D_5$, $\langle a\rangle \simeq C_{10}$ and one conjugacy class represented by $\langle a^5,s\rangle \simeq C_2 \times C_2$. The desired monodromies are:

	$\langle a^2,s\rangle$	$\langle a^2,a^5s\rangle$	$\langle a^5,s\rangle$
$a^5 \to$	$(1\ 2)$	$(1\ 2)$	$(1)(2)(3)(4)(5)$
$s \to$	$(1)(2)$	$(1\ 2)$	$(1)(2\ 5)(3\ 4)$
$sa^2 \to$	$(1)(2)$	$(1\ 2)$	$(1\ 3)(2)(4\ 5)$
$a^3 \to$	$(1\ 2)$	$(1\ 2)$	$(1\ 4\ 2\ 5\ 3)$

From the table we see that $s(\theta_{20,2}^{-1}\langle a\rangle) = (0;2,2,10,10)$, inducing the action $\theta_{10,1} : \Delta(0;2,2,10,10) \to C_{10}$ and $s(\theta_{20,2}^{-1}\langle a^2,b\rangle) = (0;2,2,2,2,5)$, inducing the action $\theta_{10,2} : \Delta(0;2,2,2,2,5) \to D_5$. $s(\theta_{20,2}^{-1}\langle a^2,a^5b\rangle) = (1;5)$, inducing $\theta_{10,3} : \Delta(1;5) \to D_5$ and $s(\theta_{20,2}^{-1}\langle a^5,b\rangle) = (0;2,.\overset{8}{.}.)$, inducing $\theta_{4,7}$.

(1) Consider the action $\theta_{10,1}$. Maximal subgroups of C_{10} are isomorphic to $\langle a^2\rangle \simeq C_5$ or $\langle a^5\rangle \simeq C_2$, inducing θ_5 and $\theta_{2,0}$.

(2) Consider the action $\theta_{10,3}$. Maximal subgroups of D_5 are isomorphic to $\langle a\rangle \simeq C_5$ or $\langle s\rangle \simeq C_2$, inducing θ_5 and $\theta_{2,3}$. □

Theorem 3.30.

$$\overline{\mathcal{M}}_5^{10,2} \subset \overline{\mathcal{M}}_5^{2,2}$$

Proof: This was proved in Theorem 3.1. □

Theorem 3.31. $\overline{\mathcal{M}}_5^{11,1}$ *is an isolated point of the branch locus and* $\overline{\mathcal{M}}_5^{22} \subset \overline{\mathcal{M}}_5^{2,0}$.

Proof: This was proved in Theorem 3.1. □

The Next corollary was proved in [1] but here we state it for the sake of completeness. It follows from Theorem 3.13, Theorem 3.24, Theorem 3.30 and Theorem 3.31.

Corollary 3.1. *The branch locus* $\mathcal{B}_5$ *is connected with the exception of a single point.*

The conditions for existence of isolated points in the branch locus are given by Kulkarni [18]. In [1] (chapter 4) general results on connectedness of branch loci are obtained.

A Riemann surface X is called *cyclic p-gonal* if it can be realized as a regular p-sheeted covering of the Riemann sphere. If $p = 2$ then X is called *hyperelliptic* and if $p = 2$ then X is called *cyclic trigonal*.

Remark 3.1. From the theorems above we see that the hyperelliptic Riemann surfaces of genus 5 are:

1. a Riemann surface X_{120} with automorphism group $A_5 \times C_2$ given in Theorem 2.12.
2. a Riemann surface $X_{96,1}$ with automorphism group $(A_4 \times C_2 \times C_2) \rtimes C_2$ given in Theorem 2.11.
3. a Riemann surface $X_{48,1}$ with automorphism group $(C_{12} \times C_2) \rtimes C_2$ given in Theorem 2.11.
4. a Riemann surface $X_{48,4}$ with automorphism group $A_4 \rtimes C_4$ given in Theorem 2.11.
5. a Riemann surface X_{40} with automorphism group $D_5 \times C_4$ given in Theorem 2.12.
6. a uniparametric family of Riemann surfaces $\overline{\mathcal{M}}_5^{24,2}$ with automorphism group $A_4 \times C_2$ given in Theorem 2.10.
7. a single Riemann surface X_{22}, which is also 11-gonal, with automorphism group C_{22} given in Theorem 2.13.
8. a uniparametric family of Riemann surfaces $\mathcal{M}_5^{20,2}$ with automorphism group D_{10} given in Theorem 2.12.
9. a uniparametric family of Riemann surfaces $\mathcal{M}_5^{16,13}$ with automorphism group$G_{4,4}$ given in Theorem 2.5.
10. a uniparametric family of Riemann surfaces $\mathcal{M}_5^{12,10}$ with automorphism group $C_3 \rtimes C_4$ given in Theorem 2.9.
11. a family of Riemann surfaces $\mathcal{M}_5^{8,4}$ with automorphism group $C_4 \times C_2$ given in Theorem 2.4, case 4.
12. a family of Riemann surfaces $\mathcal{M}_5^{8,8}$ with automorphism group $C_4 \times C_2$ given in Theorem 2.4, case 8.
13. a family of Riemann surfaces $\mathcal{M}_5^{8,13}$ with automorphism group $C_2 \times C_2 \times C_2$ given in Theorem 2.4, case 13.
14. a family of Riemann surfaces $\mathcal{M}_5^{6,1}$ with automorphism group C_6 given in Theorem 2.8, case 1.
15. a family of Riemann surfaces $\mathcal{M}_5^{4,3}$ with C_4 given in Theorem 2.3, case 3.

16. a family of Riemann surfaces $\mathcal{M}_5^{4,7}$ with automorphism group $C_2 \times C_2$ given in Theorem 2.3, case 7.

Remark 3.2. From Theorem 3.24 and Theorem 3.28 it follows that a cyclic trigonal Riemann surface of genus 5 has at most 30 automorphisms and with exception of one surface, X_{30}, at most six automorphisms. The structure of the trigonal surfaces in $\mathcal{M}_5$ is

$$X_{30} \in \overline{\mathcal{M}}_5^{6,2} \subset \overline{\mathcal{M}}_5^{3,0}$$

where X_{30} is a single surface with automorphism group $D_5 \times C_3$ given in Theorem 2.12. Observe that if $g \not\equiv 2 \; mod \; 3$ there are cyclic trigonal surfaces of genus g having $C_{3(g+1)}$ as a group of automorphisms. See [7].

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4

Paper: On the Connectedness of the Branch Locus of the Moduli Space of Riemann Surfaces of Low Genus

Authors: Gabriel Bartolini and Milagros Izquierdo.
The paper has been submitted.

The article is removed due to Copyright grounds.

Appendix A

List of Groups

Here we will list the non-abelian groups used in this thesis. We list them by their *Small-Groups Library* identity, a structure description and a presentation.

Groups of order 8

(8,3) $D_4 = \langle a, s | a^4 = s^2 = (as)^2 = 1 \rangle$

(8,4) $Q = \langle i, j | i^4 = 1, i^2 = j^2, iji = j \rangle$

Groups of order 16

(16,7) D_8

(16,11) $D_4 \times C_2$

(16,8) $C_8 \rtimes_3 C_2 = \langle a, s | a^8 = s^2 = 1, as = sa^3 \rangle$

(16,6) $C_8 \rtimes_5 C_2 = \langle a, s | a^8 = s^2 = 1, as = sa^5 \rangle$

(16,4) $C_4 \rtimes C_4 = \langle s, t | s^4 = t^4 = 1, st = ts^3 \rangle$

(16,13) $(C_4 \times C_2) \rtimes C_2 = \langle a, b, c | a^4 = b^2 = c^2 = cbca^2b = 1, [a, b] = [a, c] = 1 \rangle$

(16,3) $G_{4,4} = \langle a, b | a^4 = b^2 = (ab)^4 = [a^2, b] = 1 \rangle$

Groups of order 32

(32,27) $C_2^4 \rtimes C_2 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^4 = (ac)^4 = (bc)^2 = (abac)^2 = 1 \rangle$

(32,28) $(C_4 \times C_2 \times C_2) \rtimes C_2 = \langle a, b, c | a^4 = b^2 = c^2 = [a, c] = (bc)^4 = 1, bab = a^{-1} \rangle$

(32,43) $(D_4 \times C_2) \rtimes C_2 = \langle a, b, c | a^2 = b^2 = c^2 = (ab)^8 = (ac)^4 = (bc)^2 = (abac)^2 = 1 \rangle$

(32,2) $(C_4 \times C_2) \rtimes C_4 = \langle s,t | s^4 = t^4 = (st)^4 = 1, [s^2,t] = [t^2,s] = 1 \rangle$

(32,6) $((C_4 \times C_2) \rtimes C_2) \rtimes C_2 = \langle s,t | s^4 = t^2 = (st)^4 = [st]^2 = 1 \rangle$

(32,5) $(C_8 \times C_2) \rtimes C_2 = \langle s,t | s^8 = t^2 = 1, (st)^2 = (ts)^2, s^2t = ts^2 \rangle$

(32,7) $(C_8 \rtimes C_2) \rtimes C_2 = \langle s,t | s^8 = t^2 = (s^2t)^2 = [st]^2 = 1 \rangle$

Groups of order 64

(64,8) $((C_8 \times C_2) \rtimes C_2) \rtimes C_2 = \langle a,s,t | a^8 = s^2 = t^2 = 1, a^2s = sa^2, tat = a^7s \rangle$

(64,32) $((C_8 \rtimes C_2) \rtimes C_2) \rtimes C_2 = \langle s,t | a^8 = s^2 = t^2 = (a^2s)^2 = [as]^2 = (st)^2 = 1, tat = as \rangle$

Groups of order 6

(6,1) $D_3 = \langle a,s | a^3 = s^2 = (sa)^2 = 1 \rangle$

Groups of order 12

(12,4) $D_6 = \langle a,s | a^6 = s^2 = (sa)^2 = 1 \rangle$

(12,3) $A_4 = \langle a,b | a^3 = b^3 = (ab^2)^2 = 1 \rangle$

(12,1) $C_3 \rtimes C_4 = \langle s,t | s^4 = t^3 = 1, s^{-1}ts = t^{-1} \rangle$

Groups of order 24

(24,13) $A_4 \times C_2 = \langle a,b | a^2 = b^6 = 1, (ab)^3 = b^3 \rangle$

(24,12) $S_4 = \langle a,b | a^3 = b^4 = (ab)^2 = 1, \rangle$

(24,14) $D_6 \times C_2 = \langle a,b,c | a^6 = s^2 = b^2 = (as)^2 = (sb)^2 = 1, ab = ba \rangle$

(24,8) $(C_6 \times C_2) \rtimes C_2 = \langle a,b | a^2 = b^6 = (ab)^4 = 1, ab^2a = b^4 \rangle$

(24,7) $(C_3 \rtimes C_4) \times C_2 = \langle s,t | s^4 = t^6 = 1, s^{-1}ts = t^{-1} \rangle$

Groups of order 48

(48,14) $(C_{12} \times C_2) \rtimes C_2 = \langle a,b | a^{12} = b^2 = (ab)^4 = 1, ba^2b = a^{10} \rangle$

(48,49) $A_4 \times C_2 \times C_2 = \langle a,b | a^6 = b^2 = [a,b]^2 = 1, a^3b = ba^3 \rangle$

(48,30) $A_4 \rtimes C_4 = \langle a,b | a^3 = b^4 = 1, b^3ab = (b^2a)^2 \rangle$

(48,48) $S_4 \times C_2 \langle a,b,c | a^3 = b^4 = c^2 = (ab)^2 = 1, ac = ca, bc = cb \rangle$

Groups of order 96

(96,195) $(A_4 \times C_2 \times C_2) \rtimes C_2 = \langle a,b,c | a^6 = b^2 = c^2 = (bc)^4 = 1, (ab)^2 = (ba)^2, (ca)^3 = a^3 \rangle$

(96,3) $((C_4 \times C_2) \rtimes C_4) \rtimes C_3 = \langle s,t | s^3 = t^4 = (ts)^3 = sts^2t^2s^2tst^3 = 1 \rangle$

Groups of order 192

(192,181) $(((C_4 \times C_2) \rtimes C_4) \rtimes C_3) \rtimes C_2$
$\langle s, t | s^3 = t^8 = (ts)^4 = (st^2)^3 = (st^5)^2 = (st^6 s^2 t^7)^2 = 1 \rangle$

Groups of order 10

(10,1) $D_5 = \langle a, s | a^5 = s^2 = (sa)^2 = 1, \rangle$

Groups of order 20

(20,4) $D_{10} = \langle a, s | a^{10} = s^2 = (sa)^2 = 1, \rangle$

(20,1) $C_5 \rtimes C_4 = \langle a, b | a^4 = b^5 = a^{-1}ba = b^{-1} \rangle$

Groups of order 30

(30,2) $D_5 \times C_3 = \langle a, b | a^5 = b^6 = 1, b^{-1}ab = a^{-1} \rangle$

Groups of order 40

(40,5) $D_5 \times C_4 = \langle a, b | a^{20} = b^2 = (ba)^4 = 1, b^a b = a^9 \rangle$

Groups of order 60

(60,5) $A_5 = \langle a, b | a^5 = b^3 = (ab^2)^3 = (a^2b)^2 = 1 \rangle$

Groups of order 80

(80,49) $(C_2^4) \rtimes C_5 = \langle a, b | a^5 = b^2 = (ab)^5 = [a, b]^2 = 1 \rangle$

Groups of order 120

(120,35) $A_5 \times C_2 = \langle a^5 = b^3 = c^2 = (ab^2)^3 = (a^2b)^2 = 1, ac = ca, bc = cb \rangle$

Groups of order 160

(160,234) $((C_2^4) \rtimes C_5) \rtimes C_2 = \langle a, b | a^5 = b^4 = (ab^3)^4 = (ab)^2 = 1 \rangle$

Appendix B

GAP programs

Here we will list some sample GAP code used to generate and classify actions. First a program to find all epimorphisms to from groups with given signature to groups of given order.

```
# signature (0;2,3,8), groups of order 192

for G in AllSmallGroups(192) do
    cc2reps:=[];
    order2:=[];
    cc3reps:=[];
    order3:=[];

#sorting out elements of order 2 and 3.

    for cc in ConjugacyClasses(G) do
        if Order(Representative(cc))=2 then
            Add(cc2reps,Representative(cc));
            for g in cc do Add(order2, g); od;
        fi;
        if Order(Representative(cc))=3 then
            Add(cc3reps,Representative(cc));
            for g in cc do Add(order3, g); od;
        fi;
    od;

#for each pair of elements, g,h, of order 2 and 3 check if they
#generate G and if g*h has order 8. If true then save information
#about the group.

    for g in cc2reps do
        for h in order3 do
```

```
            if Order(Subgroup(G,[g,h]))=192 then
                if  Order(g*h)=8 then
                    AppendTo("Order192ep.txt", IdSmallGroup(G),
                    " ", StructureDescription(G), "\n");
                fi;
            fi;
        od;
    od;
od;
```

After finding the possible groups we want to find all classes of epimorphisms. The following program finds all generating vectors of a given group G and classify them up to the action of the automorphism group of G. Then we can manually find elements of $\mathcal{B}$ reducing the number of classes.

```
#choosing a suitable generating set for the group.

G:=SmallGroup(192,181);
GS:=Group(MinimalGeneratingSet(G)[1], MinimalGeneratingSet(G)[2]);
P:=PresentationViaCosetTable(GS);
SimplifyPresentation(P);
TzPrintPresentation(P);
IdSmallGroup(GS);
StructureDescription(GS);

cc2reps:=[];
order2:=[];
cc3reps:=[];
order3:=[];
cc8reps:=[];
order8:=[];
epis:=[]; #list for all epimorphisms.

#sorting out elements of order 2 and 3.

for cc in ConjugacyClasses(GS) do
    if Order(Representative(cc))=2 then
        Add(cc2reps,Representative(cc));
        for g in cc do Add(order2, g); od;
    fi;
    if Order(Representative(cc))=3 then
        Add(cc3reps,Representative(cc));
        for g in cc do Add(order3, g); od;
    fi;
od;

#for each pair of elements, g,h, of order 2 and 3 check
#if they generate G and if g*h has order 8.
#If true then save information about the epimorphism.

for g in cc2reps do
```

```
    for h in order3 do
        if Order(Subgroup(GS,[g,h]))=192 then
            if  Order(g*h)=8 then
                Add(epis, [g,h,(g*h)^-1]);
            fi;
        fi;
    od;
od;

#sort the generating vectors in classes by the action of Aut(GS).

A:=AutomorphismGroup(GS);
epiclasses:=Orbits(A, epis, OnTuples);
for epiclass in epiclasses do
    AppendTo("238to192181.txt",
        [Factorization(GS, Representative(epiclass)[1]),
        Factorization(GS, Representative(epiclass)[2]),
        Factorization(GS, Representative(epiclass)[3])],"\n");
od;
```

To find the coverings of the strata we check the induced actions of subgroups.

```
#choosing the same generating set as before

G:=SmallGroup(192,181);
GS:=Group(MinimalGeneratingSet(G)[1], MinimalGeneratingSet(G)[2]);
P:=PresentationViaCosetTable(GS);
SimplifyPresentation(P);
TzPrintPresentation(P);

#a representative of the class of generating vectors

G1:=Group(GS.1*GS.2^5, GS.1, GS.2^5);

#for every maximal subgroup H of G compute the permutations
#of the cosets of H by the elements of the generating vector.
#Listing the group information, coset permutations and coset
#representatives.

for SG in MaximalSubgroups(GS) do
    H:=Subgroup(GS, MinimalGeneratingSet(SG));
    Hgen:=[];
    for h in MinimalGeneratingSet(SG) do
        Add(Hgen, Factorization(GS, h));
    od;
    AppendTo("238subgroups.txt", IdSmallGroup(H)," ",
        StructureDescription(H), "\n");
    AppendTo("38subgroups.txt", Hgen, "\n");
    AppendTo("238subgroups.txt",
        Action(G1,RightCosets(G1,H),OnRight), "\n");
    cosetrep:=List(RightTransversal(G1,H),
```

```
        i->Factorization(GS,CanonicalRightCosetElement(H,i)));
    AppendTo("238subgroups.txt", cosetrep, "\n");
    AppendTo("238subgroups.txt", "\n", "\n");
od;
```

To show that the signature $(0;3,3,3,3)$ induces two different classes of actions we used this program.

```
#generators of the braid group, signature is of type (0;n,n,n,n)

Braid1:= function(list1)
return [list1[2],list1[2]^-1*list1[1]*list1[2],list1[3],list1[4]];
 end;;

Braid2:= function(list1)
return [list1[1],list1[3],list1[3]^-1*list1[2]*list1[3],list1[4]];
 end;;

Braid3:= function(list1)
return [list1[1],list1[2],list1[4],list1[4]^-1*list1[3]*list1[4]];
 end;;

#################################################################

G:=SmallGroup(12,3);
GS:=Group(MinimalGeneratingSet(G)[1],
    MinimalGeneratingSet(G)[1]*MinimalGeneratingSet(G)[2]);
P:=PresentationViaCosetTable(GS);
SimplifyPresentation(P);
TzPrintPresentation(P);
Display(StructureDescription(GS));

#letting Aut(G) act on the generating vector (a,a,ab,b^2)
A:=AutomorphismGroup(GS);
epiclass2 := [];
for a in A do
    Add(epiclass2, [GS.1^a, GS.1^a, (GS.1*GS.2)^a, (GS.2^-1)^a]);
od;

#letting B act on the generating vector (a,a^2,b,b^2)

braidorbit:=[];
Add(braidorbit, [GS.1, GS.1^-1, GS.2, GS.2^-1]);
for point in braidorbit do
    if (Braid1(point) in braidorbit) <> true then
        Add(braidorbit, Braid1(point));
    fi;
    if (Braid2(point) in braidorbit) <> true then
        Add(braidorbit, Braid2(point));
    fi;
    if (Braid3(point) in braidorbit) <> true then
```

```
        Add(braidorbit, Braid3(point));
    fi;
od;

#comparing the lists to see if there are any common elements

for epi in braidorbit do
    if epi in epiclass2 then
        Display(epi);
    fi;
od;
```