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Fredrik Larsson, Michael Felsberg and Per-Erik Forssén, Patch Contour Matching by Correlating Fourier Descriptors, 2009, Digital Image Computing: Techniques and Applications (DICTA), Melbourne, Australia, December 2009. IEEE Computer Society.

Postprint available at: Linköping University Electronic Press

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Patch Contour Matching by Correlating Fourier Descriptors

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Abstract—Fourier descriptors (FDs) is a classical but still popular method for contour matching. The key idea is to apply the Fourier transform to a periodic representation of the contour, which results in a shape descriptor in the frequency domain. Fourier descriptors have mostly been used to compare object silhouettes and object contours; we instead use this well established machinery to describe local regions to be used in an object recognition framework. We extract local regions using the Maximally Stable Extremal Regions (MSER) detector and represent the external contour by FDs. Many approaches to matching FDs are based on the magnitude of each FD component, thus ignoring the information contained in the phase. Keeping the phase information requires us to take into account the global rotation of the contour and shifting of the contour samples. We show that the sum-of-squared differences of FDs can be computed without explicitly de-rotating the contours. We compare our correlation based matching against affine-invariant Fourier descriptors (AFDs) and demonstrate that our correlation based approach outperforms AFDs on real world data.

Keywords—Fourier descriptors, shape matching;

I. INTRODUCTION

Fourier descriptors (FDs) [1] is a classic and still popular method for contour matching. The key idea is to apply the Fourier transform to a periodic representation of the contour, which results in a shape descriptor in the frequency domain. The low frequency components of the descriptor contain information about the general shape of the contour while the finer details are described in the high frequency components. Commonly, a one-dimensional parameterization of the boundary is used which enables the use of the 1D Fourier transform. Higher dimensional approaches have also been used, e.g. Generalized Fourier descriptors which describe a surface by 2-D Fourier transform [2]. Different ways for one-dimensional parameterization of the boundary, e.g. use of curvature, distance to the shape centroid, representing the boundary coordinates as complex numbers etc. have been used with FDs [3].

Traditionally FDs have been used to compare contours. In this paper we use this well established machinery to describe local regions to be used in an object recognition framework. A similar approach has been used by Lietner [4] who used modified FDs in parallel with SIFT features [5] for object recognition. We extract local regions using the Maximally Stable Extremal Regions (MSER) detector [6]. The contours of these regions are either sampled uniformly

according to the affine arc length criterion, see section III, or transformed with a similarity frame and then sampled in this canonical frame. We restrict the frame to similarity transformations, i.e. we roughly compensate for translation, scale and rotation, in order to keep the aspect ratio and hence to have a greater chance of separating e.g. rectangles of different aspect ratios.

In section II we review the theory behind Fourier descriptors. In section IV we address the matching of FDs and explain why matching on magnitudes only is inferior to keeping the phase information. We introduce our matching scheme and a preselection step to remove ambiguous descriptors. In section V we compare our work to the Affine-invariant Fourier descriptors (AFDs) [7] on three datasets: Leuven, Boat, and Graf. Finally, in section VI we conclude and discuss future work.

II. FOURIER DESCRIPTORS

In line with Granlund [1], the closed contour c with coordinates x and y is parameterized as a complex valued periodic function

$$c(l) = c(l + L) = x(l) + iy(l), \quad (1)$$

where L is the contour length, usually given by the number of contour samples.¹ By taking the 1D Fourier transform of c , the Fourier coefficients C are obtained as

$$C(n) = \frac{1}{L} \int_{l=0}^L c(l) \exp(-\frac{i2\pi nl}{L}) dl, \quad (2)$$

where $N \leq L$ is the descriptor length. A strength of FDs is their behavior under geometric transformations. The DC component $C(0)$ is the only one that is affected by translations c_0 of the curve $c(l) \mapsto c(l) + c_0$. By disregarding this coefficient, the remaining $N-1$ coefficients are invariant under translation. Scaling of the contour, i.e. $c(l) \mapsto ac(l)$, affects the magnitude of the coefficients and the coefficients can thus be made scale invariant by normalizing with the energy (after $C(0)$ has been removed). Without loss of generality, we assume that $\|C\|^2 = 1$ ($\|\cdot\|^2$ denoting the quadratic norm) and $C(0) = 0$ in what follows.

¹We treat contours as continuous functions here, where the contour samples can be thought as of impulses with appropriate weights.

Rotating the contour c with ϕ radians counter clockwise corresponds to multiplication of (1) with $\exp(i\phi)$, which adds a constant offset to the phase of the Fourier coefficients

$$c(l) \mapsto \exp(i\phi)c(l) \quad \Rightarrow \quad C(n) \mapsto \exp(i\phi)C(n) . \quad (3)$$

Furthermore, if the index l of the contour is shifted by Δl , a linear offset is added to the Fourier phase, i.e. the spectrum is modulated

$$c(l) \mapsto c(l-\Delta l) \quad \Rightarrow \quad C(n) \mapsto C(n) \exp\left(-\frac{i2\pi n\Delta l}{L}\right) . \quad (4)$$

When we use the term shift we always refer to a shift in the starting point, this should not be confused with translation which we use to denote spatial translation of the entire contour.

III. SAMPLING OF THE CONTOUR

We use two different approaches when sampling the contour of a region; uniform and uniform according to a first order approximation to the affine arc length. In order to use the affine arc length we reparametrize the contour according to a first order approximation [7]

$$t = \frac{1}{2} \int_l |x(l)\dot{y}(l) - y(l)\dot{x}(l)| dl . \quad (5)$$

Where $\dot{x}(l)$ and $\dot{y}(l)$ denotes the derivative in the x and y direction and $x(l), y(l)$ denotes the x and y coordinates. We then sample the contour at unit steps according to the new parameter t . We use a regularized derivative for estimating $\dot{x}(l)$ and $\dot{y}(l)$.

IV. MATCHING OF FOURIER DESCRIPTORS

Since rotation and index-shift result in modulations of the FD, it has been suggested to neglect phase information in order to be invariant to these transformations. However, as pointed out by Oppenheim and Lim [8], most information is contained in the phase and simply neglecting it means to throw away information. Matching of magnitudes ignores a major part of the signal structure such that the matching is less specific. According to (3) and (4), the phase of each FD component is modified by a rotation of the corresponding trigonometric basis function, either by a constant offset or by a linear offset. Considering the magnitudes only can be seen as finding the optimal rotation of all different components of the FD independently. That is, given a FD of length $N-1$, magnitude matching corresponds to finding $N-1$ different rotations instead of estimating two degrees of freedom (constant and slope). Due to the removal of $N-3$ degrees of freedom, two contours can be very different even though the magnitude in each FD component is the same, see figure 1.

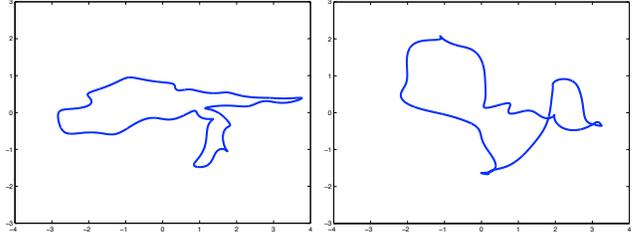


Figure 1. Both contours have the same magnitude in each Fourier coefficient. The only difference is contained in the phase. A magnitude based matching scheme would return a perfect match.

A. FD Matching Methods

Few authors made considerable efforts to really use the phase when matching FDs. In the original work [1] Granlund proposes two different methods for taking into account the global rotation. However, there is no discussion on the effect of phase changes due to shifting the starting point, which we consider to be the more interesting problem. Persoon and Fu [9] address shifting and present a technique for estimating the least squares error for rotation, scale change and shift of the starting point. As such, their approach is closely related to ours, but they compute the minimum by numerically finding the roots of the respective derivatives of the quadratic error.

Kuhl and Giardina [10] base their matching on de-rotating the FD according to the angles estimated from the first order harmonics.² Obviously, this only works if the first harmonic locus is elliptic and in case of a circular first harmonic locus, the de-rotation requires an orientation estimate from the spatial (contour) domain: The orientation of the point with maximal distance to the center point $c(l) - c_0$ is used for de-rotation. The classification into circular and elliptic loci is obviously a matter of the noise level, i.e., the method might accidentally classify a circular domain as elliptic such that the orientation becomes arbitrary. Furthermore, very thin and lengthy structures are more or less invisible to the first order harmonics, but have a huge impact on the spatial orientation estimation variant. The pathologic case is a triangle with a very thin spike at an arbitrary position. If the triangle is equilateral, the orientation depends only on the spike, and if the triangle is slightly elongated, it is given by the largest median. Changing the triangle continuously from the former to the latter case gives a discontinuity in the orientation estimate, and thus, a poor matching result between two triangles belonging to the first and second case respectively.

Bartolini et al. have a different approach of utilizing the phase information [11]. They normalize the phase information in the descriptor (similar to [10]), and when comparing two descriptors they first use the inverse Fourier transform to reconstruct the contours. They later apply dynamic time

²Actually this method has also been considered in [9].

warping in order to obtain a matching score for these reconstructed contours. In contrast to this approach, our method performs matching in the Fourier domain.

Arbter et al. proposed the Affine-Invariant Fourier descriptor [7]. They keep the phase information (depending of the order) and through a product form generate a descriptor that is invariant to affine transformations. They sample the contour uniformly according to the first order approximation of the affine arc length criterion before the descriptor is extracted. This is something we have adopted and evaluated in combination with our correlation based approach. We reimplemented the work of Arbter et al. in order to be able to compare our correlation approach to the affine-invariant Fourier descriptor. We have confirmed that our AFD implementation works as intended on synthetic data. We extracted contours from one of our test images and then applied affine transformations and index shift on each contour. On these synthetic tests we got perfect precision-recall curves even under very challenging conditions such as severe foreshortening. El Oirak et al. also propose an affine invariant normalization of FDs [12], [13] but we do not see any significant difference between their work and the work of Arbter et al.

B. Correlation-Based FD Matching

Our approach differs in two respects from the method in [10]: First, we make use of complex FDs and avoid matrix notation. The components a, b, c, d in [10] correspond to symmetric and antisymmetric parts of the real and imaginary part of the FD. Second, we do not try to de-rotate the FDs, but we aim to find the relative rotation between two FDs, such that the matching result is maximized – similar to [9], but avoiding numerical techniques. Virtually, this is done by cyclic correlation of the contours, but due to the complex-valued FDs that we use, the same effect is achieved by multiplying the FDs point-wise. We start with the complex correlation theorem [14], p. 244–245,

$$\int_{-\infty}^{\infty} \bar{c}_1(l)c_2(\Delta l+l) dl = \int_{-\infty}^{\infty} \bar{C}_1(n)C_2(n) \exp(i2\pi n\Delta l) dn \quad (6)$$

By replacing the infinite integral on the lhs with a finite integral, we obtain

$$(c_1 \star c_2)(\Delta l) \doteq \int_0^L \bar{c}_1(l)c_2(\Delta l+l) dl \quad (7)$$

$$= \sum_{n=0}^{\infty} \bar{C}_1(n)C_2(n) \exp\left(\frac{i2\pi n\Delta l}{L}\right) \quad (8)$$

If we replace the inverse Fourier series on the rhs with a truncated series of length N , we still obtain the least-squares approximation of the lhs. Surprisingly, this has never been exploited in context of FD matching before.

We use the correlation theorem to compute the least-squares match of two Fourier descriptors without explicitly

estimating the parameters. We start with assuming $c_2(l) = \exp(i\phi)c_1(l - \Delta l)$. We compute an approximation of the correlation $r_{12} = (c_1 \star c_2)$ as the finite inverse Fourier transform \mathcal{F}^{-1}

$$r_{12} \approx \mathcal{F}^{-1}\{\bar{C}_1 \cdot C_2\} \doteq \sum_{n=0}^N \bar{C}_1(n)C_2(n) \exp\left(\frac{i2\pi n\Delta l}{L}\right) \quad (9)$$

The least-squares error of matching c_1 and c_2 under rotations and shifting the origin (symbolized as \mathcal{T}) is given as ($|\cdot|$ denotes the complex modulus)

$$\min_{\mathcal{T}} \|c_1 - \mathcal{T}c_2\|^2 \approx 2 - 2 \max_l |r_{12}(l)| \quad (10)$$

If the parameters of \mathcal{T} are to be extracted, they are obtained as the position and the phase angle of the maximum:

$$\Delta l \approx \arg \max_l |r_{12}(l)| \quad (11)$$

$$\phi \approx \arg(r_{12}(\Delta l)) \quad (12)$$

All approximations become equalities in the case $N = \infty$.

We will show the previous equalities under this assumption. The approximation properties then follow from the least-squares optimality of Fourier series. We start with computing the cross-correlation of c_1 and c_2 via FDs

$$r_{12} = \mathcal{F}^{-1}\{\bar{C}_1 \cdot C_2\} \quad (13)$$

$$= \mathcal{F}^{-1}\{\bar{C}_1(n) \exp i\phi C_1(n) \exp(-\frac{i2\pi n\Delta l}{L})\} \quad (14)$$

$$= \exp i\phi \mathcal{F}^{-1}\{|C_1(n)|^2 \exp(-\frac{i2\pi n\Delta l}{L})\} \quad (15)$$

$$= \exp i\phi r_{11}(l - \Delta l) \quad (16)$$

Since the auto-correlation function r_{11} is real-valued and has its maximum at 0, the estimates for Δl and ϕ are obtained according to (11) and (12). For the least-squares error, we obtain due to the normalized FDs

$$\begin{aligned} \|c_1 - \mathcal{T}c_2\|^2 &= \|c_1\|^2 + \|\mathcal{T}c_2\|^2 - 2(c_1 \star \mathcal{T}c_2)(0) = \\ &= 2 - 2 \exp(-i\phi) (c_1 \star c_2)(\Delta l) = 2 - 2|(c_1 \star c_2)(\Delta l)| \end{aligned} \quad (17)$$

If c_2 is not a transformed c_1 , the maximum cross-correlation will be smaller than 1 and the matching result will be given by the optimal relative de-rotation and shift of the origin.

C. Preselection

Before we match the Fourier descriptors of regions in two images, we try to remove ambiguous descriptors. As a criterion for this we use the minimum error against all other regions in the same image e_{\min} . If $e_{\min} < T_{\text{err}}$ this particular FD is removed. The minimum error is given as

$$e_{\min} = \min_{i \neq j} \min_{\mathcal{T}} \|c_i - \mathcal{T}c_j\|^2 \quad (18)$$

where e_{\min} is estimated according to (10). Not only do we remove non-discriminative descriptors, we also reduce the computational time by keeping only a subset of the available descriptors.

D. Postprocessing

After having removed the ambiguous FDs within each image, we match the remaining ones between the images. Inspired by Lowe [5], we compute the error ratio e_r between the minimum error and the second to minimum error

$$e_r = \frac{e_{\min}}{e_{\text{sec}}} . \quad (19)$$

We use this error ratio as a way to remove insignificant matches. Experimentally we have found that a threshold of $T_r = 0.50$ returns 90% correct matches for FDs with our matching based on correlation. We do the matching in a symmetric way, i.e. we accept a match only if c_1 in image 1 matches with c_2 in image 2 and c_2 in image 2 matches with c_1 in image 1. The error ratio associated with c_1 is given as the higher one of the two error ratios.

V. EXPERIMENTS

A common approach for object recognition and pose estimation is to use local affine features. Features are extracted from views that are to be compared. These local features are then usually used in a voting scheme to find the object or pose hypothesis. We aim to use FDs in an object recognition framework, and evaluate our approach on the Leuven, Graf and Boat dataset [15], see Fig. 2. These are common benchmarking sets used for testing local descriptors. The homography relating two images in

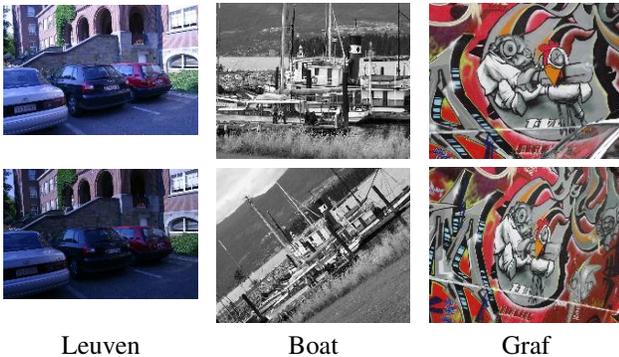


Figure 2. The first and third image from each of the three test sets used for evaluation.

a sequence are also available. This homography is used to estimate how one local region would be transformed into the corresponding view. We consider a reported match correct if it corresponds to a match given by the overlap-criterion used by Mikolajczyk et al. [15]. The given homographies are used solely for generating ground truth. The subsequent precision-recall curves were generated by varying the threshold for the error ratio e_r .

As mentioned earlier, we use MSER to detect local regions and two different approaches for sampling the contour. The first approach use the affine length criterion while the second approach transforms the region into a

canonical frame before sampling. The different steps of region detection and transformation into a canonical frame are shown in fig 3.



Figure 3. **Upper:** The original view. **Lower left:** The contour extracted in the original view. **Lower right:** The contour shown in its canonical frame.

We estimate the FDs for all MSER³ regions in each image and compare the images pairwise. We have evaluated different combinations of Fourier descriptors (Affine Fourier descriptors of order 0 and 1 (AFD0, AFD1) and ordinary Fourier descriptors with and without phase information $FD/abs(FD)$), sampling methods (Affine or Canonical) and matching methods (sum of squared differences (SSD), our new correlation method (Corr)). Hence, *FD Corr/Canonical* denotes ordinary Fourier descriptors with phase information sampled in the canonical frame and matched by our correlation method. For all methods we keep the 51 Fourier coefficients corresponding to the lowest frequencies.

Figure 4-6 show precision-recall curves for the three different data sets. We did not use the minimum error preselecting criteria when generating the precision-recall curves. We did evaluate the performance with the preselection for a few selected combinations. The resulting precision-matching curves can be seen in Fig. 7-9.

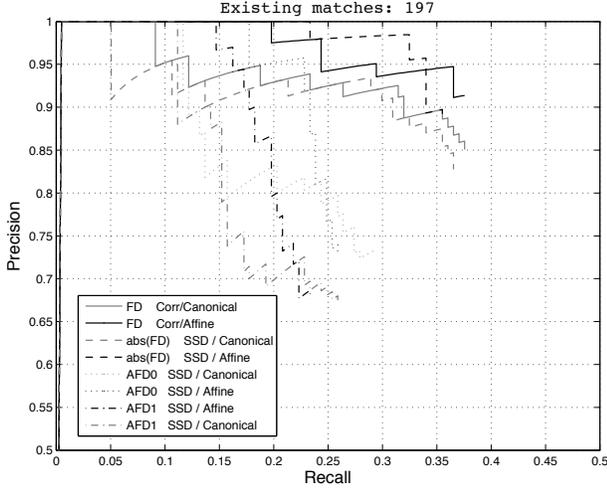


Figure 4. Precision-recall curves for the Leuven dataset.

A. Precision-recall without preselection

1) *Precision-recall on the Leuven dataset:* Fig. 4 shows the precision-recall curves for the Leuven dataset. We match FDs in the first image of the dataset versus the FDs from the other five images, one image at the time. The top performers are abs(FD) SSD / Affine followed by FD Corr / Affine. It should be noted that the Leuven dataset is supposed to test for lighting changes⁴ only. Hence, the rotation between the different images changes very little.

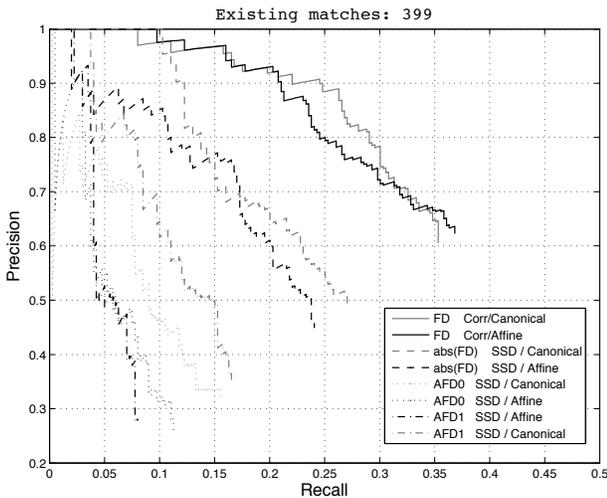


Figure 5. Precision-recall curves for the Boat dataset.

2) *Precision-recall on the Boat dataset:* Fig. 5 shows the precision-recall curve for the Boat dataset. The curves shown

³The parameters used for the MSER method were minimum margin = 30 and minimum region size = 50. We used these values for all experiments.

⁴In reality, the camera aperture and not scene illumination has been changed.

are the cumulative results when matching the first image to the other five. The Boat dataset contains transformations due to zoom and rotation. We can separate the methods into three groups. The group with lowest precision-recall result contains all the AFD versions, the second best group contains both phase neglecting versions of the original Fourier descriptors and the best performers are the original Fourier descriptors when using our new correlation based matching.

3) *Precision-recall on the Graf dataset:* Fig. 6 contains the precision-recall curves for the first image pair in the Graf dataset, which corresponds to roughly 10 degrees change. Once again, the correlation based matching schemes performs best, the Corr/Affine method followed by the FD Corr/Canonical. For larger view changes the performance decreases but the ordering of the methods stays the same. The breakdown in performance is expected and can be explained by effects such as foreshortening that is not fully compensated for despite the affine sampling.

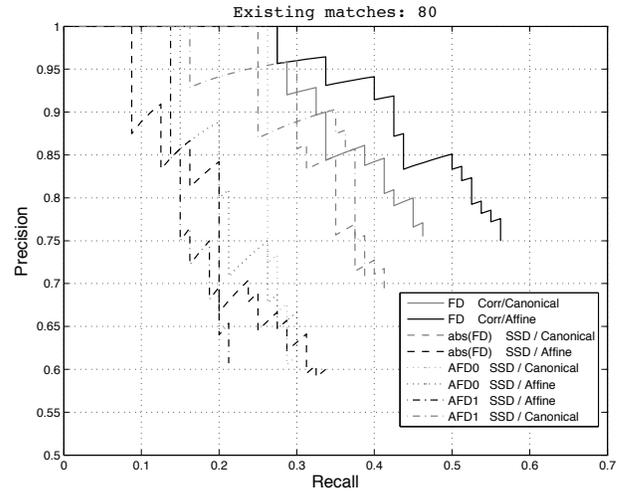


Figure 6. Precision-recall curves for the first image pair in the Graf dataset.

B. Precision with preselection

We further evaluated the performance of AFD1 SSD/Affine, AFD0 SSD/Affine, FD Corr/Affine and FD Corr/Canonical when incorporating the preselection criteria. We optimized the threshold for each method individually and the thresholds we use are $T_{\text{err}} = 10^{-4}$ for AFD0 SSD / Affine, $T_{\text{err}} = 5 \times 10^{-4}$ for AFD1 SSD/Affine, $T_{\text{err}} = 10^{-3}$ for FD Corr/Canonical and $T_{\text{err}} = 10^{-3}$ for FD Corr/Affine. Since we remove different amounts of descriptors and also descriptors belonging to different regions for each FD method, we cannot generate fair precision-recall curves. We have instead generated precision-match curves. This allows us to see the precision but also the amount of matches kept by each method.

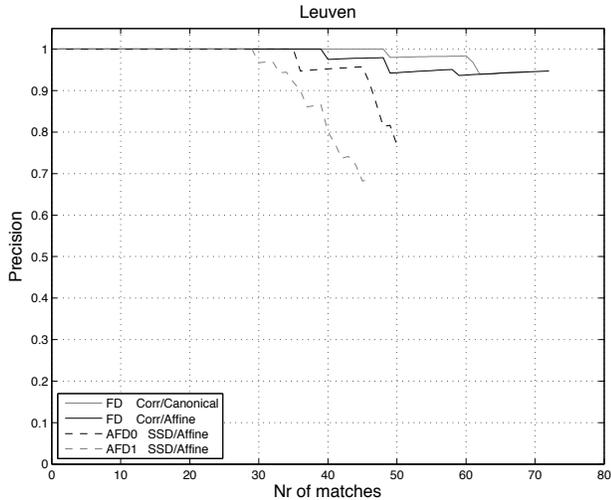


Figure 7. Precision as a function of matches for the Leuven dataset.

1) *Precision with preselection on the Leuven dataset:* We see in Fig. 7 that the precision for AFD0 and AFD1 goes below 90% around 45 and 35 matches. The correlation based FDs are performing equally well and the precision of the reported matches never go below 90%. This is also true for the Corr/Affine method even without the preselection, which can be seen in Fig. 4. However, all four methods performs well on this set.

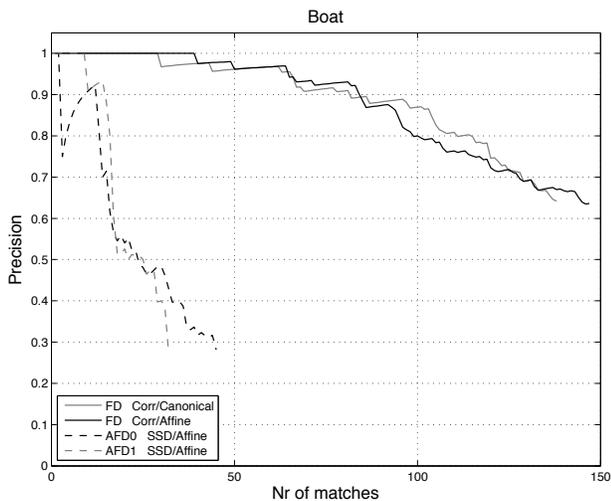


Figure 8. Precision as a function of matches for the Boat dataset.

2) *Precision with preselection on the Boat dataset:* In Fig. 8 we see a big difference between the AFD and FD corr methods. Both the precision and number of matches are well higher for the correlation based methods. We cannot see any big difference between the canonical and affine sampling method for this dataset.

3) *Precision with preselection on the Graf dataset:* The FD Corr/Affine is the method performing best while both the AFD methods perform worst, as can be seen in Fig. 9. For this dataset we can see some difference between the affine and canonical sampling approach. We also saw this trend in the Graf test without the preprocessing. The reason that we can only separate the two approaches on the Graf dataset is likely due to the fact that it is the only one that contains some, minor, amounts of foreshortening affects. The Leuven and Boat datasets are relatively free from this.

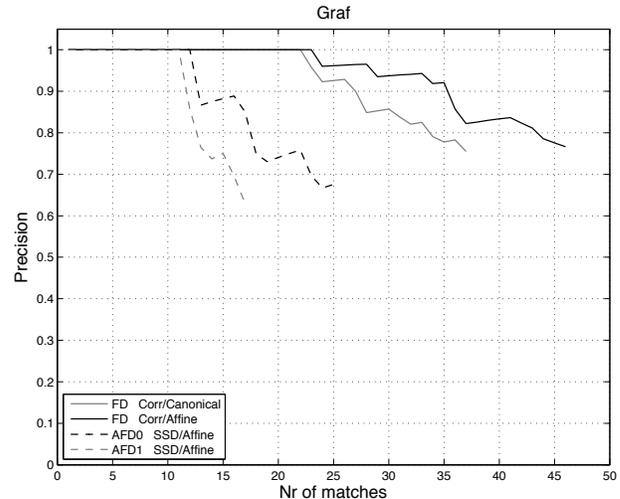


Figure 9. Precision as a function of matches for the Graf dataset.

VI. DISCUSSION AND FUTURE WORK

We show that the sum-of-squared differences of Fourier descriptors can be computed without explicitly de-rotating the contours using a correlation-based technique. We conclude that using Fourier descriptors to describe the shape of local regions is an efficient approach, both in matching precision-recall and in speed. Precision-recall is significantly boosted by keeping the phase information. Computational speed benefits from the computation in Fourier domain. We suggest FDs to be used in combination with e.g. a texture descriptor, since the latter captures different aspects of the region than the FDs.

The standard approach for matching local regions is to cut out patches and describe them, e.g., using the SIFT descriptor. However, this approach has shown to be problematic when dealing with 3-D scenes with varying background [16]. For the future, we plan to apply Fourier descriptors for region matching in 3-D scenes, where the foreground patch contours are described with FDs.

We have shown that using affine sampling in combination with our correlation based matching of Fourier descriptors outperforms affine invariant Fourier descriptors on real world data. The affine invariant Fourier descriptors achieves perfect results on synthetic data but performs poorly under real world conditions.

Concluding all our experiments we see that the canonical approach is at most as good as the affine approach. We recommend to use uniform sampling according to the affine length criterion over uniform sampling in a canonical frame, because the affine sampling approach does not require the region extraction method to produce an estimate of the canonical frame. Hence, no problems with circular regions occur and a larger choice of methods for region or contour extraction is available, such as active contours.

ACKNOWLEDGMENT

The research leading to these results has received funding from the European Community's Seventh Framework Programme (FP7/2007-2013) under grant agreement n^o 215078 DIPLECS and from the Swedish Research Council through a grant for the project *Embodied Visual Object Recognition*.

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