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Maximal order of automorphisms of trigonal Riemann surfaces

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Abstract

In this paper we find the maximal order of an automorphism of a trigonal Riemann surface of genus g , $g \geq 5$. We find that this order is smaller for generic than for cyclic trigonal Riemann surfaces, showing that generic trigonal surfaces have "less symmetry" than cyclic trigonal surfaces. Finally we prove that the maximal order is attained for infinitely many genera in both the cyclic and the generic case.

To Professor José Maria Montesinos

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1 Introduction

A closed Riemann surface X which can be realized as a 3-sheeted covering of the Riemann sphere $f : X \rightarrow \widehat{\mathbb{C}}$ is said to be *trigonal*, and such a covering f will be called a *trigonal morphism*. This is equivalent to the fact that X is represented by a curve given by a polynomial equation of the form:

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$$y^3 + yb(x) + c(x) = 0.$$

If $b(x) \equiv 0$ then the trigonal morphism is a cyclic regular covering and the Riemann surface is called *cyclic trigonal*. If $b(x) \not\equiv 0$ or equivalently when f is non-cyclic X is said to be a *generic trigonal Riemann surface*.

Let X be a trigonal surface, and let $f : X \rightarrow \widehat{\mathbb{C}}$ be the trigonal morphism. Let S be the set of singular values of f , then, for every $c \in \widehat{\mathbb{C}} - S$, $f^{-1}(c)$ consists of three points. If f is cyclic then for each point $s \in S$, $\#f^{-1}(s) = 1$, i.e. s is an order 3 singular value of f . If f is non-cyclic the points of S can be of two types: singular values of order three or simple singular values, i. e. points $s \in S$, where $\#f^{-1}(s) = 2$. If all the singular values of f are simple we say that f is a *simple trigonal morphism* or a simple covering. Simple coverings play an important role, for instance in the study of the moduli space.

It is a classic result that the maximal order of an automorphism of a Riemann surface of genus g is $4g + 2$ (see [W] and [Ha]). The same maximal order occurs if we restrict our attention to hyperelliptic Riemann surfaces instead of general Riemann surfaces (see [BCGG]). In the present work we study the maximal order of an automorphism of a trigonal Riemann surface. We obtain that such a maximal order is smaller than for general and hyperelliptic Riemann surfaces. In Proposition 3 we obtain that the order of an automorphism of a cyclic trigonal Riemann surface of genus g , $g \geq 5$, is at most $3g + 3$. Groups of automorphisms of cyclic trigonal Riemann surfaces were studied by Bujalance, et al. in [BCG] but very little is known for groups of automorphisms generic trigonal Riemann surfaces. In Proposition 4 we establish that the order of an automorphism of a generic trigonal surface of genus g , $g \geq 5$, is bounded above by $2g + 1$. Thus a generic trigonal Riemann surface has less symmetry than a cyclic trigonal surface or even a hyperelliptic Riemann surface. Among the non-cyclic trigonal surfaces the more important ones are the surfaces admitting a simple trigonal morphism, i.e the trigonal morphism is a simple covering. In this case the maximal order of an automorphism of such a surface becomes smaller, namely $g + 1$.

We provide examples of families of surfaces showing that the bounds obtained are sharp (last part of Proposition 3 and Proposition 6).

2 Preliminaries.

An essential result for our study is that, by the Severi-Castelnuovo inequality, the trigonal morphism of a trigonal Riemann surface of genus g is unique when $g \geq 5$ (see [A]).

We shall use the uniformization theory of Riemann surfaces by Fuchsian groups. A surface Fuchsian group is a Fuchsian group without elliptic or parabolic transformations. Let \mathcal{D} be the unit disc in \mathbb{C} , the following results are characterizations of trigonality by means of Fuchsian groups (see [CI])

Proposition 1 *Let X be a Riemann surface, X admits a cyclic trigonal morphism f if and only if there is a Fuchsian group Δ with signature*

$$(0, [3, 3, g+2, 3]) \quad (1)$$

and an index three normal surface subgroup Γ of Δ , such that Γ uniformizes X , i.e. $X = \mathcal{D}/\Gamma$.

In the conditions of Proposition 1 we shall denote the monodromy epimorphism by $\omega : \Delta \rightarrow C_3 = \Delta/\Gamma$.

Proposition 2 *A Riemann surface X of genus g is generic trigonal if and only if there is a Fuchsian group Δ with signature*

$$(0, [2, 2, u, 2, 3, 3, v, 3]) \text{ where } u + 2v = 2g + 4, u \equiv 0 \pmod{2}, u \neq 0, \quad (2)$$

and an index three non-normal subgroup Γ of Δ , with signature $(g, [2, 2, u, 2])$ such that \mathcal{D}/Γ is conformally equivalent to X .

The covering $f : \mathcal{D}/\Gamma \rightarrow \mathcal{D}/\Delta$ is simple if and only if in the above Proposition the signature of Δ is $(0, [2, 2, 2g+4, 2])$.

In the conditions of Proposition 2 the monodromy epimorphism of the trigonal morphism f is the representation of the action of Δ on the cosets Δ/Γ : $\omega : \Delta \rightarrow \Sigma_3$, where Σ_3 is the symmetric group of three elements $\{0, 1, 2\} \simeq \Delta/\Gamma$ and $\Gamma = \omega^{-1}(\text{Stab}(0))$.

3 Cyclic trigonal Riemann surfaces

Proposition 3 *If X is a cyclic trigonal Riemann surface of genus g , $g \geq 5$ and a is an automorphism of X of order h , then $h \leq 3g + 3$. For every integer $g \not\equiv 2 \pmod{3}$, there is a cyclic trigonal Riemann surface X_g of genus g having an automorphism of order $3g + 3$.*

Proof. Since $g \geq 5$, the trigonal morphism f is unique. The morphism f is induced by the automorphism \hat{f} of X . There is an automorphism $\hat{a} : \hat{\mathbb{C}} \rightarrow \hat{\mathbb{C}}$, of order \hat{h} , such that $f \circ a = \hat{a} \circ f$. Since the automorphism \hat{a} lifts to X then \hat{a} is an automorphism of the orbifold $X/\langle \hat{f} \rangle$. Consider the automorphism \hat{a}_*

induced by \widehat{a} on the fundamental group $\pi_1 O(X/\langle \widehat{f} \rangle)$ of the orbifold $X/\langle \widehat{f} \rangle$, since $\pi_1 O(X/\langle \widehat{f} \rangle) \simeq \Delta$ (where Δ is a Fuchsian group as in Proposition 1), we obtain that $\omega \circ \widehat{a}_* = \omega$, with ω the monodromy of the covering f . Then \widehat{a} preserves the set S of singular values of f , i. e. the $g+2$ singular values. Hence S is a union of orbits of \widehat{a} . Since the orbits of \widehat{a} consist of one point or \widehat{h} points, then \widehat{h} is at most $g+2$. Assume that \widehat{a} has order $g+2$. Thus $X/\langle a \rangle$ is uniformized by a Fuchsian group Λ with signature $(0; [3, g+2, g+2])$ and canonical presentation:

$$\langle x_1, x_2, x_3 : x_1 x_2 x_3 = 1, x_1^3 = x_2^{g+2} = x_3^{g+2} = 1 \rangle.$$

Since the covering $X \rightarrow X/\langle a \rangle = \widehat{\mathbb{C}}$ factorizes by $X \xrightarrow{3:1} \widehat{\mathbb{C}} \xrightarrow{g+2:1} X/\langle a \rangle = \widehat{\mathbb{C}}$, the monodromy $\mu : \Lambda \rightarrow C_{3(g+2)} = \langle \delta : \delta^{3(g+2)} = 1 \rangle$ of the covering $X \rightarrow X/\langle a \rangle$, must satisfy that $\mu(x_2) = \mu(x_3)^{-1}$, but that is impossible, therefore $h \leq 3(g+1)$.

Now let g be an integer such that $g \not\equiv 2 \pmod{3}$. Consider a Fuchsian group Δ_g with signature $(0; [3, g+1, 3(g+1)])$ and canonical presentation:

$$\langle x_1, x_2, x_3 : x_1 x_2 x_3 = 1, x_1^3 = x_2^{g+1} = x_3^{3(g+1)} = 1 \rangle$$

Let $C_{3(g+1)} = \langle \gamma : \gamma^{3(g+1)} = 1 \rangle$ be the cyclic group of order $3(g+1)$ with $g+1 \not\equiv 0 \pmod{3}$. We define the epimorphism:

$$\omega_g : \Delta \rightarrow C_{3(g+1)} \text{ given by } \omega_g(x_1) = \gamma^{(g+1)}, \omega_g(x_2) = \gamma^3, \omega_g(x_3) = \gamma^{2g-1}.$$

The surfaces $X_g = \mathcal{D}/\ker \omega_g$ are cyclic trigonal and have an automorphism of order $3(g+1)$. \square

4 Generic trigonal Riemann surfaces

Proposition 4 *If X is a generic trigonal Riemann surface of genus g , $g \geq 5$ and a is an automorphism of X of order h , then $h \leq 2g+1$. If the trigonal morphism $f : X \rightarrow \widehat{\mathbb{C}}$ is a simple covering then $h \leq g+1$.*

Before proving the proposition we need the following lemma:

Lemma 5 *Let X be a generic trigonal Riemann surface of genus g , $g \geq 5$, let $f : X \rightarrow \widehat{\mathbb{C}}$ be the trigonal morphism and let a be an automorphism of X . The automorphism a is the lift by f of an automorphism $\widehat{a} : \widehat{\mathbb{C}} \rightarrow \widehat{\mathbb{C}}$, moreover the order of \widehat{a} equals the order of a .*

Proof. Since $g \geq 5$ the trigonal morphism is unique and then there is an automorphism \widehat{a} of $\widehat{\mathbb{C}}$ of order \widehat{h} such that $\widehat{a} \circ f = f \circ a$. Thus a is the lift of \widehat{a} and either

$$\text{order of } a = \widehat{h} \quad \text{or} \quad \text{order of } a = 3\widehat{h}.$$

But, if order of $a = 3\widehat{h}$, then $a^{\widehat{h}}$ is an automorphism of the covering $f : X \rightarrow \widehat{\mathbb{C}}$ and this covering has not automorphisms. \square

Note. By Lemma 5 the automorphisms groups of generic trigonal Riemann surfaces are isomorphic to finite groups of $O(3)$.

Proof of the Proposition.

Let X be a generic trigonal Riemann surface. By Proposition 2, there is a Fuchsian group Δ with signature $(0, [2, 2, \dots, 2, 3, 3, \dots, 3])$ such that Δ has an index three non-normal subgroup Γ with signature $(g, [2, 2, \dots, 2])$ and \mathcal{D}/Γ is conformally equivalent to X . There are the restrictions $u + 2v = 2g + 4$, $u \equiv 0 \pmod{2}$, $u \neq 0$. Let $\omega_f : \Delta \rightarrow \Sigma_3$ be the monodromy epimorphism of the trigonal covering $f : X \rightarrow \widehat{\mathbb{C}}$.

Now the automorphism a induces an automorphism \widehat{a} of the orbifold \mathcal{D}/Δ and then an automorphism $\widehat{a}_* : \Delta \rightarrow \Delta$ of the fundamental group of the orbifold. Since \widehat{a} lifts to an automorphism a of X , the automorphism \widehat{a}_* is compatible with the monodromy epimorphism ω_f , that is $\omega_f = \omega_f \circ \widehat{a}_*$. Since the set of elliptic elements of Δ sent by ω_f to a fixed permutation of Σ_3 is a union of orbits of \widehat{a} , the maximal order of an automorphism of the orbifold \mathcal{D}/Δ satisfying $\omega_f = \omega_f \circ \widehat{a}_*$ is $2g + 2$, in the case that the signature of Δ is $(0, [2, 2, \dots, 2, 2])$. Notice that ω_f must be a transitive representation and it is not possible that all generators are sent to the same involution of Σ_3 . The quotient space $(\mathcal{D}/\Delta)/\langle \widehat{a} \rangle$ can be uniformized by a triangular Fuchsian group Ξ of signature $(0, [2, 4g + 4, 4g + 4])$. Thus we have the commutative diagram:

$$\begin{array}{ccc} X_g = \mathcal{D}/\Gamma & \xrightarrow{3:1} & \widehat{\mathbb{C}} = \mathcal{D}/\Delta \\ 2g + 2 : 1 \downarrow & & \downarrow 2g + 2 : 1 \\ X_g/\langle a \rangle = \mathcal{D}/\Lambda & \xrightarrow{3:1} & \widehat{\mathbb{C}}/\langle \widehat{a} \rangle = \mathcal{D}/\Xi \end{array}$$

Since a is the lift of \widehat{a} and \widehat{a} has the fixed points on branched values of f , the group Λ has signature $(h, [2g + 2, 2g + 2, 2g + 2, 2g + 2])$. But there is no any index three subgroup Λ of the group Ξ such that Λ has signature $(h, [2g + 2, 2g + 2, 2g + 2, 2g + 2])$. Then we must consider a signature for Δ different from $(0, [2, 2, \dots, 2, 2])$. In order to have an automorphism of \mathcal{D}/Δ

of order as big as possible we consider the signature: $(0, [2, 2, 2g+2, 2, 3])$. Consider now a group Δ with such a signature and a trigonal morphism f with monodromy ω_f . The maximal order of an automorphism \widehat{a} of the orbifold \mathcal{D}/Δ satisfying $\omega_f = \omega_f \circ \widehat{a}_*$ is $2g + 1$.

Assume now that $f : X_g \rightarrow \widehat{\mathbb{C}}$ is a simple trigonal covering. The signature of Δ must be $(0, [2, 2, 2g+4, 2])$. Again, we must consider an automorphism \widehat{a} of the orbifold \mathcal{D}/Δ satisfying $\omega_f = \omega_f \circ \widehat{a}_*$. Since we have shown that the order of \widehat{a} cannot be $2g + 2$, then the order of \widehat{a} must be at most $g + 2$. By a argument similar to the one used in the first part of the proof we can eliminate $h = g + 2$ and conclude that $h \leq g + 1$. \square

Proposition 6 *Given an integer g such that $2g + 1 \not\equiv 0 \pmod{3}$, there are generic trigonal surfaces of genus g admitting an automorphism of order $2g + 1$. For every even integer g , there is a uniparametric family of generic trigonal surfaces of genus g with simple trigonal morphism admitting an automorphism of order $g + 1$.*

Proof.

First let g be an integer such that $2g + 1 \not\equiv 0 \pmod{3}$. Let us consider a Fuchsian group Δ with signature $(0, [2, 2(2g + 1), 3(2g + 1)])$. Let

$$\langle x_1, x_2, x_3 : x_1 x_2 x_3 = 1, x_1^2 = x_2^{2(2g+1)} = x_3^{3(2g+1)} = 1 \rangle$$

be a canonical presentation for Δ .

Let $C_{2g+1} = \langle \gamma : \gamma^{2g+1} = 1 \rangle$ be the cyclic group of order $2g + 1$ and Σ_3 be the group of permutations on three symbols $\{1, 2, 3\}$. Now consider the epimorphism $\theta : \Delta \rightarrow C_{2g+1} \times \Sigma_3$ given by:

$$\theta(x_1) = (1, (1, 2)), \theta(x_2) = (\gamma, (2, 3)), \theta(x_3) = (\gamma^{-1}, (1, 3, 2)).$$

By Proposition 2, (see [CI]) the Riemann surface $X_g = \mathcal{D}/\Gamma$, with $\Gamma = \theta^{-1}(1, \text{Stab}(1))$, is a generic trigonal Riemann surface of genus g with trigonal morphism $\mathcal{D}/\Gamma \rightarrow \mathcal{D}/\theta^{-1}(1, \Sigma_3)$ having an automorphism of order $2g + 1$. This automorphism is given by the lifting of the automorphism of the cyclic covering $\mathcal{D}/\theta^{-1}(1, \Sigma_3) \rightarrow \mathcal{D}/\Delta$. Since Δ is a triangular Fuchsian group the constructed surfaces X_g are isolated points in the moduli space. If $2g + 1$ is a prime integer, this type of Riemann surfaces or complex algebraic curves have been studied by M. Homma in [Ho].

In the case of simple coverings we consider a Fuchsian group Δ with signature $(0, [2, 2, 2(g + 1), 2(g + 1)])$. Let

$$\langle x_1, x_2, x_3 : x_1 x_2 x_3 x_4 = 1, x_1^2 = x_2^2 = x_3^{2(g+1)} = x_4^{2(g+1)} = 1 \rangle$$

be a canonical presentation for Δ .

Let $C_{2g+1} = \langle \gamma : \gamma^{g+1} = 1 \rangle$ be the cyclic group of order $g + 1$ and Σ_3 be the group of permutations on three symbols $\{1, 2, 3\}$. Now consider the epimorphism $\theta : \Delta \rightarrow C_{g+1} \times \Sigma_3$ given by:

$$\theta(x_1) = (1, (1, 2)), \theta(x_2) = (1, (1, 2)), \theta(x_3) = (\gamma, (2, 3)), \theta(x_4) = (\gamma^{-1}, (2, 3)).$$

By Proposition 2, the Fuchsian group $\Gamma = \theta^{-1}(1, \text{Stab}(1))$ uniformizes a generic trigonal Riemann surface $X_g = \mathcal{D}/\Gamma$ of genus g , whose trigonal morphism is a simple covering $\mathcal{D}/\Gamma \rightarrow \mathcal{D}/\theta^{-1}(1, \Sigma_3)$. \mathcal{D}/Γ has an automorphism of order $g + 1$ given by the lifting of the automorphism of the cyclic covering $\mathcal{D}/\theta^{-1}(1, \Sigma_3) \rightarrow \mathcal{D}/\Delta$. Since the complex Teichmüller dimension for the Fuchsian groups with signature $(0, [2, 2, 2(g+1), 2(g+1)])$ is $-3 + 4 = 1$, thus the above construction yields a complex uniparametric family of surfaces. \square

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