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Maximal order of automorphisms of trigonal
Riemann surfaces

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Abstract
In this paper we find the maximal order of an automorphism of a
trigonal Riemann surface of genus $g$, $g \geq 5$. We find that this order is
smaller for generic than for cyclic trigonal Riemann surfaces, showing
that generic trigonal surfaces have "less symmetry" than cyclic trigono-
 nal surfaces. Finally we prove that the maximal order is attained for
infinitely many genera in both the cyclic and the generic case.

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 tomorphisms of Riemann Surfaces.

1 Introduction

A closed Riemann surface $X$ which can be realized as a 3-sheeted covering
of the Riemann sphere $f : X \rightarrow \hat{\mathbb{C}}$ is said to be trigonal, and such a covering
$f$ will be called a trigonal morphism. This is equivalent to the fact that $X$
is represented by a curve given by a polynomial equation of the form:

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\[ y^3 + yb(x) + c(x) = 0. \]

If \( b(x) \equiv 0 \) then the trigonal morphism is a cyclic regular covering and the Riemann surface is called \textit{cyclic trigonal}. If \( b(x) \not\equiv 0 \) or equivalently when \( f \) is non-cyclic \( X \) is said to be a \textit{generic trigonal Riemann surface}.

Let \( X \) be a trigonal surface, and let \( f : X \to \hat{\mathbb{C}} \) be the trigonal morphism. Let \( S \) be the set of singular values of \( f \), then, for every \( c \in \hat{\mathbb{C}} - S \), \( f^{-1}(c) \) consists of three points. If \( f \) is cyclic then for each point \( s \in S \), \( \#f^{-1}(s) = 1 \), i.e. \( s \) is an order 3 singular value of \( f \). If \( f \) is non-cyclic the points of \( S \) can be of two types: singular values of order three or simple singular values, i.e. points \( s \in S \), where \( \#f^{-1}(s) = 2 \). If all the singular values of \( f \) are simple we say that \( f \) is a \textit{simple trigonal morphism} or a simple covering. Simple coverings play an important role, for instance in the study of the moduli space.

It is a classic result that the maximal order of an automorphism of a Riemann surface of genus \( g \) is \( 4g + 2 \) (see [W] and [Ha]). The same maximal order occurs if we restrict our attention to hyperelliptic Riemann surfaces instead of general Riemann surfaces (see [BCGG]). In the present work we study the maximal order of an automorphism of a trigonal Riemann surface. We obtain that such a maximal order is smaller than for general and hyperelliptic Riemann surfaces. In Proposition 3 we obtain that the order of an automorphism of a cyclic trigonal Riemann surface of genus \( g \), \( g \geq 5 \), is at most \( 3g + 3 \). Groups of automorphisms of cyclic trigonal Riemann surfaces were studied by Bujalance, et al. in [BCG] but very little is known for groups of automorphisms generic trigonal Riemann surfaces. In Proposition 4 we establish that the order of an automorphism of a generic trigonal surface of genus \( g \), \( g \geq 5 \), is bounded above by \( 2g + 1 \). Thus a generic trigonal Riemann surface has less symmetry than a cyclic trigonal surface or even a hyperelliptic Riemann surface. Among the non-cyclic trigonal surfaces the more important ones are the surfaces admitting a simple trigonal morphism, i.e the trigonal morphism is a simple covering. In this case the maximal order of an automorphism of such a surface becomes smaller, namely \( g + 1 \).

We provide examples of families of surfaces showing that the bounds obtained are sharp (last part of Proposition 3 and Proposition 6).

\section{Preliminaries.}

An essential result for our study is that, by the Severi-Castelnuovo inequality, the trigonal morphism of a trigonal Riemann surface of genus \( g \) is unique when \( g \geq 5 \) (see [A]).
We shall use the uniformization theory of Riemann surfaces by Fuchsian groups. A surface Fuchsian group is a Fuchsian group without elliptic or parabolic transformations. Let $D$ be the unit disc in $\mathbb{C}$, the following results are characterizations of trigonality by means of Fuchsian groups (see [CI])

**Proposition 1** Let $X$ be a Riemann surface, $X$ admits a cyclic trigonal morphism $f$ if and only if there is a Fuchsian group $\Delta$ with signature

$$(0, [3, 3, g+2, 3])$$

and an index three normal surface subgroup $\Gamma$ of $\Delta$, such that $\Gamma$ uniformizes $X$, i.e. $X = D/\Gamma$.

In the conditions of Proposition 1 we shall denote the monodromy epimorphism by $\omega : \Delta \to C_3 = \Delta/\Gamma$.

**Proposition 2** A Riemann surface $X$ of genus $g$ is generic trigonal if and only if there is a Fuchsian group $\Delta$ with signature

$$(0, [2, 2, \ldots, 2, 3, 3, \ldots, 3]) \text{ where } u + 2v = 2g + 4, u \equiv 0 \mod 2, u \neq 0,$$

and an index three non-normal subgroup $\Gamma$ of $\Delta$, with signature $(g, [2, 2, \ldots, 2])$ such that $D/\Gamma$ is conformally equivalent to $X$.

The covering $f : D/\Gamma \to D/\Delta$ is simple if and only if in the above Proposition the signature of $\Delta$ is $(0, [2, 2, 2g+4, 2])$.

In the conditions of Proposition 2 the monodromy epimorphism of the trigonal morphism $f$ is the representation of the action of $\Delta$ on the cosets $\Delta/\Gamma$: $\omega : \Delta \to \Sigma_3$, where $\Sigma_3$ is the symmetric group of three elements $\{0, 1, 2\} \simeq \Delta/\Gamma$ and $\Gamma = \omega^{-1}(\text{Stab}(0))$.

### 3 Cyclic trigonal Riemann surfaces

**Proposition 3** If $X$ is a cyclic trigonal Riemann surface of genus $g$, $g \geq 5$ and $a$ is an automorphism of $X$ of order $h$, then $h \leq 3g + 3$. For every integer $g \neq 2 \mod 3$, there is a cyclic trigonal Riemann surface $X_g$ of genus $g$ having an automorphism of order $3g + 3$.

Proof. Since $g \geq 5$, the trigonal morphism $f$ is unique. The morphism $f$ is induced by the automorphism $\hat{f}$ of $X$. There is an automorphism $\hat{a} : \hat{C} \to \hat{C}$, of order $\hat{h}$, such that $f \circ a = \hat{a} \circ \hat{f}$. Since the automorphism $\hat{a}$ lifts to $X$ then $\hat{a}$ is an automorphism of the orbifold $X/\langle \hat{f} \rangle$. Consider the automorphism $\hat{a}$,
induced by $\hat{a}$ on the fundamental group $\pi_1(O/\langle \hat{f} \rangle)$ of the orbifold $X/\langle \hat{f} \rangle$, since $\pi_1(O/\langle \hat{f} \rangle) \simeq \Delta$ (where $\Delta$ is a Fuchsian group as in Proposition 1), we obtain that $\omega \circ \hat{a}_* = \omega$, with $\omega$ the monodromy of the covering $f$. Then $\hat{a}$ preserves the set $S$ of singular values of $f$, i.e. the $g + 2$ singular values. Hence $S$ is a union of orbits of $\hat{a}$. Since the orbits of $\hat{a}$ consist of one point or $\hat{h}$ points, then $\hat{h}$ is at most $g + 2$. Assume that $\hat{a}$ has order $g + 2$. Thus $X/\langle a \rangle$ is uniformized by a Fuchsian group $\Lambda$ with signature $(0; [3, g + 2, g + 2])$ and canonical presentation:

$$\langle x_1, x_2, x_3 : x_1x_2x_3 = 1, x_1^3 = x_2^{g+2} = x_3^{g+2} = 1 \rangle.$$ 

Since the covering $X \to X/\langle a \rangle = \hat{\mathbb{C}}$ factorizes by $X \overset{3:1}{\to} \hat{\mathbb{C}} \overset{g+2:1}{\to} X/\langle a \rangle = \hat{\mathbb{C}}$, the monodromy $\mu : \Lambda \to C_{3(g+2)} = \langle \delta : \delta^{3(g+2)} = 1 \rangle$ of the covering $X \to X/\langle a \rangle$, must satisfy that $\mu(x_2) = \mu(x_3)^{-1}$, but that is impossible, therefore $h \leq 3(g + 1)$.

Now let $g$ be an integer such that $g \not\equiv 2 \mod 3$. Consider a Fuchsian group $\Delta_g$ with signature $(0; [3, g + 1, 3(g + 1)])$ and canonical presentation:

$$\langle x_1, x_2, x_3 : x_1x_2x_3 = 1, x_1^3 = x_2^{g+1} = x_3^{3(g+1)} = 1 \rangle.$$ 

Let $C_{3(g+1)} = \langle \gamma : \gamma^{3(g+1)} = 1 \rangle$ be the cyclic group of order $3(g + 1)$ with $g + 1 \not\equiv 0 \mod 3$. We define the epimorphism:

$$\omega_g : \Delta \to C_{3(g+1)}$$

given by $\omega_g(x_1) = \gamma^{(g+1)}$, $\omega_g(x_2) = \gamma^3$, $\omega_g(x_3) = \gamma^{2g-1}$.

The surfaces $X_g = D/\ker \omega_g$ are cyclic trigonal and have an automorphism of order $3(g + 1)$. □

4 Generic trigonal Riemann surfaces

**Proposition 4** If $X$ is a generic trigonal Riemann surface of genus $g$, $g \geq 5$ and $a$ is an automorphism of $X$ of order $h$, then $h \leq 2g + 1$. If the trigonal morphism $f : X \to \hat{\mathbb{C}}$ is a simple covering then $h \leq g + 1$.

Before proving the proposition we need the following lemma:

**Lemma 5** Let $X$ be a generic trigonal Riemann surface of genus $g$, $g \geq 5$, let $f : X \to \hat{\mathbb{C}}$ be the trigonal morphism and let $a$ be an automorphism of $X$. The automorphism $a$ is the lift by $f$ of an automorphism $\hat{a} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, moreover the order of $\hat{a}$ equals the order of $a$. 

---

4
Proof. Since \( g \geq 5 \) the trigonal morphism is unique and then there is an automorphism \( \hat{a} \) of \( \hat{C} \) of order \( \hat{h} \) such that \( \hat{a} \circ f = f \circ a \). Thus \( a \) is the lift of \( \hat{a} \) and either

\[
\text{order of } a = \hat{h} \quad \text{or} \quad \text{order of } a = 3 \hat{h}.
\]

But, if order of \( a = 3 \hat{h} \), then \( \hat{a}^{\hat{h}} \) is an automorphism of the covering \( f : X \to \hat{C} \) and this covering has not automorphisms. \( \square \)

Note. By Lemma 5 the automorphisms groups of generic trigonal Riemann surfaces are isomorphic to finite groups of \( O(3) \).

Proof of the Proposition.

Let \( X \) be a generic trigonal Riemann surface. By Proposition 2, there is a Fuchsian group \( \Delta \) with signature \( (0, [2, 2, \ldots, 2, 3, 3, \ldots, 3]) \) such that \( \Delta \) has an index three non-normal subgroup \( \Gamma \) with signature \( (g, [2, 2, \ldots, 2]) \) and \( \mathcal{D}/\Gamma \) is conformally equivalent to \( X \). There are the restrictions \( u + 2v = 2g + 4 \), \( u \equiv 0 \pmod{2} \), \( u \neq 0 \). Let \( \omega_f : \Delta \to \Sigma_3 \) be the monodromy epimorphism of the trigonal covering \( f : X \to \hat{C} \).

Now the automorphism \( \hat{a} \) induces an automorphism \( \hat{a}^* : \Delta \to \Delta \) of the fundamental group of the orbifold \( \mathcal{D}/\Delta \) and then an automorphism \( \hat{a}_* : \Delta \to \Delta \) of the orbifold \( \mathcal{D}/\Delta \) satisfying \( \omega_f = \omega_f \circ \hat{a}_* \). Since the set of elliptic elements of \( \Delta \) sent by \( \omega_f \) to a fixed permutation of \( \Sigma_3 \) is a union of orbits of \( \hat{a} \), the maximal order of an automorphism of the orbifold \( \mathcal{D}/\Delta \) is \( 2g + 2, \) in the case that the signature of \( \Delta \) is \( (0, [2, 2, 2g+4, 2]) \). Notice that \( \omega_f \) must be a transitive representation and it is not possible that all generators are sent to the same involution of \( \Sigma_3 \).

The quotient space \( (\mathcal{D}/\Delta)/\langle \hat{a} \rangle \) can be uniformized by a triangular Fuchsian group \( \Xi \) of signature \( (0, [2, 4g+4, 4g+4]) \). Thus we have the commutative diagram:

\[
\begin{array}{ccc}
X_g = \mathcal{D}/\Gamma & \xrightarrow{3:1} & \hat{C} = \mathcal{D}/\Delta \\
2g + 2 : 1 & \downarrow & 2g + 2 : 1 \\
X_g/\langle a \rangle = \mathcal{D}/\Lambda & \xrightarrow{3:1} & \hat{C}/\langle \hat{a} \rangle = \mathcal{D}/\Xi
\end{array}
\]

Since \( a \) is the lift of \( \hat{a} \) and \( \hat{a} \) has the fixed points on branched values of \( f \), the group \( \Lambda \) has signature \( (h, [2g + 2, 2g + 2, 2g + 2, 2g + 2]) \). But there is no index three subgroup \( \Lambda \) of the group \( \Xi \) such that \( \Lambda \) has signature \( (h, [2g + 2, 2g + 2, 2g + 2, 2g + 2]) \). Then we must consider a signature for \( \Delta \) different from \( (0, [2, 2, 2g+4, 2]) \). In order to have an automorphism of \( \mathcal{D}/\Delta \)
of order as big as possible we consider the signature: \((0, [2, 2, 2g+2, 2, 3])\). Consider now a group \(\Delta\) with such a signature and a trigonal morphism \(f\) with monodromy \(\omega_f\). The maximal order of an automorphism \(\hat{a}\) of the orbifold \(\mathcal{D} / \Delta\) satisfying \(\omega_f = \omega_f \circ \hat{a}\) is \(2g + 1\).

Assume now that \(f : X_g \to \hat{C}\) is a simple trigonal covering. The signature of \(\Delta\) must be \((0, [2, 2, 2g+4, 2])\). Again, we must consider an automorphism \(\hat{a}\) of the orbifold \(\mathcal{D} / \Delta\) satisfying \(\omega_f = \omega_f \circ \hat{a}\). Since we have shown that the order of \(\hat{a}\) cannot be \(2g + 2\), then the order of \(\hat{a}\) must be at most \(g + 2\). By a argument similar to the one used in the first part of the proof we can eliminate \(h = g + 2\) and conclude that \(h \leq g + 1\). □

**Proposition 6** Given an integer \(g\) such that \(2g + 1 \not\equiv 0 \bmod 3\), there are generic trigonal surfaces of genus \(g\) admitting an automorphism of order \(2g + 1\). For every even integer \(g\), there is a uniparametric family of generic trigonal surfaces of genus \(g\) with simple trigonal morphism admitting an automorphism of order \(g + 1\).

Proof.

First let \(g\) be an integer such that \(2g + 1 \not\equiv 0 \bmod 3\). Let us consider a Fuchsian group \(\Delta\) with signature \((0, [2, 2, 2g+4, 2, 3])\). Let \(\langle x_1, x_2, x_3 : x_1x_2x_3 = 1, x_1^2 = x_2^{2(2g+1)} = x_3^{3(2g+1)} = 1 \rangle\) be a canonical presentation for \(\Delta\).

Let \(C_{2g+1} = \langle \gamma : \gamma^{2g+1} = 1 \rangle\) be the cyclic group of order \(2g + 1\) and \(\Sigma_3\) be the group of permutations on three symbols \(\{1, 2, 3\}\). Now consider the epimorphism \(\theta : \Delta \to C_{2g+1} \times \Sigma_3\) given by:

\[
\theta(x_1) = (1, (1, 2)), \quad \theta(x_2) = (\gamma, (2, 3)), \quad \theta(x_3) = (\gamma^{-1}, (1, 3, 2)).
\]

By Proposition 2, (see [CI]) the Riemann surface \(X_g = \mathcal{D} / \Gamma\), with \(\Gamma = \theta^{-1}(1, Stab(1))\), is a generic trigonal Riemann surface of genus \(g\) with trigonal morphism \(\mathcal{D} / \Gamma \to \mathcal{D} / \theta^{-1}(1, \Sigma_3)\) having an automorphism of order \(2g + 1\). This automorphism is given by the lifting of the automorphism of the cyclic covering \(\mathcal{D} / \theta^{-1}(1, \Sigma_3) \to \mathcal{D} / \Delta\). Since \(\Delta\) is a triangular Fuchsian group the constructed surfaces \(X_g\) are isolated points in the moduli space. If \(2g + 1\) is a prime integer, this type of Riemann surfaces or complex algebraic curves have been studied by M. Homma in [Ho].

In the case of simple coverings we consider a Fuchsian group \(\Delta\) with signature \((0, [2, 2, 2(g+1), 2(g+1)])\). Let

\[
\langle x_1, x_2, x_3 : x_1x_2x_3x_4 = 1, x_1^2 = x_2^2 = x_3^{2(g+1)} = x_4^{2(g+1)} = 1 \rangle
\]
be a canonical presentation for $\Delta$.

Let $C_{g+1} = \langle \gamma : \gamma^{g+1} = 1 \rangle$ be the cyclic group of order $g + 1$ and $\Sigma_3$ be the group of permutations on three symbols $\{1, 2, 3\}$. Now consider the epimorphism $\theta : \Delta \to C_{g+1} \times \Sigma_3$ given by:

$$\theta(x_1) = (1,(1,2)), \theta(x_2) = (1,(1,2)), \theta(x_3) = (\gamma,(2,3)), \theta(x_4) = (\gamma^{-1},(2,3)).$$

By Proposition 2, the Fuchsian group $\Gamma = \theta^{-1}(1,\text{Stab}(1))$ uniformizes a generic trigonal Riemann surface $X_g = D/\Gamma$ of genus $g$, whose trigonal morphism is a simple covering $D/\Gamma \to D/\theta^{-1}(1,\Sigma_3)$. $D/\Gamma$ has an automorphism of order $g + 1$ given by the lifting of the automorphism of the cyclic covering $D/\theta^{-1}(1,\Sigma_3) \to D/\Delta$. Since the complex Teichmüller dimension for the Fuchsian groups with signature $(0,[2,2,2(g+1),2(g+1)])$ is $-3+4 = 1$, thus the above construction yields a complex uniparametric family of surfaces. □

References


