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# On cyclic $p$ -gonal Riemann surfaces with several $p$ -gonal morphisms

Antonio F Costa <sup>\*</sup>   Milagros Izquierdo <sup>†</sup>   Daniel Ying <sup>‡</sup>

To Professor José María Montesinos

Abstract. Let  $p$  be a prime number,  $p > 2$ . A closed Riemann surface which can be realized as a  $p$ -sheeted covering of the Riemann sphere is called  $p$ -gonal, and such a covering is called a  $p$ -gonal morphism. If the  $p$ -gonal morphism is a cyclic regular covering, the Riemann surface is called a cyclic  $p$ -gonal Riemann surface. Accola showed that if the genus is greater than  $(p-1)^2$  the  $p$ -gonal morphism is unique. Using the characterization of  $p$ -gonality by means of Fuchsian groups we show that there exists a uniparametric family of cyclic  $p$ -gonal Riemann surfaces of genus  $(p-1)^2$  which admit two  $p$ -gonal morphisms. In this work we show that these uniparametric families are connected spaces and that each of them is the Riemann sphere without three points. We study the Hurwitz space of pairs  $(X, f)$ , where  $X$  is a Riemann surface in one of the above families and  $f$  is a  $p$ -gonal morphism, and we obtain that each of these Hurwitz spaces is a Riemann sphere without four points.

## 1 Introduction

Let  $p$  be a prime number and we shall always assume  $p > 2$ . A closed Riemann surface  $X$  which can be realized as a  $p$ -sheeted covering of the Riemann sphere is said to be  $p$ -gonal, and such a covering will be called a  $p$ -gonal morphism. This is equivalent to the fact that  $X$  is represented by a curve given by a polynomial equation of the form:

$$y^p + y^{p-2}a_{p-2}(x) + \dots + ya_1(x) + a_0(x) = 0.$$

If  $a_i(x) \equiv 0$ ,  $i = 1, \dots, p-2$ , then the  $p$ -gonal morphism is a cyclic regular covering and the Riemann surface is called *cyclic  $p$ -gonal*. The  $p$ -gonal Riemann surfaces and other related surfaces have been recently studied (see [2], [3], [6], [8], [9], [21], [22]).

By Lemma 2.1 in [1], if the surface  $X$  has genus  $g \geq (p-1)^2 + 1$ , then the  $p$ -gonal morphism is unique. The Severi-Castelnuovo inequality is used

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in order to prove such uniqueness, but this technique is not valid for small genera.

It is well-known that there are non-cyclic  $p$ -gonal Riemann surfaces of genus  $(p-1)^2$  admitting two  $p$ -gonal morphisms. Using the characterization of cyclic  $p$ -gonality by means of Fuchsian groups, we obtained a family of cyclic  $p$ -gonal Riemann surfaces of genus  $(p-1)^2$  admitting two cyclic  $p$ -gonal morphisms. Thus we prove that Accola's bound above is sharp even for cyclic  $p$ -gonal surfaces.

We show the existence of uniparametric families of Riemann surfaces of genera  $(p-1)^2$  admitting several cyclic  $p$ -gonal morphisms:  $\{\mathcal{M}_{(p-1)^2}^p(\lambda)\}$ . This family has been studied in [23] (pages 101-108). The existence of this family has been announced in [10] and independently found by A. Wootton in [21]. Wootton found all the automorphisms groups of  $p$ -gonal Riemann surfaces, with  $p$  prime in [22].

The surfaces  $X_{(p-1)^2}(\lambda) \in \mathcal{M}_{(p-1)^2}^p(\lambda)$  admit automorphisms groups  $Aut(X_{(p-1)^2}(\lambda)) = D_p \times D_p$  with quotients  $X_{(p-1)^2}(\lambda)/Aut(X_{(p-1)^2}(\lambda))$  which are Riemann spheres with four conic points of order 2, 2, 2 and  $p$  respectively.

Our main result establishes that the space  $\mathcal{M}_{(p-1)^2}^p$  of cyclic  $p$ -gonal surfaces of genus  $(p-1)^2$  admitting several  $p$ -gonal morphisms is always a Riemann sphere without three points, independent of the prime  $p$ . To prove that the space  $\mathcal{M}_{(p-1)^2}^p$  is connected we prove that  $\mathcal{M}_{(p-1)^2}^p$  is formed by equisymmetric Riemann surfaces. Two Riemann surfaces of genus  $g$  are called equisymmetric if the surfaces' automorphisms groups are conjugate finite subgroups of the mapping class group of genus  $g$ . The strata of equisymmetric surfaces are in 1-1 correspondence with the topological equivalence classes of actions of finite groups on genus  $g$  surfaces. See [4] and [5].

A Hurwitz space is a space formed by pairs  $(X_g, f)$ , where  $X_g$  is a Riemann surface of genus  $g$  and  $f : X \rightarrow \widehat{\mathbb{C}}$  a meromorphic function. These spaces are widely studied in algebraic geometry and mathematical physics. See, for instance, [11] and [17]. We consider the Hurwitz spaces  $\mathcal{H}_p$  of pairs  $(X, f)$ , where  $X \in \mathcal{M}_{(p-1)^2}^p$  and  $f : X \rightarrow \mathbb{C}$  is a cyclic  $p$ -gonal morphism. We obtain that  $\mathcal{H}_p$  is a two-fold covering of  $\mathcal{M}_{(p-1)^2}^p$  and that  $\mathcal{H}_p$  is a Riemann sphere without  $\{-1, 0, 1, \infty\}$ .

## 2 $p$ -gonal Riemann surfaces and Fuchsian groups

Let  $X_g$  be a compact Riemann surface of genus  $g \geq 2$ . The surface  $X_g$  can be represented as a quotient  $X_g = \mathcal{D}/\Gamma$  of the complex unit disc  $\mathcal{D}$  under the action of a (cocompact) Fuchsian group  $\Gamma$ , that is, a discrete subgroup of the group  $\mathcal{G} = Aut(\mathcal{D})$  of conformal automorphisms of  $\mathcal{D}$ . The algebraic structure of a Fuchsian group and the geometric structure of its quotient

orbifold are given by the signature of  $\Gamma$ :

$$s(\Gamma) = (g; m_1, \dots, m_r). \quad (1)$$

The orbit space  $\mathcal{D}/\Gamma$  is an orbifold with underlying surface of genus  $g$ , having  $r$  cone points. The integers  $m_i$  are called the periods of  $\Gamma$  and they are the orders of the cone points of  $\mathcal{D}/\Gamma$ . The group  $\Gamma$  is called the *fundamental group* of the orbifold  $\mathcal{D}/\Gamma$ .

A group  $\Gamma$  with signature (1) has a *canonical presentation*:

$$\langle x_1, \dots, x_r, a_1, b_1, \dots, a_g, b_g \mid x_i^{m_i}, i = 1, \dots, r, x_1 \dots x_r a_1 b_1 a_1^{-1} b_1^{-1} \dots a_g b_g a_g^{-1} b_g^{-1} \rangle \quad (2)$$

The hyperbolic area of the orbifold  $\mathcal{D}/\Gamma$  coincides with the hyperbolic area of an arbitrary fundamental region of  $\Gamma$  and equals:

$$\mu(\Gamma) = 2\pi(2g - 2 + \sum_{i=1}^r (1 - \frac{1}{m_i})), \quad (3)$$

Given a subgroup  $\Gamma'$  of index  $N$  in a Fuchsian group  $\Gamma$ , one can calculate the structure of  $\Gamma'$  in terms of the structure of  $\Gamma$  and the action of  $\Gamma$  on the  $\Gamma'$ -cosets (see [19]). The Riemann-Hurwitz formula holds:

$$\mu(\Gamma')/\mu(\Gamma) = N. \quad (4)$$

A Fuchsian group  $\Gamma$  without elliptic elements is called a *surface group* and it has signature  $(h; -)$ . Every compact Riemann surface of genus  $g \geq 2$  can be represented as the orbit space  $X = \mathcal{D}/\Gamma$ , with  $\Gamma$  a surface Fuchsian group. A finite group  $G$  is a group of automorphisms of  $X$  if and only if there exists a Fuchsian group  $\Delta$  and an epimorphism  $\theta : \Delta \rightarrow G$  with  $\ker(\theta) = \Gamma$ .

Let  $\Gamma$  be a Fuchsian group with signature (1). Then the Teichmüller space  $T(\Gamma)$  of  $\Gamma$  is homeomorphic to a complex ball of dimension  $d(\Gamma) = 3g - 3 + r$  (see [16]). The modular group  $Mod(\Gamma)$  of  $\Gamma$  is the quotient  $Mod(\Gamma) = Aut(\Gamma)/Inn(\Gamma)$ , where  $Inn(\Gamma)$  is the normal subgroup of  $Aut(\Gamma)$  consisting of all inner automorphisms of  $\Gamma$ . The *moduli space* of  $\Gamma$  is the quotient  $M(\Gamma) = T(\Gamma)/Mod(\Gamma)$  endowed with the quotient topology.

A Riemann surface  $X$  is said to be *p-gonal* if it admits a  $p$ -sheeted covering  $f : X \rightarrow \widehat{\mathbb{C}}$  onto the Riemann sphere. If  $f$  is a cyclic regular covering then  $X$  is called cyclic  $p$ -gonal. The covering  $f$  will be called the (cyclic)  $p$ -gonal morphism.

By Lemma 2.1 in [1], if the surface  $X_g$  has genus  $g \geq (p-1)^2 + 1$ , then the  $p$ -gonal morphism is unique.

We can characterize cyclic  $p$ -gonal Riemann surfaces using Fuchsian groups. Let  $X_g$  be a Riemann surface,  $X_g$  admits a cyclic  $p$ -gonal morphism  $f$  if and only if there is a Fuchsian group  $\Delta$  with signature  $(0; \overbrace{p, \dots, p}^{\frac{2g}{p-1}+2})$  and an index  $p$  normal surface subgroup  $\Gamma$  of  $\Delta$ , such that  $\Gamma$  uniformizes  $X_g$  (See [7], [8]).

**Remark 1** *It is well-known that there exist  $p$ -gonal Riemann surfaces of genus  $(p-1)^2$  admitting two  $p$ -gonal morphisms, since any smooth curve on a smooth quadric of type  $(p, p)$  has genus  $(p-1)^2$  and it has two coverings of degree  $p$  on the Riemann sphere. These coverings are in general non-regular.*

This provides the following algorithm to find the spaces  $\mathcal{M}_{(p-1)^2}^p$  of cyclic  $p$ -gonal Riemann surfaces of genera  $(p-1)^2$  with two  $p$ -gonal morphisms: Again, let  $p$  be an odd prime number. Let  $G = \text{Aut}(X_g)$ , with  $g = (p-1)^2$ , and let  $X_g = \mathcal{D}/\Gamma$  be a Riemann surface of genus  $g = (p-1)^2$  uniformized by the surface Fuchsian group  $\Gamma$ . The surface  $X_g$  admits a cyclic  $p$ -gonal morphism  $f$  if and only if there is a maximal Fuchsian group  $\Delta$  with signature  $(0; m_1, \dots, m_r)$ , an order  $p$  automorphism  $\varphi : X_g \rightarrow X_g$ , such that  $\langle \varphi \rangle \leq G$  and an epimorphism  $\theta : \Delta \rightarrow G$  with  $\ker(\theta) = \Gamma$  in such a way that  $\theta^{-1}(\langle \varphi \rangle)$  is a Fuchsian group with signature  $(0; \overbrace{p, \dots, p}^{2p})$ . Furthermore the  $p$ -gonal morphism  $f$  is unique if and only if  $\langle \varphi \rangle$  is normal in  $G$  (see [12]).

Since we assume that there are at least two  $p$ -gonal morphisms, we consider the groups  $G = D_p \times D_p$ .

**Remark 2** *It is interesting to enumerate the conjugacy classes of subgroups of order  $p$  in the group  $G = D_p \times D_p = \langle a, b, s, t/a^p = b^p = s^2 = t^2 = [a, b] = [s, b] = [t, a] = (sa)^2 = (tb)^2 = (st)^2 = 1 \rangle$ . The group  $D_p \times D_p$  contains the following conjugacy classes of subgroups of order  $p$ :*

- a)** *two conjugacy classes of normal subgroups of order  $p$ :  $\langle a \rangle$  and  $\langle b \rangle$ ,*
- b)**  *$\frac{p-1}{2}$  conjugacy classes of subgroups of order  $p$ :  $\langle a^i b \rangle$ ,  $i \in \{1, 2, \dots, p-1\}$ , where the subgroup generated by  $a^i b^j$  is conjugated to the subgroup generated by  $a^{-i} b^j$ .*

**Theorem 3** *Let  $p$  be a prime number,  $p > 2$ . There exists a uniparametric family  $\mathcal{M}_{(p-1)^2}^p$  of compact Riemann surfaces  $X_{(p-1)^2}(\lambda)$  of genus  $(p-1)^2$  admitting two cyclic  $p$ -gonal morphisms. The surfaces  $X_{(p-1)^2}(\lambda)$  have automorphisms groups  $\text{Aut}(X_{(p-1)^2}(\lambda)) = D_p \times D_p$  with quotient orbifolds  $X_{(p-1)^2}(\lambda)/G$  uniformized by the Fuchsian groups  $\Delta$  with signature  $s(\Delta) = (0; 2, 2, 2, p)$ .*

**Proof.** Let  $p$  be a prime number. Consider the finite group  $G = D_p \times D_p$ . By the Riemann-Hurwitz formula  $G$  is the automorphisms group of surfaces of genus  $(p-1)^2$  if there is an epimorphism from the Fuchsian groups  $\Delta$  with signature  $(0; 2, 2, 2, p)$  onto  $G$ . Now, consider the epimorphism  $\theta : \Delta \rightarrow D_p \times D_p$  defined by  $\theta(x_1) = s$ ,  $\theta(x_2) = t$ ,  $\theta(x_3) = stab$  and  $\theta(x_4) = a^{p-1}b^{p-1}$ . The action of  $\theta(x_4) = a^{p-1}b^{p-1}$  on the  $\langle ab \rangle$ -cosets has the following orbits:  $\{[1], [a], [a^2b], \dots, [a^{p-1}b]\}$ ,  $\{[s]\}$ ,  $\{[sa]\}$ ,  $\{[sb]\}$ ,  $\{[sa^2b]\}$ ,  $\dots$ ,  $\{[sa^{p-1}b]\}$ ,  $\{[t]\}$ ,  $\{[ta]\}$ ,  $\{[tb]\}$ ,  $\{[ta^2b]\}$ ,  $\dots$ ,  $\{[ta^{p-1}b]\}$ ,  $\{[st], [sta], [sta^2b], \dots, [sta^{p-1}b]\}$ .

Then  $s(\theta^{-1}(\langle ab \rangle))$  contains  $2p$  periods of order  $p$  and by the Riemann-

Hurwitz formula  $s(\theta^{-1}(\langle ab \rangle)) = (0; \overbrace{p, \dots, p}^{2p})$ .

The action of  $\theta(x_4) = a^{p-1}b^{p-1}$  on the  $(\langle a^{p-1}b \rangle)$ -cosets has the same orbit decomposition as the action of  $\theta(x_4) = a^{p-1}b^{p-1}$  on the  $(\langle ab \rangle)$ -cosets. Again,  $s(\theta^{-1}(\langle a^{p-1}b \rangle))$  contains  $2p$  periods equal to  $p$  and then  $s(\theta^{-1}(\langle a^{p-1}b \rangle)) =$

$(0; \overbrace{p, \dots, p}^{2p})$ . Thus the Riemann surfaces uniformized by  $Ker(\theta)$  are cyclic  $p$ -gonal Riemann surfaces that admit two different  $p$ -gonal morphisms  $f_1 : \mathcal{D}/Ker(\theta) \rightarrow \hat{\mathbb{C}}$  and  $f_2 : \mathcal{D}/Ker(\theta) \rightarrow \hat{\mathbb{C}}$  induced by the subgroups  $\langle ab \rangle$  and  $\langle a^{p-1}b \rangle$  of  $D_p \times D_p$ . The dimension of the family of surfaces  $\mathcal{D}/Ker(\theta)$  is given by the dimension of the space of groups  $\Delta$  with  $s(\Delta) = (0; 2, 2, 2, p)$ . This (complex-)dimension is  $3(0) - 3 + 4 = 1$ . ■

Note that if  $H'$  is a subgroup of order  $p$  in  $G = D_p \times D_p$  not conjugated to  $H = \langle a^i b^j \rangle$ , then the action of  $a^i b^j$  on the  $H'$ -cosets has no fixed points.

### 3 Actions of finite groups on Riemann surfaces

Our aim is to show that the spaces  $\mathcal{M}_{(p-1)^2}^p$  are connected and hence Riemann surfaces. To do that we will prove, by means of Fuchsian groups, that there is exactly one class of actions of  $D_p \times D_p$  on the surfaces  $X_{(p-1)^2}(\lambda)$ .

Each (*effective and orientation preserving*) action of  $G = D_p \times D_p$  on a surface  $X = X_{(p-1)^2}(\lambda)$  is determined by an epimorphism  $\theta : \Delta \rightarrow G$  from the Fuchsian group  $\Delta$  such that  $ker(\theta) = \Gamma$ , where  $X_{(p-1)^2}(\lambda) = \mathcal{D}/\Gamma$  and  $\Gamma$  is a surface Fuchsian group. The group  $\Delta$  has signature  $s(\Delta) = (0; 2, 2, 2, p)$  and presentation  $\langle x_1, x_2, x_3, x_4 \mid x_1^2 = x_2^2 = x_3^2 = x_4^p = x_1 x_2 x_3 x_4 = 1 \rangle$ .

**Remark 4** *The condition  $X_{(p-1)^2}(\lambda) = \mathcal{D}/\Gamma$  with  $\Gamma$  a surface Fuchsian group imposes:*

$$\begin{aligned} o(\theta(x_1)) &= o(\theta(x_2)) = o(\theta(x_3)) = 2, \\ o(\theta(x_4)) &= p \text{ and} \\ \theta(x_1)\theta(x_2)\theta(x_3) &= \theta(x_4)^{-1}. \end{aligned}$$

Two actions  $\epsilon_1, \epsilon_2$  of  $G$  on a surface  $X$ ,  $\epsilon_1, \epsilon_2 : G \rightarrow Homeo^+(X)$ , are (*weakly*) *topologically equivalent* if there is an  $w \in Aut(G)$  and an  $h \in Homeo^+(X)$  such that  $\epsilon_2(g) = h\epsilon_1 w(g)h^{-1}$ .

In terms of groups: two epimorphisms  $\theta_1, \theta_2 : \Delta \rightarrow G$  define two topologically equivalent actions of  $G$  on  $X$  if there exist automorphisms  $\phi : \Delta \rightarrow \Delta$ ,  $w : G \rightarrow G$  such that  $\theta_2 = w \cdot \theta_1 \cdot \phi^{-1}$ . In other words, let  $\mathcal{B}$  be the subgroup of  $Aut(\Delta)$  induced by orientation preserving homeomorphisms. Then two epimorphisms  $\theta_1, \theta_2 : \Delta \rightarrow G$  define the same class of  $G$ -actions if and only if they lie in the same  $\mathcal{B} \times Aut(G)$ -class. See [4], [13], [15].

We are interested in finding elements of  $\mathcal{B} \times Aut(G)$  that make our epimorphisms  $\theta_1, \theta_2 : \Delta \rightarrow G$  equivalent. We can produce the automorphism

$\phi \in \mathcal{B}$  ad hoc. In our case the elements of  $\mathcal{B}$  we need are compositions of the braid generators:  $x_j \rightarrow x_{j+1}$  and  $x_{j+1} \rightarrow x_{j+1}^{-1}x_jx_{j+1}$ , where we write down only the action on the generators moved by the automorphism. See [4].

We recall again  $G = D_p \times D_p = \langle a, b, s, t, a^p = b^p = s^2 = t^2 = (st)^2 = [a, b] = (sa)^2 = (tb)^2 = [s, b] = [t, a] = 1 \rangle$ .

**Theorem 5** *There is a unique class of actions of the finite group  $G = D_p \times D_p$  on the surfaces  $X = X_{(p-1)^2}(\lambda)$ .*

**Proof.** First of all there is an epimorphism  $\theta : \Delta \rightarrow G$  satisfying the Remark 4 if and only if  $\theta(x_4) = a^\varepsilon b^\delta$ , where  $\varepsilon, \delta \in \{1, 2, \dots, p-1\}$ . otherwise, if  $\theta(x_4) = a^i$  or  $\theta(x_4) = b^j$  then the action of  $\theta(x_4)$  on the  $\langle a \rangle$ - and  $\langle b \rangle$ -cosets leaves  $4p$  fixed cosets which is imposible.

We can now list all the surface-epimorphisms  $\theta : \Delta \rightarrow G$  in 6 cases depending on conjugacy classes of involutions of  $G$ . They are defined as follows:

- 1  $\theta(x_1) = sa^i, \quad \theta(x_2) = tb^j, \quad \theta(x_3) = sta^hb^k$
- 2  $\theta(x_1) = tb^j, \quad \theta(x_2) = sa^i, \quad \theta(x_3) = sta^hb^k$
- 3  $\theta(x_1) = tb^j, \quad \theta(x_2) = sta^ib^k, \quad \theta(x_3) = sa^h$
- 4  $\theta(x_1) = sa^i, \quad \theta(x_2) = sta^hb^j, \quad \theta(x_3) = tb^k$
- 5  $\theta(x_1) = sta^ib^j, \quad \theta(x_2) = tb^k, \quad \theta(x_3) = sa^h$
- 6  $\theta(x_1) = sta^ib^j, \quad \theta(x_2) = sa^h, \quad \theta(x_3) = tb^k$

where  $0 \leq i \leq p, 0 \leq j \leq p, i \neq h \pmod{p}$  and  $j \neq k \pmod{p}$ .

Now  $1_d \times w \in \mathcal{B} \times \text{Aut}(G)$ , where the automorphism  $w : G \rightarrow G$  is defined by  $w(s) = t, w(t) = s, w(a) = b$  and  $w(b) = a$  commutes epimorphisms of type 1 with epimorphisms of type 2; epimorphisms of type 3 with epimorphisms of type 4; and epimorphisms of type 5 with epimorphisms of type 6.

Furthermore all the surface-epimorphisms within the same case define the same action of the group  $G$  on the Riemann surface  $X$ .

In Case 1. The epimorphism  $\theta$  ( $\theta(x_1) = sa^i, \theta(x_2) = tb^j, \theta(x_3) = sta^hb^k$ , where  $0 \leq i \leq p-1, 0 \leq j \leq p-1, i \neq h \pmod{p}$  and  $j \neq k \pmod{p}$ ) is conjugated to the epimorphism  $\theta_1$  defined as  $\theta_1(x_1) = s, \theta_1(x_2) = t, \theta_1(x_3) = sta^{h-i}b^{k-j}$ , with  $h-i \neq 0$  and  $k-j \neq 0$ , by the element  $sta^{i'}b^{j'}$  of  $G$  where  $2i' \equiv i \pmod{p}$  and  $2j' \equiv j \pmod{p}$ . But,  $1_d \times w_{\frac{1}{h-i}, \frac{1}{k-j}} \in \mathcal{B} \times \text{Aut}(G)$ , where the automorphism  $w_{\frac{1}{h-i}, \frac{1}{k-j}} : G \rightarrow G$  is defined by  $w_{\frac{1}{h-i}, \frac{1}{k-j}}(s) = s, w_{\frac{1}{h-i}, \frac{1}{k-j}}(t) = t, w_{\frac{1}{h-i}, \frac{1}{k-j}}(a) = a^x$  and  $w_{\frac{1}{h-i}, \frac{1}{k-j}}(b) = b^y$ , where  $x$  and  $y$  satisfy the equations  $(h-i)x \equiv 1 \pmod{p}, (k-j)y \equiv 1 \pmod{p}$ , conjugates the epimorphism  $\theta_1$  with the epimorphism  $\theta_0$ , where  $\theta_0(x_1) = s, \theta_0(x_2) = t, \theta_0(x_3) = stab$ .

The reasoning is similar in all the others cases.

Finally we show that there are elements of  $\mathcal{B}$  conjugating the epimorphism  $\theta_0(x_1) = s$  with an epimorphism of type 3, and with an epimorphism of type 6. Consider  $\phi_{1,2} : \Delta \rightarrow \Delta$  with  $\phi_{1,2}(x_1) = x_2, \phi_{1,2}(x_2) = x_2^{-1}x_1x_2,$

and  $\phi_{2,3} : \Delta \rightarrow \Delta$  where  $\phi_{2,3}(x_2) = x_3$ ,  $\phi_{2,3}(x_3) = x_3^{-1}x_2x_3$ . The braid element  $\phi_{2,3} \cdot \phi_{1,2}$  takes the epimorphism  $\theta_0$  to the epimorphism  $\theta_3(x_1) = t$ ,  $\theta_3(x_2) = stab$ ,  $\theta_3(x_3) = sa^{p-1}$ , of type 3. The braid element  $\phi_{1,2} \cdot \phi_{2,3}$  commutes the epimorphism  $\theta_0$  to the epimorphism  $\theta_6(x_1) = stab$ ,  $\theta_6(x_2) = sa^2$ ,  $\theta_6(x_3) = tb^{p-1}$ , of type 6. ■

As a consequence of the previous theorem we obtain

**Theorem 6** *The spaces  $\mathcal{M}_{(p-1)^2}^p$  are Riemann surfaces. Furthermore each of them is the Riemann sphere with three punctures.*

**Proof.** By Theorem 5, each  $\mathcal{M}_{(p-1)^2}^p$  is a connected space of complex dimension 1. Each space  $\mathcal{M}_{(p-1)^2}^p$  can be identified with the moduli space of orbifolds with three conic points of order 2 and one of order  $p$ . Each conic point of order 2 corresponds to a conjugacy class of involutions in  $D_p \times D_p : [s], [t]$  and  $[st]$ . Using a Möbius transformation we can assume that the three order two conic points are 0 (corresponding to  $[s]$ ), 1 (corresponding to  $[t]$ ) and  $\infty$  (corresponding to  $[st]$ ). Thus each  $\mathcal{M}_{(p-1)^2}^p$  is parametrized by the position  $\lambda$  of the order three conic point and the map  $\Phi : \mathcal{M}_{(p-1)^2}^p \ni X(\lambda) \rightarrow \lambda \in \widehat{\mathbb{C}} - \{0, 1, \infty\}$  is an isomorphism. Hence each  $\mathcal{M}_{(p-1)^2}^p$  is the Riemann sphere with three punctures. ■

**Remark 7** *Consider the space  $\widehat{\mathcal{M}}_{(p-1)^2}^p$  of the moduli space  $\mathcal{M}_{(p-1)^2}$  formed by the Riemann surfaces having  $D_p \times D_p$  as a group of automorphisms, see [4]. There is one more surface in  $\mathcal{M}_{(p-1)^2}$  admitting two trigonal morphisms:*

*the surface  $Y_p$ , with  $Aut(Y_p) = (C_p \times C_p) \rtimes D_4$  and  $Y_p / (C_p \times C_p) \rtimes D_4$  uniformized by the Fuchsian group  $\Delta$  with  $s(\Delta) = (0; 2, 4, 2p)$ .  $Y_p \in \widehat{\mathcal{M}}_{(p-1)^2}^p$ . See [10], [14], [21] and [23].*

**Remark 8** *The closure  $\overline{\mathcal{M}}_{(p-1)^2}^p$  of the family  $\mathcal{M}_{(p-1)^2}^p$  inside the compactification of the moduli space is the Riemann sphere obtained by attaching to  $\mathcal{M}_{(p-1)^2}^p$  three nodal singular Riemann surfaces. Two of such nodal surfaces have  $2p$  nodal points and the remaining one has  $2p^2$  nodal points. The last surface consists of  $p^2$  spheres  $S_i$  with two punctures each and  $2p$  spheres  $S_j$  with  $p$  punctures each. The spheres are joined by nodal points in such a way that one sphere  $S_i$  and one sphere  $S_j$  meet at each nodal point. The nodal surfaces with  $2p$  nodal points consists of two isomorphic Riemann surfaces  $\Sigma_1, \Sigma_2$  with  $p$  punctures each and  $p$  spheres with two punctures. In each node meets a sphere and one of the two surfaces  $\Sigma_1, \Sigma_2$ .*

## 4 The Hurwitz space of cyclic morphisms from surfaces in $\mathcal{M}_{(p-1)^2}^p$ .

Let  $p$  be a prime number,  $p > 2$ . We consider the pairs  $(X, f)$ , where  $X \in \mathcal{M}_{(p-1)^2}^p$  and  $f : X \rightarrow \widehat{\mathbb{C}}$  is a cyclic  $p$ -gonal morphism. Two pairs  $(X_1, f_1)$  and  $(X_2, f_2)$  are equivalent if there is an isomorphism  $h : X_1 \rightarrow X_2$  such that  $f_1 = f_2 \circ h$ . The space of classes of pairs  $(X, f)$  given by the above equivalence relation and with the topology induced by the topology of  $\mathcal{M}_{(p-1)^2}^p$  is a Hurwitz space  $\mathcal{H}_p$ . See [10], where we announced our results, for the case  $p = 3$ .

**Theorem 9** *The space  $\mathcal{H}_p$  is a two-fold covering of  $\mathcal{M}_{(p-1)^2}^p$ . The space  $\mathcal{H}_p$  is a Riemann sphere without four points.*

**Proof.** By the proofs of Theorem 3 and Theorem 5 each surface of  $\mathcal{M}_{(p-1)^2}^p$  admits two cyclic  $p$ -gonal morphisms, then  $\mathcal{H}_p$  is a two-fold covering of  $\mathcal{M}_{(p-1)^2}^p$ .

We only need to prove that the covering space is connected. We need to show that the monodromy of the covering  $\mathcal{H}_p \rightarrow \mathcal{M}_{(p-1)^2}^p$  is not trivial. Each  $(X(\lambda), f) \in \mathcal{H}_p$  is given by a point  $\lambda \in \widehat{\mathbb{C}} - \{0, 1, \infty\}$  and a cyclic  $p$ -gonal morphism  $f : X \rightarrow \widehat{\mathbb{C}}$ . But the cyclic  $p$ -gonal morphism is given by the projections  $f_{ab} : X \rightarrow X/\langle ab \rangle$  or  $f_{a^{p-1}b} : X \rightarrow X/\langle a^{p-1}b \rangle$ . There is an action of  $\pi_1(\mathcal{M}_{(p-1)^2}^p) = \pi_1(\widehat{\mathbb{C}} - \{0, 1, \infty\})$  on the set of representations  $R = \{r : \pi_1(\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}) \rightarrow D_p \times D_p\}$ . The group  $\pi_1(\mathcal{M}_{(p-1)^2}^p)$  is generated by three meridians and each one acts on  $R$  as the action induced by a braid in  $\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}$ . The braid  $\Phi_{34}^{-1}\Phi_{23}^2\Phi_{34}$  (given by the action of one of the meridians of  $\mathcal{M}_{(p-1)^2}^p$ ) sends  $r_1 : \pi_1(\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}, *) \rightarrow D_p \times D_p$  defined by:  $x_1 \rightarrow s, x_2 \rightarrow t, x_3 \rightarrow stab, x_4 \rightarrow a^{p-1}b^{p-1}$

to  $r_2 : \pi_1(\widehat{\mathbb{C}} - \{0, 1, \infty, \lambda\}) \rightarrow D_p \times D_p$  defined by:  $x_1 \rightarrow s, x_2 \rightarrow tb^{-2}, x_3 \rightarrow sta^{p-1}b^{p-1}, x_4 \rightarrow ab^{p-1}$

The representations  $r_1$  and  $r_2$  are conjugated by  $sb^{p-1}$ , but such a conjugation sends  $\langle ab \rangle$  to  $\langle a^{p-1}b \rangle$ . ■

## 5 Equations of the elements of $\mathcal{M}_{(p-1)^2}^p$

Let  $p$  be a prime number,  $p > 2$ . Let  $X$  be an element of  $\mathcal{M}_{(p-1)^2}^p$ . The automorphisms group of  $X$  is  $D_p \times D_p$  with presentation:  $\langle a, b, s, t : a^p = b^p = s^2 = t^2 = [a, b] = [s, b] = [t, a] = (sa)^2 = (tb)^2 = (st)^2 = 1 \rangle$ .

The subgroup  $\langle a, b \rangle$  of  $D_p \times D_p$  is a normal subgroup of  $D_p \times D_p$  and it is isomorphic to  $C_p \times C_p$ . The quotient group  $D_p \times D_p / \langle a, b \rangle$  is isomorphic to the Klein group  $C_2 \times C_2$ . Hence we can factorize the covering  $X \rightarrow X/D_p \times D_p$  by two regular coverings:

$$\begin{array}{ccc}
C_p \times C_p & & X \\
& \swarrow & \downarrow \\
X/C_p \times C_p & & \\
& \searrow & \\
C_2 \times C_2 & & X/D_p \times D_p
\end{array}$$

The quotient space  $X/C_p \times C_p$  is a 2-orbifold with four conic points of order  $p$  and genus 0. The orbifold  $X/D_p \times D_p = (X/C_p \times C_p)/C_2 \times C_2$  has three conic points of order 2, one conic point of order  $p$  and genus 0. Using a Möbius transformation we can consider that the action of  $C_2 \times C_2$  on  $X/C_p \times C_p$  is the given by the transformations  $\{z \rightarrow \pm z, z \rightarrow \pm \frac{1}{z}\}$ . Since the set of the four conic points of order  $p$  is an orbit of the action of  $C_2 \times C_2$  on  $X/C_p \times C_p$ , then the conic points of  $X/C_p \times C_p$  are:

$$\{\pm\lambda, \pm\frac{1}{\lambda}\} \text{ for } \lambda \in \mathbb{C} - \{0, \pm 1, \pm i\}.$$

To obtain  $X$  from  $X/C_p \times C_p$  we factorize  $X \rightarrow X/C_p \times C_p$  by:

$$\begin{array}{ccc}
C_p & & X \\
& \swarrow & \downarrow \\
X/C_p & & \\
& \searrow & \\
C_p & & X/C_p \times C_p
\end{array}$$

The cyclic  $p$ -fold covering  $g : X/C_p \rightarrow X/C_p \times C_p$  branched on  $\pm\lambda$  is:

$$g(z) = \frac{-\lambda z^p + \lambda}{z^p + 1}.$$

The orbifold  $X/C_p$  has  $2p$  conic points of order  $p$  that are the preimages by  $g$  of  $\pm\frac{1}{\lambda}$ . If  $\zeta_1$  is a primitive  $p$ -root of  $\frac{\lambda^2-1}{\lambda^2+1}$  and  $\zeta_2$  is a primitive  $p$ -root of  $\frac{\lambda^2+1}{\lambda^2-1}$ , then  $X$  has equation as algebraic complex curve:

$$y^p = \prod_{i=1}^p (x - \zeta_1^i) \prod_{i=1}^p (x - \zeta_2^i)^{p-1}. \quad (5)$$

Using the uniqueness of the family  $\mathcal{M}_{(p-1)^2}^p$  (Theorem 5) of curves of genus  $(p-1)^2$  having automorphisms group  $D_p \times D_p$  one obtains another equation for this family:

$$ax^p y^p - (x^p + y^p) + a = 0, \quad (a \neq 0, \pm 1, \infty). \quad (6)$$

**Remark 10** As a consequence of equation (6) the Hurwitz space  $\mathcal{H}_p$  is the Riemann sphere without  $0, \pm 1, \infty$ . By the proof of Theorem 9 the compactification  $\overline{\mathcal{H}}_p$  of  $\mathcal{H}_p$  is the Riemann sphere which is the double covering of  $\overline{\mathcal{M}}_{(p-1)^2}^p$  branched at  $z = 0$  and  $z = \infty$ .

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