

Rank complement of diagonalizable matrices using polynomial functions

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Abstract

This report defines the rank complement of a diagonalizable matrix (i.e. a matrix which can be brought to a diagonal form by means of a change of basis) as the interchange of the range and the null space. Given a diagonalizable matrix \mathbf{A} there is in general no unique matrix \mathbf{A}_c which has a range equal to the null space of \mathbf{A} and a null space equal to the range of \mathbf{A} , only matrices of full rank have a unique rank complement; the zero matrix. Consequently, the rank complement operation is not a distinct operation, but rather a characterization of *any* operation which makes an interchange of the range and the null space. One particular rank complement operation is introduced here, which eventually leads to an implementation of rank complement operations in terms of polynomials in \mathbf{A} . The main result is that for each possible rank r of \mathbf{A} there is a polynomial in \mathbf{A} which evaluates to a matrix \mathbf{A}_c which is a rank complement of \mathbf{A} .

The report provides explicit expressions for matrix polynomials which compute a rank complement of a symmetric matrix. These results are then generalized to the case of diagonalizable matrices. Finally, a Matlab function is described that implements a rank complement operation based on the results derived.

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1 Introduction

The rank of a matrix is defined as the number of linearly independent rows or columns. An alternative and equivalent definition is available provided that a singular value decomposition (SVD) has been made on the matrix, the rank is then simply the number of non-zero singular values. For a symmetric matrix, this corresponds to the number of non-zero eigenvalues. In most practical cases the division of singular values or eigenvalues into the classes "equal to zero" and "not equal to zero" cannot be made by simply checking for equality. The reason is that numerical errors, e.g., introduced by round-off operations in an eigenvalue decomposition, make certain eigenvalues approximately equal to zero rather than identical to zero. This can be managed, e.g., by assuming that any singular value or eigenvalue of sufficiently small magnitude δ is equal to zero, and the resulting rank is then a *numerical δ rank*. However, it should be noticed that these rules are always subject to arbitrariness and in some cases also uncertainty with respect to the resulting rank. This observation means that the rank concept is somewhat ill-defined unless we have a clear understanding of what we mean by an eigenvalue being equal to zero. In the following, and unless stated otherwise, the rank concept refers to the formal definition given above, and all singular values or eigenvalues which, in some way or another, have been classified as "zero-like" are set to zero.

Let \mathbf{A} be a symmetric $n \times n$ matrix of rank r . From the definition of rank we know that $0 \leq r \leq n$. It also follows directly that

- The dimensionality of the range of \mathbf{A} , i.e., the space of all vectors $\mathbf{y} = \mathbf{A} \mathbf{x}$, is r .
- The dimensionality of the null space of \mathbf{A} , i.e., the space of vectors \mathbf{x} for which $\mathbf{A} \mathbf{x} = \mathbf{0}$, is $n - r$.

In certain applications the relation between the range and the null space for a set of symmetric matrices is of special interest. For example, in [2] Knutsson presents a method for representation of local image velocity in 2D image sequences. In this method a local 2D image velocity $\bar{\mathbf{v}}$,

$$\bar{\mathbf{v}} = \begin{pmatrix} v_x \\ v_y \end{pmatrix}, \quad (1)$$

has a homogeneous representation in terms of a spatio-temporal motion vector \mathbf{v} ,

$$\mathbf{v} = \begin{pmatrix} \bar{\mathbf{v}} \\ 1 \end{pmatrix} = \begin{pmatrix} v_x \\ v_y \\ 1 \end{pmatrix}. \quad (2)$$

The method is based on the fact that \mathbf{v} is always a *zero eigenvector*, i.e., an eigenvector of eigenvalue zero, relative to the corresponding local image structure tensor \mathbf{T} (represented

by a 3×3 positive semidefinite symmetric matrix). Moreover, the local 2D orientation of moving structures is represented by the range of the same tensor. For linear structures, such as edges or lines, where the so-called aperture problem is at hand, the situation is such that any estimated velocity vector $\bar{\mathbf{u}}$ is a linear combination

$$\bar{\mathbf{u}} = \bar{\mathbf{v}} + \alpha \hat{\mathbf{w}}, \quad \alpha \in \mathbb{R}, \quad (3)$$

where $\bar{\mathbf{v}}$ is the true velocity, and $\hat{\mathbf{w}}$ is a vector of unit length which is parallel to the linear structure. In the homogeneous representation this corresponds to

$$\mathbf{u} = \begin{pmatrix} \bar{\mathbf{u}} \\ 1 \end{pmatrix} = \begin{pmatrix} \bar{\mathbf{v}} \\ 1 \end{pmatrix} + \alpha \begin{pmatrix} \hat{\mathbf{w}} \\ 0 \end{pmatrix}. \quad (4)$$

Notice that in the homogeneous representation, any vector $\beta \mathbf{u}$, $\beta \neq 0$ is a representation of the same image velocity. Hence, the space of all spatio-temporal velocity vectors which are valid for the case of a moving linear structure is two-dimensional. For the case of a moving linear structure, the local structure tensor \mathbf{T} has indeed a null space that is two-dimensional, and we can thus see any vector of its null space as a valid homogeneous representation of the estimated image velocity.

For a set of edges which has the same motion but different orientation, e.g., the edges near a corner in motion, this means that there is a unique motion vector (apart from a constant multiplier) which is a zero eigenvector relative to all local structure tensors of the edges. This can be used by forming a new tensor as the sum, or weighted average, of the local structure tensors near the corner. This sum must then have a range of dimension 2, and a null space of dimension 1, where the latter represents the true image velocity of the corner.

The above example uses the fact that the null space of a sum of symmetric and positive semidefinite matrices is obtained by taking the intersection of their null spaces, i.e, a vector is an element of the resulting null space if and only if it is an element of all the null spaces of the matrices in the sum. However, this summation also implies that the range of the resulting matrix is the direct sum of the ranges of each of the matrices, i.e., a vector is an element of the resulting range if and only if it is the sum of elements from the ranges of the matrices in the sum.

This correspondence between taking the intersection or the direct sum of certain vector spaces and the simple operation of matrix summation is indeed convenient and can be used in many applications. However, there is one problem, namely that a matrix summation always corresponds to taking the intersection of the null spaces and the direct sum of the ranges. If we need to compute the direct sum of the null spaces, or the intersection of the ranges, a simple matrix summation will not do.

An obvious solution to this problem is at hand if the range and the null space of a symmetric matrix could be interchanged. If we wanted to compute the intersection of the ranges of some matrices, we could first interchange the range and null space for each

matrix, then compute the matrix sum, and finally interchange range and null space to get a resulting range which is the intersection of all ranges of the original matrices.

2 Definition of rank complement

The operation of interchanging range and null space of a matrix \mathbf{A} is here referred to as *rank complement*, and the result is a new matrix which has a null space equal to the range of \mathbf{A} , and a range equal to the null space of \mathbf{A} . Obviously, the rank of the result is $n - r$ if the rank of \mathbf{A} is r . It should be noted that the operation of rank complement is defined from a qualitative point of view, since the interchange of range and null space in general can be made in many different ways, so that there is no unique matrix which corresponds to the rank complement of \mathbf{A} . The only exception are matrices of full rank which have a rank complement equal to the zero. Consequently, there may be different implementations of a rank complement operation which give different results from a quantitative point of view. To simplify the presentation, we will restrict \mathbf{A} to be a symmetric matrix. In Section 7, we will generalize the results to more general types of matrices.

One possible implementation of the rank complement operation is to first compute an eigenvalue decomposition of the symmetric matrix \mathbf{A} , according to

$$\mathbf{A} = \mathbf{E} \mathbf{D} \mathbf{E}^T, \quad (5)$$

where \mathbf{E} is an orthogonal matrix, containing the eigenvectors of \mathbf{A} in its columns, and \mathbf{D} is a diagonal matrix which holds the corresponding eigenvalues. If \mathbf{A} is of rank r , then r eigenvalues are non-zero, and the corresponding eigenvectors span the range of \mathbf{A} . Furthermore, $n - r$ eigenvalues are equal to zero, and the corresponding eigenvectors span the null space of \mathbf{A} . To get a rank complement of \mathbf{A} , we could compute a new diagonal matrix \mathbf{D}_c according to

$$\mathbf{D}_{c,ii} = \begin{cases} 0, & \mathbf{D}_{ii} \neq 0, \\ 1, & \mathbf{D}_{ii} = 0, \end{cases} \quad (6)$$

and set

$$\mathbf{A}_c = \mathbf{E} \mathbf{D}_c \mathbf{E}^T. \quad (7)$$

The matrix \mathbf{A}_c is then a rank complement relative to \mathbf{A} in the above sense. Notice that instead of setting $\mathbf{D}_{c,ii} = 1$ for $\mathbf{D}_{ii} = 0$, any non-zero value will do.

There are a few advantages with the above implementation of the rank complement operation. It is rather straightforward to understand and to implement, and it describes one single operation which works for any rank of \mathbf{A} . However, in some application, in particular when n is large, the computational effort of computing a full eigenvalue

decomposition may be too heavy or does not comply with a processing structure which ideally should consist of simple operations. In particular, the elements of the resulting matrix, \mathbf{A}_c , are not simple functions of the elements of the original matrix \mathbf{A} , and there is also the arbitrariness regarding which eigenvalues are considered to be non-zero and which are not.

Instead of using an eigenvalue decomposition, an alternative is to see each element of the matrix \mathbf{A}_c as some simple function, e.g., a polynomial, of the elements of \mathbf{A} and then find out which such functions make \mathbf{A}_c a rank complement of \mathbf{A} . As will be shown in the following, there are indeed polynomial functions which can make the necessary transformation from \mathbf{A} to \mathbf{A}_c , in fact they represent polynomial matrix functions from \mathbf{A} to \mathbf{A}_c . On the other hand, these functions are dependent on the rank of \mathbf{A} , i.e., a specific polynomial works as a rank complement only for a specific rank of \mathbf{A} .

In the following sections, a method for obtaining polynomial functions for rank complement is presented. The method is to a large extent based on classical results from linear algebra.

3 Preliminaries

Consider the fraction

$$\frac{\epsilon}{d + \epsilon}. \quad (8)$$

If we evaluate the fraction as a limit when $\epsilon \rightarrow 0$, the result is

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{d + \epsilon} = \begin{cases} 0, & d \neq 0, \\ 1, & d = 0. \end{cases} \quad (9)$$

Using this observation, and the discussion related to Equations (6) and (7), we may define the rank complement of \mathbf{A} as

$$\mathbf{A}_c = \mathbf{E} \lim_{\epsilon \rightarrow 0} \epsilon (\mathbf{D} + \epsilon \mathbf{I})^{-1} \mathbf{E}^T = \lim_{\epsilon \rightarrow 0} \epsilon (\mathbf{A} + \epsilon \mathbf{I})^{-1}. \quad (10)$$

As is shown in the following sections, this definition of rank complement has the interesting character that the limit value is essentially a polynomial in \mathbf{A} , which consequently can be used as a closed form expression for computing a rank complement of \mathbf{A} . However, it should be noted already here that the resulting expression for the limit value depends on the rank of \mathbf{A} .

Let us illustrate the idea by means of an example. Assume that \mathbf{A} is a symmetric 2×2 matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \quad (11)$$

from which follows

$$\mathbf{A} + \epsilon \mathbf{I} = \begin{pmatrix} a_{11} + \epsilon & a_{12} \\ a_{12} & a_{22} + \epsilon \end{pmatrix}, \quad (12)$$

and, using the simple rule for inversion of a 2×2 matrix,

$$\epsilon (\mathbf{A} + \epsilon \mathbf{I})^{-1} = \frac{\epsilon}{\det(\mathbf{A} + \epsilon \mathbf{I})} \begin{pmatrix} a_{22} + \epsilon & -a_{12} \\ -a_{12} & a_{11} + \epsilon \end{pmatrix}. \quad (13)$$

Expanding the matrix in the right hand side of the last equation into a polynomial in ϵ gives

$$\begin{pmatrix} a_{22} + \epsilon & -a_{12} \\ -a_{12} & a_{11} + \epsilon \end{pmatrix} = (a_{11} + a_{22} + \epsilon) \mathbf{I} - \mathbf{A}, \quad (14)$$

and similarly for the determinant in the denominator

$$\det(\mathbf{A} + \epsilon \mathbf{I}) = \det(\mathbf{A}) + (a_{11} + a_{22}) \epsilon + \epsilon^2. \quad (15)$$

Together, this means that

$$\epsilon (\mathbf{A} + \epsilon \mathbf{I})^{-1} = \frac{\epsilon}{\det(\mathbf{A}) + (a_{11} + a_{22}) \epsilon + \epsilon^2} [(a_{11} + a_{22} + \epsilon) \mathbf{I} - \mathbf{A}]. \quad (16)$$

Now, let us examine the three interesting cases;

- \mathbf{A} has full rank, i.e., $r = 2$,
- \mathbf{A} has rank $r = 1$, and
- \mathbf{A} has rank $r = 0$.

rank(\mathbf{A}) = 2 In this case, $\det(\mathbf{A}) \neq 0$ and, consequently,

$$\mathbf{A}_c = \lim_{\epsilon \rightarrow 0} \epsilon (\mathbf{A} + \epsilon \mathbf{I})^{-1} = \mathbf{0}. \quad (17)$$

In view of the above definition of a rank complement, this is correct since \mathbf{A}_c has a null space equal to the range of \mathbf{A} , and vice versa.

rank(\mathbf{A}) = 1 In this case, there exists an eigenvector \mathbf{e}_1 of \mathbf{A} which has an eigenvalue $\lambda_1 \neq 0$, and one eigenvector \mathbf{e}_2 of \mathbf{A} which has an eigenvalue $\lambda_2 = 0$. We get $\det(\mathbf{A}) = \lambda_1 \lambda_2 = 0$, and $a_{11} + a_{22} = \lambda_1 + \lambda_2 = \lambda_1 \neq 0$. Consequently,

$$\mathbf{A}_c = \lim_{\epsilon \rightarrow 0} \epsilon (\mathbf{A} + \epsilon \mathbf{I})^{-1} = \frac{1}{a_{11} + a_{22}} [(a_{11} + a_{22}) \mathbf{I} - \mathbf{A}] = \frac{1}{\lambda_1} [\lambda_1 \mathbf{I} - \mathbf{A}]. \quad (18)$$

To see that this is indeed a rank complement, notice that the range of \mathbf{A} is spanned by \mathbf{e}_1 alone, and that

$$\mathbf{A}_c \mathbf{e}_1 = \frac{1}{\lambda_1} (\lambda_1 \mathbf{e}_1 - \lambda_1 \mathbf{e}_1) = \mathbf{0}, \quad (19)$$

which means that \mathbf{e}_1 is in the null space of \mathbf{A}_c . Correspondingly, the null space of \mathbf{A} is spanned by \mathbf{e}_2 alone, and we get

$$\mathbf{A}_c \mathbf{e}_2 = \frac{1}{a_{11} + a_{22}} [(a_{11} + a_{22}) \mathbf{I} - \mathbf{A}] \mathbf{e}_2 = \mathbf{e}_2, \quad (20)$$

which means that \mathbf{e}_2 is an element of the range of \mathbf{A}_c . To summarize, the null and ranges of \mathbf{A} have been interchanged in \mathbf{A}_c .

rank(\mathbf{A}) = 0 This is the same as $\mathbf{A} = \mathbf{0}$, and we get

$$\mathbf{A}_c = \lim_{\epsilon \rightarrow 0} \epsilon (\mathbf{A} + \epsilon \mathbf{I})^{-1} = \mathbf{I}, \quad (21)$$

which again corresponds to a rank complement of \mathbf{A} .

From this simple example it is evident that, at least for the case of a 2×2 matrix, we can find simple expressions for \mathbf{A}_c , the rank complement of \mathbf{A} . For example, if \mathbf{A} has $r = 1$, then

$$\mathbf{A}_c = \text{trace}(\mathbf{A}) \mathbf{I} - \mathbf{A} \quad (22)$$

is a rank complement of \mathbf{A} , according to Equation (18) (omitting the denominator does not change the rank complement property). However, it should be noted that this expression for \mathbf{A}_c evaluates to a rank complement only when $r = 1$, which means that we have to know the rank of \mathbf{A} to choose the appropriate polynomial for the rank complement.

The following sections present a method to compute polynomials for rank complement of matrices of known rank.

4 $(\mathbf{A} + \epsilon \mathbf{I})^{-1}$ as a rational function of ϵ

This section derives an expression of $(\mathbf{A} + \epsilon \mathbf{I})^{-1}$ in terms of a rational function which later is used to evaluate the limit expression of Equation (10). The result of the derivations is Equation (52).

Let \mathbf{A} be an $n \times n$ symmetric matrix. The characteristic polynomial of \mathbf{A} is defined as

$$P(\lambda) = \det(\mathbf{A} - \lambda \mathbf{I}), \quad (23)$$

which is an n -th degree polynomial in λ and can therefore also be expressed as

$$P(\lambda) = \sum_{k=0}^n p_k \lambda^k, \quad (24)$$

where each p_k is an $(n - k)$ -th degree polynomial in the elements of \mathbf{A} . Note that

$$p_0 = \det(\mathbf{A}), \quad \text{and} \quad p_n = (-1)^n. \quad (25)$$

A well-known property of the characteristic polynomial is that its roots are the eigenvalues of \mathbf{A} . Another interesting property is that \mathbf{A} satisfies its own characteristic polynomial (the Cayley-Hamilton theorem), i.e.,

$$P(\mathbf{A}) = \sum_{k=0}^n p_k \mathbf{A}^k = \mathbf{0}, \quad (26)$$

which means, e.g., that \mathbf{A}^n is linearly dependent of all lower degrees of \mathbf{A} . From this follows immediately that any polynomial or power series of \mathbf{A} can be reduced to a polynomial of degree less than or equal to $n - 1$.

Let us now consider the characteristic polynomial of $\mathbf{A} + \epsilon \mathbf{I}$,

$$Q(\lambda) = \det(\mathbf{A} + \epsilon \mathbf{I} - \lambda \mathbf{I}) = \det(\mathbf{A} - (\lambda - \epsilon) \mathbf{I}) = P(\lambda - \epsilon). \quad (27)$$

Insertion of Equation (24) into the right hand side of Equation (27), and a binomial expansion of $(\lambda - \epsilon)^l$ gives

$$Q(\lambda) = \sum_{l=0}^n p_l (\lambda - \epsilon)^l = \sum_{l=0}^n p_l \sum_{k=0}^l \binom{l}{k} \lambda^k (-\epsilon)^{l-k}, \quad (28)$$

and by changing the order of summation, we get

$$Q(\lambda) = \sum_{k=0}^n \left[\sum_{l=k}^n p_l \binom{l}{k} (-\epsilon)^{l-k} \right] \lambda^k. \quad (29)$$

The expression in the bracket is an $(n - k)$ -th degree polynomial in ϵ and in the elements of \mathbf{A} , and we define

$$q_k(\epsilon) = \sum_{l=k}^n p_l \binom{l}{k} (-\epsilon)^{l-k} \quad (30)$$

to get

$$Q(\lambda) = \sum_{k=0}^n q_k(\epsilon) \lambda^k. \quad (31)$$

In the same way as \mathbf{A} is a root of P , so must $\mathbf{A} + \epsilon \mathbf{I}$ be a root of Q , i.e.,

$$Q(\mathbf{A} + \epsilon \mathbf{I}) = \sum_{k=0}^n q_k(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k = \mathbf{0}. \quad (32)$$

From this follows immediately that

$$q_0(\epsilon) + \sum_{k=1}^n q_k(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k = \mathbf{0}, \quad (33)$$

$$q_0(\epsilon) = - \sum_{k=1}^n q_k(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k, \quad (34)$$

$$q_0(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^{-1} = - \sum_{k=0}^{n-1} q_{k+1}(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k, \quad (35)$$

$$(\mathbf{A} + \epsilon \mathbf{I})^{-1} = \frac{-1}{q_0(\epsilon)} \sum_{k=0}^{n-1} q_{k+1}(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k, \quad (36)$$

provided that the matrix inverse of the left hand side of the last equation exists. This result shows that $(\mathbf{A} + \epsilon \mathbf{I})^{-1}$ can be written essentially as a finite sum of terms, where each term is a polynomial in $(\mathbf{A} + \epsilon \mathbf{I})$. In the remaining part of this section, this is expanded into a rational function of ϵ .

Since \mathbf{A} and \mathbf{I} commute, the sum in the right hand side of the last equation can be expanded into a binomial sum, which after insertion of the expression for q_{k+1} from Equation (30) results in

$$\begin{aligned} \sum_{k=0}^{n-1} q_{k+1}(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k &= \sum_{k=0}^{n-1} q_{k+1}(\epsilon) \sum_{m=0}^k \binom{k}{m} \epsilon^{k-m} \mathbf{A}^m = \\ &= \sum_{k=0}^{n-1} \sum_{l=k+1}^n p_l \binom{l}{k+1} (-\epsilon)^{l-k-1} \sum_{m=0}^k \binom{k}{m} \epsilon^{k-m} \mathbf{A}^m. \end{aligned} \quad (37)$$

A change of the summation order for indices k and l , and using the fact that for integers $k \geq 0$ and m it is the case that

$$\binom{k}{m} = 0 \quad (38)$$

for $m < 0$ and $m > k$, gives

$$\sum_{k=0}^{n-1} q_{k+1}(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k = \sum_{l=1}^n \sum_{k=0}^{l-1} p_l \binom{l}{k+1} \binom{k}{m} (-1)^{l-k-1} \epsilon^{l-m-1} \mathbf{A}^m, \quad (39)$$

or, after a change of the summation order,

$$\sum_{k=0}^{n-1} q_{k+1}(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k = \sum_{l=1}^n p_l \sum_m \mathbf{A}^m \epsilon^{l-m-1} (-1)^{l-1} \sum_{k=0}^{l-1} \binom{l}{k+1} \binom{k}{m} (-1)^k. \quad (40)$$

Appendix A presents a proof of the following identity

$$\sum_{k=0}^{l-1} \binom{l}{k+1} \binom{k}{m} (-1)^k = \begin{cases} (-1)^m, & 0 \leq m \leq l-1, \\ 0, & \text{otherwise,} \end{cases} \quad (41)$$

which allows us to write

$$\sum_{k=0}^{n-1} q_{k+1}(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k = \sum_{l=1}^n p_l \sum_{m=0}^{l-1} \mathbf{A}^m (-\epsilon)^{l-m-1} = \sum_{l=1}^n p_l \sum_{m=0}^{l-1} \mathbf{A}^{l-m-1} (-\epsilon)^m. \quad (42)$$

A final change of summation order gives

$$\begin{aligned} \sum_{k=0}^{n-1} q_{k+1}(\epsilon) (\mathbf{A} + \epsilon \mathbf{I})^k &= \sum_{m=0}^{n-1} \sum_{l=m+1}^n p_l \mathbf{A}^{l-m-1} (-\epsilon)^m = \\ &= \sum_{m=0}^{n-1} \left[\sum_{l=0}^{n-m-1} p_{l+m+1} \mathbf{A}^l \right] (-\epsilon)^m. \end{aligned} \quad (43)$$

Defining the expression in the bracket as $R_{m+1}(\mathbf{A})$, i.e.,

$$R_m(\mathbf{A}) = \sum_{l=0}^{n-m} p_{l+m} \mathbf{A}^l = \sum_{l=m}^n p_l \mathbf{A}^{l-m}, \quad (44)$$

allows us to finally insert the derived result into Equation (36), and get

$$(\mathbf{A} + \epsilon \mathbf{I})^{-1} = -\frac{1}{q_0(\epsilon)} \sum_{m=0}^{n-1} R_{m+1}(\mathbf{A}) (-\epsilon)^m. \quad (45)$$

It should be noted that from Equation (44) it follows immediately that

$$R_0(\mathbf{A}) = \sum_{l=0}^n p_l \mathbf{A}^l = P(\mathbf{A}) = \mathbf{0}, \quad (46)$$

and, for $m > 0$,

$$\begin{aligned} R_m(\mathbf{A}) &= p_m \mathbf{I} + \sum_{l=1}^{n-m} p_{l+m} \mathbf{A}^l = p_m \mathbf{I} + \sum_{l=0}^{n-m-1} p_{l+m+1} \mathbf{A}^{l+1} = \\ &= p_m \mathbf{I} + R_{m+1}(\mathbf{A}) \mathbf{A}. \end{aligned} \quad (47)$$

Hence, there is a recursive relation between the polynomials R_m according to

$$R_m(\mathbf{A}) = p_m \mathbf{I} + \mathbf{A} R_{m+1}(\mathbf{A}), \quad (48)$$

where the recursion starts with

$$R_n(\mathbf{A}) = p_n \mathbf{I} = (-1)^n \mathbf{I}. \quad (49)$$

Finally, since

$$q_0(\epsilon) = \sum_{l=0}^n p_l \binom{l}{0} (-\epsilon)^l = \sum_{l=0}^n p_l (-\epsilon)^l = P(-\epsilon) = \det(\mathbf{A} + \epsilon \mathbf{I}), \quad (50)$$

we arrive at an expression for $(\mathbf{A} + \epsilon \mathbf{I})^{-1}$ in terms of a rational function in ϵ ,

$$(\mathbf{A} + \epsilon \mathbf{I})^{-1} = -\frac{\sum_{m=0}^{n-1} R_{m+1}(\mathbf{A}) (-\epsilon)^m}{\sum_{l=0}^n p_l (-\epsilon)^l}. \quad (51)$$

The limit value defined in Equation (10) can now be written as

$$\mathbf{A}_c = \lim_{\epsilon \rightarrow 0} \epsilon \left[-\frac{\sum_{m=0}^{n-1} R_{m+1}(\mathbf{A}) (-\epsilon)^m}{\sum_{l=0}^n p_l (-\epsilon)^l} \right] = \lim_{\epsilon \rightarrow 0} \frac{\sum_{m=0}^n R_m(\mathbf{A}) (-\epsilon)^m}{\sum_{l=0}^n p_l (-\epsilon)^l}. \quad (52)$$

5 Limit values of Equation (10)

This section derives an explicit expression for a practical implementation of a rank complement operation, by computing limit values of Equation (10) using Equation (52). The result of this derivation is presented in Equation (70) as an *m-th order rank complement*. However, before any limit values can be derived, some properties of the expressions $R_m(\mathbf{A})$ and p_l , the coefficients of the rational function in Equation (52), need to be established.

First, we note directly that $R_m(\mathbf{A})$ is an $n - m$ degree polynomial in \mathbf{A} , and that p_l is an $(n - l)$ degree polynomial in the elements of \mathbf{A} . Second, if the rank of \mathbf{A} is r , the following properties are true

P1 $p_l = 0$ for $l < n - r$.

P2 $p_{n-r} \neq 0$.

P3 $R_m(\mathbf{A}) = 0$ for $m < n - r$.

P4 The ranges and null spaces of $R_{n-r}(\mathbf{A})$ and \mathbf{A} are interchanged.

Proofs of properties **P1**–**P4** are provided below.

P1 + P2 If the rank of \mathbf{A} is r , then there are $n - r$ eigenvalues of \mathbf{A} which are zero. Since these eigenvalues are roots of $P(\lambda)$, it follows that

$$P(\lambda) = \lambda^{n-r} \left(\sum_{l=n-r}^n p_l \lambda^{l-n+r} \right) = \lambda^{n-r} S_r(\lambda), \quad (53)$$

where $S_r(\lambda)$ has only non-zero roots, or

$$\begin{cases} p_l = 0 & \text{for } l < n - r, \\ p_l \neq 0 & \text{for } l = n - r. \end{cases} \quad (54)$$

P3 From Equation (44) follows

$$R_m(\mathbf{A}) = \sum_{l=m}^n p_l \mathbf{A}^{l-m} = \sum_{l=0}^n p_l \mathbf{A}^{l-m} - \sum_{l=0}^{m-1} p_l \mathbf{A}^{l-m}, \quad (55)$$

and, consequently,

$$\mathbf{A}^m R_m(\mathbf{A}) = \sum_{l=0}^n p_l \mathbf{A}^l - \sum_{l=0}^{m-1} p_l \mathbf{A}^l = R_0(\mathbf{A}) - \sum_{l=0}^{m-1} p_l \mathbf{A}^l = - \sum_{l=0}^{m-1} p_l \mathbf{A}^l. \quad (56)$$

Since in this case $m < n - r$, **P1** implies that $p_l = 0$ for $0 \leq l \leq m - 1$ and, consequently,

$$\mathbf{A}^m R_m(\mathbf{A}) = \mathbf{0}. \quad (57)$$

Let us assume that $R_m(\mathbf{A}) \neq \mathbf{0}$, from which follows that there exists an eigenvector \mathbf{e} of $R_m(\mathbf{A})$ such that

$$R_m(\mathbf{A}) \mathbf{e} = \sigma \mathbf{e}, \quad \sigma \neq 0. \quad (58)$$

Consequently,

$$\mathbf{0} = \mathbf{A}^m R_m(\mathbf{A}) \mathbf{e} = \sigma \mathbf{A}^m \mathbf{e}, \quad (59)$$

which implies that \mathbf{e} is a zero eigenvector relative to \mathbf{A}^m , and therefore also a zero eigenvector relative to \mathbf{A} . But since $R_m(\mathbf{A})$ is a polynomial in \mathbf{A} , Equation (44), it follows that

$$R_m(\mathbf{A}) \mathbf{e} = p_m \mathbf{e} \quad \Rightarrow \quad \sigma = p_m. \quad (60)$$

However, this is a contradiction since, in this case, $p_m = 0$. The contradiction is only resolved by having $R(\mathbf{A}) = \mathbf{0}$, and we have thus proved that $R(\mathbf{A}) = \mathbf{0}$ for $m < n - r$.

P4 Consider first the case that \mathbf{A} has full rank, i.e., $r = n$. Then, $R_{n-r}(\mathbf{A}) = R_0(\mathbf{A}) = \mathbf{0}$, which obviously has rank $n - r = 0$.

Second, consider the case $r = 0$, i.e., $\mathbf{A} = \mathbf{0}$. Then, $R_{n-r}(\mathbf{A}) = R_n(\mathbf{A}) = p_n \mathbf{I} = (-1)^n \mathbf{I}$, which obviously has rank $n - r = n$.

Third, consider the case $0 < r < n$. From Equations (44) and (53) we get

$$R_{n-r}(\mathbf{A}) = S_r(\mathbf{A}). \quad (61)$$

From the construction of $S_r(\lambda)$ it follows that

$$S_r(\lambda) = p_n (\lambda - \lambda_1) (\lambda - \lambda_2) \dots (\lambda - \lambda_r), \quad a \neq 0, \quad (62)$$

where $p_n = (-1)^n$, and $\lambda_1, \dots, \lambda_r$ are the r non-zero eigenvalues of \mathbf{A} . Hence,

$$R_{n-r}(\mathbf{A}) = (-1)^n (\mathbf{A} - \lambda_1 \mathbf{I}) (\mathbf{A} - \lambda_2 \mathbf{I}) \dots (\mathbf{A} - \lambda_r \mathbf{I}). \quad (63)$$

Let \mathbf{e} be an arbitrary non-zero vector of the null space of \mathbf{A} , i.e., \mathbf{e} is a zero eigenvector of \mathbf{A} . From Equation (63) follows then

$$R_{n-r}(\mathbf{A}) \mathbf{e} = (-1)^n (-\lambda_1) (-\lambda_2) \dots (-\lambda_r) \mathbf{e} \neq \mathbf{0}. \quad (64)$$

which proves that \mathbf{e} is a non-zero eigenvector of $R_{n-r}(\mathbf{A})$, i.e., it lies in the range of $R_{n-r}(\mathbf{A})$. Assume instead that \mathbf{e} is an eigenvector of \mathbf{A} , with eigenvalue $\lambda_j \neq 0$, i.e., an element of the range of \mathbf{A} . From Equation (63) follows now that

$$R_{n-r}(\mathbf{A}) \mathbf{e} = \mathbf{0}. \quad (65)$$

which proves that \mathbf{e} is in the null space of $R_{n-r}(\mathbf{A})$.

We may summarize this by stating that the range of \mathbf{A} is the null space of $R_{n-r}(\mathbf{A})$, and the null space of \mathbf{A} is the range of $R_{n-r}(\mathbf{A})$.

Having proved properties **P1–P4**, we may now derive a limit value of \mathbf{A}_c , as defined in Equation (10), using its expression in terms of a rational function of ϵ , Equation (52). From **P1–P3** we get immediately

$$\begin{aligned} \mathbf{A}_c &= \lim_{\epsilon \rightarrow 0} \frac{(-\epsilon)^{n-r} \sum_{m=0}^r R_{m+n-r}(\mathbf{A}) (-\epsilon)^m}{(-\epsilon)^{n-r} \sum_{l=0}^r p_{l+n-r} (-\epsilon)^l} = \\ &= \lim_{\epsilon \rightarrow 0} \frac{R_{n-r}(\mathbf{A}) + \sum_{m=1}^r R_{m+n-r}(\mathbf{A}) (-\epsilon)^m}{p_{n-r} + \sum_{l=1}^r p_{l+n-r} (-\epsilon)^l} = \frac{R_{n-r}(\mathbf{A})}{p_{n-r}}. \end{aligned} \quad (66)$$

Notice that $p_{n-r} \neq 0$ according to **P2**, and that $R_{n-r}(\mathbf{A})$ is indeed a rank complement of \mathbf{A} according to **P4**. Since the denominator of this expression is a scalar, we may omit it without destroying the rank complement property of \mathbf{A}_c . Hence, given that \mathbf{A} has rank r , the rank complement of \mathbf{A} may be defined as

$$\mathbf{A}_c = R_{n-r}(\mathbf{A}). \quad (67)$$

If the eigenvalues of \mathbf{A} are all positive or zero, we may be interested in preserving this property also for its rank complement. From Equation (64), and from the above discussion, it follows that if we define

$$\mathbf{A}_c = (-1)^{n-r} R_{n-r}(\mathbf{A}), \quad (68)$$

then the eigenvalues of \mathbf{A}_c are either zero or

$$\lambda_c = \lambda_1 \lambda_2 \dots \lambda_r > 0, \quad (69)$$

where $\lambda_1, \dots, \lambda_r$ are the r non-zero eigenvalues of \mathbf{A} .

It should be noted that the above given implementation of a rank complement operation is dependent on the rank of the original matrix, i.e., we have to use different expressions for the rank complement operation for different ranks. From a formal point of view, we should therefore make the following definition.

- Given a symmetric $n \times n$ matrix \mathbf{A} , the matrix expression $\mathbf{A}_{c,m}$ given by

$$\mathbf{A}_{c,m} = (-1)^{n-m} R_{n-m}(\mathbf{A}). \quad (70)$$

is referred to as an m -th order rank complement of \mathbf{A} . The previous discussion shows that $\mathbf{A}_{c,m}$ is a rank complement of a \mathbf{A} if $m = r$ where r is the rank of \mathbf{A} , and it is in fact only in this case that $\mathbf{A}_{c,m}$ corresponds to a proper rank complement operation.

With this definition at hand, it is interesting to consider also what type of matrix $\mathbf{A}_{c,m}$ represents if $m \neq r$. From **P3** it follows directly that $\mathbf{A}_{c,m} = \mathbf{0}$ for the case that $m > r$. The case $m < r$ is more difficult, in particular if there is no limitation on the eigenvalues of \mathbf{A} . In this case, $\mathbf{A}_{c,m}$ is a matrix that has a rank which does not correspond to the rank of \mathbf{A} in any straightforward way. In fact, $\mathbf{A}_{c,m}$ may have arbitrary rank independent of the rank of \mathbf{A} .

6 Examples

This section exemplifies the results derived in the last section by deriving an explicit expression m -th order rank complement matrix $\mathbf{A}_{c,m}$ for small values of n .

6.1 The 2×2 case

For $n = 2$ and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}, \quad (71)$$

we get

$$P(\lambda) = a_{11} a_{22} - a_{12}^2 - (a_{11} + a_{22}) \lambda + \lambda^2. \quad (72)$$

From this follows directly that

$$p_0 = a_{11} a_{22} - a_{12}^2 = \det(\mathbf{A}), \quad (73)$$

$$p_1 = -(a_{11} + a_{22}) = -\text{trace}(\mathbf{A}), \quad (74)$$

$$p_2 = 1, \quad (75)$$

and

$$R_0(\mathbf{A}) = p_0 \mathbf{I} + p_1 \mathbf{A} + p_2 \mathbf{A}^2 = \mathbf{0}, \quad (76)$$

$$R_1(\mathbf{A}) = p_1 \mathbf{I} + p_2 \mathbf{A} = -\text{trace}(\mathbf{A}) \mathbf{I} + \mathbf{A}, \quad (77)$$

$$R_2(\mathbf{A}) = p_2 \mathbf{I} = \mathbf{I}. \quad (78)$$

From the last section we know that $\mathbf{A}_{c,m} = (-1)^{n-m} R_{n-m}(\mathbf{A})$ is a rank complement of \mathbf{A} if $m = r$. Let us check:

$$A_{c,2} = (-1)^0 R_0(\mathbf{A}) = \mathbf{0}, \quad (79)$$

$$A_{c,1} = (-1)^1 R_1(\mathbf{A}) = \text{trace}(\mathbf{A}) \mathbf{I} - \mathbf{A}, \quad (80)$$

$$A_{c,0} = (-1)^2 R_2(\mathbf{A}) = \mathbf{I}. \quad (81)$$

These are exactly the expressions which were derived for the 2×2 case in Section 3.

6.2 The 3×3 case

For $n = 3$ and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, \quad (82)$$

we get

$$P(\lambda) = p_0 + p_1 \lambda + p_2 \lambda^2 + p_3 \lambda^3, \quad (83)$$

where

$$p_0 = \det(\mathbf{A}), \quad (84)$$

$$p_1 = a_{12}^2 + a_{13}^2 + a_{23}^2 - a_{11} a_{22} - a_{11} a_{33} - a_{22} a_{33}, \quad (85)$$

$$p_2 = a_{11} + a_{22} + a_{33} = \text{trace}(\mathbf{A}), \quad (86)$$

$$p_3 = -1. \quad (87)$$

This leads to

$$R_0(\mathbf{A}) = p_0 \mathbf{I} + p_1 \mathbf{A} + p_2 \mathbf{A}^2 + p_3 \mathbf{A}^3 = \mathbf{0}, \quad (88)$$

$$R_1(\mathbf{A}) = p_1 \mathbf{I} + p_2 \mathbf{A} + p_3 \mathbf{A}^2, \quad (89)$$

$$R_2(\mathbf{A}) = p_2 \mathbf{I} + p_3 \mathbf{A} = \text{trace}(\mathbf{A}) \mathbf{I} - \mathbf{A}, \quad (90)$$

$$R_3(\mathbf{A}) = p_3 \mathbf{I} = -\mathbf{I}, \quad (91)$$

from which the rank complements of different orders are computed as

$$A_{c,3} = (-1)^0 R_0(\mathbf{A}) = \mathbf{0}, \quad (92)$$

$$A_{c,2} = (-1)^1 R_1(\mathbf{A}) = -p_1 \mathbf{I} - \text{trace}(\mathbf{A}) \mathbf{A} + \mathbf{A}^2, \quad (93)$$

$$A_{c,1} = (-1)^2 R_2(\mathbf{A}) = \text{trace}(\mathbf{A}) \mathbf{I} - \mathbf{A}, \quad (94)$$

$$A_{c,0} = (-1)^3 R_3(\mathbf{A}) = \mathbf{I}. \quad (95)$$

6.3 The 4×4 case

For $n = 4$ and

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{12} & a_{22} & a_{23} & a_{24} \\ a_{13} & a_{23} & a_{33} & a_{34} \\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}, \quad (96)$$

we get

$$P(\lambda) = p_0 + p_1 \lambda + p_2 \lambda^2 + p_3 \lambda^3 + p_4 \lambda^4, \quad (97)$$

where

$$p_0 = \det(\mathbf{A}), \quad (98)$$

$$\begin{aligned} p_1 = & a_{12}^2(a_{33} + a_{44}) + a_{13}^2(a_{22} + a_{44}) + a_{14}^2(a_{22} + a_{33}) + a_{23}^2(a_{11} + a_{44}) + \\ & + a_{24}^2(a_{11} + a_{33}) + a_{34}^2(a_{11} + a_{22}) - a_{11} a_{22} a_{33} - a_{11} a_{22} a_{44} - \\ & - a_{11} a_{33} a_{44} - a_{22} a_{33} a_{44} - 2 a_{12} a_{13} a_{23} - 2 a_{12} a_{14} a_{24} - \\ & - 2 a_{13} a_{14} a_{34} - 2 a_{23} a_{24} a_{34}, \end{aligned} \quad (99)$$

$$\begin{aligned} p_2 = & -a_{12}^2 - a_{13}^2 - a_{14}^2 - a_{23}^2 - a_{24}^2 - a_{34}^2 + a_{11} a_{22} + a_{11} a_{33} + a_{11} a_{44} + \\ & + a_{22} a_{33} + a_{22} a_{44} + a_{33} a_{44}, \end{aligned} \quad (100)$$

$$p_3 = -a_{11} - a_{22} - a_{33} = -\text{trace}(\mathbf{A}), \quad (101)$$

$$p_4 = 1. \quad (102)$$

This leads to

$$R_0(\mathbf{A}) = p_0 \mathbf{I} + p_1 \mathbf{A} + p_2 \mathbf{A}^2 + p_3 \mathbf{A}^3 + p_4 \mathbf{A}^4 = \mathbf{0}, \quad (103)$$

$$R_1(\mathbf{A}) = p_1 \mathbf{I} + p_2 \mathbf{A} + p_3 \mathbf{A}^2 + p_4 \mathbf{A}^3, \quad (104)$$

$$R_2(\mathbf{A}) = p_2 \mathbf{I} + p_3 \mathbf{A} + p_4 \mathbf{A}^2, \quad (105)$$

$$R_3(\mathbf{A}) = p_3 \mathbf{I} + p_4 \mathbf{A} = -\text{trace}(\mathbf{A}) \mathbf{I} + \mathbf{A}, \quad (106)$$

$$R_4(\mathbf{A}) = p_4 \mathbf{I} = \mathbf{I}, \quad (107)$$

from which the rank complements of different orders are computed as

$$A_{c,4} = (-1)^0 R_0(\mathbf{A}) = \mathbf{0}, \quad (108)$$

$$A_{c,3} = (-1)^1 R_1(\mathbf{A}) = -p_1 \mathbf{I} - p_2 \mathbf{A} - p_3 \mathbf{A}^2 - p_4 \mathbf{A}^3, \quad (109)$$

$$A_{c,2} = (-1)^2 R_2(\mathbf{A}) = p_2 \mathbf{I} + p_3 \mathbf{A} + p_4 \mathbf{A}^2, \quad (110)$$

$$A_{c,1} = (-1)^3 R_3(\mathbf{A}) = \text{trace}(\mathbf{A}) \mathbf{I} - \mathbf{A}, \quad (111)$$

$$A_{c,0} = (-1)^4 R_4(\mathbf{A}) = \mathbf{I}. \quad (112)$$

7 Generalizations

The results in the previous sections are derived under the assumption that \mathbf{A} , the matrix for which we want to compute a rank complement, is *real* and *symmetric* simply because this refers to most of the practical implementations. However, from a mathematical point of view it may be interesting also to see how the results can be generalized.

For a general $n \times n$ matrix \mathbf{A} all we can say about its range and null space is that their dimensionalities add to n . In fact, they do not even have to be disjoint. A class of matrices for which this phenomena occurs is nil-potent matrices, i.e., matrices \mathbf{A} for which $\mathbf{A}^l = \mathbf{0}$ for some integer $l > 1$. If \mathbf{A} is a nil-potent matrix, and l is the smallest integer such that $\mathbf{A}^l = \mathbf{0}$, then $\mathbf{A}^{l-1} \neq \mathbf{0}$ and there must be a vector \mathbf{e} such that $\mathbf{A}^{l-1} \mathbf{e} \neq \mathbf{0}$. Hence, $\mathbf{A}^{l-1} \mathbf{e}$ lies in the range of \mathbf{A} . However, since $\mathbf{A}^l \mathbf{e} = \mathbf{0}$, it must also be the case that $\mathbf{A}^{l-1} \mathbf{e}$ lies in the null space of \mathbf{A} . Since matrices with overlapping range and null space are considered of little practical use for the applications considered here, we will not try to generalize the presented results for these types matrices.

The next issue must then be to establish which matrices have disjoint range and null spaces. However, to characterize this class of matrices falls outside the scope of this presentation, and instead we will consider a smaller class of matrices which still covers most of the practical cases. This is the class of diagonalizable matrices, i.e., matrices which can be brought to a diagonal form by means of a change of basis. Any such matrix can be written as

$$\mathbf{A} = \mathbf{E} \mathbf{D} \mathbf{E}^{-1}, \quad (113)$$

where \mathbf{E} is a non-singular matrix which represents the change of basis, and \mathbf{D} is the corresponding diagonal matrix. An $n \times n$ matrix of this type has always a set of eigenvectors which forms an n -dimensional basis (the columns of \mathbf{E}), and the diagonal elements of \mathbf{D} are the corresponding eigenvalues. As a consequence, an eigenvector of \mathbf{A} must either lie in the range or in the null space of \mathbf{A} depending on whether its eigenvalue is non-zero or zero, respectively. From this follows immediately that the range and null space must be disjoint.

It is a simple exercise to show that all the results which have been derived for the case of symmetric matrices are valid also for diagonalizable matrices. The major differences relative to the case of symmetric matrices are that both eigenvalues and eigenvectors may be now complex valued, and that the eigenvectors may form non-orthogonal bases. Since no assumptions about these properties have been made in the derivations, all results can be proved also for this class of matrices.

8 Short summary

If \mathbf{A} is $n \times n$ and of rank r , define

$$\mathbf{A}_c = (-1)^{n-r} R_{n-r}(\mathbf{A}),$$

where

$$R_m = \sum_{l=0}^{n-m} p_{l+m} \mathbf{A}^l$$

and p_k are the coefficients of the characteristic polynomial of \mathbf{A} ,

$$\det(\mathbf{A} - \lambda \mathbf{I}) = \sum_{k=0}^n p_k \lambda^k.$$

It follows that \mathbf{A}_c defined in this way constitutes a rank complement of \mathbf{A} . Furthermore, \mathbf{A}_c has all non-zero eigenvalues equal to

$$\lambda = \lambda_1 \lambda_2 \dots \lambda_r,$$

where $\lambda_1 \dots \lambda_r$ are the r non-zero eigenvalues of \mathbf{A} . Hence, if \mathbf{A} is positive semi-definite, then so is \mathbf{A}_c .

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A Proof of identity in Equation (41)

To prove the identity presented in Equation (41):

$$\sum_{k=0}^{l-1} \binom{l}{k+1} \binom{k}{m} (-1)^k = \begin{cases} (-1)^m, & 0 \leq m \leq l-1, \\ 0, & \text{otherwise,} \end{cases} \quad (114)$$

where $l \geq 0$ and m are integers, we use the summation formula (5.24) from Table 169 of [1],

$$\sum_k \binom{l}{m+k} \binom{s+k}{n} (-1)^k = (-1)^{l+m} \binom{s-m}{n-l}, \quad (115)$$

where $l \geq 0, m$, and n are all integers. Before we can use this formula we need to extend the summation over k to all integers. Since $\binom{l}{k+1}$ is zero outside the interval $-1 \leq k \leq l-1$ it suffices to include $k = -1$ in the sum. We get

$$\begin{aligned} \sum_{k=0}^{l-1} \binom{l}{k+1} \binom{k}{m} (-1)^k &= -\binom{l}{0} \binom{-1}{m} (-1)^{-1} + \sum_k \binom{l}{k+1} \binom{k}{m} (-1)^k \\ &= \binom{l}{0} \binom{-1}{m} + (-1)^{l+1} \binom{-1}{m-l}. \end{aligned} \quad (116)$$

To simplify this expression we notice that

$$\binom{l}{0} = 1, \quad \text{and} \quad \binom{-1}{k} = \begin{cases} (-1)^k & k \geq 0, \\ 0 & k < 0. \end{cases} \quad (117)$$

Thus we get

$$\sum_{k=0}^{l-1} \binom{l}{k+1} \binom{k}{m} (-1)^k = \begin{cases} 0, & m < 0, \\ (-1)^m, & 0 \leq m < l, \\ (-1)^m + (-1)^{l+1+m-l} = 0, & m \geq l, \end{cases} \quad (118)$$

as stated in Equation (114).

B Matlab function for rank complement

The following Matlab function computes an m -th order rank complement. It should be noted that Matlab defines the characteristic polynomial of the matrix \mathbf{A} as $\det(\lambda \mathbf{I} - \mathbf{A})$, which differs a factor $(-1)^n$ relative to the polynomial used in the previous presentation.

```
function Rj = rankcompl(A, m)
% This function computes a rank complement of order m of a symmetric
% matrix A. If the rank of A is m, the result is another symmetric
% matrix Ac whose range is equal to the null space of A, and whose
% null space is equal to the range of A. Also, the non-zero
% eigenvalues of Ac are all equal to the product of the non-zero
% eigenvalues of A.
s = size(A);
if (length(s) ~= 2),
    'A must be an n times n matrix'
    return;
end
if (s(1) ~= s(2)),
    'A must be an n times n matrix'
    return;
end
n = s(1);
% Compute the coefficients of the characteristic polynomial.
% Matlab uses det(x I - A) instead of det(A - x I) as
% characteristic polynomial, hence the factor (-1)^n.
% The order of the elements in p is such that
% p(k) = p_{n + 1 - k} or p_{1} = p(n + 1 - 1).
p = (-1)^n * poly(A);
% Compute the R_{n - m} polynomial recursively
I = eye(n);
Rj = (-1)^n * I;
j = n;
while (j ~= n - m),
    j = j - 1;
    Rj = p(n + 1 - j) * I + A * Rj;
end
% Multiply with (-1)^(n - m) to get right sign of eigenvalues
Rj = (-1)^(n - m) * Rj;
return
```

References

- [1] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics*. Addison-Wesley, 1989.
- [2] G. H. Granlund and H. Knutsson. *Signal Processing for Computer Vision*. Kluwer Academic Publishers, 1995. ISBN 0-7923-9530-1.