Local Curvature from Gradients of the Orientation Tensor Field

Klas Nordberg   Hans Knutsson   Gösta Granlund

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Klas Nordberg Hans Knutsson Gösta Granlund
Computer Vision Laboratory,
Department of Electrical Engineering
Linköping University, S-581 83 LINKÖPING, SWEDEN
Phone: +46-13-28 16 34, Telefax: +46-13-13 85 26
Email: klas@isy.liu.se

Abstract

This paper presents an algorithm for estimation of local curvature from gradients of a tensor field that represents local orientation. The algorithm is based on an operator representation of the orientation tensor, which means that change of local orientation corresponds to a rotation of the eigenvectors of the tensor. The resulting curvature descriptor is a vector that points in the direction of the image in which the local orientation rotates anti-clockwise and the norm of the vector is the inverse of the radius of curvature. Two coefficients are defined that relate the change of local orientation with either curves or radial patterns.

1 Introduction

Two-dimensional curvature can be defined in different ways. For example, curvature can be defined as the rate of change of the tangent angle with respect to the curve length. An other way is to first estimate local orientation, resulting in a field of orientation descriptors, and then consider its variation with respect to the image coordinates. For a curve, the orientation descriptor varies according to the change of local orientation *along* the curve and it is constant perpendicular to it. This latter technique has been used by Bårman [1] who considered orientation descriptors in 2D and 3D, and estimated their local variation by means of so-called quasi-quadrature filters. This paper presents an algorithm that is similar to Bårman's in that it operates on orientation descriptors rather than on the image itself. Compared to the approach by Bårman, simple gradients of the orientation descriptors are used instead.

In the following, we will consider descriptors of local orientation that are second order tensors. This type of tensors were introduced by Knutsson [2] as a means to obtain a continuous representation of local orientation in multi-dimensional images. Taking a 2D image as an example, any point in the image is represented by a symmetric 2×2 tensor, according to

$$\mathbf{T} = \lambda_1 \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T + \lambda_2 \, \hat{\mathbf{e}}_2 \, \hat{\mathbf{e}}_2^T \tag{1}$$

where $\lambda_1 \geq \lambda_2 \geq 0$ are the eigenvalues of $\mathbf{T}$, and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are corresponding eigenvectors. The tensor is local and describes what type of local image structure is found in an neighbourhood around a particular image point. The set of all tensors that result by analyzing all image points will then form a tensor field as a function of the image coordinates. For a neighbourhood that contains a clear linear structure, the tensor is characterized by $\lambda_1 > \lambda_2 = 0$, and $\hat{\mathbf{e}}_1$ is perpendicular to the linear structure. This is referred to as $\mathbf{T}$ being anisotropic. For an isotropic neighbourhood, with no dominant orientation, the tensor is characterized by $\lambda_1 = \lambda_2 > 0$. In this case is $\mathbf{T}$ proportional to the identity tensor $\mathbf{I}$, and $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are therefore ambiguous. For a general case will $\mathbf{T}$ have neither of these two characteristic shapes, but we can always say that it is the eigenvector corresponding to the largest eigenvalue that describes the local dominant orientation. Knutsson describes how tensors with the above properties can be obtained, locally, from linear combinations of filter responses of spherically separable quadrature filters, where the directional function is carefully chosen.

2 The algorithm

From the above it is clear that the information of local orientation is carried by the eigenvectors of $\mathbf{T}$. Hence, any local variation of orientation is reflected in a variation of the eigenvectors of the local orientation tensor. In particular, if we consider rank one tensors, for which $\lambda_1 > 0$ and the other eigenvalues are zero, we are interested in the variation of the eigenvector which has largest eigenvalue. According to the above, these tensors describe linear structures. Since a curve, locally, appears as a linear structure, the orientation tensor must be of rank one in the vicinity of the curve, and the eigenvector $\hat{\mathbf{e}}_1$ is perpendicular to the curve. However, if we move along the curve, the eigenvector $\hat{\mathbf{e}}_1$ must change as a consequence of the curvature. In fact, since $\hat{\mathbf{e}}_1$ is normalized, it must rotate, and its angular speed is proportional to the inverse of the radius of curvature. However, translation across the curve will leave $\mathbf{T}$ unaffected. This is proved by the following.

Let an image contain a circular pattern, centered at the origin. The local orientation tensor $\mathbf{T}$ is then expressed as

$$\mathbf{T} = \lambda \, \frac{\mathbf{x}\,\mathbf{x}^T}{r^2} \tag{2}$$

or

$$\mathbf{T}_{ij} = \lambda \, \frac{x_i \, x_j}{r^2} \tag{3}$$

where $\mathbf{x}$ is the global image coordinates, and $r = |\mathbf{x}|$. Differentiation of $\mathbf{T}$ with respect to the image coordinates then gives

$$\frac{d\mathbf{T}_{ij}}{dx_k} = \lambda \, \frac{(\delta_{ik} \, x_j + x_i \, \delta_{jk}) \, r^2 - 2 \, x_i \, x_j \, x_k}{r^4} \tag{4}$$

At point $\mathbf{x}$ in the image, a translation along the circular curves will be in the orientation given by

$$\mathbf{y} = \mathbf{H}\,\mathbf{x} \tag{5}$$

where

$$\mathbf{H} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \tag{6}$$

since the anti-Hermitian operator $\mathbf{H}$ maps $\mathbf{x}$ to a perpendicular vector $\mathbf{y}$, corresponding to an anti-clockwise rotation by $\frac{\pi}{2}$. Hence, $|\mathbf{y}| = |\mathbf{x}| = r$. A translation along the curve of length p, and in the direction of $\mathbf{y}$, will then, to first order of approximation, cause a variation of $\mathbf{T}$ that amounts to

$$\frac{p}{r} y_k \, \frac{d\mathbf{T}_{ij}}{dx_k} = \frac{\lambda \, p}{r^5} \left[(y_i \, x_j + x_i \, y_j) \, r^2 - 2 \, x_i \, x_j \, x_k \, y_k \right] = \frac{\lambda \, p}{r^3} \left[y_i \, x_j + x_i \, y_j \right] \tag{7}$$

or, in matrix notation

$$\frac{p}{r} \, \frac{\mathbf{y}\,\mathbf{x}^T + \mathbf{x}\,\mathbf{y}^T}{r^2} = \frac{p}{r} \, \frac{\mathbf{H}\,\mathbf{x}\,\mathbf{x}^T - \mathbf{x}\,\mathbf{x}^T\,\mathbf{H}}{r^2} = \frac{p}{r} \left[\mathbf{H}\,\mathbf{T} - \mathbf{T}\,\mathbf{H} \right] \tag{8}$$

A translation perpendicular to the curve, at point $\mathbf{x}$, is made in the direction of $\mathbf{x}$. Hence, to first order of approximation, the variation of $\mathbf{T}$ is given by

$$\frac{p}{r} x_k \frac{d\mathbf{T}_{ij}}{dx_k} = \frac{\lambda\, p}{r^5} \left[(x_i\, x_j + x_i\, x_j)\, r^2 - 2\, x_i\, x_j\, x_k\, x_k \right] = \frac{\lambda\, p}{r^5} \left[2x_i\, x_j\, r^2 - x_i\, x_j\, r^2 \right] = 0, \tag{9}$$

which proves the last part of the previous statement. It remains to show that the translation along the curve corresponds to a rotation of the eigenvectors of $\mathbf{T}$, with an angular speed proportional to the inverse of the radius of curvature, here given by r. We will do so by using the anti-Hermitian operator $\mathbf{H}$ as generator of the group of 2D rotations by means of the matrix exponential function, see Nordberg [3]. It is straightforward to show that

$$e^{\alpha \mathbf{H}} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \tag{10}$$

where the right hand side is an anti-clockwise 2D rotation by an angle α. Hence, an anti-clockwise rotation of the dominant eigenvector of $\mathbf{T}$, an angle α, is given by

$$\mathbf{T}(\alpha) = \frac{\lambda}{r^2} \left(e^{\alpha \mathbf{H}}\, \mathbf{x} \right) \left(e^{\alpha \mathbf{H}}\, \mathbf{x} \right)^T = \frac{\lambda}{r^2}\, e^{\alpha \mathbf{H}}\, \mathbf{x}\, \mathbf{x}^T\, e^{-\alpha \mathbf{H}} = e^{\alpha \mathbf{H}}\, \mathbf{T}\, e^{-\alpha \mathbf{H}} \tag{11}$$

Differentiation of $\mathbf{T}(\alpha)$ with respect to α gives

$$\frac{d\mathbf{T}(\alpha)}{d\alpha} = \mathbf{H}(e^{\alpha \mathbf{H}}\, \mathbf{T}\, e^{-\alpha \mathbf{H}}) - (e^{\alpha \mathbf{H}}\, \mathbf{T}\, e^{-\alpha \mathbf{H}})\mathbf{H} = \mathbf{H}\, \mathbf{T}(\alpha) - \mathbf{T}(\alpha)\, \mathbf{H} \tag{12}$$

Comparing Equations (8) and (12) we see that translation along the curves causes a rotation of the dominant eigenvector by an angel $\frac{p}{r}$, i.e. the angular speed is $\frac{1}{r}$.

Having derived an expression for the variation of $\mathbf{T}$ for circular curves, we simply assume that arbitrary curves at each point have a well-defined radius of curvature, which means that the above result is valid locally. To make this model more general, we will assume that, locally, there exists a function f of the local image coordinates $\boldsymbol{\xi}$, such that

$$e^{f(\boldsymbol{\xi})\, \mathbf{H}} \tag{13}$$

is a valid description of the rotation group that changes the eigenvectors of $\mathbf{T}$. Hence, we can assume that $f(\mathbf{0}) = 0$, which means that the local description of $\mathbf{T}$ can be written

$$\mathbf{T}(\boldsymbol{\xi}) = e^{f(\boldsymbol{\xi})\, \mathbf{H}}\, \mathbf{T}(\mathbf{0})\, e^{-f(\boldsymbol{\xi}\, \mathbf{H})} \tag{14}$$

The interesting component of f is its gradient with respect to the image coordinates. According to the above, it will point in direction in which the eigenvectors of $\mathbf{T}$ rotates fastest and anti-clockwise, and its norm is the angular speed in that direction. The gradient of f can be computed from the gradient of $\mathbf{T}$ according to the following

First, compute the gradient of $\mathbf{T}$ with respect to the image coordinates

$$\nabla_k \mathbf{T} = \frac{d\mathbf{T}}{d\xi_k} = \frac{df}{d\xi_k} \mathbf{H}\, \mathbf{T} - \mathbf{T}\, \frac{df}{d\xi_k}\, \mathbf{H} = \frac{df}{d\xi_k} [\mathbf{H}\, \mathbf{T} - \mathbf{T}\, \mathbf{H}] = \nabla_k\, f\, [\mathbf{H}\, \mathbf{T} - \mathbf{T}\, \mathbf{H}] \tag{15}$$

Let $V_{\mathbf{T}}$ denote the vector space of 2×2 symmetric tensors, i.e. $\mathbf{T} \in V_{\mathbf{T}}$. Define $\mathcal{H} : V_{\mathbf{T}} \to V_{\mathbf{T}}$ as the linear mapping

$$\mathcal{H}\{\mathbf{X}\} = \mathbf{H}\, \mathbf{X} - \mathbf{X}\, \mathbf{H}, \qquad \mathbf{X} \in V_{\mathbf{T}} \tag{16}$$

Thus

$$\nabla_k \mathbf{T} = \nabla_k f \cdot \mathcal{H}\{\mathbf{T}\} \tag{17}$$

To find an explicit expression for ∇f, we must introduce a scalar product in $V_{\mathbf{T}}$ according to

$$< \mathbf{X} \mid \mathbf{Y} > = \text{trace}(\mathbf{X}^T \, \mathbf{Y}), \tag{18}$$

which, in fact, correspond to standard linear combination of the elements of $\mathbf{X}$ and $\mathbf{Y}$. It follows then immediately that

$$\nabla_k f = \frac{< \nabla_k \mathbf{T} \mid \mathcal{H}\{\mathbf{T}\} >}{< \mathcal{H}\{\mathbf{T}\} \mid \mathcal{H}\{\mathbf{T}\} >} = \frac{< \nabla_k \mathbf{T} \mid \mathcal{H}\{\mathbf{T}\} >}{|\mathcal{H}\{\mathbf{T}\}|^2} \tag{19}$$

This is one of the main results of this paper. It shows that the gradient of f can be computed from the gradient of the orientation tensor field, $\nabla \mathbf{T}$, simply by first computing the mapping $\mathcal{H}\{\mathbf{T}\}$, its norm squared, and the scalar product between $\nabla \mathbf{T}$ and $\mathcal{H}\{\mathbf{T}\}$. The gradient of f is then given by a simple ratio. Furthermore, this gradient points along any curve, in the direction given by an anti-clockwise rotation of the local orientation, its norm is proportional to the inverse of the radius of curvature.

The above derivation for an expression of ∇f is based on a rank one tensor $\mathbf{T}$, i.e. $\lambda_1 > \lambda_2 = 0$. However, since the eigenvectors of $\mathbf{T}$ must transform in the same way, and a general tensor is a linear combination of two rank one tensors, the above result hold even for the case that $\lambda_2 > 0$. Furthermore, ∇f, defined by Equation (19), will have the mentioned properties even though $\nabla \mathbf{T}$ contains components other than those cause by variation of orientation. Except for variation of its eigenvectors, also the two eigenvalues of $\mathbf{T}$ may vary. Rewriting the expression given for $\mathbf{T}$ according to Equation (1) as

$$\mathbf{T} = (\lambda_1 - \lambda_2) \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T + \lambda_2 \, (\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T + \hat{\mathbf{e}}_2 \, \hat{\mathbf{e}}_2^T) = (\lambda_1 - \lambda_2) \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T + \lambda_2 \, \mathbf{I}, \tag{20}$$

then differentiation of the eigenvalues (denoted by a prime sign) gives

$$\mathbf{T}' = (\lambda_1' - \lambda_2') \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T + \lambda_2' \, \mathbf{I}. \tag{21}$$

And, using that $\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T \, \mathbf{T} = \mathbf{T} \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T$, we see that

$$\begin{aligned}
&< \mathbf{T}' \mid \mathcal{H}\{\mathbf{T}\} > = < (\lambda_1' - \lambda_2')\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T - \lambda_2' \, \mathbf{I} \mid \mathbf{H} \, \mathbf{T} - \mathbf{T} \, \mathbf{H} > = \\
&\text{trace}[((\lambda_1' - \lambda_2')\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T - \lambda_2' \, \mathbf{I})^T \, (\mathbf{H} \, \mathbf{T} - \mathbf{T} \, \mathbf{H})] = \\
&\text{trace}[((\lambda_1' - \lambda_2')\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T - \lambda_2' \, \mathbf{I}) \, (\mathbf{H} \, \mathbf{T} - \mathbf{T} \, \mathbf{H})] = \\
&(\lambda_1' - \lambda_2') \, (\text{trace}[\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T \, \mathbf{H} \, \mathbf{T}] - \text{trace}[\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T \, \mathbf{T} \, \mathbf{H}]) + \lambda_2' \, (\text{trace}[\mathbf{I} \, \mathbf{H} \, \mathbf{T}] - \text{trace}[\mathbf{I} \, \mathbf{T} \, \mathbf{H}]) = \\
&(\lambda_1' - \lambda_2') \, (\text{trace}[\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T \, \mathbf{H} \, \mathbf{T}] - \text{trace}[\mathbf{T} \, \hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T \, \mathbf{H}]) + \lambda_2' \, (\text{trace}[\mathbf{H} \, \mathbf{T}] - \text{trace}[\mathbf{T} \, \mathbf{H}]) = \\
&(\lambda_1' - \lambda_2') \, (\text{trace}[\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T \, \mathbf{H} \, \mathbf{T}] - \text{trace}[\hat{\mathbf{e}}_1 \, \hat{\mathbf{e}}_1^T \, \mathbf{H} \, \mathbf{T}]) + \lambda_2' \, (\text{trace}[\mathbf{H} \, \mathbf{T}] - \text{trace}[\mathbf{H} \, \mathbf{T}]) = 0
\end{aligned} \tag{22}$$

Hence, ∇f is invariant to variations of the two eigenvalues. This includes variation of the norm of $\mathbf{T}$ as well as transitions between the isotropic to the anisotropic states.

It should be noted that this method is based on describing local variation of orientation by means of operators that transform the orientation tensor, as opposed to methods that consider the content of the image itself. Hence, it does not matter if the image, locally, looks like a curve segment or as radial pattern, as illustrated in Figure 1. Both cases will give the same curvature

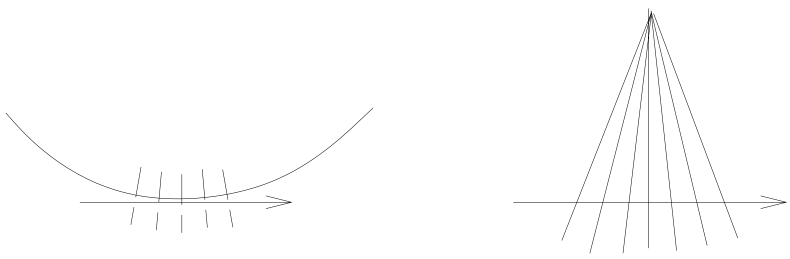

Figure 1: Left: The direction of ∇f for a curve. Right: The direction of ∇f for a radial pattern.

descriptor, and which case it is can be determined by comparing ∇f with the dominant eigenvector of $\mathbf{T}$. For example, the two coefficients c_1 and c_2, given by

$$c_1 = \frac{< \nabla f \nabla^T f | - \mathbf{H}\,\mathbf{T}\,\mathbf{H} >}{|\nabla f \nabla^T f|\; |\mathbf{H}\,\mathbf{T}\mathbf{H}|} = -\frac{\nabla^T f\, \mathbf{H}\,\mathbf{T}\,\mathbf{H}\, \nabla f}{|\nabla f|^2\; |\mathbf{T}|} \tag{23}$$

and

$$c_2 = \frac{< \nabla f \nabla^T f | \mathbf{T} >}{|\nabla f \nabla^T f|\; |\mathbf{T}|} = \frac{\nabla^T f\; \mathbf{T}\, \nabla f}{|\nabla f|^2\; |\mathbf{T}|} \tag{24}$$

are such that

$$c_1 + c_2 = \frac{\text{trace}\mathbf{T}}{|\mathbf{T}|} \tag{25}$$

and $c_1 = 1$ for the case that the variation of is caused by a curved segment, and $c_2 = 1$ for the case that the pattern instead is radial.

3 Results

The algorithm has been tested on two-dimensional images. Each image image is first input to an algorithm that estimates the local orientation and represents it by an orientation tensor. The result is an orientation tensor which is a function of the image coordinates, or a tensor valued image. This algorithm is described by Knutsson, [2]. This tensor field is then subject to differentiation with respect to the image coordinates, corresponding to estimation of the local gradient for each component of the tensor. Since the orientation tensor is invariant to phase, we can assume that the tensor field varies significantly slower than the original image itself. The gradient should therefore in general be quite well-defined and even quite crude gradient estimators will do the job. Here, we have used gradient filters that are defined from the first order term of a Taylor series, using a technique called normalized convolution, see Westin [4].

The first image, Figure 2 upper left, contains a circular pattern for which the local orientation tensor is described by Equation (2). Upper right defines a representation of directions in terms of colour. Using this representation, the lower left figure shows the direction of the estimated gradient ∇f. As see, the gradient points along the curves and in the direction corresponding to an anti-clockwise rotation of the local orientation. The norm of the gradient should be the inverse of the radius of curvature. The lower right figure is a plot taken along a horizontal line right through the centre of the image and shows the norm of ∇f times r. As seen, this product is close to unit, at least when the curvature is not to high and when then not close to the borders of the image.

The second image, Figure 3 upper left, shows two patterns, both with the same type of orientation variation. The local orientation changes in the same way for both the outer circular and the inner radial pattern, the only difference is that the underlying local orientation is rotated $\frac{\pi}{2}$ between the two patterns. Upper right shows the direction of ∇f, which is the same for both patterns. To distinguish between the pattern, the coefficients c_1 and c_2 are calculated, and displayed in the lower left and lower right images, respectively. As shown, the two coefficients discriminate the patters.

4 Summary

We have presented a method for obtaining a curvature descriptor, in the shape of a vector that points in the direction of anti-clockwise rotation of local orientation and with a norm that is the inverse of the radius of curvature. The descriptor is calculated by simple operations on the tensor field, and on the gradient of the tensor field. The descriptor is invariant to all other variations of the tensor field. We have also defined coefficients c_1 and c_2 that describe the type of underlying structure relative to the change of orientation, allowing discrimination between curves and radial patterns.

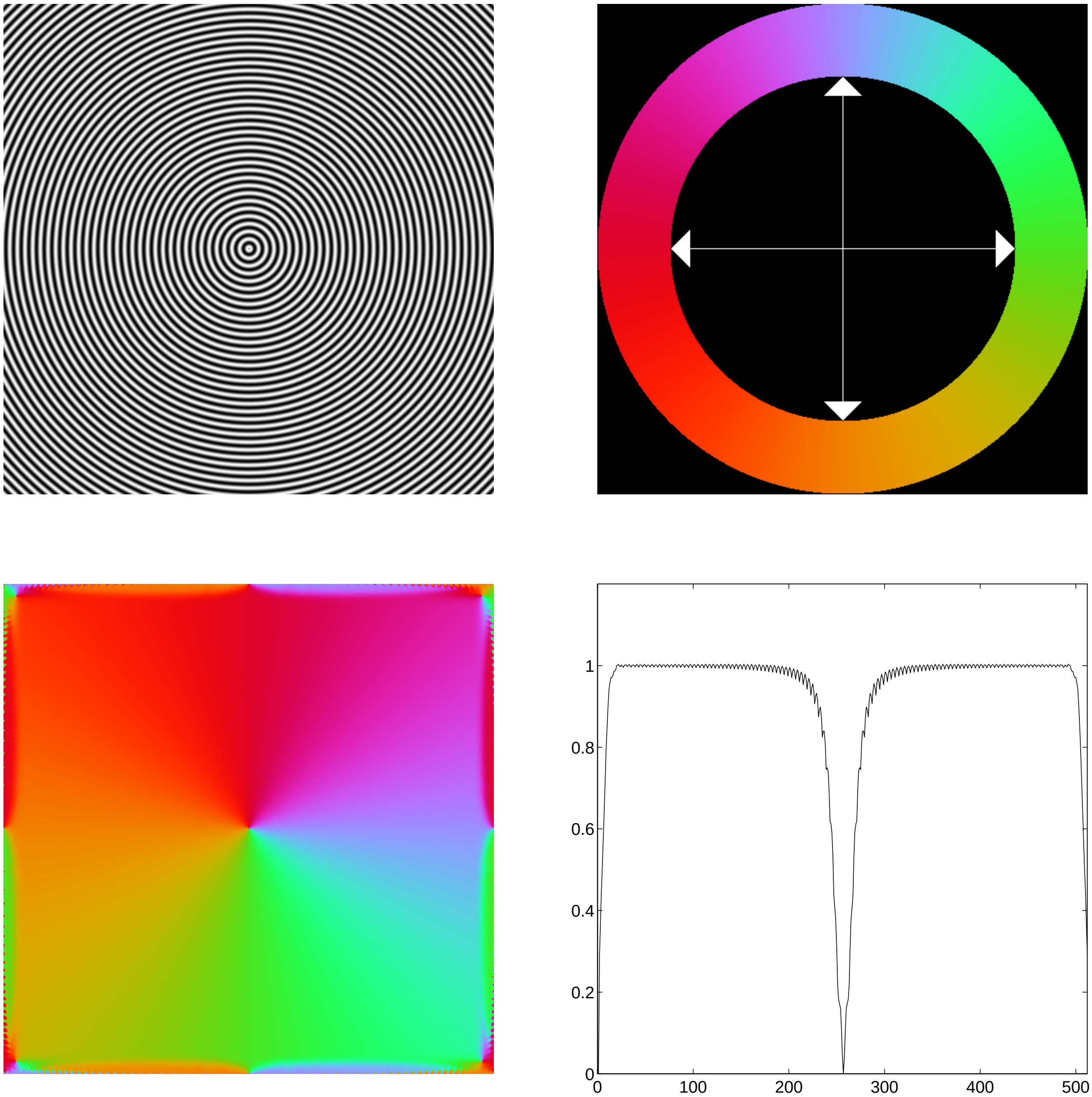

Figure 2: Upper left: A test image with a circular pattern. Upper right: A colour representation of directions. Lower left: The direction of ∇f for the test pattern. Lower right: The norm of $\nabla f \cdot r$ for a horizontal line across the test image and through the origin.

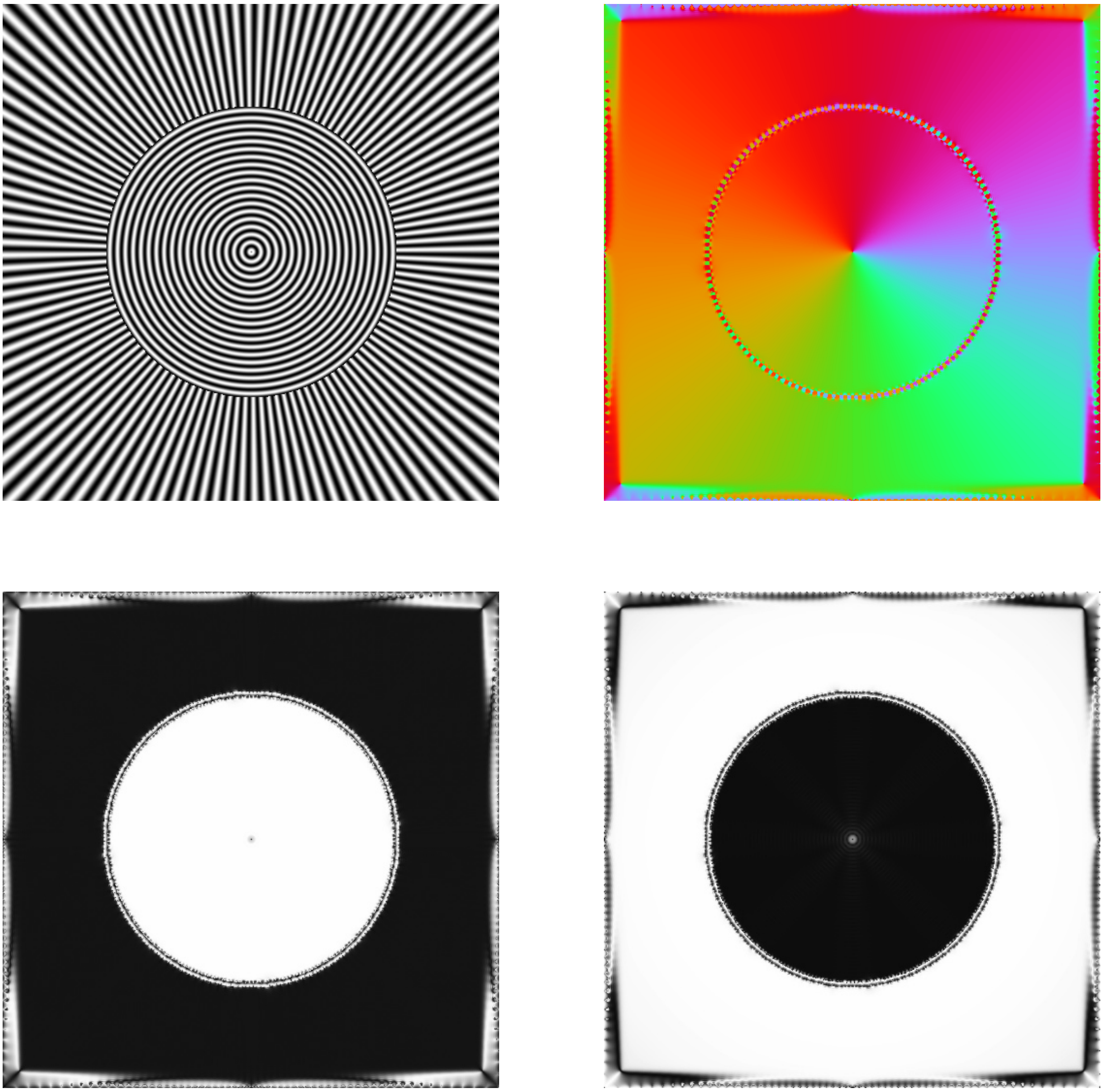

Figure 3: Upper left: A test image with both a circular and a radial pattern. Upper right: The direction of ∇f for the test pattern. Lower left: The coefficient c_1. Lower right: The coefficient c_2.

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