The GET Operator

Michael Felsberg*
Linköping University, Computer Vision Laboratory,
SE-58183 Linköping, Sweden,
mfe@isy.liu.se, http://www.isy.liu.se/cvl/

October 4, 2004

LiTH-ISY-R-2633

Abstract

In this paper we propose a new operator which combines advantages of monogenic scale-space and Gaussian scale-space, of the monogenic signal and the structure tensor. The gradient energy tensor (GET) defined in this paper is based on Gaussian derivatives up to third order using different scales. These filters are commonly available, separable, and have an optimal uncertainty. The response of this new operator can be used like the monogenic signal to estimate the local amplitude, the local phase, and the local orientation of an image, but it also allows to measure the coherence of image regions as in the case of the structure tensor.

1 Introduction

In this paper we define a way to compute features of the monogenic scale-space [6] of an image from Gaussian derivatives. The advantages of the proposed method are:

- Many people have implementations of Gaussian derivatives available so that they can use monogenic features without implementing new basis filters.
- The Gaussian derivatives are separable and decay faster than the Poisson filter and its Riesz transform resulting in more efficient computational schemes.
- The additional feature (coherence) of the derivative-based method directly indicates the validity of the monogenic phase model which is based on the assumption of locally 1D signals.

A key assumption of this paper is of course that the local phase is useful for the processing and analysis of images. Although most of the discussions focus on images, the reflections about phase based signal processing generalize to signals of arbitrary dimension.

*This work has been supported by EC Grant IST-2002-002013 MATRIS and EC Grant IST-2003-004176 COSPAL.

1.1 The 2D Energy Tensor

For continuous, 2D bandpass signals $b(\mathbf{x})$, the 2D energy tensor is defined as [4]

$$\Psi_c[b(\mathbf{x})] = [\nabla b(\mathbf{x})][\nabla b(\mathbf{x})]^T - b(\mathbf{x})[\mathbf{H}b(\mathbf{x})] \ , \quad (1)$$

where $\nabla = (\partial_x, \partial_y)^T$ indicates the gradient and $\mathbf{H} = \nabla\nabla^T$ indicates the Hessian. Switching to the Fourier domain, this equals

$$\int \Psi_c[b(\mathbf{x})] \exp(-i2\pi\mathbf{u}^T\mathbf{x})\, d\mathbf{x} = 4\pi^2\{-[\mathbf{u}B(\mathbf{u})]*[\mathbf{u}B(\mathbf{u})]^T + B(\mathbf{u})*[\mathbf{u}\mathbf{u}^T B(\mathbf{u})]\} \ , \quad (2)$$

where $B(\mathbf{u})$ ($\mathbf{u} = (u, v)^T$) is the 2D Fourier transform of $b(\mathbf{x})$. If the local signal is approximated by an impulse spectrum $B(\mathbf{u}) = A\delta(\mathbf{u} - \mathbf{u}_0) + \bar{A}\delta(\mathbf{u} + \mathbf{u}_0)$, the left part of (2), i.e., the structure / orientation tensor according to [7, 1] (but without spatial averaging), yields

$$-[\mathbf{u}B(\mathbf{u})]*[\mathbf{u}B(\mathbf{u})]^T = -\mathbf{u}_0\mathbf{u}_0^T(A^2\delta(\mathbf{u}-2\mathbf{u}_0) - 2A\bar{A}\delta(\mathbf{u}) + \bar{A}^2\delta(\mathbf{u}+2\mathbf{u}_0)) \ . \quad (3)$$

The right part of (2) gives the same expression, but with a positive sign for the second term,

$$B(\mathbf{u}) * [\mathbf{u}\mathbf{u}^T B(\mathbf{u})] = \mathbf{u}_0\mathbf{u}_0^T(A^2\delta(\mathbf{u} - 2\mathbf{u}_0) + 2A\bar{A}\delta(\mathbf{u}) + \bar{A}^2\delta(\mathbf{u} + 2\mathbf{u}_0)) \ , \quad (4)$$

such that

$$\int \Psi_c[b(\mathbf{x})] \exp(-i2\pi\mathbf{u}^T\mathbf{x})\, d\mathbf{x} = 16\pi^2\mathbf{u}_0\mathbf{u}_0^T|A|^2\delta(\mathbf{u}) \ . \quad (5)$$

The energy tensor is a second order symmetric tensor like the structure tensor. The latter is included in the energy operator, but it is combined with a product of even filters, which assures the phase invariance as it can be seen in (5). The energy tensor can hence be classified as a phase invariant, orientation equivariant second order tensor [13]. Same as the 2D structure tensor, the energy operator can be converted into a complex double angle orientation descriptor [2]:

$$o(\mathbf{x}) = \Psi_c[b(\mathbf{x})]_{11} - \Psi_c[b(\mathbf{x})]_{22} + i2\Psi_c[b(\mathbf{x})]_{12} \quad (6)$$

which is equivalent to the 2D energy operator defined in [12]. As one can easily show, $|o(\mathbf{x})| = \lambda_1(\mathbf{x}) - \lambda_2(\mathbf{x})$, where $\lambda_1(\mathbf{x}) > \lambda_2(\mathbf{x})$ are the eigenvalues of the energy tensor. Since the trace of the tensor is given by the sum of eigenvalues, we obtain $2\lambda_{1,2} = \mathrm{tr}(\Psi_c[b(\mathbf{x})]) \pm |o(\mathbf{x})|$, which can be subject to the same analysis in terms of coherence as suggested in [10, 8] or for the Harris detector [9]. However, a minor problem might occur in the case of not well defined local frequencies: the second term in (1), i.e., the tensor based on even filters, can become positive, corresponding to reduced or negative eigenvalues of the energy tensor. Furthermore, the operator (1) cannot be applied directly since natural images $I(\mathbf{x})$ are typically no bandpass signals $b(\mathbf{x})$. For these reasons and in order to compute the derivatives for discrete data, the operator has to be regularized. This regularization gives rise to the GET operator, which is defined in the subsequent section.

2 The GET Operator

As pointed out above, the energy tensor needs to be regularized. For this purpose, we prefer Gaussian functions due to their high localization in both domains. However, Gaussian filters are not DC-free, which is a central requirement in context of the energy tensor. If we consider a difference of Gaussian filters as in [4], we implicitly lift the level of differentiation by two. According to the equation of linear diffusion [11], the scale derivative of a Gaussian filter is equivalent to the Laplacian of the Gaussian, i.e., a combination of second order derivatives. Applying the Hessian to the Laplacian of the Gaussian means to consider fourth order derivatives instead of second order derivatives. The operator that we propose below is a compromise between these two cases: it makes use of Gaussian derivatives up to order three, but avoids the zeroth order Gaussian, i.e., the DC-component is removed.

2.1 The Gradient Energy Tensor

The idea to define the gradient energy tensor (GET) is straightforward after the previous considerations. We introduce the tensor in three steps. First, we plug the gradient of the image into (1) and use tensor notation instead of matrix notation:

$$\begin{aligned}
\mathrm{GET}\,\{I(\mathbf{x})\} &= \Psi_c[\nabla I(\mathbf{x})] \\
&= [\nabla\otimes\nabla I(\mathbf{x})]\otimes[\nabla\otimes\nabla I(\mathbf{x})] \\
&\quad -\frac{1}{2}([\nabla I(\mathbf{x})]\otimes[\nabla\otimes\nabla\otimes\nabla I(\mathbf{x})]+[\nabla\otimes\nabla\otimes\nabla I(\mathbf{x})]\otimes[\nabla I(\mathbf{x})])
\end{aligned} \tag{7}$$

where we symmetrized the tensor by replacing the second term by the corresponding anticommutator term[1]. The obtained operator has 16 coefficients, where 6 can be omitted due to symmetry and one further coefficient is a linear combination of two others. Hence, 9 independent coefficients are left.

In a second step, we contract the tensor. This becomes possible, since there is no gain from the coefficients that are omitted in the contraction:

$$\begin{aligned}
\mathrm{GET}\,\{I(\mathbf{x})\} &= [\nabla\otimes\nabla I(\mathbf{x})]\cdot[\nabla\otimes\nabla I(\mathbf{x})] \\
&\quad -\frac{1}{2}([\nabla I(\mathbf{x})]\otimes[\nabla\cdot\nabla\otimes\nabla I(\mathbf{x})]+[\nabla\otimes\nabla\cdot\nabla I(\mathbf{x})]\otimes[\nabla I(\mathbf{x})]) \\
&= [\mathbf{H}I(\mathbf{x})][\mathbf{H}I(\mathbf{x})]-\frac{[\nabla I(\mathbf{x})][\nabla\Delta I(\mathbf{x})]^T+[\nabla\Delta I(\mathbf{x})][\nabla I(\mathbf{x})]^T}{2}
\end{aligned} \tag{8}$$

In this formula $\Delta=\nabla^T\nabla$ denotes the Laplacian. To understand why there is no gain in preserving the other coefficients, one has to consider certain different cases. Assuming that $I(\mathbf{x})=\cos(u\,x+v\,y+\phi)$, we obtain for the full tensor

$$\mathrm{GET}\,\{I(\mathbf{x})\}=\begin{bmatrix}\begin{bmatrix}u^4 & u^3 v\\ u^3 v & u^2 v^2\end{bmatrix} & \begin{bmatrix}u^3 v & u^2 v^2\\ u^2 v^2 & u v^3\end{bmatrix}\\ \begin{bmatrix}u^3 v & u^2 v^2\\ u^2 v^2 & u v^3\end{bmatrix} & \begin{bmatrix}u^2 v^2 & u v^3\\ u v^3 & v^4\end{bmatrix}\end{bmatrix}=\begin{bmatrix}u^2\begin{bmatrix}u^2 & u v\\ u v & v^2\end{bmatrix} & u v\begin{bmatrix}u^2 & u v\\ u v & v^2\end{bmatrix}\\ u v\begin{bmatrix}u^2 & u v\\ u v & v^2\end{bmatrix} & v^2\begin{bmatrix}u^2 & u v\\ u v & v^2\end{bmatrix}\end{bmatrix}$$

[1] We appreciate this suggestion given by Ullrich Köthe.

and for the contracted tensor

$$\mathrm{GET}\left\{I(\mathbf{x})\right\} = \begin{bmatrix} u^2\left(u^2+v^2\right) & u\,v\left(u^2+v^2\right) \\ u\,v\left(u^2+v^2\right) & v^2\left(u^2+v^2\right) \end{bmatrix} \quad . \tag{9}$$

Hence, no information is lost by the contraction under the assumed signal model. If we extend the model to two different frequencies in the same direction, the tensor coefficients are multiplied by a modulation factor. However, this modulation is the same for all coefficients, and therefore, the full tensor does not provide additional information. By repeating this procedure for more frequencies in the same direction, the result will always be the same, and hence, of locally 1D signals there is no gain from the full tensor.

If we assume the signal to contain two perpendicular oscillations, i.e., $I(\mathbf{x}) = \cos(u\,x + v\,y + \phi) + \cos(l\,v\,x - l\,u\,y + \psi)$, the contracted tensor shows no modulations:

$$\mathrm{GET}\left\{I(\mathbf{x})\right\} = \begin{bmatrix} u^4 + \left(1+l^4\right)u^2\,v^2 + l^4\,v^4 & -\left(\left(-1+l^4\right)u\,v\left(u^2+v^2\right)\right) \\ -\left(\left(-1+l^4\right)u\,v\left(u^2+v^2\right)\right) & \left(u^2+v^2\right)\left(l^4\,u^2+v^2\right) \end{bmatrix} \quad , \tag{10}$$

but the full tensor does. Performing the contraction, the modulations exactly compensate each other, which is a final and very strong argument to use the tensor in its contracted form. As it can be seen from this example, the 2×2 tensor obtained from (8) allows to estimate two perpendicular oscillations with different frequencies at the same time, i.e., it covers the same model as the structure multivector in [3].

2.2 Regularization and Gaussian Derivatives

The results from the previous section are obtained for idealized, continuous signals. In practice, however, we have to deal with non-ideal, noisy, and discrete signals. The most common thing to do is therefore to regularize the derivative operators from (8) with Gaussian kernels. A Gaussian regularization is the optimal choice if nothing is known about the signal and its noise characteristic. Therefore, we replace the derivatives in (8) with Gaussian derivatives of order one to three.

2.3 Extraction of Monogenic Features

The monogenic signal provides three features: local amplitude, local phase, and local orientation [5]. In case signals with intrinsic dimensionality one, i.e., $I(\mathbf{x}) = s(\mathbf{n}^T\mathbf{x})$ ($s : \mathrm{R} \to \mathrm{R}$, $|\mathbf{n}| = 1$), the GET is of rank one:

$$\begin{aligned} \mathrm{GET}\left\{I(\mathbf{x})\right\} &= [\mathbf{n}\mathbf{n}^T\ddot{s}(\mathbf{n}^T\mathbf{x})][\mathbf{n}\mathbf{n}^T\ddot{s}(\mathbf{n}^T\mathbf{x})] \\ &\quad - \frac{[\mathbf{n}\dot{s}(\mathbf{n}^T\mathbf{x})][\mathbf{n}\dddot{s}(\mathbf{n}^T\mathbf{x})]^T + [\mathbf{n}\dddot{s}(\mathbf{n}^T\mathbf{x})][\mathbf{n}\dot{s}(\mathbf{n}^T\mathbf{x})]^T}{2} \\ &= \mathbf{n}\mathbf{n}^T[\ddot{s}(\mathbf{n}^T\mathbf{x})^2 - \dddot{s}(\mathbf{n}^T\mathbf{x})\dot{s}(\mathbf{n}^T\mathbf{x})] \quad . \end{aligned}$$

The first eigenvector of this expression is $\pm\mathbf{n}$, i.e., the local orientation of the signal. The first eigenvalue (or its trace, aka the second eigenvalue is zero) of the GET is more difficult to analyze, except for the single-frequency case, where we obtain according to (9) $16\pi^4|\mathbf{u}|^4A^2$ for an oscillation with amplitude A.

Much more interesting is the extraction of the local phase, which is obtained in two steps. First, we consider the two addends of the GET separately. The first one represents the symmetric (even) parts of the signal, whereas the second one represents the antisymmetric (odd) parts of the signal. However, both parts are quadratic expressions, such that we have to consider their square-roots:

$$q_{\text{even}} = \pm\sqrt{\text{trace}(\mathbf{T}_{\text{even}})} \qquad \text{and} \qquad q_{\text{odd}} = \pm\sqrt{\text{trace}(\mathbf{T}_{\text{odd}})} \qquad \text{where}$$

$$\mathbf{T}_{\text{even}} = [\mathbf{H}I(\mathbf{x})][\mathbf{H}I(\mathbf{x})] \qquad \text{and} \tag{11}$$

$$\mathbf{T}_{\text{odd}} = -\frac{[\nabla I(\mathbf{x})][\nabla\Delta I(\mathbf{x})]^T + [\nabla\Delta I(\mathbf{x})][\nabla I(\mathbf{x})]^T}{2} \quad . \tag{12}$$

In a second step, the correct signs for both parts are selected, such that $\arg(q_{\text{even}} + iq_{\text{odd}})$ gives the local phase of the signal. A careful comparison of the signs in different quadrant results in the following procedure. Let $\mathbf{T} = \mathbf{T}_{\text{even}} + \mathbf{T}_{\text{odd}}$ denote the GET response, $z = \mathbf{T}_{11} - \mathbf{T}_{22} + i2\mathbf{T}_{12}$ its complex double angle orientation representation [2], and $\mathbf{o} = (\text{real}(\sqrt{z}), \text{imag}(\sqrt{z}))^T$ the orientation vector. We then define the two signs as

$$s_{\text{even}} = -\text{sign}\,(\mathbf{o}^T[\mathbf{H}I(\mathbf{x})]\mathbf{o}) \qquad \text{and} \qquad s_{\text{odd}} = -\text{sign}\,(\mathbf{o}^T\nabla I(\mathbf{x})) \tag{13}$$

such that

$$\varphi = \arg(q_{\text{even}} + iq_{\text{odd}}) = \arg(s_{\text{even}}\sqrt{\text{trace}(\mathbf{T}_{\text{even}})} + is_{\text{odd}}\sqrt{\text{trace}(\mathbf{T}_{\text{odd}})}). \tag{14}$$

If the underlying signal is non-simple, i.e., it has intrinsic dimensionality two, the analysis becomes more difficult. Following the strategy of the structure multivector in [3], the first eigenvector is extracted from $\mathbf{T}$. Then, the even tensor and the odd tensor are projected onto the first eigenvector and onto the orthogonal vector (aka the second eigenvector). This gives two even components and two odd components, which are then combined with appropriate signs to extract two phases for the two perpendicular orientations.

Note also that in the latter case not a single amplitude is obtained, but two eigenvalues, which correspond to the local amplitudes of the two perpendicular components. These eigenvalues can then be used for coherence analysis or corner detection likewise the eigenvalues of the structure tensor.

3 Conclusion

In this paper we have described an alternative way of extracting the image features of the monogenic signal, i.e., local amplitude, local phase, and local orientation, by using a quadratic form. The proposed method of the gradient energy tensor is the contraction of a fourth order tensor built from image derivatives of order one to three. The new tensor is compatible to the structure tensor concerning eigensystem analysis, but it is phaseinvariant without spatial averaging. Using Gaussian regularization of the derivatives leads to a connection of monogenic scale-space and Gaussian scale-space via the quadratic form. We provided formulas to extract the local phase from the two different parts of the GET. For non-simple signals, it even provides the two additional features of second eigenvalue and second phase, which makes it comparable to the much slower structure multivector.

References

[1] Bigün, J., and Granlund, G. H. Optimal orientation detection of linear symmetry. In *Proceedings of the IEEE First International Conference on Computer Vision* (London, Great Britain, June 1987), pp. 433–438. Report LiTH-ISY-I-0828, Computer Vision Laboratory, Linköping University, Sweden, 1986.

[2] Bigün, J., Granlund, G. H., and Wiklund, J. Multidimensional orientation estimation with applications to texture analysis and optical flow. *IEEE Transactions on Pattern Analysis and Machine Intelligence 13*, 8 (August 1991), 775–790.

[3] Felsberg, M. *Low-Level Image Processing with the Structure Multivector.* PhD thesis, Institute of Computer Science and Applied Mathematics, Christian-Albrechts-University of Kiel, 2002. TR no. 0203, available at `http://www.informatik.uni-kiel.de/reports/2002/0203.html`.

[4] Felsberg, M., and Granlund, G. POI detection using channel clustering and the 2D energy tensor. In *26. DAGM Symposium Mustererkennung, Tübingen* (2004). accepted.

[5] Felsberg, M., and Sommer, G. The monogenic signal. *IEEE Transactions on Signal Processing 49*, 12 (December 2001), 3136–3144.

[6] Felsberg, M., and Sommer, G. The monogenic scale-space: A unifying approach to phase-based image processing in scale-space. *Journal of Mathematical Imaging and Vision 21* (2004), 5–26.

[7] Förstner, W., and Gülch, E. A fast operator for detection and precise location of distinct points, corners and centres of circular features. In *ISPRS Intercommission Workshop, Interlaken* (June 1987), pp. 149–155.

[8] Granlund, G. H., and Knutsson, H. *Signal Processing for Computer Vision.* Kluwer Academic Publishers, Dordrecht, 1995.

[9] Harris, C. G., and Stephens, M. A combined corner and edge detector. In *4th Alvey Vision Conference* (1988), pp. 147–151.

[10] Jähne, B. *Digitale Bildverarbeitung.* Springer, Berlin, 1997.

[11] Koenderink, J. J. The structure of images. *Biological Cybernetics 50* (1984), 363–370.

[12] Larkin, K. G., Oldfield, M. A., and Bone, D. J. Demodulation and phase estimation of two-dimensional patterns. Australian patent AU 200110005 A1, 2001.

[13] Nordberg, K. *Signal Representation and Processing using Operator Groups.* PhD thesis, Linköping University, Sweden, SE-581 83 Linköping, Sweden, 1995. Dissertation No 366, ISBN 91-7871-476-1.

[14] Nordberg, K., and Farnebäck, G. A framework for estimation of orientation and velocity. In *Proceedings of the International Conference on Image Processing 2003* (Barcelona, Spain, 2003).